Efforts external of pressure in great displacements

Summary:

A loading of pressure in great displacements is a following loading. By employing elements of skin, one is brought to calculate, on the one hand, a second member to which calculation is close to that in small displacements, and on the other hand, a term of additional rigidity which is not, in general, symmetrical.

Contents

1 Introduction

The taking into account of loadings of type pressure (keyword `PRES_REP` in the order `AFFE_CHAR_MECA` [U4.44.01]) raises a certain number of difficulties in the absence of the assumption of small displacements. Indeed, unlike dead loads evoked in [R5.03.01], the pressure depends on displacements since it is about an effort whose direction is normal with the field; one speaks then about following forces, activated by the keyword `TYPE_CHARGE=' SUIV'` in the order `STAT_NON_LINE` [U4.51.03]. Nevertheless, the choice of the current configuration like configuration of reference (Lagrangian updated) carried to simple expressions - with the help of some concepts of differential geometry - work of the efforts of pressure and its variation first compared to displacement, the latter being a nonsymmetrical bilinear form.

2 Writing continues problem

2.1 Elements of kinematics in great transformations

A solid is considered Ω subjected to great deformations (see figure 2-1). That is to say $\boldsymbol{F}$ the tensor gradient of the transformation ϕ making pass the initial configuration Ω_0 with the deformed current configuration Ω_t . One notes $\boldsymbol{X}$ the position of a point in Ω_0 and $\boldsymbol{x}$ the position of this same point after deformation in Ω_t. $\boldsymbol{u}$ is then the EPD.lacing enters the two configurations. One thus has:

$$\boldsymbol{x}=\boldsymbol{X}+\boldsymbol{u} \quad (1)$$

The tensor gradient of the transformation is written:

$$\boldsymbol{F}=\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}}=\boldsymbol{I}+\nabla\times\boldsymbol{u} \quad (2)$$

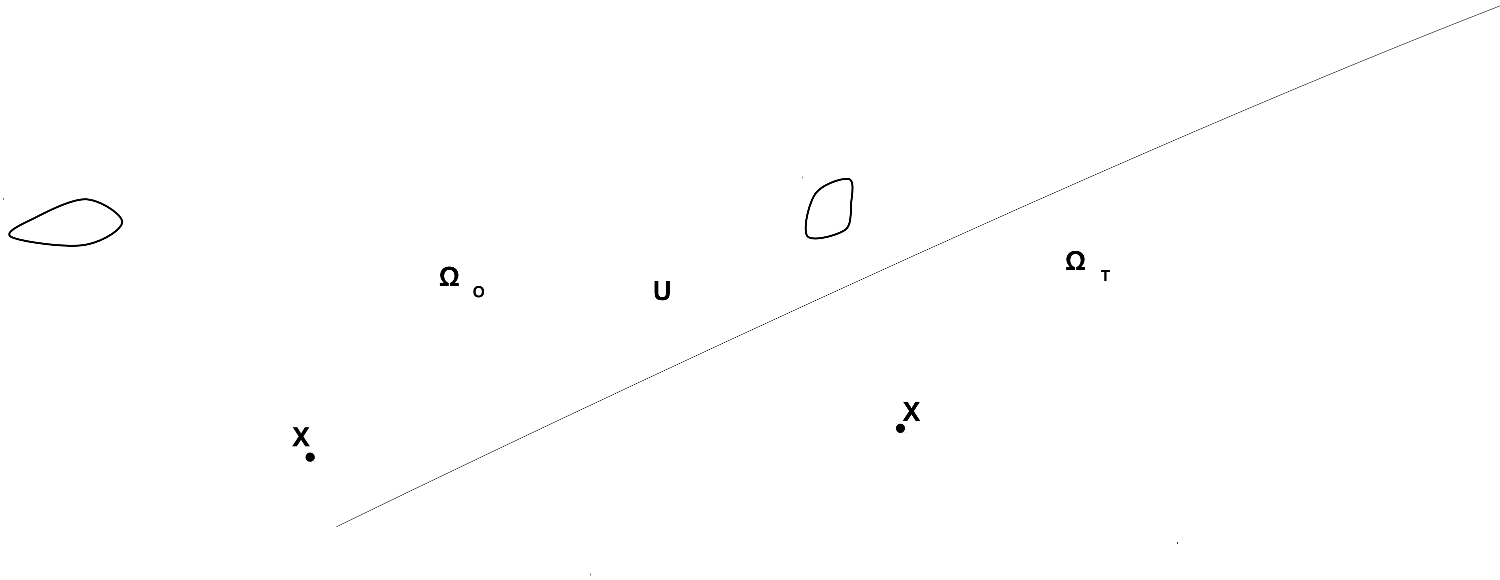


Figure 2-1: Solid in great transformations

2.2 Virtual work of the efforts external of pressure

A pressure is considered P normal on the surface in the configuration of reference. This pressure is written p in the current configuration.

