Finite element method isoparametric

Summary:

This document presents the bases of the isoparametric finite elements introduced into *Code_Aster* for the modeling of the continuous mediums 2D and 3D. One first of all recalls the passage of a strong formulation to a variational formulation, then one details the discretization by finite elements: use of an element of reference, calculation of the functions of form and evaluation of the elementary terms. One also briefly describes the principle of the assembly of these terms and the imposition of the boundary conditions, and one evokes the methods of matric resolution used. Finally are exposed the principal stages of a calculation by finite elements such as it are conceived and established in *Code_Aster*.

Contents

1 Introduction

The finite element method is employed in many scientific disciplines to solve partial derivative equations. It makes it possible to build a simple approximation of the unknown factors to transform these continuous equations into a system of equations of finished size, which one can schematically write in the following form:

$$[A].\{U\}=\{L\} \quad (1)$$

where $\{U\}$ is the vector of the unknown factors, $[A]$ a matrix and $\{L\}$ a vector.

Initially, one transforms the partial derivative equations into an integral formulation (or formulation ***strong*** problem), often this first integral form is modified (weakened) by using the formula of Green (one obtains a formulation then ***weak***). The approximate solution is sought like linear combination of functions given. These functions must be simple but enough general to be able "well" to approach the solution. They must in particular make it possible to generate a space of finished size which is as close as one wants space of functions in which the solution is. From this old idea (method of the balanced residues), the various ways of choosing these functions cause various digital methods (collocation, methods spectral, finite elements, etc).

The originality of the finite element method is to take as functions of approximation of the polynomials which are worthless on almost all the field, and thus take part in calculation only in the vicinity of a particular point. Thus, the matrix $[A]$ is very hollow, containing only the terms of interaction between "close points", which reduces the computing time and the place memory necessary to storage. Moreover, the matrix $[A]$ and the vector $\{L\}$ can be built by assembly of matrices and elementary vectors, calculated locally.

2 Obtaining a variational formulation

One can obtain the variational formulation of a problem starting from the partial derivative equations, by multiplying those by functions tests and while integrating by parts. In mechanics of the solids, the weak formulation then obtained is identical to that given by the Principle of Virtual Work and in the conservative case, the minimization of the total potential energy of the structure. Let us note however that for certain problems, the equations of the model are easier to establish within the variational framework (case of the plates and the hulls for example).

2.1 Modeling of the physical problem – Principles and notation

A physical system is generally modelled by partial derivative equations which act on unknown factors $\mathbf{u}$ who can be:

- A scalar like the temperature in the problems of thermics;
- A vector like displacements in the problems of mechanics;
- A tensor like the constraints in the problems of mechanics;

One can also use several fields of unknown factors simultaneously, connected by partial derivative equations. They are problems *coupled*. In Code_Aster, one can quote as example the problems of thermo-hydro-mechanics which couple displacements, pressure and temperature.
The fields of unknown factors are parameterized by:

- The space, which can be described by a Cartesian frame of reference or any other type of parameterization. In the continuation of the document, one it will note x ;
- Time, noted t ;

2.2 Equations of the system

A continuous physical system can be represented by one *system* partial derivative equations which one will write in the field Ω :

$$\mathbf{L}(\mathbf{u})+\mathbf{f}=\mathbf{0} \quad \text{dans} \quad \Omega \quad (2)$$

This system is associated with the boundary conditions on the border Γ field Ω :

$$\mathbf{C}(\mathbf{u})=\mathbf{h} \quad \text{sur} \quad \Gamma=\partial\Omega \tag{3}$$

The differential operator can express himself on several partial derivative equations. ON could write:

$$\begin{aligned} L_1(\mathbf{u})+f_1&=0\\ L_2(\mathbf{u})+f_2&=0\\ &\dots \end{aligned} \tag{4}$$

$L_i(\mathbf{u})$ is a differential operator acting on the vector of the unknown factors $\mathbf{u}$. In a more general way, the differential operator $L_i(\mathbf{u})$ is written according to the partial unknown factors and of their derivative:

$$L_i\left(\mathbf{u},\frac{\partial\mathbf{u}}{\partial x_1},\dots,\frac{\partial^2\mathbf{u}}{\partial x_1.\partial x_2},\dots,\frac{\partial^m\mathbf{u}}{\partial x_\alpha^m},t,\frac{\partial\mathbf{u}}{\partial t},\dots,\frac{\partial^p\mathbf{u}}{\partial t^p},\right) \tag{5}$$

Such an operator is known as of order m in space and of order p in time. If it does not depend on time (and its derivative), it is said that the problem is stationary. In the continuation of the document one will consider only the stationary problems.

2.3 Method of the balanced residues – strong integral Formulation

One will define it residue $\mathbf{R}(\mathbf{u})$ as being quantity cancelling itself when $\mathbf{u}$ is the solution of the physical problem:

$$\mathbf{R}(\mathbf{u})=\mathbf{L}(\mathbf{u})-\mathbf{f}=\mathbf{0} \quad \text{dans} \quad \Omega \tag{6}$$

method of the balanced residues consists:
1/ To build an approximate solution $\mathbf{u}$ by the linear combination of judiciously selected functions

$$\mathbf{u}(\boldsymbol{x})=\sum_{i=1}^{N} c_i.\boldsymbol{\phi}_i(\boldsymbol{x}) \tag{7}$$

Where $\boldsymbol{\phi}_i(\boldsymbol{x})$ are the functions of form of the approximation and c_i coefficients to be identified.
2/ To solve the system in integral form:

$$\begin{aligned} &\text{Trouver} \quad \mathbf{u}\in E_u \quad \text{tel que} \quad \forall\,\mathbf{P}\in E_P\\ &\text{Avec} \quad W=\int_\Omega \mathbf{R}(\mathbf{u}).\mathbf{P}(\mathbf{u}).d\Omega+\int_\Gamma[\mathbf{C}(\mathbf{u})-\mathbf{h}].\mathbf{P}(\mathbf{u}).d\Omega=0 \end{aligned} \tag{8}$$

We used the same weight functions for the principal system and the limiting conditions, but it is not obligatory. $\mathbf{P}(\mathbf{u})$ are the weight functions belonging to a set of functions E_P. The solution $\mathbf{u}$ belongs to space E_u regular functions “sufficiently” (derivable until the order *m*).
The choice of the weight functions $\mathbf{P}(\mathbf{u})$ allows to create several methods:

- If the function $\mathbf{P}(\mathbf{u})$ is a distribution of Dirac, one obtains the method of collocation by points.
- If the function $\mathbf{P}(\mathbf{u})$ is constant on under-fields, one obtains the method of collocation by under-fields.
- If weight functions $\mathbf{P}(\mathbf{u})$ use the same functions of form $\boldsymbol{\phi}_i(\boldsymbol{x})$ that the approximation of the solution (7), one obtains the method of Galerkin.

One obtains thus ***strong integral form***.

2.4 Weak integral formulation

The integral formulation (8) requires derivable spaces of function to the order m for E_u. The weak formulation consists in carrying out an integration by parts (by application of the formula of Green) of the system (8). On the other hand one increases the requirements for regularity on the weight functions $\mathbf{P}(\mathbf{u})$. Lformula of Green has is stated as follows:

$$\int_\Omega \mathbf{u}.\nabla.\mathbf{P}.d\Omega=-\int_\Omega \mathbf{P}.\nabla.\mathbf{u}.d\Omega+\int_\Gamma \mathbf{u}.\mathbf{P}.\mathbf{n}.d\Gamma \tag{9}$$

where $\boldsymbol{n}$ is the outgoing normal at the border Γ field.

