Hexahedral element at a point of integration, stabilized by the method "Assumed Strain"

Summary:

The hexahedral element with 8 standard nodes under-integrated into 1 point of integration introduces parasitic modes associated with a worthless energy (modes of sand glass) and can lead to a singularity of the matrix of total stiffness for certain boundary conditions. The deficiency of the row of the matrix of stiffness, due to under-integration, must thus be filled by adding with elementary rigidity a matrix known as of stabilization. It is the object of method ASM (Assumed Strain Method) developed here.

The main feature of this method is that the operator discretized gradient B necessarily does not derive from the field of displacement and the classical relations connecting the deformation to displacement. Indeed, this method ASM consists in projecting the operator gradient discretized on a suitable subspace in order to avoid the various types of blocking.

Contents

Contents

1 Introduction

In calculations finite elements, the recourse to the methods of under-integration makes it possible to reduce times calculations to a significant degree, which explains their success. The other objective of these methods is to draw aside the various blockings met in the numeric work implementation of the finite elements.

However, this under-integration does not have only advantages: it unfortunately introduces parasitic modes associated with a worthless energy, which lead to modes of sand glass, which will deform the grid in an unrealistic way and end up exploding the solution. This is due to a deficiency of the row of the matrix of stiffness due to under-integration. One cures it by adding elementary rigidity a matrix known as of stabilization. The core of the new rigidity obtained by this means must be reduced to the only modes corresponding to the rigid movements of body.

These last years, certain authors developed various elements based on technique ASM (Assumed Strain Method). The main feature of this method is that the operator discretized gradient B necessarily does not derive from the field of displacement and the classical relations connecting the deformation to displacement. Indeed, this method ASM consists in projecting the operator gradient discretized on a suitable subspace in order to avoid the various types of blocking. This technique was largely used recently and led to several stabilized finite elements of type quadrangles to 4 nodes or hexahedrons with 8 nodes [1], [2], [3].

It is the element hexahedron with 8 nodes under-integrated into 1 point of integration and stabilized by method ASM, which had in Belytschko and Bindeman [2], which we describe in this document.

2 Formulation of element HEXA8 at a point of integration

2.1 Field of displacement and operator discretized gradient

In the hexahedral element, the space coordinates x_i are connected to the nodal coordinates x_{iI} by means of the isoparametric functions of form N_I by the formulas:

$$x_i=x_{iI} N_I(\xi,\eta,\zeta)=\sum_{I=1}^{8} N_I(\xi,\eta,\zeta)x_{iI}$$ **éq 1.1-1**

One will use in the continuation the convention of summation for the repeated indices. Indices into tiny i vary from 1 to 3 and represent the space directions. Indices in capital letter I vary from 1 to 8 and correspond to the nodes of the element.

The same functions of form are used to define the field of displacement of the element u_i according to nodal displacements u_{iI} :

$$u_i=u_{iI} N_I(\xi,\eta,\zeta)$$

Since the same functions of form apply to the coordinates and displacements, their material derivative is cancelled and the field speed can be given by:

$$v_i=v_{iI} N_I(\xi,\eta,\zeta)$$ **éq 1.1-2**

The preceding interpolation of the field speed will make it possible to define the rate of deformation and to write the relations connecting the deformations at the nodal speeds. The gradient $v_{i,j}$ field speed is written:

$$v_{i,j}=v_{iI}N_{I,j}$$

By convention, a comma preceding an index into tiny represents a differentiation compared to the space coordinates. The tensor rate of deformation D_{ij} is given then by the symmetrical part of the gradient speed: $D_{ij}=\frac{1}{2}(v_{i,j}+v_{j,i})$

Trilinear isoparametric functions of form are given $N_I(\xi,\eta,\zeta)$ defined by:

$$N_I(\xi,\eta,\zeta)=\frac{1}{8}(1+\xi_I\xi)(1+\eta_I\eta)(1+\zeta_I\zeta)$$

$\xi,\eta,\zeta\in[-1,1], I=1,\ldots,8$ with ξ_I,η_I,ζ_I being worth 1 or – 1 **éq 1.1-3**