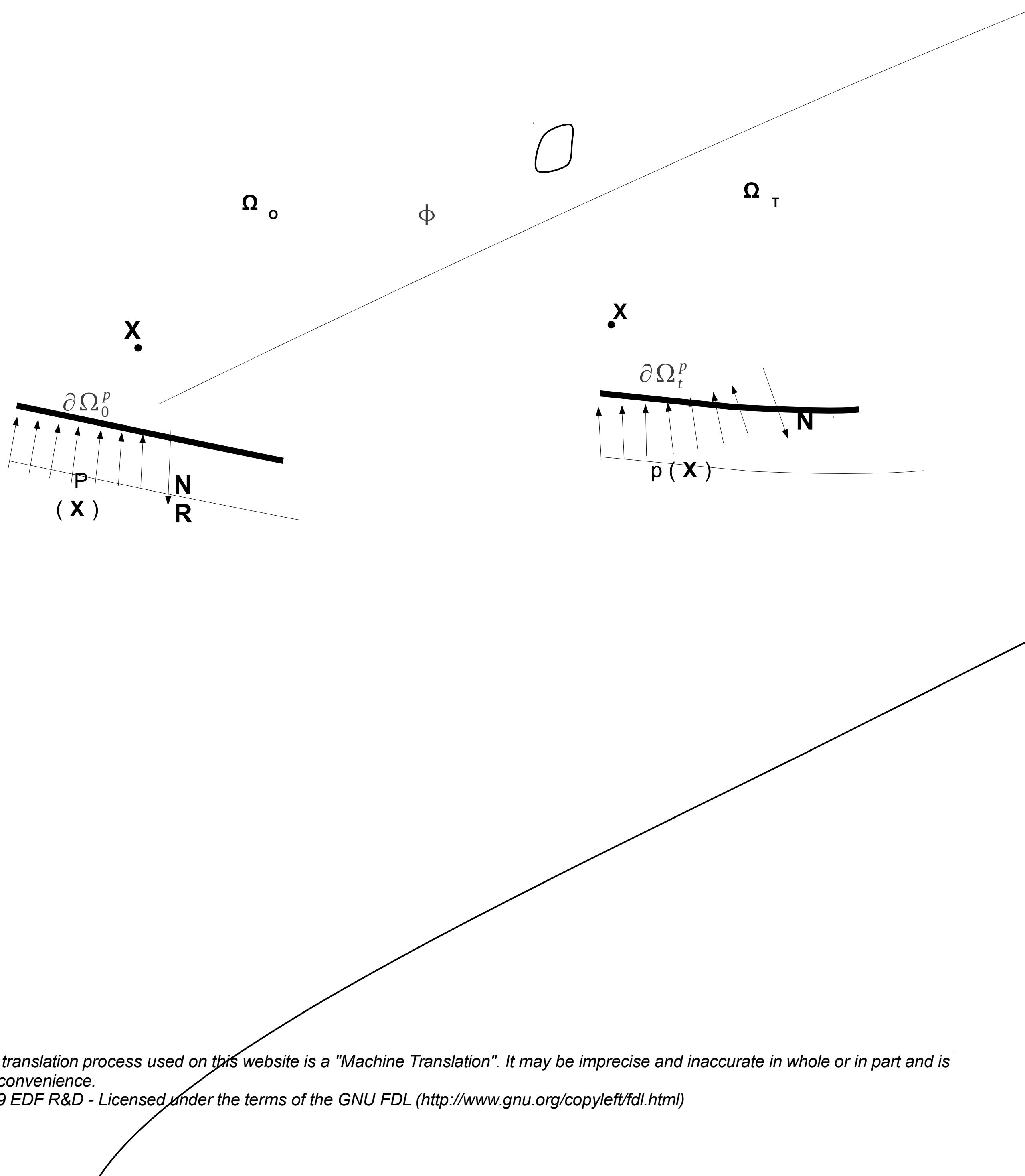

Ω o
ϕ
Ω T
X
X
$\partial\Omega_0^p$
$\partial\Omega_t^p$
P
(X)
N
R
N
p (X)