3 Finite element method

3.1 Principles generals

The search for a suitable approximate function on all the field becomes difficult in the case general of a geometry of an unspecified form. The idea of the finite element method is thus to build this approximation in two times:

- To identify under-fields Ω_e geometrically simple which paves the field;
- To define a function approached on each under-field;

A certain number of characteristics of this construction are thus had a presentiment of:

- The paving of the field Ω_e by the under-fields Ω_e must be as precise as possible;
- The function approached on the under-field must observe conditions of continuity between the various under-fields;
- The function approached on the under-field must have coherent properties with the conditions of derivability and in keeping with the physical description of the solution (what can imply to use a weakened formulation for example).

3.2 Approximation of the geometry

3.2.1 Principle

They are identified N_e under-fields (or *elements*) Ω_e who pave space Ω solid:

$$\Omega = \sum_{e=1}^{N_e} \Omega_e \tag{10}$$

Let us note $x_{\alpha=1,3}$ punctual coordinates $\boldsymbol{x}$ in the absolute reference mark. The geometry of the under-field is built with a nodal approximation, that is to say for one *element* with N_{nd} nodes:

$$\boldsymbol{x}^e = \sum_{i=1}^{N_{nd}} \boldsymbol{x}_i^e . \bar{\boldsymbol{N}}_i^e \text{ or } x_\alpha^e = \sum_{i=1}^{N_{nd}} x_{\alpha,i}^e . \bar{N}_i^e \tag{11}$$

This paving (grid) is an operation being able to be complex, especially in 3D. There exist algorithms generals to net. Triangles or quadrangles in 2D and tetrahedrons or hexahedrons in 3D are used (more some elements being used as connections). The triangles and tetrahedrons give grids what is called *free* , the quadrangles and the hexahedrons form grids *regulated* . The free grids are relatively easy to build thanks to largely tested techniques: cells of Voronoï building a triangulation of Delaunay or methods of propagation (methods known as frontal), the regulated grids are much more delicate to generate. The grid induces necessarily a geometrical error of discretization For example, on the figure (1), it is seen that a curved border only is imperfectly approached by linear elements.

Illustration 1: Geometrical error of discretization

In the same way the grid must be in conformity: no holes or of covering (see figure (2)).

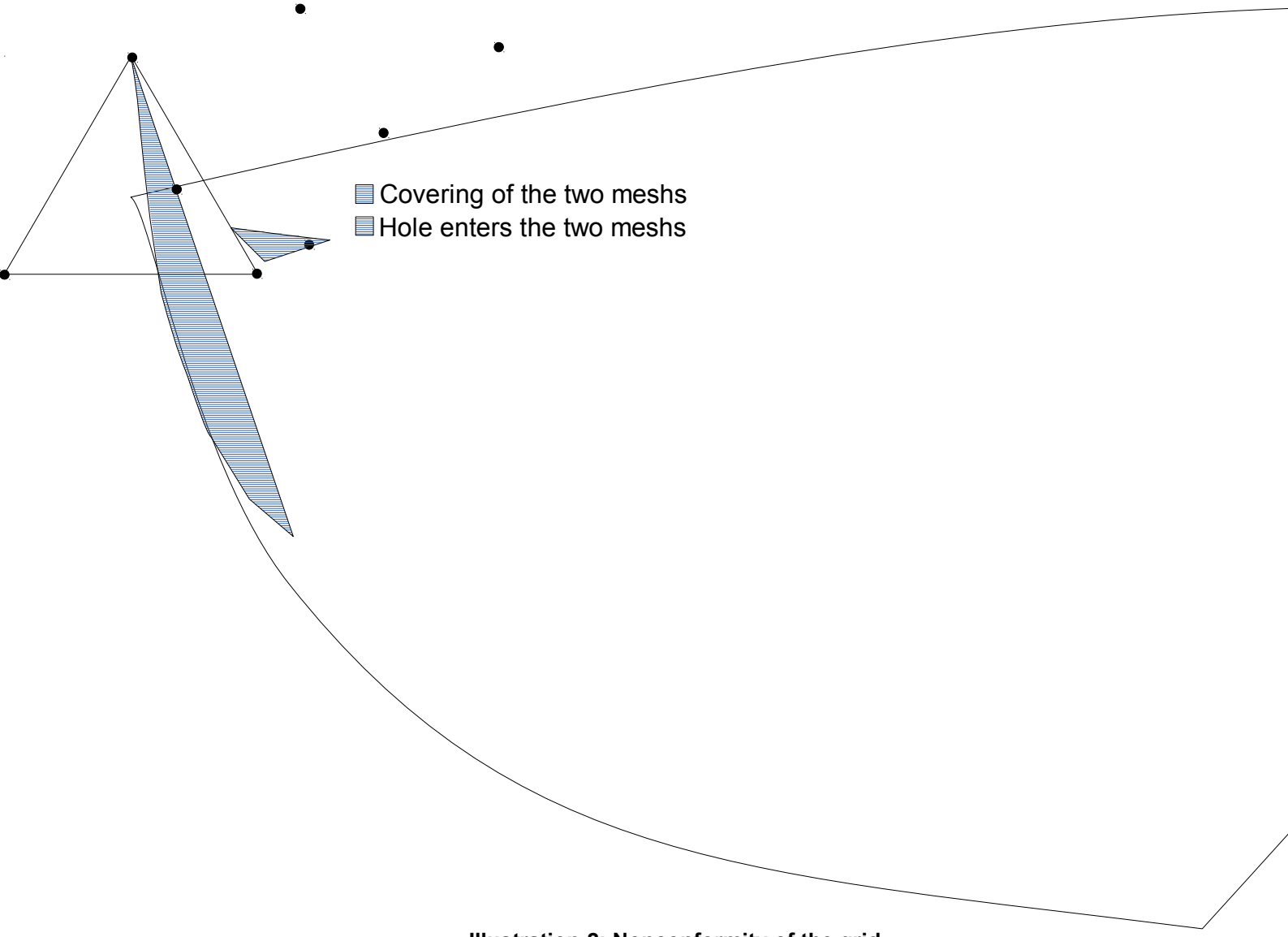

Illustration 2: Nonconformity of the grid

To observe this condition of *conformity*, it is enough to two rules:

1. Each element must be defined in a single way starting from the coordinates of its geometrical nodes (and not those of its neighbors!);
2. The border of an element must be defined in a single way starting from the nodes of these borders, these nodes being common between the elements dividing this border.

These conditions of conformity are an important difference compared to finished volumes which do not have these requirements. The paving of the field makes it possible to apply the finite element method to complex geometries, contrary to the methods by finished differences. The geometrical paving of the field induces a first error: it is not possible, in the case general, to represent a real geometry by a grid by regular polygons, in particular on the border of the field.

