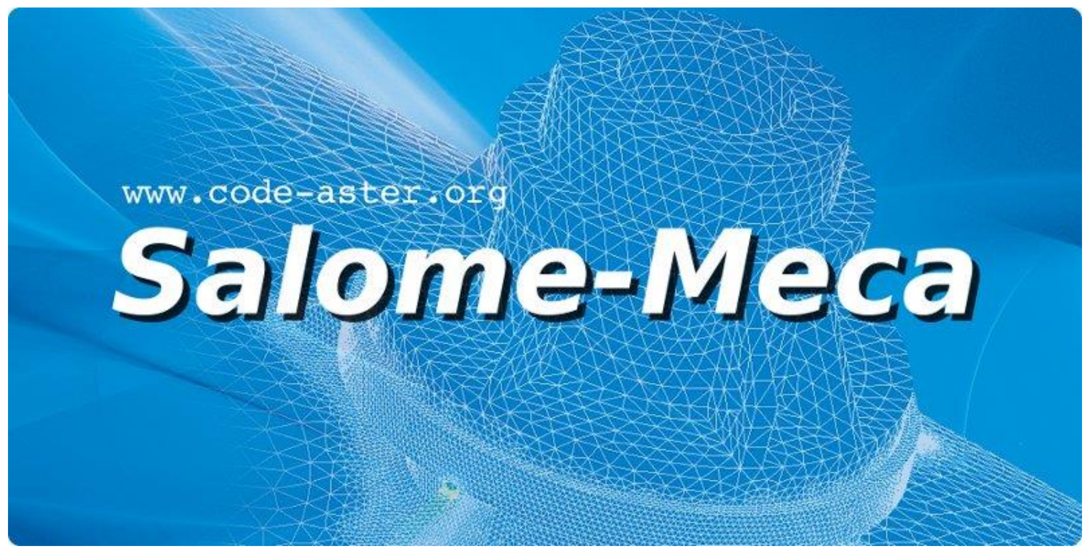
# Limit analysis




**Code_Aster, Salome-Meca course material**

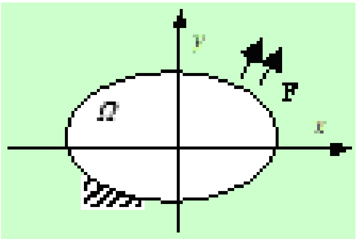
# Outline

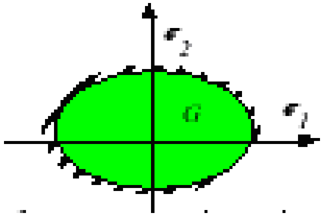
- Purpose of the limit analysis
- Theory
- Usage in *Code_Aster*
- Example
- Documentation

# Purpose of the limit analysis

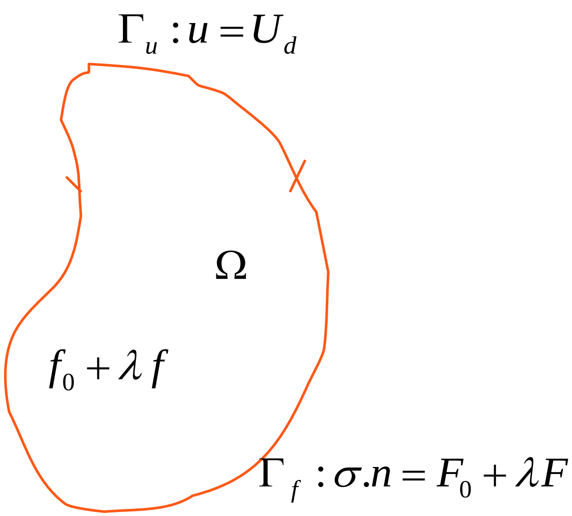
- The objectives of the limit analysis include :
  - the safety study of a structure subjected to severe loads (Ultimate Limit State, ULS) ;
  - the rapid estimation of a limit load without describing in-depth the failure process ;
  - the understanding of the energetic process of the failure and its mechanisms ;
  - the simple characterization of the non-linear behaviour of the structure.