Figure 2-2: Configuration of reference and current configuration

In the current configuration, the virtual work of the efforts external of pressure W^p is simply written (see figure 2-2):

$$W^p(\boldsymbol{u}).\delta\boldsymbol{v}=\int_{\partial\Omega_t^p(\boldsymbol{u})} -p.\boldsymbol{n}.\delta\boldsymbol{v}.ds \tag{3}$$

Moreover, it is supposed henceforth that the value of the pressure does not depend explicitly on displacement but only on the material point of application:

$$p(\boldsymbol{x})=P(\boldsymbol{\phi}(\boldsymbol{X})) \tag{4}$$

The with dimensions follower of the force comes from the dependence of *normal* with displacement. In this case, one can then express the virtual work of the efforts of pressure in the configuration of reference (change of variable in the integral):

$$W^p(\boldsymbol{u}).\delta\boldsymbol{v}=\int_{\partial\Omega_0^p} -P\det(\boldsymbol{F})\left[\boldsymbol{F}^{-T}.\boldsymbol{N}\right].\delta\boldsymbol{v}(\varphi(\boldsymbol{X})).dS \tag{5}$$

On the practical level, one will use the formula (3) to calculate the work of the efforts of pressure. However, the formula (5) is best adapted to a derivation compared to the displacement, for which one will see the need in the following paragraph.

2.3 Variation of the virtual work of the efforts external of pressure

In the optics of a resolution of the problem of balance of the structure by a method of Newton, one is brought to express the variation of the virtual work of the efforts external of pressure compared to displacement, in a way similar to what was made for the virtual work of the interior efforts in [R5.03.01]. The field of integration being fixed in the expression (5), derivation under the sign nap is licit, (cf. [2]):

$$\frac{\partial W^p(\boldsymbol{u})}{\partial\boldsymbol{u}}.\delta\boldsymbol{u}.\delta\boldsymbol{v}=\int_{\partial\Omega_0^p} -P.\frac{\partial}{\partial\boldsymbol{u}}\left[\det(\boldsymbol{F}).\boldsymbol{F}^{-T}\right].\delta\boldsymbol{u}.\boldsymbol{N}.\delta\boldsymbol{v}.dS \tag{6}$$

We decide to choose like configuration of reference the current configuration, for which $\boldsymbol{F}=\boldsymbol{I}$. This choice led to a simple expression of the derivative of the term between hooks:

$$\frac{\partial}{\partial\boldsymbol{u}}\left[\det(\boldsymbol{F}).\boldsymbol{F}^{-T}\right].\delta\boldsymbol{u}=\text{div }(\delta\boldsymbol{u}).\boldsymbol{I}-\nabla^T\times\delta\boldsymbol{u} \tag{7}$$