These functions of form transform a unit cube in space (ξ,η,ζ) in an unspecified hexahedron in space $(x_1,x_2,x_3)=(x,y,z)$. By combining the equations **1.1-1, 1.1-2, 1.1-3**, one manages to develop the field speed as the sum of a constant term, linear terms in x_i , and of the terms utilizing functions h_α :

$$v_i=a_{0i}+a_{1i}x_1+a_{2i}x_2+a_{3i}x_3+c_{1i}h_1+c_{2i}h_2+c_{3i}h_3+c_{4i}h_4 \quad \text{I = 1,2,3} \quad \textbf{éq 1.1-4}$$

$$h_1=\eta\zeta \qquad h_2=\xi\zeta \qquad h_3=\eta\xi \qquad h_4=\xi\eta\zeta$$

Indeed, the equation **1.1-1** allows to write ξ,η,ζ according only to x_i , h_α and of a constant parameter. By then injecting these last relations in the equation **1.1-2** , the expressions are found **1.1-4** searched.

By evaluating the equation **1.1-4** at the nodes of the element, one arrives at the 3 systems of 8 equations following:

$$\dot{d}_i=a_{0i}s+a_{1i}x_1+a_{2i}x_2+a_{3i}x_3+c_{1i}h_1+c_{2i}h_2+c_{3i}h_3+c_{4i}h_4 \qquad i=1,2,3 \quad \textbf{éq 1.1-5}$$

In the preceding equation and the continuation, the bold characters indicate tensors of order at least 1. Thus, vectors $\dot{\mathbf{d}}_i$ and x_i represent, respectively, the nodal speeds and coordinates and are given by:

$$\dot{d}_i^t=(v_{i1},v_{i2},v_{i3},\ldots,v_{i8})$$

$$x_i^t=(x_{i1},x_{i2},x_{i3},\ldots,x_{i8})$$

Vectors s and $h_\alpha(\alpha=1,\ldots,4)$ are given as for them by

$$s^t=(1,1,1,1,1,1,1,1)$$

$$h_1^t=(1,1,-1,-1,-1,-1,1,1)$$

$$h_2^t=(1,-1,-1,1,1,-1,1,1,-1)$$

$$h_3^t=(1,-1,1,-1,1,-1,1,-1)$$

$$h_4^t=(-1,1,-1,1,1,-1,1,-1)$$

To arrive at an advantageous writing of the operator discretized gradient $\mathbf{B}$, the 3 vectors are introduced b_i defined by: $b_i^T=N_{,i}(0)=\frac{\partial N}{\partial x_i}|_{\xi=\eta=\zeta=0}$ $\quad i=1,2,3$ **éq 1.1-6** where N represent $(N_1, N_2, \ldots, N_8)$. These vectors b_i represent the simplest shape of the operator discretized gradient under-integrated introduced by Hallquist and which is based on the evaluation of derived from the isoparametric functions of form at the origin of the reference frame (ξ, η, ζ). They can be explicitly given. Moreover, one can check the following conditions of orthogonality:

$$b_i^T . h_\alpha = 0 \quad , \quad b_i^T . s = 0 \quad , \quad b_i^T . x_j = \delta_{ij}$$

$$h_\alpha^T . s = 0 \quad , \quad h_\alpha^T . h_\beta = 8 \delta_{\alpha\beta} \qquad \textbf{éq 1.1-7}$$

$$i = 1 \, à \, 3 \quad , \quad \alpha, \beta = 1 \, à \, 4$$

where δ indicate the symbol of Kronecker.

This stage, one can determine the constant unknown factors which intervene in the writing of the field speed (**éq 1.1-4)** by multiplying scalairement the equation (**éq 1.1-5)** by $\mathbf{b_j^T}$ and $\mathbf{h_\alpha^T}$, respectively, and by using the relations of orthogonality above. One obtains:

$$a_{ij} = b_j^T . \dot{d}_i \quad , \quad \boldsymbol{c}_{\alpha i} = \boldsymbol{\gamma}_\alpha^T . \dot{\boldsymbol{d}}_i \qquad \text{with } \gamma_\alpha = \frac{1}{8} \left[h_\alpha - \sum_{j=1}^{3} (h_\alpha^T . x_j) b_j \right]$$

The field speed is put finally in the following very practical form:

$$v_i = a_{0i} + (x_1 b_1^T + x_2 b_2^T + x_3 b_3^T + h_1 \gamma_1^T + h_2 \gamma_2^T + h_3 \gamma_3^T + h_4 \gamma_4^T) . \dot{d}_i \quad i = 1,2,3 \qquad \textbf{éq 1.1-8}$$