- Details about the structure are required :
  - its exact geometry and measures ;
  - the direction of the loads ;
  - a yield limit for the material.



# Theory




## Loads

- Fixed loads : F0, f0
- Limit loads (continuation) : F, f

## Resistance criteria

- Von Mises criteria for a rigid-plastic material

$$g(\sigma) = J(\sigma) - \sigma_y = \sqrt{3/2}.\sqrt{\sigma^D.\sigma^D} - \sigma_y$$

$$= \frac{\sqrt{2}}{2}.\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2} - \sigma_y$$

$\sigma_y$ Yield limit

$\sigma^D$ Deviatoric stress

## Definition

- The limit load of the structure is the highest value satisfying the following equation :

$$\lambda_{\lim} = \sup_{\lambda \geq 0} \lambda \quad \backslash \quad g(F_0 + \lambda F, f_0 + \lambda f) = 0$$

# Limit load estimation (1)

- There are two ways to the limit load estimation :
  - Static approach : internal estimation (infimum) arising from statically admissible stress fields
  - Kinematic approach : external estimation (supremum) arising from kinematically admissible displacement fields
    - *This is the preferred approach in Code_Aster since it is more suitable in a finite element framework*

- Definitions
  - Normalized space for kinematically admissible displacement fields

$$\mathcal{V}_a^1 = \left\{ \text{v admissible, } \text{v} = 0 \text{ sur } \Gamma_u \text{ , } \mathcal{L}(\text{v}) = \int_\Omega \text{f.v}\, d\Omega + \int_{\Gamma_f} \text{F.v}\, ds = 1 \right\}$$

$\mathcal{L}_0(\text{v})$ Power of the fixed loads $(F_0 , f_0)$

  - Statically admissible stress fields

$$G_{(\mathbf{x})} = \left\{ \sigma(\mathbf{x}) \text{ , } \text{g}(\sigma(\mathbf{x})) \leq 0 \right\}$$

  - Indicator function

$$\pi(\varepsilon) = \underset{\sigma \in \mathrm{IR}^6}{\text{Sup}} \left[ \sigma . \varepsilon - \Psi_G(\sigma) \right]$$

  - Support function

$$\Psi_G(\sigma(\mathbf{x})) = \begin{cases} 0 \text{ ,} si\, \sigma(\mathbf{x}) \in G_{(\mathbf{x})} \\ +\infty, si\, \sigma(\mathbf{x}) \notin G_{(\mathbf{x})} \end{cases}$$

# Limit load estimation (2)

## Property

- The limit load obtained by the kinematic approach satisfies :

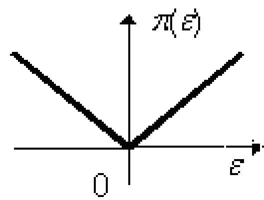
$$\lambda_{\lim} = \operatorname*{Inf}_{\mathbf{v}\in\mathcal{V}_a^1} \mathcal{S}_e(\mathbf{v})$$

*where* $\mathcal{S}_e(\mathbf{v}) = \int_\Omega \pi(\varepsilon(\mathbf{v}))\, d\Omega - \mathcal{L}_0(\mathbf{v})$ *is the maximal resistant power*

## Example (1D)

- Indicator function

$$\pi(\varepsilon) = \operatorname*{Sup}_{\sigma\in\mathrm{IR}^6} \left[\sigma.\varepsilon - \Psi_G(\sigma)\right] = \sigma_y \left|\varepsilon\right|$$




- Power

$$\mathcal{S}_e(v) = \int_\Omega \pi(\varepsilon(v))\, d\Omega = \int_\Omega \sigma_y \left|\varepsilon\right| d\Omega = \sigma_y \frac{\left|v(L)\right|}{L} SL = \sigma_y S \left|v(L)\right|$$