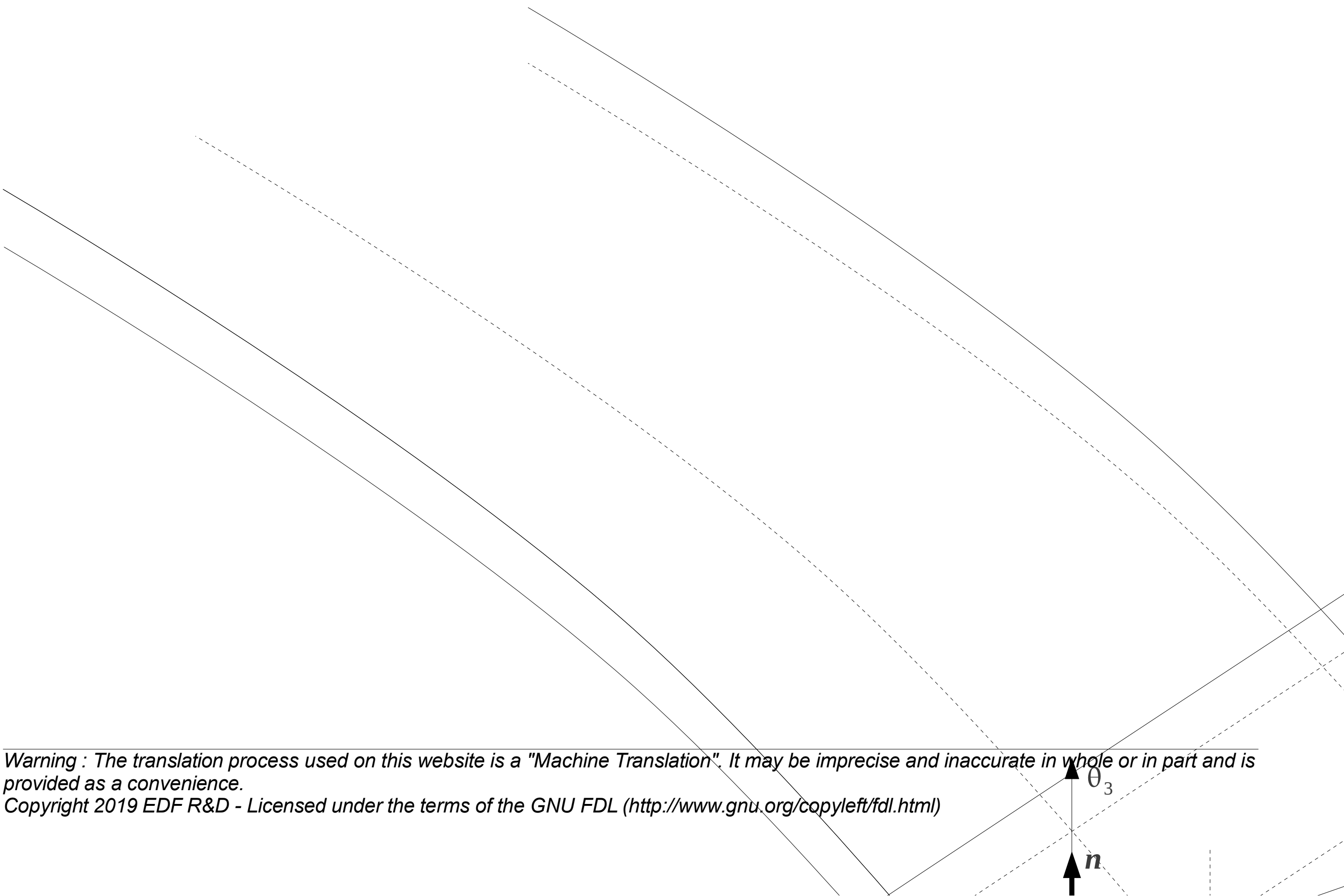
Finally, the variation of the virtual work of the efforts external of pressure is written in the current configuration:

$$\frac{\partial W^p(\boldsymbol{u})}{\partial\boldsymbol{u}}.\delta\boldsymbol{u}.\delta\boldsymbol{v}=\int_{\partial\Omega_t^p(\boldsymbol{u})} -p.\frac{\partial}{\partial\boldsymbol{u}}\left[\text{div }(\delta\boldsymbol{u}).\boldsymbol{I}-\nabla^T\times\delta\boldsymbol{u}\right].\boldsymbol{n}.\delta\boldsymbol{v}.ds \tag{8}$$

In the expression (8) remain a difficulty. Indeed, one expects to obtain a primarily surface size whereas the intégrande reveals terms of normal derivation on the surface. In other words, it is necessary to know the expression of virtual displacements not only on surface of the field but also inside this one (in a vicinity of surface to be able to express the derivative normals). This disadvantage is not pain-killer since in *Code_Aster*, to calculate the elementary terms due to the surface efforts, one employs elements of skin for which a normal variation does not have a direction.

2.4 Adoption of a curvilinear parameter setting of surface

To cure the problem mentioned previously, it is necessary to seek to express the relation (8) using surface sizes only. For that, one resorts to elements of differential geometry, [1], of which one adopts the notations in particular (, one adopts the convention of summation of the repeated indices where the Greek indices take the values 1 and 2 while the Latin indices take the values 1 with 3).