By differentiating the formula above compared to x_j, the gradient speed is obtained:

$$v_{i,j} = (b_j^T + \sum_{\alpha=1}^{4} h_{\alpha,j} \gamma_\alpha^T) . \dot{d}_i = (b_j^T + h_{\alpha,j} \gamma_\alpha^T) . \dot{d}_i$$

The operator discretized gradient connecting the tensor rate of deformation to the vector nodal speeds by:

$$\nabla_s v = B . \dot{d}$$

where:

$$\nabla_s \boldsymbol{v} = \begin{bmatrix} v_{x,x} \\ v_{y,y} \\ v_{z,z} \\ v_{x,y} + v_{y,x} \\ v_{x,z} + v_{z,x} \\ v_{y,x} + v_{z,y} \end{bmatrix}, \qquad \dot{\boldsymbol{d}} = \begin{bmatrix} \dot{d}_x \\ \dot{d}_y \\ \dot{d}_z \end{bmatrix}$$

takes the practical matric shape then:

$$B=\begin{bmatrix} \boldsymbol{b}_x^T+\boldsymbol{h}_{\alpha,x}\boldsymbol{\gamma}_\alpha^T & 0 & 0 \\ 0 & \boldsymbol{b}_y^T+\boldsymbol{h}_{\alpha,y}\boldsymbol{\gamma}_\alpha^T & 0 \\ 0 & 0 & \boldsymbol{b}_z^T+\boldsymbol{h}_{\alpha,z}\boldsymbol{\gamma}_\alpha^T \\ \boldsymbol{b}_y^T+\boldsymbol{h}_{\alpha,y}\boldsymbol{\gamma}_\alpha^T & \boldsymbol{b}_x^T+\boldsymbol{h}_{\alpha,x}\boldsymbol{\gamma}_\alpha^T & 0 \\ \boldsymbol{b}_z^T+\boldsymbol{h}_{\alpha,z}\boldsymbol{\gamma}_\alpha^T & 0 & \boldsymbol{b}_x^T+\boldsymbol{h}_{\alpha,x}\boldsymbol{\gamma}_\alpha^T \\ 0 & \boldsymbol{b}_z^T+\boldsymbol{h}_{\alpha,z}\boldsymbol{\gamma}_\alpha^T & \boldsymbol{b}_y^T+\boldsymbol{h}_{\alpha,y}\boldsymbol{\gamma}_\alpha^T \end{bmatrix}$$ **éq 1.1-9**

Vectors γ_α who intervene in $\boldsymbol{B}$ check the following conditions of orthogonality:

$$\boldsymbol{\gamma}_\alpha^T.\boldsymbol{x}_j=0 \quad , \quad \boldsymbol{\gamma}_\alpha^T\boldsymbol{h}_\beta=\delta_{\alpha\beta}$$

An element based on this formulation is convergent when it is evaluated exactly. However, the evaluation of this operator $\boldsymbol{B}$ in each point of integration this element too expensive for the applications makes practices, and the simplified shape of this element is essential.

2.2 Variational formulation of the problem

The extension of the weak form of the variational principle of Hu-Washizu to the case of the mechanics of the nonlinear solids is written for a simple element Ω_e

$$\delta\pi(v,\bar{\boldsymbol{\epsilon}},\bar{\boldsymbol{\sigma}})=\int_{\Omega_e}\delta\bar{\boldsymbol{\epsilon}}^T.\ \boldsymbol{\sigma}\ d\Omega+\delta\int_{\Omega_e}\bar{\boldsymbol{\sigma}}^T\ .\ \left(\nabla_s v-\bar{\boldsymbol{\epsilon}}\right)d\Omega-\delta\dot{d}^T\ .\ f^{ext}=0$$ **éq 1.2-1**

where δ represent a variation, v the field speed, $\bar{\boldsymbol{\epsilon}}$ the rate of applied deformation, $\bar{\sigma}$ the applied constraint, σ the constraint evaluated by the constitutive law, $\dot{d}$ nodal speeds, f^{ext} external nodal forces and $\nabla_s v$ the symmetrical part of the gradient of the field speed.