- Kinematically admissible displacement

$$\mathcal{L}(\mathrm{v}) = \int_{\Gamma_f} F.\mathrm{v}\, ds = F.v(L)S \;\Rightarrow\; \text{si } v \in \mathcal{V}_a^1 \text{ alors } v(L) = \frac{1}{FS}$$

- Limit load

$$\lambda_{\lim} = \operatorname*{Inf}_{v\in\mathcal{V}_a^1} \mathcal{S}_e(v) = \operatorname*{Inf}_{v \backslash v(L) = 1/SF} \sigma_y S \left|v(l)\right| = \boxed{\frac{\sigma_y}{\left|F\right|}}$$


Cantilever beam (Cross-section $S$, Young modulus $E$, length $L$) subjected to tension *(no fixed loads)*

Yield limit : $\sigma_y$

Displacement : $v(x) = \frac{F^v}{ES} x$

Strain : $\varepsilon(v) = \frac{F^v}{ES} = \frac{v(L)}{L}$

# Limit load estimation (3)

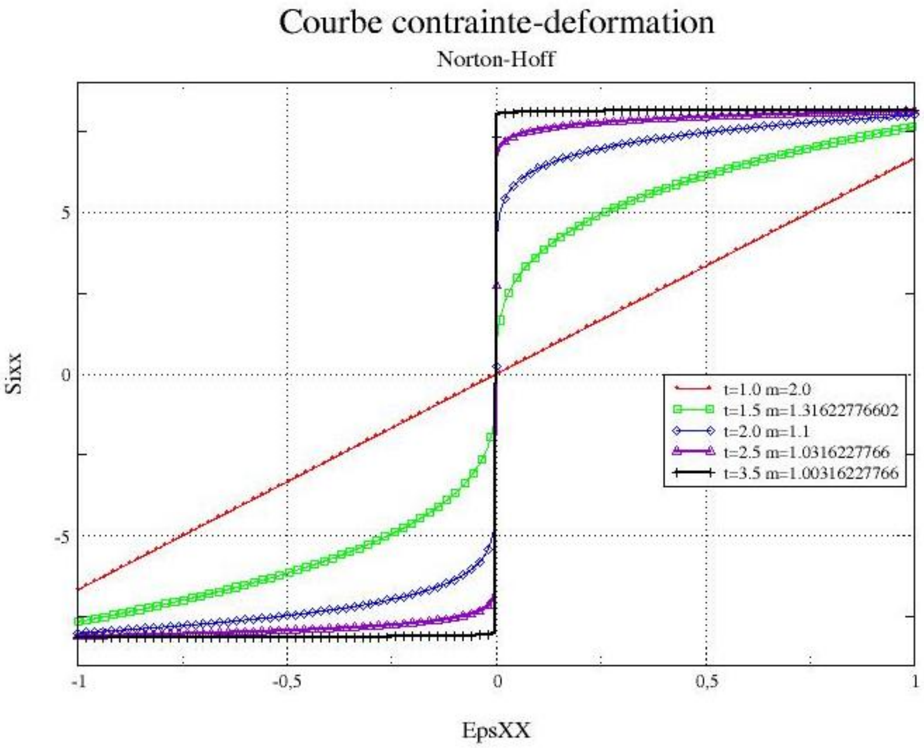
- **Numerical application**
  - The power function is non-differentiable

  ➔ The Norton-Hoff-Friaâ regularization is used
  - One substitutes the indicator function

    with $\pi^{NH}(\varepsilon) = \dfrac{k^{1-m}}{m}(\pi(\varepsilon))^m$

    where $1 \le m \le 2$ is a regularization parameter

    such that $S_e^m(\mathbf{v}) = \int_\Omega \dfrac{k^{1-m}}{m}\pi(\varepsilon(\mathbf{v}))^m \, d\Omega - \mathcal{L}_0(\mathbf{v})$




