Digital modeling of the mean structures: axisymmetric thermoelastoplastic hulls

Summary:

One presents a digital formulation for the modeling of the structures to average surface of particular geometry: hulls with symmetry of revolution around the axis Oy (modeling COQUE_AXIS).

One completely describes the isotropic thermoelastoplastic case, within the framework as of theories of COILS - KIRCHHOFF and of HENCKY-MINDLIN-REISSNER, as well as the various studied loadings, for the selected isoparametric finite element.

Contents

1 Introduction

One is interested in what follows to the mechanical modeling of mean structures to average surface of particular geometry: hulls with symmetry of revolution around the axis Oy

More particularly, one limits oneself if the mechanical parameters (materials, loadings) are independent of a direction of space (the circumference for the hulls of revolution).

For the resolution of chained thermomechanical problems, one must use before the finite element of thermal hull describes in [R3.11.01] according to the case in his axisymmetric version.

One gives hereafter first of all a progress report on the description of the mechanical model: kinematics, thermoelastoplastic law of behavior. Then one presents the selected finite element, the interpolation and the method of integration.

One gives finally some digital results of application, by comparison with analytical solutions.

2 Continuous problem

The geometry is defined in a unidimensional way by the meridian line in the plan $(O\,xy)$ for a hull of revolution.

2.1 Description of the geometry, kinematics

One considers a hull of revolution of axis Oy . For this hull, average surface is defined by the curve $\omega = AB$ in the plan $O\,xz$: ω is a meridian line for the hull of revolution.

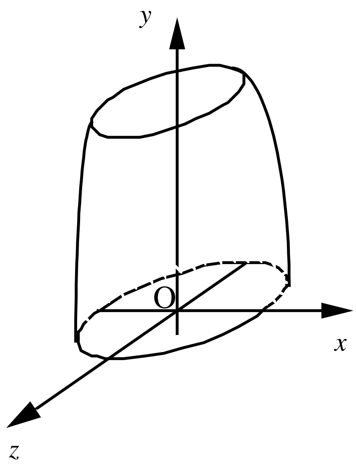


Figure 2.1-1: Hull of revolution

The curve $\omega = AB$ is parameterized by the curvilinear X-coordinate s . One will note the derivative partial $\frac{\partial A}{\partial s}$ of a quantity A by $A_{,s}$. In a point m of ω the local reference mark is defined (n, t, e_z) by:

$$t = \frac{\mathbf{Om}_{,s}}{\|\mathbf{Om}_{,s}\|} \quad \text{and} \quad n \wedge t = e_z \tag{1}$$

One notes also the angle α such as:

$$n = \cos\alpha e_x + \sin\alpha e_y \quad (2)$$

Curve of ω is defined by:

$$\frac{1}{R} = -n \cdot t_{,s} = \alpha_{,s} \quad (3)$$

The position on the parallel passing by m is noted θ. The tangent vector on this parallel is e_θ. For the meridian line located in the plan Oxz, $\theta = 0$ and $e_\theta = -e_z$. The radius of curvature of the parallel in m is:

$$R_\theta = \frac{r}{\cos\alpha} \quad (4)$$

Where r is the X-coordinate x point m of ω. The transformations kinematics of the hull are defined by displacement U point m average surface, like by rotation β_s normal n at the point m. The vector U can be expressed in local base:

$$U_{(s)} = U_{(s)} \cdot t_{(s)} + W_{(s)} \cdot n_{(s)} \quad (5)$$

Or in Cartesian base:

$$U_{(s)} = u_x(s) \cdot e_x + u_y(s) \cdot e_y \quad (6)$$

Deformations of the hull associated with this transformation (U, β_s) are determined by:

- a tensor of membrane deformation E,
- a tensor of variation of curve K,
- a vector of deformation of distortion tranverse γ.