The formulation "Assumed strain" retained in the continuation to build the element is based on a simplified form of the variational principle of Hu-Washizu which is based on the fact that the applied constraint is selected orthogonal with the difference between the symmetrical part of the gradient speed and the rate of applied deformation. Thus the second term of equation 1.2-1 is eliminated and one obtains:

$$\delta\pi(\bar{\boldsymbol{\epsilon}})=\int_{\Omega_e}\delta\bar{\boldsymbol{\epsilon}}^T.\ \sigma\ d\Omega-\delta\dot{d}^T\ .\ f^{ext}=0$$ **éq 1.2-2**

In this form, the variational principle is independent of the interpolation of the constraint, since the applied constraint does not intervene any more and thus need does not have to be defined. The discretized equations thus require the only interpolation speed v and of the rate of applied deformation $\bar{\boldsymbol{\epsilon}}$ in the element. If N_I indicate the isoparametric functions of form of the element, one a:

$\boldsymbol{v}(x,t)=\sum_{I=1}^{m}N_I(x)\dot{d}_I(t)$ where m indicate the number of nodes. One from of deduced:

$\nabla_s\boldsymbol{v}(x,t)=\boldsymbol{B}(x).\dot{\boldsymbol{d}}(t)$

The rate of applied deformation $\bar{\epsilon}$ is defined as for him by: $\bar{\epsilon}(x,t)=\bar{B}\ (x).\ \dot{d}\ (t)$ **éq 1.2-3**

By replacing the equation **1.2-3** in the variational principle **1.2-2**, one obtains:

$$\delta \dot{d}^T \,.\, \int_{\Omega_e} \bar{B}^T \,.\, \sigma \; d\Omega - \delta \dot{d}^T \,.\, f^{ext} = 0$$

Like $\delta \dot{d}^T$ can be arbitrarily selected, the preceding equation leads to:

$$f^{\text{int}} = f^{ext}$$

with $$f^{\text{int}} = \int_{\Omega_e} \bar{B}^T \,.\, \sigma\left(\bar{\dot{\varepsilon}}\right) d\Omega$$

In the equation above, it is well specified that the constraint σ is calculated by the law constitutive starting from the rate of applied deformation $\bar{\dot{\varepsilon}}$. For the nonlinear problems, σ can also be a function of the integral of the rate of applied deformation and other internal variables. The formulation thus obtained is valid for problems including the two types of non-linearities: geometrical and material.

In the case of linear problems, one a:

$$\boldsymbol{\sigma} = \boldsymbol{C} \,.\, \bar{\boldsymbol{\varepsilon}} = \boldsymbol{C} \,.\, \bar{\boldsymbol{B}} \,.\, \boldsymbol{d}$$

The internal forces of the element are written:

$$\boldsymbol{f}^{\text{int}} = \boldsymbol{K}_e \,.\, \boldsymbol{d} \qquad \text{with} \qquad \boldsymbol{K}_e = \int_{\Omega_e} \bar{\boldsymbol{B}}^T \,.\, \boldsymbol{C} \,.\, \bar{\boldsymbol{B}} \, d\Omega$$

In a standard approach in displacement, the rate of applied deformation is identified with the symmetrical part of the gradient speed, which amounts replacing $\bar{\boldsymbol{B}}$ by $\boldsymbol{B}$ in the preceding expressions. One obtains:

$$\boldsymbol{K}_e = \int_{\Omega_e} \boldsymbol{B}^T \,.\, \boldsymbol{C} \,.\, \boldsymbol{B} \, d\Omega$$

2.3 Modes of "hourglass" associated with a worthless energy

The matric writing **éq 1.1-9** from the operator discretized gradient will allow to understand the origin of the modes of hourglass, or modes of sand glass. As one will see it, these kinematics modes are due to under-integration and are associated with a worthless energy whereas they induce a nonworthless deformation. This anomaly is explained by the difference that induced under-integration, between the core of the operator of rigidity discretized and the core of the continuous operator of rigidity.
Let us notice initially that the operator under-integrated discretized gradient (i.e associated with only one point of integration located at the center of the element) is reduced to:

$$\boldsymbol{B} = \begin{bmatrix} b_x^T & 0 & 0 \\ 0 & b_y^T & 0 \\ 0 & 0 & b_z^T \\ b_y^T & b_x^T & 0 \\ b_z^T & 0 & b_x^T \\ 0 & b_z^T & b_y^T \end{bmatrix} \qquad \textbf{éq 1.3-1}$$

Indeed, terms $h_{\alpha,i}$ equation **éq 1.1-9** cancel themselves at the point of integration $|\xi = \eta = \zeta = 0$.

Now let us analyze the core of the matrix of rigidity obtained by integration. In the linear case, this elementary matrix is written: $\boldsymbol{K}_e = V\, \boldsymbol{B}^T \,.\, \boldsymbol{C} \,.\, \boldsymbol{B}$ where V indicate the volume of the element.