θ3
n

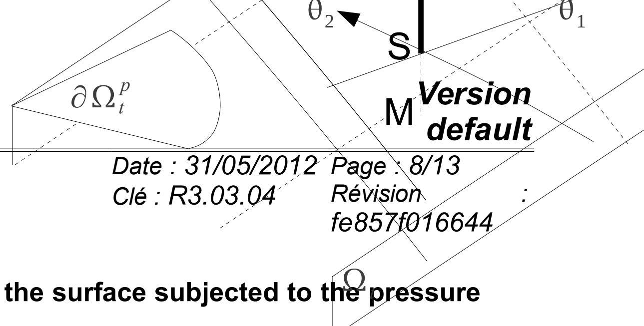


Figure 2-3: Curvilinear parameter setting of the vicinity of the surface subjected to the pressure

That is to say (θ^1, θ^2) an acceptable parameter setting of surface. To describe volume made up of a vicinity of this surface, one associates a third variable to him, θ^3, which measures the progression according to the unit normal $\boldsymbol{n}$ in (θ^1, θ^2). One has thus (see figure 2-3):

$$\boldsymbol{OM}(\theta^1, \theta^2, \theta^3) = \boldsymbol{OS}(\theta^1, \theta^2) + \theta^3 . \boldsymbol{n}(\theta^1, \theta^2) \tag{9}$$

With this choice of parameter setting, the natural base covariante $(\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3)$ is written:

$$\boldsymbol{g}_i = \frac{\partial \boldsymbol{OM}}{\partial \theta^i} \tag{10}$$

While the metric tensor $\boldsymbol{g}$ is worth:

$$g_{ij} = \boldsymbol{g}_i \cdot \boldsymbol{g}_j = \begin{bmatrix} g_{11} & g_{12} & 0 \\ g_{21} & g_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{11}$$

In this curvilinear parameter setting, the intégrande (8) has as an expression:

$$-p . g_{ij} . n^i . \left[\delta u^k|_k . \delta v^j - \delta u^j|_k . \delta v^k \right] \tag{12}$$

This term is simplified considerably. Indeed, one can already note that when $j = k$, the term between hook is null. Moreover, in the adopted curvilinear system, the components contravariantes of $\boldsymbol{n}$ are:

$$n^1 = 0, n^2 = 0, n^3 = 1 \tag{13}$$

Lastly, by taking account of the particular form of $\mathbf{g}$ (i.e. $g_{13} = 0$, $g_{23} = 0$ and $g_{33} = 1$), the variation of work is written simply:

$$\frac{\partial W^p(\boldsymbol{u})}{\partial \boldsymbol{u}} . \delta \boldsymbol{u} . \delta \boldsymbol{v} = \int_{\partial \Omega_t^p(\boldsymbol{u})} -p . \left[\delta u^\alpha|_\alpha . \delta v^3 - \delta u^3|_\alpha . \delta v^\alpha \right] . ds \tag{14}$$

On this expression, it is noted that only intervene of the surface differential operators (derivation covariante compared to θ^1 and θ^2 only), which is well the sought-after goal. By introducing the base contravariante $(\boldsymbol{g}^1, \boldsymbol{g}^2, \boldsymbol{g}^3 = \boldsymbol{n})$, also called bases dual and which is expressed starting from the base covariante by:

$$\boldsymbol{g}^i = \left[\boldsymbol{g}^{-1} \right]^{ij} . \boldsymbol{g}_j \tag{15}$$

One can free oneself from the curvilinear components:

$$\frac{\partial W^p(\boldsymbol{u})}{\partial \boldsymbol{u}} . \delta \boldsymbol{u} . \delta \boldsymbol{v} = \int_{\partial \Omega_t^p(\boldsymbol{u})} -p . \left[\left(\frac{\partial \delta \boldsymbol{u}}{\partial \theta^\alpha} . \boldsymbol{g}^\alpha \right) . (\delta \boldsymbol{v} . \boldsymbol{n}) - \left(\frac{\partial \delta \boldsymbol{u}}{\partial \theta^\alpha} . \boldsymbol{n} \right) . (\delta \boldsymbol{v} . \boldsymbol{g}^\alpha) \right] . ds \tag{16}$$

It is henceforth the expression (16) who will be used to calculate the variation of the virtual work of the efforts of pressure.

2.5 Typical case of a structure subjected to an internal or external pressure constant

In the typical case of a constant pressure in a cavity (see figure 2-4), it is shown that the efforts of pressure derive from a potential Ξ who is not other than the product of the pressure by the volume of the cavity. This result extends to the case from a structure plunged in a fluid with constant pressure.