This last appears in the theory of hulls of HENCKY-MINDLIN-NAGHDI and not in that of COILS. According to displacement U and of rotation β_s, these sizes are expressed (cf [bib1]) if U is expressed in local base (n, t, e_z) :

$$\begin{gathered} E_{ss} = U_{,s} + \frac{W}{R} \quad E_{\theta\theta} = \frac{1}{r}(-U\sin\alpha + W\cos\alpha) \\ K_{ss} = \beta_{s,s} \quad K_{\theta\theta} = -\frac{\sin\alpha}{r}\beta_s \\ \gamma_s = \beta_s + W_{,s} - \frac{U}{R} \end{gathered} \quad (7)$$

And for the case where U is expressed in total base (e_x, e_y, e_z) :

$$\begin{gathered} E_{ss} = u_{y,s}\cos\alpha - u_{x,s}\sin\alpha \quad E_{\theta\theta} = \frac{u_x}{r} \\ K_{ss} = \beta_{s,s} \quad K_{\theta\theta} = -\frac{\sin\alpha}{r}\beta_s \\ \gamma_s = \beta_s + u_{x,s}\cos\alpha + u_{y,s}\sin\alpha \end{gathered} \quad (8)$$

Note:

> *Change of direction of the curvilinear X-coordinate s do not modify the values of: β_s E_{ss}* and *$E_{\theta\theta}$ but the sign changes of α, U, W, R, K_{ss} and $K_{\theta\theta}$.*

Within the framework of the theory of COILS, the condition $\gamma_s = 0$ (the normals with the hull remain it after deformation) results in a direct relationship between rotations β_s and the slope $W_{,s}$. The components of the tensor variation of curve are according to displacement in the local base:

$$K_{ss} = -W_{,ss} + \frac{U_{,s}}{R} - U\frac{R_{,s}}{R^2} \text{ and } K_{\theta\theta} = \frac{\sin\alpha}{r}\left(W_{,s} - \frac{U}{R}\right) \quad (9)$$

If displacement is expressed in total base:

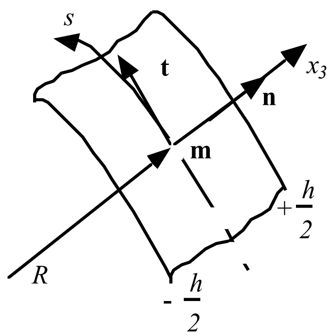
$$K_{ss}=\frac{1}{R}\left(u_{x,s}\sin\alpha-u_{y,s}\cos\alpha\right)-u_{x,ss}\cos\alpha-u_{y,ss}\sin\alpha$$

and (10)

$$K_{\theta\theta}=\frac{\sin\alpha}{r}\left(u_{x,s}\cos\alpha+u_{y,s}\sin\alpha\right)$$

It is noticed that the expression of the variations of curve according to displacement in theory of COILS is rather complicated and that it utilizes derivative second. If one requires an interpolation conforms i.e. here C^1 , this requires the use of finite elements of high degree.

Tensors E , K and γ allow to express the three-dimensional deformation ε in the thickness. One indicates by x_3 the position in the thickness $\left]-\frac{h}{2},\frac{h}{2}\right[$ compared to average fibre, at the point m , of curvilinear X-coordinate s on ω .



In a point thickness, displacement is expressed in total reference mark:

$$U(s,x_3)=\left(u_x(s)-\beta_s(s).x_3\ \sin\alpha(s)\right).e_x+\left(u_y(s)+\beta_s(s).x_3\cos\alpha(s)\right).e_y \quad (11)$$

In order to take account of the variation of metric in the thickness (due to the curve of average surface), the functions are defined:

$$\rho_s(x_3)=1+\frac{x_3}{R} \text{ and } \rho_\theta(x_3)=1+\frac{x_3}{r}.\cos\alpha \quad (12)$$

For a sufficiently thin hull, this correction is negligible:

$$\rho_s\approx 1 \text{ and } \rho_\theta\approx 1 \quad (13)$$

In practice this correction carried out if `MODI_METRIQUE=' OUI'` in `AFFE_CARA_ELEM` [U4.42.01] is useless if the reports $\frac{h}{R}$ and $\frac{h}{R_\theta}$, when they exist, are lower than $\frac{1}{15}$.