" ***A beautiful grid is a good grid "***

3.2.2 Elements of reference

The calculation of the functions of form for an unspecified element can be rather complicated. This is why one often prefers to bring back oneself to an element known as of reference, from which one can generate all the elements of the same family by a geometrical transformation. The functions of form are then calculated on this noted generic element Ω_r, and the transport of the sizes on the real element Ω_e is carried out thanks to the knowledge of the geometrical transformation.

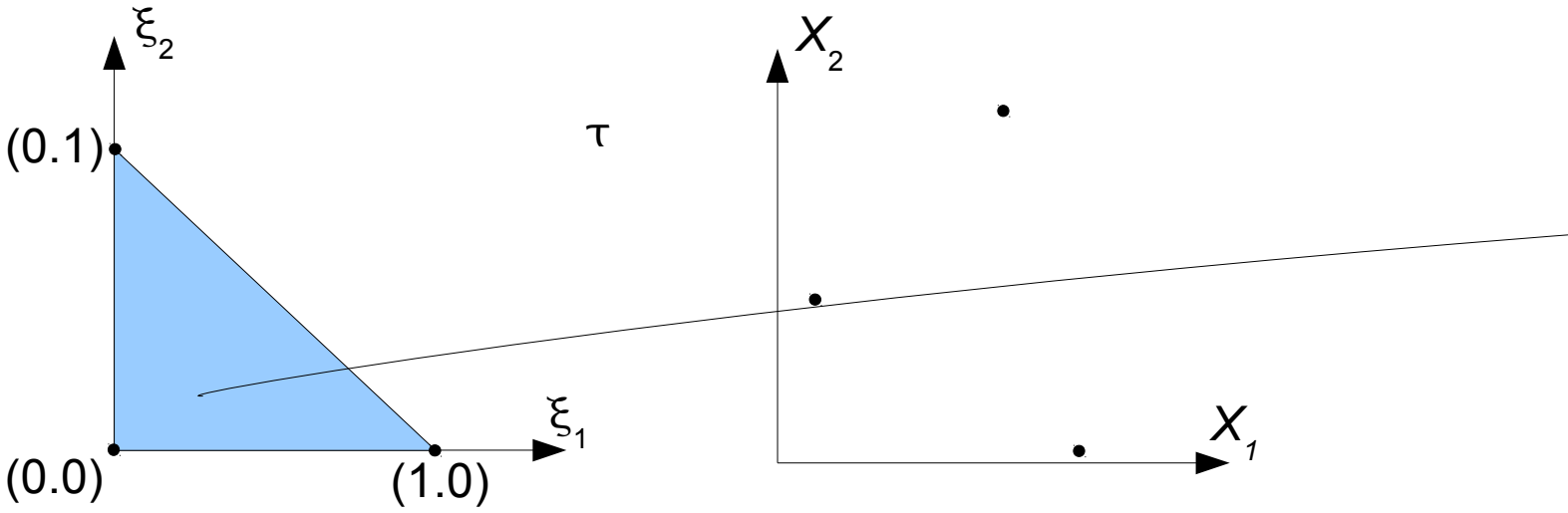

Illustration 3: Passage of the space of reference to real space

The points of the element of reference will be described in parametric terms of coordinates $\xi_{\alpha=1,3}$. The transformation τ must be bijective and transform the tops and sides of the element of reference into tops and sides of the real element:

$$\xi_\alpha \overset{\tau}{\rightarrow} x_\alpha \tag{12}$$

3.2.3 Functions of geometrical interpolation

The geometry of the element is thus approximate by the means of functions of geometrical interpolation. These noted functions $\bar{N}(\boldsymbol{\xi})$ are defined on the element of reference; they make it possible to know the coordinates x_α of an unspecified point of the real element starting from its coordinates ξ_α of its antecedent in the element of reference and the coordinates x_α^i nodes (of local number I) real element:

$$\boldsymbol{x}^e = \sum_{i=1}^{N_{nd}} \boldsymbol{x}_i^e . \bar{\boldsymbol{N}}_i^e \text{ or } x_\alpha^e = \sum_{i=1}^{N_{nd}} x_{\alpha,i}^e . \bar{N}_i^e \tag{13}$$

3.2.4 Matrix jacobienne of the transformation

The jacobienne of the transformation is the matrix of the derivative partial of the real coordinates x_α compared to the coordinates ξ_α in the element of reference:

$$J_{\alpha\beta} = \frac{\partial x_\alpha}{\partial \xi_\beta} \tag{14}$$

By taking account of the definition of the coordinates x_α according to the coordinates $x_{\alpha,i}$ nodes, one obtains an equivalent expression of the matrix jacobienne:

$$J_{\alpha\beta} = \sum_{i=1}^{N_{nd}} \frac{\partial \bar{N}_i}{\partial \xi_\beta} . x_{\alpha,i} \tag{15}$$

Where $\frac{\partial \bar{N}_i}{\partial \xi_\beta}$ are the terms of the matrix $\left[\frac{\partial \bar{N}}{\partial \boldsymbol{\xi}}\right]^T$, of which the number of lines is the number of directions of space, and the number of columns the number of nodes of the element. Let us note that the matrix $\left[\frac{\partial \bar{N}}{\partial \boldsymbol{\xi}}\right]^T$ depends only on the definition of the element of reference and not of that of the real element. LE determining of the matrix jacobienne, useful in calculations which will follow, is called the jacobien of the geometrical transformation. It is nonnull when the transformation τ who makes pass from the element of reference to the real element is bijective, and positive when τ respect the orientation of space.

$$J=\det\left[\frac{\partial \bar{N}}{\partial \xi}\right]\geq 0 \tag{16}$$

3.3 Representation of the unknown factors

To solve the problem, one considers an approximation by finite elements of an unknown field. Spaces E_P and E_u are represented by spaces discrete E^h. There are two equivalent ways to represent the unknown factors in an element: by the coefficients of their polynomial approximation, or by their nodal values. These two possibilities correspond to the two manners complementary to define an element: by the data of a base of students' rag processions, or by the data of the functions of form associated with the nodes. In a general way, one builds the function approached by writing the following linear relation on each element:

$$\mathbf{u}^e(\xi)=\sum_{i=1}^{N_{nd}} a_i^e.\boldsymbol{\phi}_i^e(\xi) \tag{17}$$

Where them $\boldsymbol{\phi}_i^e(\xi)$ are independent linear functions. They constitute ***base*** approximation, the parameters generals of the approximation being coefficients a_i.

3.3.1 Nodal approximation

The first idea of the finite element method is to build approximation of a nodal type for which coefficients $u_i=a_i$ correspond to the solution in these nodes:

$$\mathbf{u}^e(\xi)=\sum_{i=1}^{N_{nd}} u_i^e.\boldsymbol{N}_i^e(\xi) \tag{18}$$

One obtains a nodal approximation then with $\boldsymbol{N}_i^e(\xi)$ functions of interpolation on the element of reference. On each one of these under-fields one builds an approximate function different from one under-field to another. The approximation finite elements is *elementary* because the function depends only on the nodal values constituting the element:

$$\mathbf{u}^e(\boldsymbol{x})=\sum_{i=1}^{N_{nd}} u_i^e.\boldsymbol{N}_i^e(\boldsymbol{x}) \tag{19}$$

An element is *isoparametric* when it is based on identical interpolations for its geometry and its unknown factors: $\bar{N}(\xi)=N(\xi)$.
To ensure the continuity of the solution on the element and, possibly, the continuity of its derivative, it is necessary that the functions $N_i^e(\xi)$ are continuous and, possibly, with derivative continuous.
In the same way if one wants to ensure the continuity of the solution and of its derivative at the borders of the elements (conformity of the approximation), it is necessary that the solution and its derivative depend in a single way of the nodal variables on the nodes of the border.