- **Example (1D)**
  - Where the regularized function $S_e^m(v)$ is differentiable

$$\lambda_{\lim} = \lim_{m \to 1}\left(S_e^m(\mathrm{u}_m)\right)$$

# Limit load estimation (4)

## Upper bound of the solution

- Let $(\lambda_m, \mathrm{u}_m)$ the solution obtained for each calculated instant (that is for each value of the regularization parameter *m*)
- Then the limit load is upper-bounded such that

$$\lambda_{\lim} \leq \hat{\lambda}_m = \int_\Omega \sigma_y \sqrt{\tfrac{2}{3}\varepsilon(\mathrm{u}_m).\varepsilon(\mathrm{u}_m)}\, d\Omega - \mathcal{L}_0(\mathrm{u}_m)$$

- Property : $(\hat{\lambda}_m)$ is a monotonous sequence when *m*➔1 which converges towards $\lambda_{\lim}$

## Lower-bound of the solution

- When there are no fixed loads, the solution may be lower-bounded

$$\underline{\lambda}_m = \int_\Omega \frac{A(m)}{m}.\left(\sqrt{\varepsilon(\mathbf{u}_m).\varepsilon(\mathbf{u}_m)}\right)^m d\Omega \,.\left(\operatorname*{Sup}_{\mathbf{x}\in\Omega}\left(\frac{\sqrt{\frac{3}{2}\sigma^D(\mathbf{u}_m).\sigma^D(\mathbf{u}_m)}}{\sigma_y}\right)\right)^{-1} \leq \hat{\lambda}_m$$

# Practical use in Code_Aster (1)

- Finite element model
  - The limit load analysis is carried out with **finite elements taking into account incompressibility**

```
MO=AFFE_MODELE(MAILLAGE=MA,
               AFFE=_F(TOUT = 'OUI',
                       PHENOMENE = 'MECANIQUE',
                       MODELISATION =
               )
```

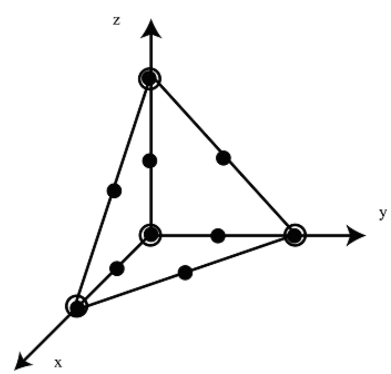
→ 'AXIS_INCO' / 'D_PLAN_INCO' / '3D_INCO'

  - *Note : since the material properties and the loading direction are fixed, the upper-bound obtained in the kinematical approach in plane strain is higher than the exact limit load for plane stress* $\lambda^{+}_{D\_PLAN} \geq \lambda^{\lim}_{C\_PLAN}$

- Mesh
  - Only **second order meshes** support finite elements taking into account incompressibility

● Displacements DX, DY, DZ

○ Pressure PRES and swelling GONF



# Practical use in Code_Aster (2)

## Incompressibility

- A linear relation must be added for the swelling

```
INCOMP = AFFE_CHAR_MECA(MODELE = MO,
                        DDL_IMPO = _F(GROUP_NO = 'SOMMETS',
                                      GONF     = 0)
                       )
```

A group of nodes that may be created with
```
DEFI_GROUP
 CREA_GROUP_NO
  CRIT_NOEUD='SOMMET'
```

## Material properties

- An elastic perfectly plastic material following Von Mises criteria must be assigned

```
MATPLAQ=DEFI_MATERIAU( ECRO_LINE=_F(SY=100,
                                    D_SIGM_EPSI = 0)
                     )
```

## Loadings

- The loads supported in the limit analysis are : forces, surfacic and volumic loads

# Practical use in Code_Aster (3)

## Definition of the pseudo-time stepping

- The instants are used to control the regularization parameter $m$ in Norton law

$$m = 1 + 10^{1-t}$$

| $t$ | 1.0 | 1.5 | 2.0 | 2.5 | 3.0 | $\infty$ |
|---|---|---|---|---|---|---|
| $m = 1 + 10^{1-t}$ | 2.00 | 1.30 | 1.10 | 1.03 | 1.01 | 1 |