In theory of HENCKY-MINDLIN-NAGHDI, the components of the tensor of deformation ε are:

$$\begin{cases}\varepsilon_{ss}(s,x_3)=\dfrac{1}{\rho_s}\left(E_{ss}+x_3K_{ss}\right)\\ \varepsilon_{\theta\theta}(s,x_3)=\dfrac{1}{\rho_\theta}\left(E_{\theta\theta}+x_3K_{\theta\theta}\right)\\ \varepsilon_{sx_3}(s,x_3)=\dfrac{1}{2\rho_s}\gamma_s\end{cases} \quad (14)$$

2.2 Thermoelastoplastic balance

It is considered that the material constitutive of the hull is thermoelastoplastic isotropic. The usually allowed assumption is made that the transverse normal constraint is worthless $\sigma_{x_3 x_3} \equiv 0$. The most general law of behavior is written then:

$$\begin{pmatrix} s_{11} \\ s_{22} \\ s_{1x_3} \end{pmatrix} = \begin{pmatrix} C_{1111} & C_{1122} & 0 \\ C_{2211} & C_{2222} & 0 \\ 0 & 0 & C_{11x_3x_3} \end{pmatrix} \begin{pmatrix} \varepsilon_{11}-\varepsilon_{11}^{th} \\ \varepsilon_{22}-\varepsilon_{22}^{th} \\ \varepsilon_{1x_3} \end{pmatrix} \quad (15)$$

Where $C(\varepsilon, \mu)$ components C_{ijkl} is the local matrix of behavior in plane constraints and μ represent the whole of the internal variables when the behavior is non-linear. In the continuation the index 1 refers to the curvilinear X-coordinate s and 2 with θ . With the above definite three-dimensional deformations, one associates the components of the tensor then forced σ :

$$\begin{cases} \sigma_{ss} = C_{ssss}(\varepsilon_{ss}-\varepsilon_{ss}^{th}) + C_{ss\theta\theta}(\varepsilon_{\theta\theta}-\varepsilon_{\theta\theta}^{th}) \\ \sigma_{\theta\theta} = C_{\theta\theta ss}(\varepsilon_{ss}-\varepsilon_{ss}^{th}) + C_{\theta\theta\theta\theta}(\varepsilon_{\theta\theta}-\varepsilon_{\theta\theta}^{th}) \\ \sigma_{sx_3} = \quad C_{ssx_3x_3}\varepsilon_{sx_3} \end{cases} \quad (16)$$

One draws the expression from it from the elastic energy of deformation, which one will deduce the matrix from rigidity according to the kinematics of hull seen in the paragraph [§2.1]:

$$W^{\text{él}} = \frac{1}{2}\int_{\omega}\int_0^{2\pi}\int_{-h/2}^{h/2} \begin{bmatrix} C_{ssss}\varepsilon_{ss}^2 + \\ C_{\theta\theta\theta\theta}\varepsilon_{\theta\theta}^2 + \\ (C_{\theta\theta}+C_{\theta\theta ss})\varepsilon_{ss}\varepsilon_{\theta\theta} + \\ 2\mathrm{C}_{ssx_3x_3}\varepsilon_{sx_3^2} \end{bmatrix} (\rho_s+\rho_\theta-1)\, r\, ds\, d\theta\, dx_3 \quad (17)$$

Note:

In thermoelasticity, if one notes E the YOUNG modulus and ν the Poisson's ratio, one has $C_{iiii} = \frac{E}{1-\nu^2}$, $C_{iijj} = \frac{E\nu}{1-\nu^2} \forall (i,j) \in [1,2]$ *and* $C_{11x_3x_3} = \frac{E}{1+\nu}$

The following sizes are defined:

- membrane rigidity:

$$[C_{ij}] = \int_{-h/2}^{h/2} \frac{\rho_s+\rho_\theta-1}{\rho_i\rho_j} . \begin{bmatrix} C_{ssss} & C_{ss\theta\theta} \\ C_{\theta\theta ss} & C_{\theta\theta\theta\theta} \end{bmatrix} dx_3 \quad (18)\ \mathrm{Q}$$

  who is worth $\frac{Eh}{1-\nu^2}\begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix}$ in elasticity and absence of correction of metric in the thickness;