3.3.2 Base polynomial

The way simplest to define an element is to choose a polynomial base made up of a certain number of independent students' rag processions. For a given unknown factor, the number of students' rag processions used must be equal to the number of nodal variables, i.e. with the number of nodes used to represent the unknown factor. One generally defines the polynomial base on the element of reference; it contains students' rag processions of the form $\xi_1^{\gamma}.\xi_2^{\delta}.\xi_3^{\varepsilon}$, where γ, δ and ε are positive or worthless whole exhibitors. The degree of such a students' rag procession is the entirety $\gamma+\delta+\varepsilon$. The base is known as complete of degree n when all students' rag processions of degree n are present. In certain cases, incomplete bases are employed. One notes $P_p(\xi)$ $p^{ième}$ students' rag procession of the base (which understands some m). Components of the vector displacement $\mathbf{u}(\xi)$ in the element are then given by the formula:

$$u_\alpha(\xi)=\sum_{p=1}^{m} a_{\alpha,p}.P_p(\xi) \quad (20)$$

One will note Π the matrix giving the values taken by the students' rag processions of the polynomial base on the nodes of the element of reference:

$$\Pi_{Ip}=P_p(\xi_I) \quad (21)$$

where p is the sequence number of the students' rag procession in the base, I the number of the node locally to the element and ξ_I coordinates of the node I in the element of reference. This matrix is square, its dimension is the square amongst nodes of the element.
With the node I displacement u_α^I is worth:

$$u_{I,\alpha}=a_{\alpha,p}.\Pi_{Ip} \quad (22)$$

One distinguishes three great types of finite elements frequently used:

- the finite elements of Lagrange which rest on bases polynomial complete and different standard from geometries (symplectic for the triangles and the tetrahedrons, with tensorial structure for the quadrangles and the hexahedrons or of prismatic type);
- the finite elements of Serendip type, which are finite elements of Lagrange with incomplete bases;
- the finite elements of Hermit, of utmost precision, which use the nodal unknown factors and their derivative;

Finite elements of Lagrange symplectic

To determine if a polynomial base is complete with the elements symplectic, it is enough to use the triangle of Pascal:

Order					
Constant			1		
Linear		ξ_1		ξ_2	
Quadratic	$(\xi_1)^2$		$\xi_1.\xi_2$		$(\xi_2)^2$
Cubic	$(\xi_1)^3$	$(\xi_1)^2.\xi_2$		$\xi_1.(\xi_2)^2$	$(\xi_2)^3$
Order 4	$(\xi_1)^4$	$(\xi_1)^3.\xi_2$	$(\xi_1)^2.(\xi_2)^2$	$\xi_1.(\xi_2)^3$	$(\xi_2)^4$

A complete polynomial base of order two comprises six students' rag processions: $\{1;\xi_1;\xi_2;\xi_1^2;\xi_2^2;\xi_1.\xi_2\}$ and thus the geometric standard support will be a triangle with six nodes.

Finite elements of Lagrange with tensorial structure

To describe finite elements quadrangular (or hexahedral), it is enough to take complete polynomials of the order given and to make the product of it.

Order	Constant	Linear	Quadratic	Cubic
Constant	1	ξ_1	$(\xi_1)^2$	$(\xi_1)^3$
Linear	ξ_2	$\xi_1.\xi_2$	$(\xi_1)^2.\xi_2$	$(\xi_1)^3.\xi_2$
Quadratic	$(\xi_2)^2$	$\xi_1.(\xi_2)^2$	$(\xi_1)^2.(\xi_2)^2$	$(\xi_1)^3.(\xi_2)^2$
Cubic	$(\xi_2)^3$	$\xi_1.(\xi_2)^3$	$(\xi_1)^2.(\xi_2)^3$	$(\xi_1)^3.(\xi_2)^3$

A polynomial base “on” - complete of order two for a quadrangular element comprises nine students' rag processions: $\{1;\xi_1;\xi_1^2;\xi_2;\xi_2^2;\xi_1.\xi_2;\xi_1.\xi_2^2;\xi_1^2.\xi_2;\xi_1^2.\xi_2^2\}$, which means nine nodes. Such an element comprises terms of order 3 and 4.

Finite elements of Serendip

Elements of Serendip, for a polynomial of order s, exclude the cross terms from degree higher than $s+1$ not to have nodes inside the elements. For example, for an element of Serendip of order two, the students' rag processions will be $\{1;\xi_1;\xi_1^2;\xi_2;\xi_2^2;\xi_1.\xi_2;\xi_1.\xi_2^2;\xi_1^2.\xi_2\}$, that is to say eight nodes.

3.3.3 Functions of form

An equivalent way to define a finite element is to give, for each unknown factor, the expression of the functions of form of the element. For a given scalar unknown factor (component of displacement according to there for example), there is as much as nodes where the unknown factor must be calculated. In much of case, one uses the same functions of form for all the components of an unknown vector, but it is not obligatory. In what follows, it will be supposed however to simplify the writings that it is the case.

The functions of form can be defined on the real element Ω_e : they then are noted $N^e(\mathbf{x})$, they depend on the geometry of the real element, and are thus different from one element to another. It is simpler to express them on the element of reference, which gives the functions $N(\boldsymbol{\xi})$ independent of the geometry of the real element. Let us recall that these functions are polynomial on the element, and that the function of form associated with a given node there the value one takes, whereas it is cancelled in all the other nodes of the element. The unknown factors are expressed then like linear combination of the functions of form, the coefficients $u_{\alpha,i}$ combination being called nodal variables:

$$u_\alpha(\boldsymbol{\xi}) = \sum_{i=1}^{N_{nd}} u_{\alpha,i} . N_i(\boldsymbol{\xi}) \tag{23}$$

By using the transformation τ enter the element of reference and the real element:

$$\xi_\alpha \overset{\tau}{\rightarrow} x_\alpha \tag{24}$$

One a:

$$u_\alpha(\boldsymbol{\xi}) = \sum_{i=1}^{N_{nd}} u_{\alpha,i} . N_i\left(\tau^{-1}(\boldsymbol{x})\right) \tag{25}$$

3.3.4 Correspondence between polynomial base and functions of form

There are two relations. The first comes from the approximation of the solution by a polynomial base:

$$u_\alpha(\boldsymbol{\xi}) = \sum_{p=1}^{m} a_{\alpha,p} . P_p(\boldsymbol{\xi}) \tag{26}$$

The second is the nodal approximation:

$$u_\alpha(\boldsymbol{\xi}) = \sum_{i=1}^{N_{nd}} u_{\alpha,i} . N_i(\boldsymbol{\xi}) \tag{27}$$

The matrix giving the values taken by the students' rag processions of the polynomial base on the nodes of the element of reference:

$$\Pi_{Ip} = P_p(\boldsymbol{\xi}_I) \tag{28}$$

In a node I , the following polynomial approximation was written;

$$u_{I,\alpha} = a_{\alpha,p} . \Pi_{Ip} \tag{29}$$

By injecting the equation (29) in the nodal expression (27), one obtains:

$$u_\alpha(\boldsymbol{\xi}) = \sum_{i=1}^{N_{nd}} a_{\alpha,p} . \Pi_{Ip} . N_i(\boldsymbol{\xi}) \tag{30}$$

By comparison with the polynomial approximation (26), one from of deduced the following relation between the polynomial base and the functions from form:

$$\Pi_{Ip} . N_i(\boldsymbol{\xi}) = P_p(\boldsymbol{\xi}) \tag{31}$$

In practice, one will find in the literature the writings of the nodal functions of form for the most current elements, according to the choice of the polynomial base.