- When $m \rightarrow 1$, the material behaviour tends to be rigid and perfectly plastic
- One usually begins with a constant time step between 1 and 2 then gradually increase the instant towards 3 to refine the estimation (convergence is slower)

## Calculation

```
RESU1=STAT_NON_LINE(MODELE = MO,
            CHAM_MATER = CHMAT,
            EXCIT = (_F(CHARGE = CONDLIM),
                     _F(CHARGE = INCOMP),
                     _F(CHARGE = CH1,  TYPE_CHARGE = 'FIXE_PILO'),),
            PILOTAGE = _F(TYPE = 'ANA_LIM'),
            COMP_INCR = _F( RELATION = 'NORTON_HOFF'),
  INCREMENT = _F(LIST_INST = L_INST,SUBD_METHODE='UNIFORME',),)
```

Load to be used for the limit analysis

Continuation method dedicated to limit analysis

Norton-Hoff constitutive law

# Practical use in Code_Aster (4)

- Post-processing
  - The limit load is retrieved in a table using `POST_ELEM` operator

```
ECHL1=POST_ELEM(CHAR_LIMITE=_F(CHAR_CSTE= /'OUI'
                                          /'NON',),
                RESULTAT=RESU1,);
```

→ Specifies whether there were any fixed loads or not

- Example without any fixed load (lower bound is available)

#TABLE_SDASTER

| NUME_ORDRE | INST | CHAR_LIMI_SUP | CHAR_LIMI_ESTIM |
|---|---|---|---|
| 1 | 5.00000E-01 | 1.38675E+01 | 3.33171E+00 |
| 2 | 1.00000E+00 | 1.38675E+01 | 6.93375E+00 |
| 3 | 1.50000E+00 | 1.38675E+01 | 1.05358E+01 |
| 4 | 2.00000E+00 | 1.38675E+01 | 1.26068E+01 |

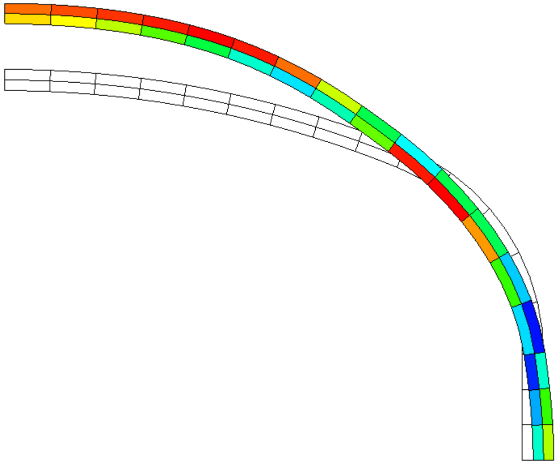
- Example with a fixed load (no lower bound)