- the rigidity of coupling membrane-inflection:

$$[B_{ij}] = \int_{-\frac{h}{2}}^{\frac{h}{2}} x_3 . \frac{\rho_s+\rho_q-1}{\rho_i\rho_j} . \begin{bmatrix} C_{ssss} & C_{ss\theta\theta} \\ C_{\theta\theta ss} & C_{\theta\theta\theta\theta} \end{bmatrix} dx_3 \quad (19)$$

  who is worthless in elasticity and absence of correction of metric in the thickness;

- the rigidity of inflection:

$$[D_{ij}] = \int_{-\frac{h}{2}}^{\frac{h}{2}} x_3^2 . \frac{\rho_s+\rho_\theta-1}{\rho_i\rho_j} . \begin{bmatrix} C_{ssss} & C_{ss\theta\theta} \\ C_{\theta\theta ss} & C_{\theta\theta\theta\theta} \end{bmatrix} dx_3 \quad (20)$$

who is worth $\frac{Eh^3}{12(1-\nu^2)}\begin{bmatrix}1 & \nu\\ \nu & 1\end{bmatrix}$ in elasticity and absence of correction of metric in the thickness;

- the transverse rigidity of distortion:

$$G_{sx_3}=\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\rho_s+\rho_\theta-1}{\rho_s^2}.C_{ssx_3x_3}dx_3 \tag{21}$$

who is worth $\frac{Eh}{1+\nu}$ in elasticity and absence of correction of metric in the thickness.

In elasticity, coefficients C_{ss}^C, B_{ss}^C and D_{ss}^C are the products of the coefficients C_{ss}^D, B_{ss}^D and D_{ss}^D by $1-\nu^2$. One can thus express elastic energy according to the tensors of deformations of hull E, K and γ by:

$$\begin{aligned}W^{él}=\frac{1}{2}\int_\omega\int_0^{2\pi}[&C_{ss}E_{ss}^2+2B_{ss}E_{ss}K_{ss}+D_{ss}K_{ss}^2+C_{\theta\theta}E_{\theta\theta}^2+2B_{\theta\theta}E_{\theta\theta}K_{\theta\theta}+D_{qq}K_{\theta\theta}^2\\&+2\left(C_{s\theta}E_{ss}E_{\theta\theta}+B_{s\theta}\left(E_{ss}K_{\theta\theta}+E_{\theta\theta}K_{ss}\right)+D_{s\theta}K_{ss}.K_{\theta\theta}\right)\\&+\frac{G_{sx_3}}{2}\gamma_s^2]r.ds.d\theta\end{aligned} \tag{22}$$

With these expressions, it is necessary to add the potential associated with the thermal stresses, which will be a contribution to the second member (that one will express below in total reference mark):

$$L_{(V)}^{th}=\int_\omega\int_0^{2\pi}\int_{-h/2}^{h/2}\left[\alpha\left(T-T^{réf}\right)\left(\left(C_{ssss}+C_{ss\theta\theta}\right)\varepsilon_{ss}+\left(C_{\theta\theta ss}+C_{\theta\theta\theta\theta}\right)\varepsilon_{\theta\theta}\right)\right]r\,d\theta\,dx_3\,ds \tag{23}$$

This expression for an isotropic elastic behavior becomes:

$$L_{(V)}^{th}=\int_\omega\int_0^{2\pi}\int_{-h/2}^{h/2}\left[\frac{E\alpha}{1-\nu}\left(T-T^{réf}\right)\left(\frac{v_x}{r}-v_{x,s}\sin\alpha+v_{y,s}\cos\alpha+x_3\left(\beta_{s,s}-\frac{\sin\alpha}{r}\beta_s\right)\right)\right]r\,d\theta\,c \tag{24}$$