3.4 Results of existence and unicity

One can write the problem in a more abstract way:

$$\begin{aligned} &\text{Trouver } \mathbf{u}\in E_u \text{ tel que } \forall \mathbf{v}\in E_v \\ &a(\mathbf{u},\mathbf{v})=f(\mathbf{v}) \end{aligned} \quad (32)$$

E_u and E_v are vector spaces of functions defined on Ω. They are spaces of Hilbert.
$a(\mathbf{u},\mathbf{v})$ is a bilinear form on $E_u\times E_v$ (it was supposed that $\mathbf{L}(\mathbf{u})$ represent a linear physical problem compared to $\mathbf{u}$).
$f(\mathbf{v})$ is a linear and continuous form on E_v.
To establish the conditions of existence and unicity, one applies the theorem of Lax-Milgram. Initially, it is supposed that the solution belongs to the same space as the functions test $E_u=E_v$
If the form $a(\mathbf{u},\mathbf{v})$ is coercive i.e.:

$$\forall \mathbf{u}\in E_u \quad a(\mathbf{u},\mathbf{u})\geq c.\|\mathbf{u}\|^2_{E_u} \quad avec\ c>0 \quad (33)$$

Then the problem:

$$\begin{aligned} &\text{Trouver } \mathbf{u}\in E_u \text{ tel que } \forall \mathbf{v}\in E_u \\ &a(\mathbf{u},\mathbf{v})=f(\mathbf{v}) \end{aligned} \quad (34)$$

admits one and only one solution.

4 Method of Ritz

The method of Galerkine, in certain cases, is equivalent making stationary a functional calculus. It is the case if the bilinear form $a(\mathbf{u},\mathbf{v})$ is symmetrical and positive:

$$\forall \mathbf{u},\mathbf{v}\in E_u \quad a(\mathbf{u},\mathbf{v})=a(\mathbf{v},\mathbf{u}) \text{ et } a(\mathbf{u},\mathbf{u})\geq 0 \quad (35)$$

In this case the problem (34) admits one and only one solution $\mathbf{u}$ minimize on E_u the following functional calculus:

$$\pi(\mathbf{u})=\frac{1}{2}.a(\mathbf{u},\mathbf{u})-f(\mathbf{u}) \quad (36)$$

From the mechanical point of view, that means that the principle of the virtual powers can be also written like the minimization of a scalar size: the total energy of the structure. This manner of writing balance is very frequently employed. We here will have some results of them.

We point out initially that a functional calculus is a function of a set of functions (and of its derivative). One will write π this functional calculus. One will limit oneself to the formulations in displacement, knowing that there is the different one. In this case, the functional calculus π will be written:

$$\pi(\boldsymbol{u})=\pi\left(\boldsymbol{u},\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}}\right) \quad (37)$$

For the conservative problems, one can show that to write that the first variation of π is worthless (condition of stationnarity of the functional calculus) is equivalent applying the principle of virtual work, or using the method of Galerkine by taking virtual displacements like weight function. One calls that the method of Galerkine consists starting from the problem with the derivative partial establishing balance of the structure, that is to say:

$$\boldsymbol{L}(\boldsymbol{u})+\boldsymbol{f}=0 \text{ dans } \Omega \text{ avec } \boldsymbol{\sigma}.\boldsymbol{n}=\boldsymbol{g} \text{ sur } \Gamma_N \text{ et } \boldsymbol{u}=\boldsymbol{u}^D \text{ sur } \Gamma_D \quad (38)$$

One then seeks to solve the problem in integral form by using weight functions which are of the same nature as the approximate solution:

$$W=\int_{\Omega}[\boldsymbol{L}(\boldsymbol{u})+\boldsymbol{f}].\psi(\boldsymbol{u}).d\Omega=0$$
$$\text{Avec}\ \boldsymbol{\sigma}.\boldsymbol{n}=\boldsymbol{g}\ \text{sur}\ \Gamma_N\ \text{et}\ \boldsymbol{u}=\boldsymbol{u}^D\ \text{sur}\ \Gamma_D \quad (39)$$

If one chooses like weight function the variation of the unknown factors $\psi=\delta\boldsymbol{u}$ and after having integrated by parts once, one obtains:

$$\delta\pi(\boldsymbol{u})=W(\boldsymbol{u})=0\ avec\ \boldsymbol{u}=\boldsymbol{u}^D\ sur\ \Gamma_D \quad (40)$$

To find the form exact of the functional calculus is not immediate in the case general. In mechanics, for the conservative cases, it is that this functional calculus is equivalent to the total potential energy of the system. After discretization of the functional calculus (by an approximation finite elements), one finds oneself with a matric system strictly equivalent to that of the method of Galerkine (or its mechanical principle are equivalent, the method of the virtual powers).

Intuitively, it is understood that a weak variation $\delta\boldsymbol{u}$ solution is a field which can be kinematically acceptable and which thus corresponds well to the assumptions of the method of the virtual powers.

5 Construction of the matric system

We now will present the various ingredients leading to the construction of the matric system which will make it possible to solve the problem.

5.1 New notation (notation of Voigt)

In order to understand well the construction of the discrete terms in the finite element method, we will use a more compact notation:

$\langle V\rangle$ is a vector line
$\{V\}$ is a vector column
$[A]$ is a matrix

Thus the geometrical interpolation is written according to three dimensions of space:

$$\begin{aligned}x^e=x_1^e=\langle x_{1,i}^e\rangle.\{N_i^e\}=\langle N_i^e\rangle.\{x_{1,i}^e\}\\ y^e=y_1^e=\langle y_{1,i}^e\rangle.\{N_i^e\}=\langle N_i^e\rangle.\{y_{1,i}^e\}\\ z^e=z_1^e=\langle z_{1,i}^e\rangle.\{N_i^e\}=\langle N_i^e\rangle.\{z_{1,i}^e\}\end{aligned} \quad (41)$$

Or in a more compact way in vectorial form:

$$\{\boldsymbol{x}^e\}=[N_i^e].\{\boldsymbol{x}_i^e\}=\langle\boldsymbol{x}_i^e\rangle.[N_i^e]^T \quad (42)$$

With the matrix N functions of form. By considering an element with two nodes, one obtains in developed form:

$$\begin{Bmatrix}x^e\\y^e\\z^e\end{Bmatrix}=\begin{bmatrix}N_1^e&0&0&N_2^e&0&0\\0&N_1^e&0&0&N_2^e&0\\0&0&N_1^e&0&0&N_2^e\end{bmatrix}.\begin{Bmatrix}x_1^e\\y_1^e\\z_1^e\\x_2^e\\y_2^e\\z_2^e\end{Bmatrix} \quad (43)$$