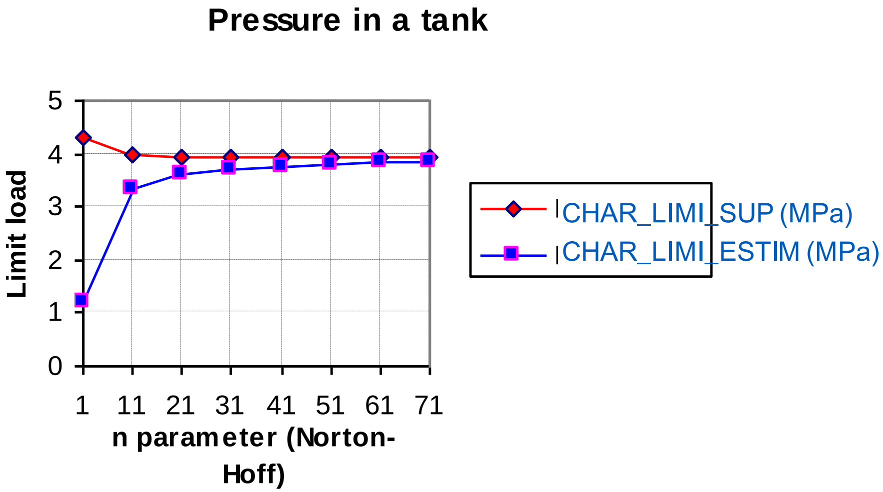
#TABLE_SDASTER

| NUME_ORDRE | INST | CHAR_LIMI_SUP | PUIS_CHAR_CSTE |
|---|---|---|---|
| 1 | 1.00000E+00 | 1.46838E+01 | -2.50000E-01 |
| 2 | 1.69897E+00 | 1.46838E+01 | -2.50000E-01 |

# Example (1)

- Test-case SSNV146
  - Geometry : axisymmetric tank (2mm thickness, 49mm internal radius)
  - Load : internal pressure (1 MPa)
  - Yield limit : 100 MPa






Calculation convergence with n parameter

$$n = \frac{1}{m-1}$$

# Example (2)

## Mesh refinement influence

Initial mesh
(34 QUAD8 elements)

| INST | | CHAR_LIMI_SUP $\hat{\lambda}_m$ | CHAR_LIMI_ESTIM $\underline{\lambda}_m$ | Ecart (%) |
|---|---|---|---|---|
| 1.00000 | $m = 2{,}000$ | 4.30614 | 1.19383 | 72.3 |
| 2.00000 | $m = 1{,}100$ | 3.95811 | 3.27022 | 17.4 |
| 2.20000 | $m = 1{,}0631$ | 3.94001 | 3.48360 | 11.6 |
| 2.30000 | $m = 1{,}0501$ | 3.93413 | 3.56501 | 9.4 |

*No convergence for t>2,3s*

Refined mesh
(136 QUAD8 elements)

| INST | | CHAR_LIMI_SUP $\hat{\lambda}_m$ | CHAR_LIMI_ESTIM $\underline{\lambda}_m$ | Ecart (%) |
|---|---|---|---|---|
| 1.00000 | $m = 2{,}000$ | 4.30541 | 1.14181 | 73.5 |
| 2.00000 | $m = 1{,}100$ | 3.97022 | 3.25097 | 18.1 |
| 2.49136 | $m = 1{,}0322$ | 3.94019 | 3.69005 | 6.3 |
| 2.70757 | $m = 1{,}0196$ | 3.93640 | 3.78280 | 3.9 |
| 2.85126 | $m = 1{,}0141$ | 3.93515 | 3.82449 | 2.8 |

- A mesh refinement may be needed to reach higher *m* values and further improve the convergence between `CHAR_LIMI_SUP` and `CHAR_LIMI_ESTIM`

# Conclusion

- The Norton-Hoff law implemented in Code_Aster enables bounding of the limit load of a structure

- Benchmarks show that a sufficient precision can be reached with this method
  - Results are comparable with incremental elasto-plastic calculations up to failure (using continuation methods such as arc-length)
  - The limit load analysis is easier to apply to a given structure

- The convergence of the bounding estimation is controlled by the time-stepping
  - To improve the precision, one needs to conduct calculations for higher values of the instant (and possibly refine the stepping)

# Documentation

- **Must-read**
  - How-to conduct a limit load analysis [U2.05.04]
- User manual
  - `STAT_NON_LINE` [U4.51.03] and `POST_ELEM` [U4.81.22]
- Theory manual
  - Limit load analysis with Norton-Hoff-Friaâ method [R7.07.01]
- Test-cases
  - SSNV124 [V6.04.124]
  - SSNV146 [V6.04.146]

# End of presentation

Is something missing or unclear in this document?
Or feeling happy to have read such a clear tutorial?

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Do not hesitate to share with us your comments on the Code_Aster forum dedicated thread.