The examination of the core of under-integrated rigidity returns under investigation from the row of the matrix $\boldsymbol{B}$. It is thus necessary to seek the modes speed $\dot{\boldsymbol{d}}$ with worthless deformation, i.e. checking:

$$\nabla_s \boldsymbol{v} = \boldsymbol{B} . \dot{\boldsymbol{d}} = 0 \qquad \textbf{éq 1.3-2}$$

One must find in the core of K_e the modes associated with the movements with rigid body, is in 3D, 3 translations and 3 rotations.
The core of the continuous operator is thus of dimension 6 and is reduced to the 6 vectors:

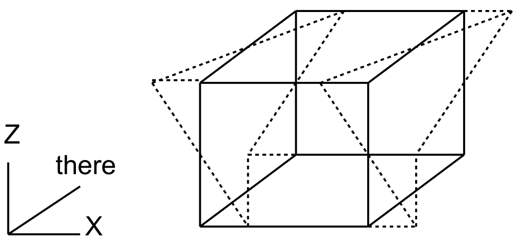
$$\begin{pmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{pmatrix} \begin{pmatrix} y & z & 0 \\ -x & 0 & z \\ 0 & -x & -y \end{pmatrix}$$

The first 3 vectors column check **1.3-2** because of $\boldsymbol{b}_i^T . \boldsymbol{s} = 0$, the 3 last because of $b_i^T . x_j = \delta_{ij}$.

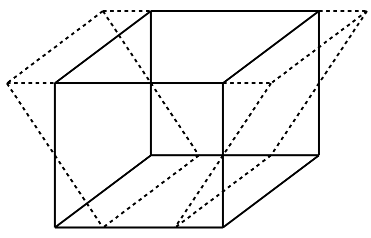
But in addition to the preceding rigid modes, the operator discretized gradient given in **1.3-1** cancel them 12 other following vectors:

$$\begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_1 & 0 \\ 0 & 0 & h_1 \end{pmatrix} \begin{pmatrix} h_2 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_2 \end{pmatrix} \begin{pmatrix} h_3 & 0 & 0 \\ 0 & h_3 & 0 \\ 0 & 0 & h_3 \end{pmatrix} \begin{pmatrix} h_4 & 0 & 0 \\ 0 & h_4 & 0 \\ 0 & 0 & h_4 \end{pmatrix}$$

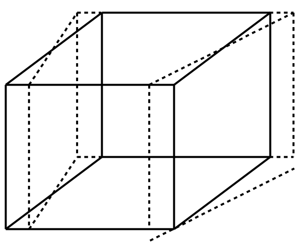
because $b_i^T . h_\alpha = 0$
The vectors columns above are called modes of sand glass or modes of hourglass.
4 modes according to Ox are below illustrated:



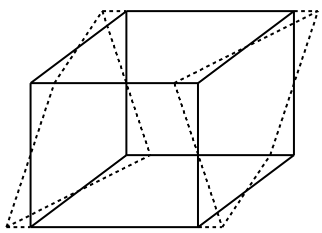
$h_1 warping$



$h_2 bending$



$h_3 bending$



$h_4 nonphysique$

Modes of sand glass according to Ox according to Belytschko and Bindeman

The operator discretized gradient given in **1.3-1** and that integrated exactly given in **1.1-9** the gradients of the linear fields calculate correctly and give identical results when they are applied to the 6 modes of rigid body. On the other hand, the exact formula of B calculate the other modes properly h_α of deformation, whereas the under-integrated form gives erroneous results.

Deficiency in the row of the operator B under-integrated (row 6 only instead of 18 for B integrated exactly) results in unrealistic oscillations of the grid leading to solutions which explode for certain boundary conditions.

These under-integrated elements thus require techniques known as of stabilization.

3 Stabilization of the type "Assumed Strain Method" (ASM)

The approach followed to stabilize the under-integrated hexahedral element is that of Belytschko and Bindeman [2]. Method developed to control the fashions of hourglass in an under-integrated element is known as Assumed Strain Method. It implies moreover the construction of the suitable shape of the matrix B making it possible to avoid digital blockings.