In this expression, one deliberately neglected the correction of metric in the thickness (terms in ρ_s and ρ_θ seen for rigidity). Moreover the temperature T who appears is defined by the thermal model of hull in three fields (cf [R3.11.01]):

$$T(s,x_3)=T^m(s).\left(1-\left(\frac{x_3}{h}\right)^2\right)+T^s(s)\frac{x_3}{2h}\left(1+\frac{x_3}{h}\right)+T^i(s)\frac{x_3}{2h}\left(-1+\frac{x_3}{h}\right) \tag{25}$$

From this expression, one deduces the tensors from generalized efforts N and M (normal efforts and bending moments) associated with the generalized deformations E and K by the principle of virtual work. They are related to the tensor of the constraints $\tau_{\alpha\beta}$ three-dimensional by:

$$N_{\alpha\beta}=\int_{-h/2}^{h/2}\tau_{\alpha\beta}dx_3 \text{ and } M_{\alpha\beta}=\int_{-h/2}^{h/2}x_3.\tau_{\alpha\beta}dx_3 \tag{26}$$

Where one neglected the variations of metric in the thickness.

Note:

Transverse energy of shearing

The model of hull presented above, said HENCKY-MINDLIN-NAGHDI, rests on a kinematic assumption: parameters W and β_s indicate the normal displacement of the point m average surface ω and the rotation of the normal vector n.

One also frequently finds the model known as of REISSNER which rests on a static assumption of the distribution of stresses shear transverse. The parameters kinematics deduced W and β_S in this model are weighted averages in the thickness of normal displacement and local rotations. If one wishes to place oneself within this framework, it is enough to affect the coefficient $\kappa=5/6$ at the end of transverse energy of shearing (in γ_s^2). (cf [bib7], [bib9]).

Lastly, if one wants, for a thin hull, to be located within the framework of the model of COILS - KIRCHHOFF, one can neutralize the energy of shearing with a great value of κ (which penalizes the condition $\gamma_s=0$), for example $10^6 h/R$, where h is the thickness and R a characteristic radius of curvature or a distance characteristic of the loadings: (cf [feeding-bottle 2]). In practice the user can inform the value of κ under the keyword `A_CIS` order `AFFE_CARA_ELEM` [U4.42.01].

3 Formulation of the finite element and discretization

3.1 Description of the selected finite element

3.1.1 Motivations

The choice of framework HENCKY-MINDLIN-NAGHDI to describe the kinematics of hull, presented to the paragraph [§2], led to expressions of the deformations where the derivative are limited to order 1, contrary to the model of LOVE-KIRCHHOFF. This offers the advantage of being able to use a finite element of a nature limited while ensuring conformity. The natural choice is the element of LAGRANGE $P2$, isoparametric, which makes it possible to have a fine representation of a curved geometry and good estimates of the constraints.

The degrees of freedom are of course displacements and rotations.

As it is known as previously, the model of LOVE-KIRCHHOFF can be recovered by penalization for a parameter κ very large affecting the transverse energy of shearing.

This formulation joined the category of the finite elements of hulls known as “degenerated”, i.e. built by injecting the kinematics of hull in elements of three-dimensional continuous mediums: cf [bib10].

As for all the finite elements of hulls, of the particular aspects must be analyzed: the taking into account of the rigid modes and the risks of blocking of membrane or shearing.

In the case of the axisymmetric hull of revolution, there is only one rigid mode: translation according to the axis of symmetry Oy .

So that the finite element is performing, it is necessary that the approximations retained for the description of displacement ensure an exact representation of the state of worthless deformations (rigid mode). In practice, as the concept of rigid mode is defined compared to the total reference mark one thus decides to describe displacements in total base (e_x, e_y) , in which the rigid modes (functions closely connected) are represented by the selected interpolation.

As for the risks of blocking out of membrane and transverse shearing, the usual treatment consists in a selective digital integration (cf [bib2]), but the practice reveals that these phenomena seldom appear for the hulls of revolution.

3.1.2 General presentation of the element

The selected element of reference is quadratic, isoparametric with three nodes and three degrees of freedom per node. These degrees of freedom are (see figure 3.1.2-1):

u_x, u_y : components of displacement U in total reference mark,

β_s : rotation around e_z normal n .