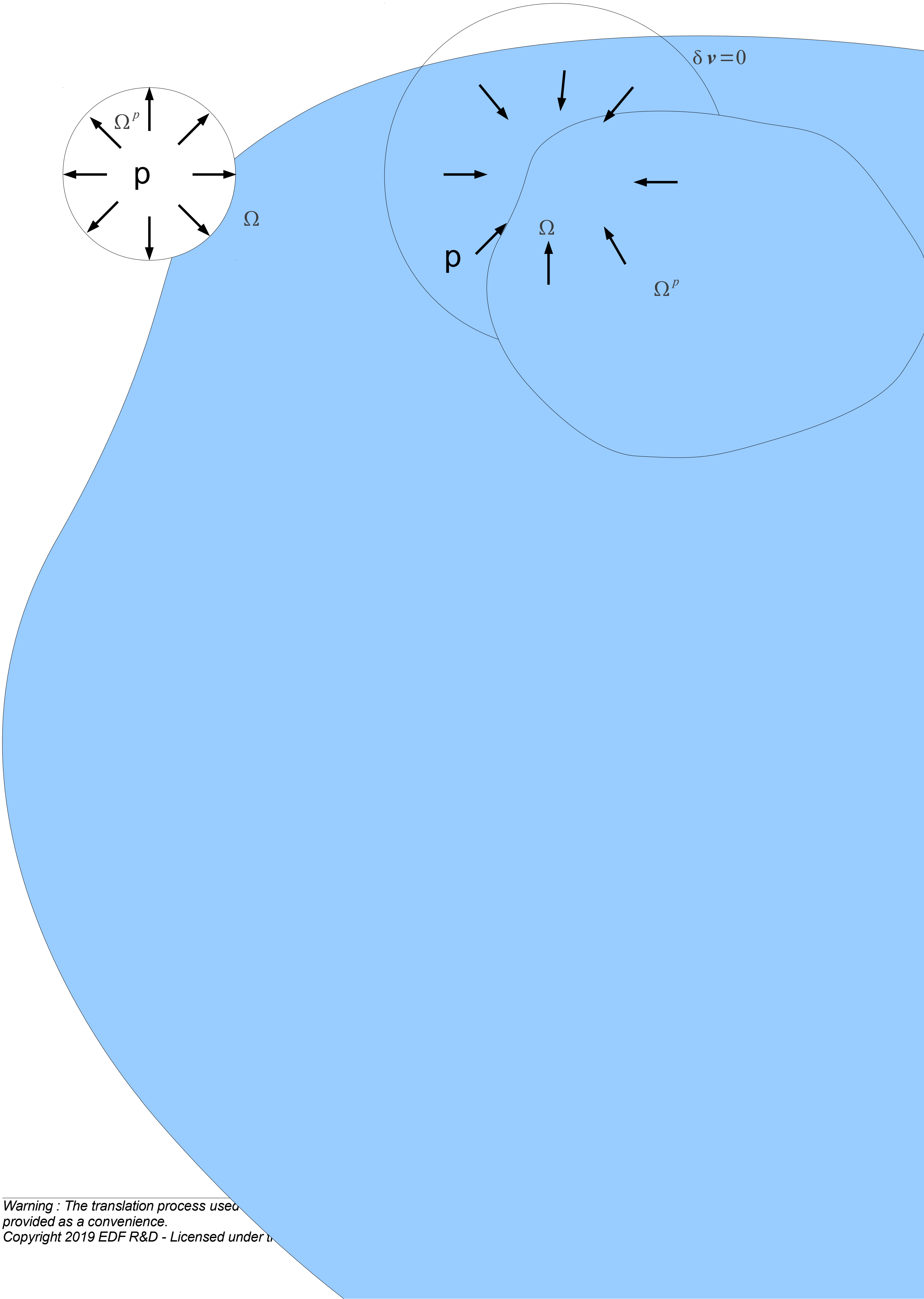

Ω^p
p
Ω
$\delta v = 0$
p
Ω
Ω^p

Figure 2-4: Structure under internal or external pressure constant

This potential is written:

$$\Xi = p.\int_{\partial\Omega_t^p} d\Omega_t = p.\int_{\partial\Omega_0^p} \det \boldsymbol{F}.d\Omega_0 \quad (17)$$

Again, one chooses like configuration of reference the current configuration. Variation of Ξ conduit then well with the virtual work of the efforts external of pressure:

$$\frac{\partial\Xi}{\partial\boldsymbol{u}}.\delta\boldsymbol{v} = p.\int_{\Omega_t^p} \operatorname{div}(\delta\boldsymbol{v}).d\Omega_t = -\int_{\partial\Omega_t^p} p.\delta\boldsymbol{v}.\boldsymbol{n}.ds = W_p.\delta\boldsymbol{v} \quad (18)$$