3.1 Formulas of Hallquist and Flanagan-Belytschko

Up to now, to establish the formulas **1.1-9** of the operator B, one used the vectors $\boldsymbol{b}_i$ based on the derivation of the functions of form in the beginning (equation **1.1-6**). This writing of B is known as form of Hallquist. One second form is considered now, based on the median values of the derivative of the functions of form of the element. This form is due to Flanagan and Belytschko and is written:

$$\hat{\boldsymbol{b}}_i^T = \frac{1}{V} \int_{\Omega_e} \boldsymbol{N}_{,i}(\xi, \eta, \zeta)\, d\Omega ¿ \qquad i=1,2,3$$

These vectors can be calculated explicitly. They can also be integrated numerically exactly.
The conditions of orthogonality of the §1.2 remain checked excluded the first on the elements nonparallelipipedic:

$$\hat{\boldsymbol{b}}_i^T . \boldsymbol{h}_j \begin{cases} =0 & \text{sur les parallélipipèdes} \\ \neq 0 & \text{sinon} \end{cases}$$

$\hat{\boldsymbol{b}}_i^T . \boldsymbol{h}_4 = 0$ on all the elements.

By supposing still true this relation of orthogonality, the gradient of displacement is written like previously:

$$\boldsymbol{v}_{i,j} = (\hat{\boldsymbol{b}}_j^T + \boldsymbol{h}_{\alpha,j} \hat{\boldsymbol{\gamma}}_\alpha^T) . \dot{\boldsymbol{d}}_i \qquad \text{with } \hat{\boldsymbol{\gamma}}_\alpha = \frac{1}{8}\left[\boldsymbol{h}_\alpha - \sum_{j=1}^{3} (\boldsymbol{h}_\alpha^T . \boldsymbol{x}_j) \hat{\boldsymbol{b}}_j\right]$$

One can continue to write: $\nabla_s v = \hat{B} \cdot \dot{d}$
where $\hat{B}$ appears as the sum of a constant term $\hat{B}_c$ and of a nonconstant term $\hat{B}_n$ defined by:

$$\hat{\boldsymbol{B}} = \hat{\boldsymbol{B}}_c + \hat{\boldsymbol{B}}_n$$

$$\hat{\boldsymbol{B}}_c=\begin{bmatrix}\hat{b}_x^T & 0 & 0\\ 0 & \hat{b}_y^T & 0\\ 0 & 0 & \hat{b}_z^T\\ \hat{b}_y^T & \hat{b}_x^T & 0\\ \hat{b}_z^T & 0 & \hat{b}_x^T\\ 0 & \hat{b}_z^T & \hat{b}_y^T\end{bmatrix}\qquad \hat{\boldsymbol{B}}_n=\begin{bmatrix}\hat{X}_{1234}^T & 0 & 0\\ 0 & \hat{Y}_{1234}^T & 0\\ 0 & 0 & \hat{Z}_{1234}^T\\ \hat{Y}_{1234}^T & X_{1234}^T & 0\\ \hat{Z}_{1234}^T & 0 & X_{1234}^T\\ 0 & \hat{Z}_{1234}^T & \hat{Y}_{1234}^T\end{bmatrix}$$

with $\hat{X}_{1234}^T=\sum_{\alpha=1}^{4} h_{\alpha,x}\hat{\boldsymbol{\gamma}}_{\boldsymbol{\alpha}}^T$ $\quad\hat{Y}_{1234}^T=\sum_{\alpha=1}^{4} h_{\alpha,y}\hat{\boldsymbol{\gamma}}_{\boldsymbol{\alpha}}^T$ $\quad\hat{Z}_{1234}^T=\sum_{\alpha=1}^{4} h_{\alpha,z}\hat{\boldsymbol{\gamma}}_{\boldsymbol{\alpha}}^T$

This formulation, even if it is less rigorous than that of Hallquist (since it supposes true all the relations of orthogonality) gives better results in terms of precision and convergence.

3.2 Projection on a field of deformation $\bar{\boldsymbol{B}}$

One applies a projection to the operator discretized gradient $\hat{\boldsymbol{B}}$ to deduce an operator from it $\bar{\boldsymbol{B}}$ having certain good properties. The objective of projection is double:
•It makes it possible, on the one hand, to eliminate voluminal blocking from the finite element in the incompressible case
•It avoids, in addition, blocking due under the excessive terms of transverse shearing in the problems with dominant inflection.

