Calculation of the constraints to the nodes by local smoothing

Summary:

One presents a local method of calculation of constraints to the nodes starting from the constraints at the points of GAUSS. It is used in the options SIGM_ELNO and SIEF_ELNO.

This method is summarized to calculate the constraints at the tops of an element by multiplying the constraints at the points of GAUSS by a matrix of smoothing, constant for each type of element.

For the isoparametric elements of degree 2, the constraints with the nodes mediums are obtained by average of the values of the constraints at the 2 tops of the edge.

This method of smoothing has two advantages:

- the nodal constraints obtained have an order of precision moreover than by direct calculation with the nodes,
- the method is inexpensive in time CPU.

This method was generalized:

- with calculations of the deformations (option EPSI_ELNO) and of the internal variables (option VARI_ELNO) with the nodes in mechanics,
- with the calculation of flows (option FLUX_ELNO) with the nodes in thermics.

Contents

1 Preliminaries

This method is based on the observation [bib1] which it exists points where the calculation of the constraints, starting from displacements in a primal formulation in displacements, is more precise.

In the case of isoparametric finite elements of order 2 (`SEG3` in 1D, `QUAD8` and `QUAD9` in 2D, `HEXA20` in 3D), one shows that the points of GAUSS of the formula of squaring with 2^n points (n : dimension of space) are such as one can hope, without that being formally shown, for the calculation of $\boldsymbol{\sigma}$ the same order of precision as for the calculation of the field of displacement $\mathbf{u}$.

The idea of the method is to calculate for each element the constraints $\hat{\boldsymbol{\sigma}}$ with the nodes from $\boldsymbol{\sigma}^{\mathbf{k}}$ at the points of GAUSS, these last being calculated on each element by the formula:

$$\boldsymbol{\sigma}^{\mathbf{k}}=\mathbf{D}\,\mathbf{B}^{\mathbf{k}}\,\mathbf{u}=\mathbf{D}\sum_{i=1}^{NNO}\mathbf{B}_{\mathbf{i}}^{\mathbf{k}}\mathbf{U}_{\mathbf{i}}$$

where:

$\mathbf{D}$ is the matrix of elasticity,
$\mathbf{B}^{\mathbf{k}}$ is the matrix connecting the deformations to displacements at the point of GAUSS k ,
$\mathbf{U}_{\mathbf{i}}$ are nodal displacements (NNO nodes)

2 Local method of minimization by least squares

Generally, one wishes to approximate, within the meaning of least squares, the space distribution of the constraints by $\boldsymbol{\sigma}(x)$ a polynomial function:

$$\hat{\boldsymbol{\sigma}}(x)=\sum_{i=0,\dots,p} \mathbf{a_i} P^i(x)$$

The problem amounts finding the coefficients $\mathbf{a_i}$ who minimize the functional calculus:

$$\chi=\int\int(\boldsymbol{\sigma}-\hat{\boldsymbol{\sigma}})^2 dx\,dy$$

Values of the function $\boldsymbol{\sigma}$ are known here only at the points of Gauss: $\boldsymbol{\sigma}^{\mathbf{k}}=\boldsymbol{\sigma}(x_k)$

The minimum will be reached if and only if:

$$\frac{\partial\chi}{\partial\mathbf{a_i}}=0 \qquad \forall i=0,\dots,p$$

Within the framework of the finite element method in displacement, one chooses the following function of smoothing:

$$\hat{\boldsymbol{\sigma}}(x)=\sum_{i=1}^{n} N_i(x)\hat{\boldsymbol{\sigma}}_{\mathbf{i}}$$

where:

N_i is the function of form associated with the node i on the finite element considered,
$\hat{\boldsymbol{\sigma}}_{\mathbf{i}}$ is the value of the constraint to the node i sought,
n the number of nodes retained for smoothing.

One must thus solve the system:

$$\frac{\partial\chi}{\partial\hat{\boldsymbol{\sigma}}_{\mathbf{i}}}=0 \qquad \forall i=1,\dots,n \qquad \text{éq 2-1}$$

One can choose between two methods of local smoothing: continuous smoothing or discrete smoothing.