5.2 Discretized system

One places oneself in the case hyperelastic in small deformations, the problem of mechanics to be solved one writes in a more compact way:

$$\text{To find } \boldsymbol{u}\in E^h \text{ such as } \forall \tilde{\boldsymbol{u}}\in E^{\hat{h}} \quad \text{with } a(\boldsymbol{u},\tilde{\boldsymbol{u}})+l(\tilde{\boldsymbol{u}})=0 \tag{44}$$

With $a(\boldsymbol{u},\tilde{\boldsymbol{u}})$ a bilinear, symmetrical form which represents the potential energy of the structure and $l(\tilde{\boldsymbol{u}})$ potential[1] voluminal and surface efforts:

$$\begin{aligned} a(\boldsymbol{u},\tilde{\boldsymbol{u}}) &= \int_{\Omega^h} \boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}):\boldsymbol{\sigma}(\boldsymbol{u}).d\Omega^h \\ l(\tilde{\boldsymbol{u}}) &= \int_{\Omega^h} \boldsymbol{f}.\tilde{\boldsymbol{u}}.d\Omega^h + \int_{\Gamma_N^h} \boldsymbol{g}.\tilde{\boldsymbol{u}}.d\Gamma^h \end{aligned} \tag{45}$$

The discretization consists in choosing a base of space Ω^h and to calculate the terms of the matrix numerically $\boldsymbol{A}$ and of the vector $\boldsymbol{L}$. For that, the bilinear form is expressed $a(.,.)$ and the linear form $l(.)$ like a sum on elements, defined by basic field division:

$$\begin{cases} a(u_i,\tilde{u}_j) &= \sum_{\text{éléments } \Omega^e} \int_{\Omega^e} \sigma_{kl}(u_i).\varepsilon_{kl}(\tilde{u}_j).d\Omega^e \\ l(\tilde{u}_i) &= \sum_{\text{éléments } \Omega^e} \int_{\Omega^e} f_i.\tilde{u}_i.d\Omega^e + \int_{\Gamma_N^e} g_i.\tilde{u}_i.d\Gamma_N^e \end{cases} \tag{46}$$

Terms A_{ij}, which represents the interaction between two degrees of freedom i and j are built in *assembling* (the noted operation $\sum_{\text{éléments } \Omega^e}(\ldots)$) contributions coming from each element which contains the corresponding nodes; one proceeds in the same way to build the vector second member L_i. These contributions, called elementary terms, are calculated at the time of a loop on the elements and depend only on the only variables of the element Ω^e :

$$\begin{cases} a^e &= \int_{\Omega^e} \sigma_{kl}.\varepsilon_{kl}.d\Omega^e \\ l^e &= \int_{\Omega^e} f_i.w_i.d\Omega^e + \int_{\Gamma_N^e} g_i.w_i.d\Gamma_N^e \end{cases} \tag{47}$$

The relation between the tensor of the constraints of Cauchy $\boldsymbol{\sigma}$ and displacements $\mathbf{u}$ is given by the relation of behavior, and is independent of the writing of the variational formulation. In the elastic case, one a:

$$\sigma_{ij}(w_i)=\Lambda_{ijkl}.\varepsilon_{ku}(w_i) \tag{48}$$

Λ_{ijkl} is the tensor of elasticity of Hooke. This tensorial form is not very practical, one preferentially uses the notation of Voigt, which makes it possible to write:

$$\boldsymbol{\sigma}:\boldsymbol{\varepsilon}=\langle\sigma\rangle.\{\varepsilon\} \tag{49}$$

In Cartesian coordinates, one a:

$$\langle\sigma\rangle=\left\langle \sigma_{xx} \quad \sigma_{yy} \quad \sigma_{zz} \quad \sigma_{xy} \quad \sigma_{xz} \quad \sigma_{yz} \right\rangle \tag{50}$$

And the modified shape of the components of deformation to make it possible to express the contracted product, is:

$$\langle\varepsilon\rangle=\left\langle \varepsilon_{xx} \quad \varepsilon_{yy} \quad \varepsilon_{zz} \quad 2.\varepsilon_{xy} \quad 2.\varepsilon_{xz} \quad 2.\varepsilon_{yz} \right\rangle \tag{51}$$

Notice important:

In the integration of the laws of behavior, components of shearing of the constraints and deformations used by Code_Aster *are:*

$$\langle\sigma\rangle=\left\langle \sigma_{xx} \quad \sigma_{yy} \quad \sigma_{zz} \quad \sqrt{2}.\sigma_{xy} \quad \sqrt{2}.\sigma_{xz} \quad \sqrt{2}.\sigma_{yz} \right\rangle$$

1 The potential of the efforts external does not depend on displacement of the structure, it is what is called a loading *died* or *not-follower*. In the cases of the great deformations, the loadings of type pressure cannot respect this assumption.

$$\langle \varepsilon \rangle = \langle \varepsilon_{xx} \quad \varepsilon_{yy} \quad \varepsilon_{zz} \quad \sqrt{2}.\varepsilon_{xy} \quad \sqrt{2}.\varepsilon_{xz} \quad \sqrt{2}.\varepsilon_{yz} \rangle$$

The product of these two vectors gives the same result well as the double contracted product (49).

With this new notation, we have in elasticity:

$$\{\sigma\} = [A].\{\varepsilon\} \tag{52}$$

We set out again of writing EF of the field of displacements:

$$\left\{ \boldsymbol{u}^e \right\} = \left[N_i^e \right].\left\{ \boldsymbol{u}_i^e \right\} = \left\langle \boldsymbol{u}_i^e \right\rangle.\left[N_i^e \right]^T \tag{53}$$

And, in similar manner, the field of virtual displacements:

$$\left\{ \tilde{\boldsymbol{u}}^e \right\} = \left[N_i^e \right].\left\{ \tilde{\boldsymbol{u}}_i^e \right\} = \left\langle \tilde{\boldsymbol{u}}_i^e \right\rangle.\left[N_i^e \right]^T \tag{54}$$

By preoccupation with a simplification of the notations, one will omit the reference to the element. It is necessary of xprimer the tensor of the deformations (virtual or real):

$$\left\{ \boldsymbol{\varepsilon} \right\} = \left[B \right].\left\{ \boldsymbol{u} \right\} = \left\langle \boldsymbol{u} \right\rangle.\left[B \right]^T \text{ and } \left\{ \tilde{\boldsymbol{\varepsilon}} \right\} = \left[B \right].\left\{ \tilde{\boldsymbol{u}} \right\} = \left\langle \tilde{\boldsymbol{u}} \right\rangle.\left[B \right]^T \tag{55}$$

One obtains then for the matrix relating to the bilinear form:

$$[a] = \left\langle \tilde{\boldsymbol{u}} \right\rangle.\int_{\Omega^e} \left[B \right]^T.[\Lambda].\left[B \right].d\Omega^e.\left\{ \boldsymbol{u} \right\} \tag{56}$$

Matrices $\left[B \right]$ and $[\Lambda]$ contain the possible non-linearity of the behavior and will depend on displacements:

1. $\left[B \right]$ is a function of displacements if one is in the situation of the great deformations or the great transformations (great rotations and/or great displacements).
2. $[\Lambda]$ is the matrix of behavior. It becomes dependent on displacements in the case of (and other variables) the non-linear and/or inelastic behaviors.