This element is a generalization of the element of plane beam curved. It is well adapted to the discretization of the hulls with meridian curve R variable, cf [bib2].

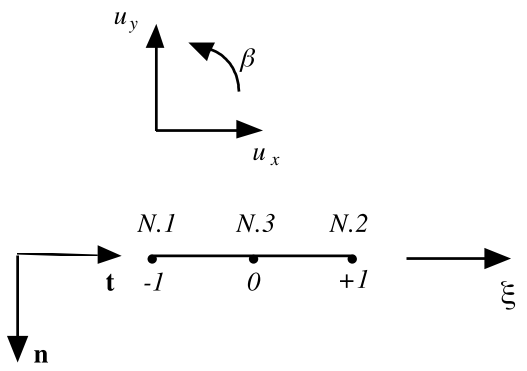
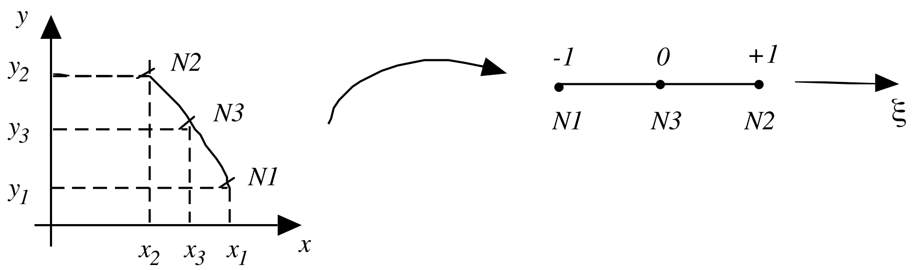


Figure 3.1.2-1: Élément of reference

The functions of form (basic) are the polynomials of LAGRANGE:

$$\hat{N}_1(\xi)=\xi\frac{-1+\xi}{2} \quad \hat{N}_2(\xi)=\xi\frac{1+\xi}{2} \quad \hat{N}_3(\xi)=1-\xi^2 \tag{27}$$

3.1.3 Transformations finite element towards finite element of reference



The geometry is interpolated using the coordinates $(x_i,\ y_i)$ of the three nodes $N1$, $N3$ and $N2$:

$$x(\xi)=\sum_{i=1}^{3} x_i\hat{N}_i(\xi) \text{ and } y(\xi)=\sum_{i=1}^{3} y_i\hat{N}_i(\xi) \tag{28}$$

In the same way using the degrees of freedom $(u_{x_i},\ u_{y_i},\ \beta_{s_i})$ on the nodes, one a:

$$u_x(\xi)=\sum_{i=1}^{3} u_{x_i}\hat{N}_i(\xi) \quad u_y(\xi)=\sum_{i=1}^{3} u_{y_i}\hat{N}_i(\xi) \quad \beta_s(\xi)=\sum_{i=1}^{3} \beta_{s_i}\hat{N}_i(\xi) \tag{29}$$

One also needs the jacobien of the transformation:

$$m(\xi)=\frac{ds}{d\xi}(\xi)=\sqrt{(x_{,\xi})^2+(y_{,\xi})^2} \tag{30}$$

And of the vectors of the local base:

$$t(\xi)=\frac{1}{m(\xi)}(x_{,\xi}e_x+y_{,\xi}e_z) \text{ and } n(\xi)=\frac{1}{m(\xi)}(y_{,\xi}e_x-x_{,\xi}e_z) \tag{31}$$

Finally:

$$\cos\alpha=\frac{y_{,\xi}}{m(\xi)} \text{ and } \sin\alpha=-\frac{x_{,\xi}}{m(\xi)} \tag{32}$$

The meridian curve is obtained by:

$$\frac{1}{R}=-(n.t_{,\xi}).\frac{d\xi}{ds}=\frac{1}{m^3(\xi)}(x_{,\xi}\cdot y_{,\xi\xi}-y_{,\xi}\cdot x_{,\xi\xi}) \tag{33}$$