3 Methods of local smoothing (ref. [bib2] and [bib3])

3.1 Continuous local smoothing

This kind of smoothing led to solve the system [éq 2-1] with the functional calculus defined on the finite element running:

$$\chi=\int_e (\boldsymbol{\sigma}-\hat{\boldsymbol{\sigma}})^2=\int_e \left(\boldsymbol{\sigma}-\sum_{i=1}^{n} N_i \hat{\boldsymbol{\sigma}}_{\mathbf{i}}\right)^2$$

Minimization leads to $M^e \hat{\boldsymbol{\sigma}}=F^e$

with:

$$M_{ij}^e=\int_e N_i N_j \, dx\, dy=\sum_{k=1}^{npg} \bar{N}_i(\xi_k) N_j(\xi_k)(det\,\mathbf{J})_k\, \omega_k$$

$$F_i^e=\int_e N_i \boldsymbol{\sigma}\, dx\, dy=\sum_{k=1}^{npg} \bar{N}_i(\xi_k)\boldsymbol{\sigma}_{\mathbf{k}}(det\,\mathbf{J})_k \omega_k$$

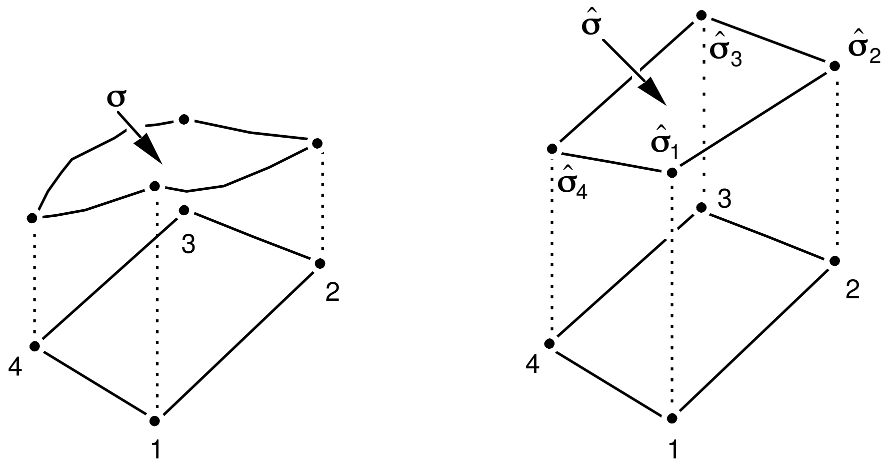
where ξ_k are the points of GAUSS in the element of reference

$(det\,\mathbf{J})_k$ the jacobien of the geometrical transformation enters the element of reference and the element running to the point ξ_k.

ω_k : the weight associated with the point ξ_k

ξ_k : the constraint at the point ξ_k

$\bar{N}_i(\xi_k)$: the value of the function of form in the element of reference to the point ξ_k



calcul direct des contraintes contraintes lissées

Note:

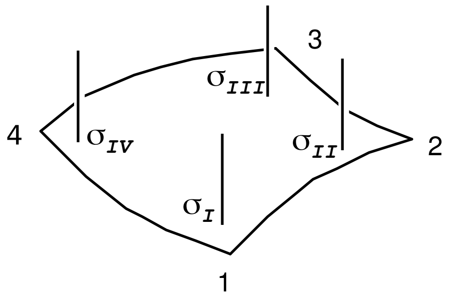
- *If spaces of interpolation of* $\boldsymbol{\sigma}$ *and of* $\hat{\boldsymbol{\sigma}}$ *are the same ones, one has* $\boldsymbol{\sigma}=\hat{\boldsymbol{\sigma}}$ *. In practice, one retains for space of* $\hat{\boldsymbol{\sigma}}$ *a space smaller than that where is defined* $\boldsymbol{\sigma}$ *by the finite element.*
- *One sees the link between the approximation at the points of GAUSS of* $\boldsymbol{\sigma}$ *where* $\boldsymbol{\sigma}$ *thus converge better and this process of smoothing whose justification is on the contrary continuous.*
- *The way in which* $\boldsymbol{\sigma}$ *is calculated at the points of GAUSS does not intervene. Generalization with the nonlinear problems is thus obvious, although it cannot concern the same justification.*

This method is however not adopted because it requires a resolution of system linear for each calculation of $\hat{\boldsymbol{\sigma}}$.