In these two cases, the process of resolution of the equations will imply a specific treatment (generally, a linearization of the Newton-Raphson type). In a similar way, one will easily obtain the elementary form for the second member.

5.3 Calculation of the elementary terms

The elementary terms to calculate are form:

$$\int_{\Omega_e} f\left(\mathbf{u}(\mathbf{x}), \frac{\partial \mathbf{u}(\mathbf{x})}{\partial \mathbf{x}} \right).d\boldsymbol{x} \tag{57}$$

Three types of operations are to be carried out:

1. the transformation of the derivative compared to $\mathbf{x}$ in derived compared to $\boldsymbol{\xi}$;
2. the passage of an integration on the real element with an integration on the element of reference,
3. the digital realization of this integration which is generally made by a formula of squaring.

5.3.1 Transformation of the derivative

The transformation of the derivative is carried out thanks to the matrix jacobienne $\boldsymbol{J}$, according to the rule of derivation in chain:

$$\frac{\partial u_\alpha}{\partial x_\beta} = \frac{\partial \xi_\gamma}{\partial x_\beta}.\frac{\partial u_\alpha}{\partial \xi_\gamma} = \boldsymbol{J}^{-1}.\left[\frac{\partial \boldsymbol{N}}{\partial \boldsymbol{\xi}} \right]^T.\boldsymbol{u}_\alpha^{\text{nod}} \tag{58}$$

where $\boldsymbol{u}_\alpha^{\text{nod}}$ is the vector of the nodal values of the component α displacement.

5.3.2 Change of field of integration

The passage to integration on the element of reference is carried out by multiplying the intégrande by the determinant of the matrix jacobienne, called jacobien:

$$\int_{\Omega_e} f\left(\mathbf{u}(\mathbf{x}), \frac{\partial \mathbf{u}(\mathbf{x})}{\partial \mathbf{x}}\right).d\,\boldsymbol{x} = \int_{\Omega_r} f\left(\boldsymbol{u}(\boldsymbol{\xi}), \frac{\partial \boldsymbol{u}(\boldsymbol{\xi})}{\partial \boldsymbol{\xi}}\right).\det(\boldsymbol{J}(\boldsymbol{\xi})).d\,\boldsymbol{\xi} \quad (59)$$

The passage of the element of reference to the real element implies the bijectivity of the transformation τ. It is thus necessary $\det(\boldsymbol{J}(\boldsymbol{\xi}))\neq 0$, which implies that the element should not be turned over or degenerate (for example it is not necessary that the quadrangle degenerates into triangle).

5.3.3 Digital integration

In certain typical cases, one can calculate the integrals analytically. For example, for a triangle in two dimensions, Jacobien is constant on the triangle, and the intégrandes are brought back to students' rag processions which one can integrate exactly thanks to the formula of digital integration known as "of Gauss"[2] :

$$\int_0^1 \int_0^{1-\xi} \xi_1^{\alpha}.\xi_2^{\beta}.d\,\xi_1.d\,\xi_2 = \frac{\alpha!\,\beta!}{(\alpha+\beta+2)!} \quad (60)$$

However, these typical cases are rare, and one prefers to evaluate the integrals numerically by calling on formulas of squaring. Those give an approximation of the integral in the form of a balanced sum of the values of the intégrande in a certain number of points of the element called points of integration:

$$\int_{\Omega_r} g(\boldsymbol{\xi}).d\,\boldsymbol{\xi} \approx \sum_{g=1}^{r} \omega_g.g(\xi_g) \quad (61)$$

Scalars ω_g the weights of integration, and the coordinates are called ξ_g are the coordinates of r points of integration in the element of reference.
In the methods of integration of Gauss, the points and weights of integration are given so as to integrate exactly polynomials of a nature given. It is this kind of method which one uses in *Code_Aster*, the points of integration are called then points of Gauss.
The number of points of selected Gauss makes it possible to integrate exactly in the element of reference. In fact, because of the possible non-linearity of the geometrical transformation or the space dependence of the coefficients (for example for elements deformed or of second order), integration is not exact in the real element.
For each element Ω_e, one knew to calculate the terms known as elementary: elementary matrix $\mathbf{A}^e$ and elementary vector $\mathbf{L}^e$. The matrix A and the vector L are obtained by a procedure that one calls the assembly of the elementary terms.
If one regains the elementary shape of rigidity:

$$[a] = \int_{\Omega^e} \{\sigma\}.\langle\varepsilon\rangle.d\,\Omega^e \quad (62)$$

Digital integration implies that one evaluates the constraints and the deformations at the points of integration:

$$[a] = \int_{\Omega^e} \{\sigma\}.\langle\varepsilon\rangle.d\,\Omega^e \approx \sum_{g=1}^{r} \omega_g.\{\sigma(\xi_g)\}.\langle\varepsilon(\xi_g)\rangle \quad (63)$$

What means that the constraints and the deformations are most exact (or the least false) at the points of integration (fields known as "ELGA" in *Code_Aster*). The simple fact of extrapolating these values with the nodes for posting introduces an error. It is besides about a method evaluation of the error, called indicator of error of Zhu-Zienkiewicz.
In elasticity 2D, a triangle displaying a jacobien constant, only one point of Gauss is sufficient to integrate exactly the terms of the matrix and the second member (if it is constant).
The cost calculation increases with the number of points of integration, particularly for the non-linear laws of behavior. For example, a hexahedron with 27 nodes needs 27 points of Gauss to integrate the quantities. It thus arrives frequently that one "under-just", i.e. that one uses less points of integration than the required minimum, thus making a mistake that one will possibly compensate by a finer grid. Besides this systematic

2 By abuse language, one frequently calls the digital diagrams of integration "diagrams of Gauss" although there are several kinds (Hammer for the triangle, Gauss-Radau, Newton-Dimensions, etc).

error, this under-integration must be made with precaution because it can produce defects of row of the matrix and thus make the system linear noninvertible.

6 Resolution of the matric system

One thus obtains a linear system to solve:

$$\langle \tilde{\boldsymbol{u}} \rangle . [\boldsymbol{A}] . \{\boldsymbol{u}\} + \langle \tilde{\boldsymbol{u}} \rangle . \{\boldsymbol{L}\} = 0 \quad (64)$$

Whatever the field of virtual displacements, therefore:

$$[\boldsymbol{A}] . \{\boldsymbol{u}\} = \{\boldsymbol{L}\} \quad (65)$$

6.1 Imposition of the boundary conditions kinematics

Treatment of the boundary conditions kinematics of the type $\mathbf{u} = \mathbf{u}^{\mathbf{D}}$ is done in two different ways:

1. "Kinematic" method (AFFE_CHAR_CINE in Code_Aster) consists in modifying the matrix and the second member. This method is fast and does not introduce additional variables. On the other hand, it is not general and does not allow to apply complex limiting conditions of the style $\sum \boldsymbol{u}_i . a_i = \boldsymbol{u}^{\boldsymbol{D}}$.
2. Method by dualisation (AFFE_CHAR_MECA in Code_Aster) consists in introducing a vector of multipliers (or parameters) of Lagrange $\boldsymbol{\lambda}$, which increases the number of unknown factors but makes it possible to treat all the cases.