Because of the interpolation $P2$, the derivative second which appears below express using the coordinates of the three nodes by:

$$x_{,\xi\xi}=x_1+x_2-2x_3 \text{ and } y_{,\xi\xi}=y_1+y_2-2y_3 \quad (34)$$

3.1.4 Surface digital integration

For digital integrations along the element one uses a digital formula of integration at four points of GAUSS, single for all the terms to be integrated. This formula reveals the blockings mentioned in the paragraph [§3.1.1] in the event of extremely localised plasticization. One thus advises to avoid the use of these elements in plasticity for the moment. The digital formula of integration at four points of Gauss will be replaced later on by a formula at two points of Gauss supposed to avoid these nuisances.

3.1.5 Digital integration in the thickness

For an elastic behavior, insofar as it is admitted that one limits oneself to uniform elastic characteristics in the thickness, rigidities $[C_{ij}]$, $[B_{ij}]$, $[D_{ij}]$ and G_{sx_3} defined in the paragraph [§2.2] are calculated exactly.

For a non-linear behavior, one subdivides the initial thickness in N layers thicknesses identical numbered in the direction of the normal to the average surface of the element. For each layer one uses three points of integration. The points of integration are located in higher skin of layer, in the middle of the layer and in lower skin of layer. For N layers, the number of points of integration is of $2N+1$. One advises to use from 3 to 5 layers in the thickness for a number of points of integration being worth 7.9 and 11 respectively.
For each layer, the state of the constraints is calculated $(\sigma_{11,}\sigma_{22,}\sigma_{12})$ and the whole of the internal variables, in the middle of the layer and in skins higher and lower of layer, starting from the local plastic behavior and of the local field of deformation $(\varepsilon_{11,}\varepsilon_{22,}\varepsilon_{12})$. The positioning of the points of integration enables us to have the rightest estimates, because not extrapolated, in skins lower and higher of layer, where it is known that the constraints are likely to be maximum. The plastic behavior does not understand for the moment the terms of transverse shearing which are treated in an elastic way, because transverse shearing is uncoupled from the membrane behavior in plane constraints.

Coordinates of the points	Weight ω_l
ξ_1=-1	$1/3$
$\xi_2=0$	$4/3$
$\xi_3=+1$	$1/3$
$\int_{-1}^{1} y(\xi)d\xi=$	$\sum_{i=1}^{n}\omega_i y(\xi_i)$

Digital formula of integration for a layer in the thickness in plasticity

For a thermoelastic behavior, one uses integration, by layer in the thickness $\left]-\frac{h}{2},+\frac{h}{2}\right[$ described previously in the non-linear field, of the thermomechanical terms seen in the paragraph [§2.2]. It is then necessary to use `STAT_NON_LINE` with an elastic behavior.

Note:

> *One already mentioned with [the §2.2]. and in [R3.07.04] that the value of the coefficient of correction in transverse shearing for the elements of hull was obtained by identification of elastic complementary energies after resolution of balance 3D. This method is not more usable in elastoplasticity and the choice of the coefficient of correction in transverse shearing*

is posed then. The transverse terms of shearing are thus not affected by plasticity and are treated elastically, for want of anything better. If one places oneself in theory of Coils-Kirchhoff for a value of this coefficient of $10^6 h/R$ (h being the thickness of the hull and R its average radius of curvature) the transverse terms of shearing become negligible and the approach is more rigorous.

3.2 Formulation of the elementary terms

3.2.1 Mass, centre of gravity, matrix of inertia

In the case of the hulls of revolution, the mass is worth:

$$\int_\omega \int_0^{2\pi} \int_{-h/2}^{h/2} \rho(\rho_s+\rho_\theta-1)\, dx_3\, r\, d\theta\, ds = \int_\omega \int_0^{2\pi} \rho\, h\, r\, d\theta\, ds = 2\pi\rho h \int_\omega r\, ds \tag{35}$$