In this case, typical case the variation of virtual work is also the second variation of the potential Ξ , i.e. a bilinear form **symmetrical** :

$$\frac{\partial W^p(\boldsymbol{u})}{\partial\boldsymbol{u}}.\delta\boldsymbol{u}.\delta\boldsymbol{v} = \frac{\partial^2\Xi(\boldsymbol{u})}{\partial\boldsymbol{u}^2}.\delta\boldsymbol{u}.\delta\boldsymbol{v} \quad (19)$$

3 Discretization

3.1 Introduction into *Code_Aster*

In *Code_Aster*, finite elements of skin (surface elements plunged in a three-dimensional space) are employed to discretize real and virtual displacements intervening in surface expressions such as (3) and (16). These last make it possible respectively to express the vector second member and the matrix of rigidity due to the pressure, of which employment by the algorithm of `STAT_NON_LINE` is specified in [R5.03.01]
Four options are thus developed:

1. `RIGI_MECA_PRSU_R` : matrix of rigidity for a following pressure like real constant
2. `RIGI_MECA_PRSU_F` : matrix of rigidity for a following pressure like real function
3. `CHAR_MECA_PRSU_R` : vector second member for a following pressure like real constant
4. `CHAR_MECA_PRSU_F` : vector second member for a following pressure like real function

These options are developed for the elements `3D`, `D_PLAN` and `AXIS`. One can apply a normal following pressure but not following tangent shearing. The pressure can be a real function of time or a real constant.

3.2 Discretization of the terms of differential geometry

The first two vectors of the base covariante $\{\boldsymbol{G}_{\alpha=1,2}\}$ is calculated starting from the displacement and of the derivative of the functions of form $[\boldsymbol{B}_\alpha]$:

$$\{\boldsymbol{G}_\alpha\}=[\boldsymbol{B}_\alpha].\{\boldsymbol{u}\} \tag{20}$$

The normal $\{\boldsymbol{N}\}$ is calculated like product it vector of these the first two vectors $\{\boldsymbol{G}_{\alpha=1,2}\}$:

$$\boldsymbol{n}=\frac{\boldsymbol{g}_1\wedge\boldsymbol{g}_2}{\|\boldsymbol{g}_1\wedge\boldsymbol{g}_2\|} \tag{21}$$

One can also calculate the metric tensor $[\boldsymbol{G}_{\alpha\beta}]$:

$$[\boldsymbol{G}_{\alpha\beta}]=\{\boldsymbol{G}_\alpha\}\langle\boldsymbol{G}_\beta\rangle \tag{22}$$

E T its jacobien:

$$J=\det[\boldsymbol{G}_{\alpha\beta}] \tag{23}$$

One can calculate the metric matrix contravariante:

$$[\boldsymbol{G}^{\delta\gamma}]=[\boldsymbol{G}_{\alpha\beta}]^{-1} \tag{24}$$

And finally to extract the base contravariante $\{\boldsymbol{G}^\delta\}$:

$$\{\boldsymbol{G}^\delta\}=[\boldsymbol{G}^{\delta\gamma}]\{\boldsymbol{G}_\alpha\} \tag{25}$$

3.3 Vector of the following forces

The calculation of the virtual work of the efforts of pressure (3) is in fact identical to that carried out in small displacements, with the help of a preliminary reactualization of the geometry. One leaves the expression of virtual work:

$$W^p(\boldsymbol{u}).\delta\boldsymbol{v}=\int_{\partial\Omega_t^p(\boldsymbol{u})}-p.\boldsymbol{n}.\delta\boldsymbol{v}.ds \tag{26}$$

In discretized form:

$$W_p(\boldsymbol{u}).\delta\boldsymbol{v}=\langle\delta\boldsymbol{V}\rangle\{F^p(\boldsymbol{u})\} \tag{27}$$

The variation of displacements is written starting from the functions of form:

$$\delta\boldsymbol{v}=\langle\boldsymbol{\varphi}\rangle.\{\delta\boldsymbol{V}\} \tag{28}$$

One did not discretize all the terms of differential geometry in the preceding paragraph, it but does not remain us any more to discretize the integral by using a diagram of Gauss with the weights $\omega_{i_{pg}}$:

$$\int_{\partial\Omega_t^p(\boldsymbol{u})} A.\,ds = \sum_{i_{pg}} A_{\xi_{pg}}.\,\omega_{i_{pg}} \qquad (29)$$

The diagrams of integration used are summarized in the table below.