$$\begin{cases} [\boldsymbol{A}] . \{\boldsymbol{u}\} + [\boldsymbol{Q}]^T \{\boldsymbol{\lambda}\} = \{\boldsymbol{L}\} \\ [\boldsymbol{Q}] . \{\boldsymbol{u}\} = \{\boldsymbol{u}^{\boldsymbol{D}}\} \end{cases} \quad (66)$$

6.2 Resolution

The linear system can be solved by a certain number of digital methods. Methods used in *Code_Aster* are a factorization LDL^T by blocks, a multi-frontal method (or its equivalent with swivelling, MUMPS), a combined gradient prepacked as well as the collection of iterative solveurs PETSC.
The methods of resolution are divided into two categories:

- The direct methods which solve exactly (with the digital errors near)
- The iterative methods which build a vector series converging towards the solution

The matrices resulting from the finite element method are very hollow (they comprise a majority of worthless terms). In practice, on systems of standard size (a few tens of thousands of equations), the density of nonworthless terms seldom exceeds the 0.01%. They are thus stored in form digs (or "sparse") and take little place in memory. A contrario, the matrices are not built to be used effectively with the mathematical libraries of programs optimized dedicated to the full matrices (booksellers BLAS for example). Solveurs are thus developed specifically for these problems.
A direct solvor has as a principle of breaking up the matrix into a product of particular matrices of form. For example, decomposition LDL^T :

$$[A] = [L] . [D] . [L]^T \quad (67)$$

Where the matrix D is diagonal and the matrix L is triangular lower. This decomposition is valid only for the symmetrical matrices. If it is not the case, other decompositions should be used.
The principle is the following:

- From the initial matrix (very hollow), one builds a product of remarkable matrices. It is the operation of *factorization*.
- These remarkable matrices make it possible to solve the very fast problem of manner. It is the phase of *descent-increase.*

The phase of factorization is most expensive. For the most spread decompositions, the cost machine is in n^3 where n is the number of equations. The cost report will depend on the profile of the matrix (of the classification of the finite elements). Automatic processes seek to optimize this classification to have a structure as compact as possible. Even with this optimization, it is frequent that the factorized matrix take

several hundreds of times, even several thousands of times more memory than the initial matrix. The direct solveurs thus consume much memory and that becomes crippling about it starting from several hundreds of thousands of degrees of freedom, even on the most powerful machines. On the other hand, these direct methods are particularly robust. The problems in mechanics of the structures and the solids very often lead to matrices with a bad conditioning (it is particularly the case of all the last digital innovations which use mixed methods with many multipliers of Lagrange).
When it is possible, iterative methods whose principle consists in finding an approximation of the reverse of the matrix and to proceed then to an iterative resolution, not by step, which uses only products matrix-vectors, very effective and inexpensive in memory are preferentially used.
However, these iterative methods have several defects:

- They are less robust than the direct methods, particularly when conditioning is bad
- The methods of prepacking are very numerous and there are some as much as different problems (even several possible by problem). What obliges the user to juggle with the various methods, without never being assured to get a result at the end.
- They are iterative methods, which implies a criterion of stop of the process, and thus a parameter to be managed but also problems of office plurality of rounding errors.

7 Organization of a calculation by finite elements in *Code_Aster*

One very briefly describes how and at which place the aspects evoked in this document are established in *Code_Aster*.

7.1 Concept of finite element in *Code_Aster*

A kind of finite element is defined by:

- a kind of mesh
- a list of nodes
- functions of form
- options of calculation

An element in the grid is defined by a kind of mesh, a geometry (coordinated nodes) and a topology (ordered list of the nodes). It is the type of modeling chosen in the command file which makes it possible to assign to each mesh of the grid a kind of finite element. The order AFFE_MODELE [U4.22.01] assigns to each mesh a kind of finite element corresponding to the modeling specified for this mesh.

Notice important:

> *One should not forget to assign finite elements to the meshs of edge which one needs to impose the boundary conditions and loadings, and that one will have taken care to create during the manufacturing of the grid.*

The operator AFFE_CHAR_MECA [U4.44.01], which affects boundary conditions and loadings, also will create finite elements, for example the finite elements which will carry the degrees of freedom of LAGRANGE used in the dualisation of the boundary conditions [R3.03.01].
The operator AFFE_CARA_ELEM [U4.42.01] allows to define additional characteristics for certain types of elements: for example, the thickness of the hulls, orientation of the beams, matrices of mass and rigidity of the discrete elements.
An option of calculation indicates the elementary type of calculation that the element is able to calculate. For example RIGI_MECA relate to the calculation of the elementary matrix of mechanical rigidity:

$$A^e_{\alpha\beta} = \int_{\Omega^e} \Lambda_{ijkl} . \varepsilon_{ij}(N^e_\alpha(\mathbf{x})) . \varepsilon_{kl}(N^e_\beta(\mathbf{x})) . d\Omega^e \quad (68)$$

The “data” of this option are the geometry Ω^e and the material Λ, supplemented by the temperature if the material depends on it.
Let us recall that to apply the loadings of border, one uses finite elements of edge individuals, and not the borders of the finite elements of volume (3D) or surface (2D).

Note:

A developer can sometimes have the choice between creating a new finite element or adding an option of calculation to an existing element; the choice between these two solutions in general takes account of criteria of data-processing facility (e.g. elements under - integrated).

7.2 Initializations of the elements

The use of elements of reference makes it possible to once and for all carry out a certain number of calculations at the beginning of the execution. One defines, for each type of element of reference:

- The number of nodes and their coordinates;
- The number of families of points of GAUSS;
- The number of points of GAUSS;
- Weights of integration ω_g ;
- Values of the functions of form at the points of Gauss $N_i(\boldsymbol{\xi}_\mathbf{g})$;
- Values of the derivative of the functions of form at the points of Gauss $\frac{\partial N_i(\boldsymbol{\xi}_\mathbf{g})}{\partial \boldsymbol{\xi}}$.

For a given element, one inevitably does not integrate all the elementary terms with the same number of points of Gauss: for example, one in general uses more points of Gauss for the matrix of mass than for the matrix of rigidity, because the products of functions of form are of degree higher than the products of their derivative. Another example is the under-integration used in certain cases. One calls family of points of Gauss each whole of points of Gauss likely to be used.

7.3 Calculation of the elementary terms

During the calculation of the elementary terms (in the routines TE….), one carries out for each point of Gauss the following operations:

- Calculation of the derivative of the functions of form on the real element starting from the coordinates of the nodes of the element and the derivative of the functions of form on the element of reference;
- Calculation of the matrix jacobienne;
- Recovery of the weight of integration multiplied by Jacobien at the point of GAUSS considered;
- Evaluation of the intégrande (according to the calculated option).

The elementary term is calculated by nap on the points of Gauss while balancing by the weights of integration.

7.4 Total resolution

The total resolution takes place in the routines OP…. high level corresponding to the orders user (MECA_STATIQUE [U4.51.01], STAT_NON_LINE [U4.51.03], THER_LINEAIRE [U4.54.01], etc).

8 Bibliography

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9 Description of the versions of the document

Version Aster	Author (S) Organization (S)	Description of the modifications
3	I.VAUTIER	Initial text
10.2	M.Abbas	Partial rewriting, “anonymization” of the concepts compared to mechanics