The operator $\hat{\boldsymbol{B}}$ is replaced by an operator $\bar{\boldsymbol{B}}$ such as:

$$\begin{cases}\hat{\boldsymbol{B}}=\hat{\boldsymbol{B}}_c+\hat{\boldsymbol{B}}_n\\ \bar{\boldsymbol{B}}=\hat{\boldsymbol{B}}_c+\bar{\boldsymbol{B}}_n\end{cases}$$

Only the nonconstant part $\hat{\boldsymbol{B}}_n$ is projected, the constant part $\hat{\boldsymbol{B}}_c$ remain unchanged.

Two different projections lead to the 2 following finite elements:

•Element ASQBI (Assumed Strain Quintessential Bending Incompressible)!

$$\bar{\boldsymbol{B}}_n=\begin{bmatrix}\hat{X}_{1234}^T & -\bar{\nu}\,\hat{Y}_3^T-\nu\,\hat{Y}_{24}^T & -\bar{\nu}\,\hat{Z}_2^T-\nu\,\hat{Z}_{34}^T\\ -\bar{\nu}\,\hat{X}_3^T-\nu\,\hat{X}_{14}^T & \hat{Y}_{1234}^T & -\bar{\nu}\,\hat{Z}_1^T-\nu\,\hat{Z}_{34}^T\\ -\bar{\nu}\,\hat{X}_2^T-\nu\,\hat{X}_{14}^T & -\bar{\nu}\,\hat{Y}_1^T-\nu\,\hat{Y}_{24}^T & \hat{Z}_{1234}^T\\ \hat{Y}_{12}^T & \hat{X}_{12}^T & 0\\ \hat{Z}_{13}^T & 0 & \hat{X}_{13}^T\\ 0 & \hat{Z}_{23}^T & \hat{Y}_{23}^T\end{bmatrix}\qquad \text{éq 2.2-1}$$

Where: $\bar{\nu}=\dfrac{\nu}{1-\nu}$, $\quad\hat{\boldsymbol{X}}_{14}^T=h_{1,x}\hat{\boldsymbol{\gamma}}_{\boldsymbol{1}\boldsymbol{T}}+h_{4,x}\hat{\boldsymbol{\gamma}}_{\boldsymbol{4}\boldsymbol{T}}$ $\quad\hat{\boldsymbol{Z}}_2^T=h_{2,z}\hat{\boldsymbol{\gamma}}_{\boldsymbol{2}\boldsymbol{T}}$

•Element ADS (Assumed Deviatoric Strain):

$$\bar{\boldsymbol{B}}_n=\begin{bmatrix} \frac{2}{3}\hat{\boldsymbol{X}}_{1234}^T & -\frac{1}{3}\hat{\boldsymbol{Y}}_{1234}^T & -\frac{1}{3}\hat{\boldsymbol{Z}}_{1234}^T \\ -\frac{1}{3}\hat{\boldsymbol{X}}_{1234}^T & \frac{2}{3}\hat{\boldsymbol{Y}}_{1234}^T & -\frac{1}{3}\hat{\boldsymbol{Z}}_{1234}^T \\ -\frac{1}{3}\hat{\boldsymbol{X}}_{1234}^T & -\frac{1}{3}\hat{\boldsymbol{Y}}_{1234}^T & \frac{2}{3}\hat{\boldsymbol{Z}}_{1234}^T \\ \hat{\boldsymbol{Y}}_{12}^T & \hat{\boldsymbol{X}}_{12}^T & 0 \\ \hat{\boldsymbol{Z}}_{13}^T & 0 & \hat{\boldsymbol{X}}_{13}^T \\ 0 & \hat{\boldsymbol{Z}}_{23}^T & \hat{\boldsymbol{Y}}_{23}^T \end{bmatrix} \quad \textbf{éq 2.2-2}$$

The diagrams of projection are detailed in [2]. One will indicate nevertheless the broad outlines of obtaining their operators $\bar{\boldsymbol{B}}_n$.

For the element ADS, that initially amounts breaking up the operator $\hat{\boldsymbol{B}}_n$ in the sum of a spherical term and a term deviatoric. Then, the spherical part under-is integrated (i.e evaluated into the point $\xi=\eta=\zeta=0$, which comes down cancelling it). This procedure thus makes it possible to treat the diagonal terms of the deformation, therefore voluminal blocking, and one can check in **éq 2.2-2** that the sum of the first three terms in each vector column is worthless. To avoid blocking in transverse shearing, it is necessary to act this time on the nondiagonal terms of the deformation, and thus to cancel those responsible for an excessive shearing. The result of this stage results in the suppression of certain terms in the three last lines of the operator $\hat{\boldsymbol{B}}_n$ (to compare the formulas giving $\hat{\boldsymbol{B}}_n$ and $\bar{\boldsymbol{B}}_n$).