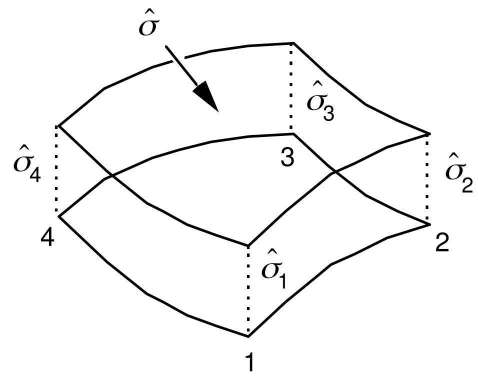
3.2 Discrete local smoothing

In this case, the functional calculus χ is replaced by the summation:

$$\tilde{\chi}=\sum_{k=1}^{npg}(\boldsymbol{\sigma}(\xi_k)-\hat{\boldsymbol{\sigma}}(\xi_k))^2=\sum_{k=1}^{npg}\left(\boldsymbol{\sigma}(\xi_k)-\sum_{i=1}^{n}\hat{\boldsymbol{\sigma}}_{\boldsymbol{i}}N_i(\xi_k)\right)^2$$



contraintes aux points de GAUSS



contraintes lissées

The system to be solved is written there still: $\dfrac{\partial\tilde{\chi}}{\partial\hat{\boldsymbol{\sigma}}_{\mathbf{i}}}=0$ that is to say:

$$\sum_{k=1}^{npg}\sum_{j=1}^{n}N_i(\xi_k)N_j(\xi_k)\hat{\boldsymbol{\sigma}}_{\mathbf{j}}=\sum_{k=1}^{npg}N_i(\xi_k)\boldsymbol{\sigma}(\xi_k)\quad\forall i=1,\ldots,n$$

maybe in matric form: $\mathbf{M}\{\hat{\boldsymbol{\sigma}}_{\mathbf{noeud}}\}=\mathbf{P}\{\boldsymbol{\sigma}_{\mathbf{GAUSS}}\}$

Matrices $\mathbf{M}$ (square $n\times n$) and $\mathbf{P}$ (rectangular $n\times npg$) are then independent of the current element e .

They can thus be calculated once and for all on the element of reference.

Note:

- *This method is more economic than the preceding one and gives comparable results [bib2],*
- *There still, manner thus $\boldsymbol{\sigma}^{\mathbf{k}}$ is calculated in each point of GAUSS is indifferent (since the number of points of GAUSS used for the calculation of $\boldsymbol{\sigma}$ and that of $\hat{\boldsymbol{\sigma}}$ is the same one). One will be able to thus use this method into nonlinear.*

4 Application of the method to calculation of the constraints to the nodes for various elements

The local smoothing adopted in *Code_Aster* is the discrete local smoothing [§2.2], which makes it possible to avoid the calculation of integrals on the element.

On all the elements of continuous medium 2D and 3D, one chose a space of smoothing being based on the functions of form relating to the tops of the element.

The method thus makes it possible to obtain the constraints at the tops. In the case of the elements of order 2, one calculates the constraints with the nodes mediums by taking the arithmetic median value of the two tops “framing” the node medium considered.

One gives hereafter the matrices of passage allowing to calculate the constraints with the nodes tops starting from the constraints at the points of GAUSS. These matrices can be square or rectangular. Indeed, matrices of passage $\mathbf{M}^{-1}\mathbf{P}$ are calculated once and for all with the initialization of each type of finite element (in `AFFE_MODELE`). Two types of matrices exist:

- matrices $\mathbf{M}^{-1}\mathbf{P}$ square, which is to be used when the number of points of GAUSS used for the calculation of the constraints at the points of GAUSS $\boldsymbol{\sigma}^{\mathbf{k}}$ is identical to the number of nodes tops,
- matrices $\mathbf{M}^{-1}\mathbf{P}$ rectangular, which is to be used when the number of points of GAUSS of $\boldsymbol{\sigma}^{\mathbf{k}}$ is different (in general higher) with the number of nodes tops.