Geometrical mesh	Diagram of Gauss (see [R3.01.01])
3D	
TRIA3	FPG3
TRIA6	FPG4
QUAD4	FPG4
QUAD8	FPG9
QUAD9	FPG9
D_PLAN	
SEG2	FPG2
SEG3	FPG4
AXIS	
SEG2	FPG2
SEG3	FPG4

In *Code_Aster*, pressure p data by `AFFE_CHAR_MECA` is localised with the nodes, one thus must by interpolating the pressure p nodes $p^{i_{no}}$ towards the point of Gauss $p_{\xi_{pg}}$:

$$p_{\xi_{pg}} = \sum_{i_{no}} N_{\xi_{pg}}^{i_{no}}.\,p^{i_{no}} \qquad (30)$$

Finally:

$$\{F^p(\boldsymbol{u})\} = -\sum_{i_{pg}} p_{\xi_{pg}}.\,\omega_{i_{pg}}.\,n_{\xi_{pg}}.\,J_{\xi_{pg}}.\,\{\boldsymbol{\varphi}_{\xi_{pg}}\} \qquad (31)$$

3.4 Matrix of the following forces

The calculus of the variation of the virtual work of the efforts of pressure (16) is worth:

$$\frac{\partial W^p(\boldsymbol{u})}{\partial \boldsymbol{u}}.\,\delta\boldsymbol{u}.\,\delta\boldsymbol{v} = \int_{\partial\Omega_t^p(\boldsymbol{u})} -p.\left[\left(\frac{\partial\delta\boldsymbol{u}}{\partial\theta^\alpha}.\,\boldsymbol{g}^\alpha\right).(\delta\boldsymbol{v}.\,\boldsymbol{n}) - \left(\frac{\partial\delta\boldsymbol{u}}{\partial\theta^\alpha}.\,\boldsymbol{n}\right).(\delta\boldsymbol{v}.\,\boldsymbol{g}^\alpha)\right].ds \qquad (32)$$

It makes it possible to extract the tangent matrix from the following forces:

$$\frac{\partial W^p(\boldsymbol{u})}{\partial \boldsymbol{u}}.\,\delta\boldsymbol{u}.\,\delta\boldsymbol{v} = \langle\delta\boldsymbol{V}\rangle[-\boldsymbol{K}^p(\boldsymbol{u})]\{\delta\boldsymbol{U}\} \qquad (33)$$

The minus sign comes owing to the fact that the contribution of the matrix is to the first member.

Finally:

$$[\boldsymbol{K}^p(\boldsymbol{u})] = \sum_{i_{pg}} p_{\xi_{pg}}.\,\omega_{i_{pg}}.\,J_{\xi_{pg}}.\,[\boldsymbol{B}_{\xi_{pg}}]^T.\left(\{\boldsymbol{G}^\delta_{\xi_{pg}}\}.\langle\boldsymbol{n}_{\xi_{pg}}\rangle - \{\boldsymbol{n}_{\xi_{pg}}\}.\langle\boldsymbol{G}^\delta_{\xi_{pg}}\rangle\right).[\boldsymbol{\varphi}_{\xi_{pg}}] \qquad (34)$$

3.5 Choice of the matrix

In general, the matrix is not **not symmetrical** (except typical case of a structure subjected to an internal or external pressure constant, cf §2.5). It is noted also in practice, that for strong variations of the geometry (by using a behavior very-rubber band in great deformations like ELAS_HYPER), the fact to symmetrize this matrix is not a good strategy (failure of convergence). One thus decides to keep this not-symmetrical matrix, in spite of (light) the overcost induced by the factorization of such a matrix.

4 Bibliography

[1] "Foundations of solid mechanics", Fung Y.C., Prentice Hall. 1965, pp 31-57.

[2] "Calculation of the derivative of a size compared to a bottom of crack by the method theta", Mialon P., EDF - Bulletin of the Management of the Studies and Research - Series C - n° 3. 1988, pp 1-28.

History of the versions of the document

Version Aster	Author (S) Organization (S)	Description of the modifications
3	E.Lorentz	Initial text
11	**M.Abbas**	Systematic use of the not-symmetrical matrix, harmonization of the notations, notes on the discretization, new figures