For element ASQBI, the treatment of blocking in shearing is the same one as for element ADS. It results from it that the three last lines from the operator $\bar{\boldsymbol{B}}_n$ are identical in the two elements. On the other hand, the treatment of voluminal blocking is a little different for element ASQBI (see **éq 2.2-1**). One can check however that, when $\nu\rightarrow\frac{1}{2}$, therefore $\bar{\nu}\rightarrow 1$, the sum of the first three terms in each vector column of $\bar{\boldsymbol{B}}_n$ is worthless.

3.3 Choice of the finite elements

One compared two finite elements ADS and ASQBI on a certain number of tests. It arises that ASQBI gives better results that ADS in elasticity whereas ADS gives better results in plasticity, its behavior being a little too flexible in the elastic cases. The case test of the beam comforts in inflection [V6.04.196] in is a good illustration. These observations are in good conformity with those of Belytschko and Bindeman [2].

One thus chose to establish in Aster element ASQBI in elasticity and element ASD in plasticity to profit in each case from the best element. Moreover, one avoids with the user having to choose the finite element. It is pointed out that the only difference between the two elements is the matrix $\bar{\boldsymbol{B}}_n$.

4 Integration of the element in *Code_Aster*

4.1 Description and use

This element is pressed on the voluminal meshs 3D HEXA8.

4.1.1 Modeling

Modeling is affected 3D_SI with the meshs HEXA8 indicated. Usual elements of face of modeling 3D are affected on the meshs QUAD4.

4.1.2 Material

All the coefficients material relating to the valid laws of behavior in small deformations for modelings 3D are usable.

4.1.3 Boundary conditions and loading

All the loadings and boundary conditions available on the elements 3D and the elements of face are available.

4.1.4 Options of postprocessing

All the options of postprocessing usually available for modelings 3D are usable.

4.1.5 Calculation in linear buckling

The option RIGI_MECA_GE being activated in the catalogue of the element, it is possible to carry out a classical calculation of buckling after assembly of the matrices of elastic and geometrical rigidity.

4.1.6 Calculations nonlinear

All the behaviors available for modeling 3D usual are usable, in small deformations. With regard to the great deformations, the only option usable is the 'approximation PETIT_REAC.

Digital integration is carried out with a point of Gauss, just like in nonlinear material.

4.2 Establishment

The options are activated in the catalogue meca_hexs8.catastrophes.

4.3 Validation

The tests validating this element are, in version 9 of *Code_Aster* :

- SSNV196 : Beam 3D in inflection in elasticity and plasticity
- HSNV125G, PERFE01A, SDNV103, SSND105 use only one element to validate nonlinear behaviors, thus allowing time-savings calculation (1 only point of Gauss)

5 Conclusion

The element hexahedron with 8 nodes under-integrated into a point of integration and stabilized by method ASM gives correct results on problems where HEXA8 standard blocks (inflection, shearing), as CAS-test SSNV196 shows it. Moreover it gives significant profits in time calculation.

It presents however a disadvantage in statics (the same one as the similar element 2D QUAD4 under-integrated stabilized): when the elements are not parallelepipedic, the results are affected, even false if the elements have angles distant from 90°. The recommended solution is then to resort on these meshs to HEXA8 standard (at 8 points of Gauss). In dynamics clarifies on the other hand, this disadvantage does not remain. But the current establishment relates to only statics.

6 Bibliography

[1] T. BELYTSCHKO and L.P. BINDEMAN. Quadrilateral Assumed strain stabilization of the 4-node with 1-point squaring for nonlinear problems. Comput. Methods Appl. Mech. Eng., 88:311 - 340, 1991.

[2] T. BELYTSCHKO and L.P. BINDEMAN. Assumed strain stabilization of the 8-node hexahedral element. Comput. Methods Appl. Mech. Eng., 105:225 - 260, 1993.

[3] F.ABED-MERAÏM and A.COMBESCURE. Stabilization of the under-integrated finite elements. *Report interns n°247 LMT-Cachan, January 2001.*

7 Description of the versions of the document:

Version Aster	Author (S), Organization (S)	Description of the modifications
9.5	X Desroches EDF R & D AMA	Initial version