4.1 Square matrices of passage

These matrices are used in the elements for which the number of points of GAUSS of the calculation of `SIEF_ELGA/SIGM_ELGA` is equal to the number of tops. The option calculates in first the constraints in a number of points of GAUSS equal to the number of tops. Then matrices $\mathbf{M}^{-1}\mathbf{P}$ (given afterwards) are used to calculate the constraints with the nodes. They is the elements:

- in 2D: QUAD4, TRIA6, under-integrated QUAD8,
- in 3D: TETRA4, PENTA6, HEXA8, PYRAM5 and under-integrated HEXA20.

4.1.1 Square matrices of passage for the elements 2D

4.1.1.1 Triangles

$$\mathbf{M}^{-1}\mathbf{P}=\frac{1}{3}\begin{bmatrix}5 & -1 & -1\\ -1 & 5 & -1\\ -1 & -1 & 5\end{bmatrix}$$

4.1.1.2 Quadrangles

$$\mathbf{M}^{-1}\mathbf{P}=\begin{bmatrix} 1+\frac{\sqrt{3}}{2} & -\frac{1}{2} & 1-\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1+\frac{\sqrt{3}}{2} & -\frac{1}{2} & 1-\frac{\sqrt{3}}{2} \\ 1-\frac{\sqrt{3}}{2} & -\frac{1}{2} & 1+\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1-\frac{\sqrt{3}}{2} & -\frac{1}{2} & 1+\frac{\sqrt{3}}{2} \end{bmatrix}$$

4.1.2 Square matrices of passage for the elements 3D

4.1.2.1 Tetrahedrons

$$\mathbf{M}^{-1}\mathbf{P}=\frac{1}{a-b}\begin{bmatrix} a & a & a-1 & a \\ a & a-1 & a & a \\ a-1 & a & a & a \\ a & a & a & a-1 \end{bmatrix}$$

avec $a=\frac{5-\sqrt{5}}{20}$ $b=\frac{5+3\sqrt{5}}{20}$

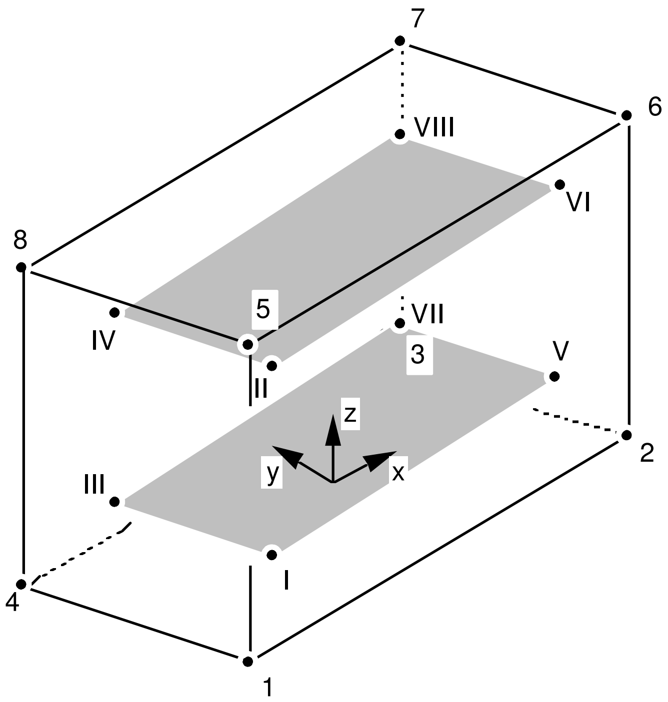
4.1.2.2 Pentahedrons

$$\mathbf{M}^{-1}\mathbf{P}=\begin{bmatrix} \alpha & -\alpha & \alpha & 1-\alpha & \alpha-1 & 1-\alpha \\ \alpha & \alpha & -\alpha & 1-\alpha & 1-\alpha & \alpha-1 \\ -\alpha & \alpha & \alpha & \alpha-1 & 1-\alpha & 1-\alpha \\ 1-\alpha & \alpha-1 & 1-\alpha & \alpha & -\alpha & \alpha \\ 1-\alpha & 1-\alpha & \alpha-1 & \alpha & \alpha & -\alpha \\ \alpha-1 & 1-\alpha & 1-\alpha & -\alpha & \alpha & \alpha \end{bmatrix}$$

$$\alpha=\frac{\sqrt{3}+1}{2}$$

4.1.2.3 Hexahedrons

$$\mathbf{M}^{-1}\mathbf{P}=\begin{bmatrix} a & b & b & c & b & c & c & d \\ b & c & c & d & a & b & b & c \\ c & d & b & c & b & c & a & b \\ b & c & a & b & c & d & b & c \\ b & a & c & b & c & b & d & c \\ c & b & d & c & b & a & c & b \\ d & c & c & b & c & b & b & a \\ c & b & b & a & d & c & c & b \end{bmatrix} \quad a=\frac{5+3\sqrt{3}}{4} \quad b=\frac{1+\sqrt{3}}{4} \quad c=\frac{\sqrt{3}-1}{4} \quad d=\frac{5-3\sqrt{3}}{4}$$



4.1.2.3 figure - has: Classification of the points of GAUSS on the hexahedron with 8 nodes

4.2 Matrices of passage $\mathbf{M}^{-1}\mathbf{P}$ rectangular

Into nonlinear for certain types of elements (TRIA3, QUAD8 and QUAD9 in 2D, TETRA10, PENTA15 and HEXA20 in 3D), the internal constraints and variables at the points of GAUSS are calculated on a family of points of richer GAUSS (9 points for the quadrangles, 15 points for the tetrahedrons, 21 points for the pentahedrons, 27 points for the hexahedrons).

Discrete local smoothing is then carried out starting from these fields and transport with the nodes utilizes matrices different from the preceding ones. They are not square any more, because of dimension (many tops, many points of GAUSS). Matrices of passage $\mathbf{M}^{-1}\mathbf{P}$ are not calculated explicitly, in particular $\mathbf{M}$ is reversed by *Code_Aster*.

In the typical case of the triangle with 3 nodes, the fields are supposed to be constant by element (only one point of GAUSS) and:

$$\mathbf{M}^{-1}\mathbf{P}=\begin{bmatrix}1\\1\\1\end{bmatrix}$$

For example, the calculation carried out by the option `SIGM_ELNO` is then the following:

If the constraints were calculated on a family having a number of points of GAUSS higher than the number of tops (for announced elements Ci - above). $\mathbf{M}^{-1}\mathbf{P}$ is then rectangular, and $\hat{\boldsymbol{\sigma}}_{\mathbf{i}}=\sum_{i=1}^{\text{nb sommets}}\sum_{k=1}^{\text{nb pts Gauss}}(\mathbf{M}^{-1}\mathbf{P})_{ik}\boldsymbol{\sigma}^{\mathbf{k}}$

If not, if the number of points of GAUSS is equal to the number of tops, $\mathbf{M}^{-1}\mathbf{P}$ is then square. One calculates $\hat{\boldsymbol{\sigma}}_{\mathbf{i}}=(\mathbf{M}^{-1}\mathbf{P})_{ik}\boldsymbol{\sigma}^{\mathbf{k}}$ [§4.1].

5 Other options of calculation using the same method

The method described previously is used in *Code_Aster* to calculate the deformations, internal variables and flows with the nodes.

The produced fields are `cham_elem` with the nodes.

6 Other methods of smoothing of constraints

There exist two other methods of smoothing, relating only to the constraints, used by the estimators of Zhu-Zienkiewicz version 1 and 2 [R4.10.01 §3].

The stress fields to the produced nodes are then `cham_no`.

The corresponding options of calculation are accessible by the order `CALC_ERREUR` [U4.81.06].

7 Bibliography

1) BARLOW J. - Optimal stress hirings in finite element models - International Newspaper for Numerical Methods in Engineering Vol.10 p 243 - 251 (1976).

2) HINTON E., CAMPBELL JJ. - Total room and smoothing of discontinuous finite element functions using has least public gardens method - International Newspaper for Numerical Methods in Engineering Vol.8 p 461 - 480 (1974).

3) HINTON E., SCOTT F.C., RICKETTS R.E. - Room least public gardens stress smoothing for parabolic isoparametric elements - Int. J. for Num. Meth. in Eng. Flight 9 p 235 - 256 (1975)

8 Description of the versions of the document

Version Aster	**Author (S) Organization (S)**	**Description of the modifications**
10.2	*X. DESROCHES (EDF/IMA/T62)*	Small corrections