Where ρ is the presumedly constant density of the element. The position of the centre of inertia is given in the reference mark $Oxyz$ [§2.1] by:

$$\begin{cases} x_G = 0 \\ y_G = \dfrac{\int_\omega \mathrm{yr}\, ds + \dfrac{h^2}{12}\int_\omega \sin\alpha\left(\dfrac{1}{R}+\dfrac{\cos\alpha}{r}\right) r ds}{\int_\omega r\, ds} \\ z_G = 0 \end{cases} \tag{36}$$

Terms of the matrix of inertia compared to O in the reference mark $Oxyz$ [§2.1] have then as an expression:

$$\begin{aligned} I_{xx/O} &= 2\pi\rho \int_\omega \left[h\left(\frac{x^2}{2}+y^2\right) + \frac{h^3}{12}\left(\sin^2\alpha + \frac{\cos^2\alpha}{2} + \delta\, x\cos\alpha + 2\,\delta\, y\sin\alpha\right)\right] r ds \\ I_{yy/O} &= 2\pi\rho \int_\omega \left[hx^2 + \frac{h^3}{12}\left(\cos^2\alpha + 2\,\delta\, x\cos\alpha\right)\right] r ds \\ I_{zz/O} &= 2\pi\rho \int_\omega \left[h\left(\frac{x^2}{2}+y^2\right) + \frac{h^3}{12}\left(\sin^2\alpha + \frac{\cos^2\alpha}{2} + \delta\, x\cos\alpha + 2\,\delta\, y\sin\alpha\right)\right] r ds \end{aligned} \tag{37}$$

Where $\delta = \dfrac{1}{R} + \dfrac{\cos\alpha}{r}$.

3.2.2 Matrix of mass

The kinetic term of energy is treated by considering the density ρ constant in the thickness and the correction of metric due to the negligible curve:

$$\int_\omega \int_0^{2\pi} \int_{-h/2}^{h/2} \rho\, v(s, x_3) . \bar{v}(s, x_3)\, r\, dx_3\, d\theta\, ds \tag{38}$$

The intégrande is burst in three terms. Kinetic energy of translation:

$$\rho h \left(u_x . \bar{u}_x + u_y . \bar{u}_y\right) \tag{39}$$

Kinetic energy of rotation:

$$\rho \frac{h^3}{12} \beta_s . \bar{\beta}_s \tag{40}$$

And kinetic energy of coupling:

$$\rho \frac{h^3}{12} \delta \left(-\sin\alpha\left(u_x \bar{\beta}_s + \bar{u}_x \beta_s\right) + \cos\alpha\left(u_y \bar{\beta}_s + \bar{u}_y \beta_s\right)\right) \tag{41}$$

With

$$\delta=\frac{1}{R}+\frac{\cos\alpha}{r}.$$

3.2.3 Second member of centrifugal force

In the case of the hulls of revolution, a vector rotation is considered $\Omega=\omega_2.e_y$, carried by the axis of revolution. The term of the second corresponding member is:

$$\begin{aligned}&\int_\omega\int_0^{2\pi}\int_{-h/2}^{h/2}\rho\,\omega_2^2.r\left(\bar{u}_x-\bar{\beta}_s.x_3\,\sin\alpha\right)dx_3\,r\,d\theta\,ds\\&=\int_\omega\int_0^{2\pi}h\,\rho\,\omega_2^2r^2.\bar{u}_x\,d\theta\,ds\end{aligned}\qquad(42)$$

By neglecting the correction of metric in the thickness). The second member is then:

$$\int_\omega h\,\rho\,\omega_3^2\left(x.\bar{u}_x+y.\bar{u}_y\right)ds\qquad(43)$$

3.2.4 Second member of gravity

Gravity is directed according to e_y . The second member is:

$$\int_\omega\int_0^{2\pi}\rho\,g\,h\bar{u}_y r\,d\theta\,ds\qquad(44)$$

And thus:

$$\int_\omega\rho\left(g_x.e_x+g_y.e_y\right)ds\qquad(45)$$

3.2.5 Second member of distributed loads

These distributed loads can be two forces in the plan (xOy) and couples it M_z carried by the axis Oz . The two forces, which one considers that they are applied to average surface ω , could be provided in total reference mark $(e_x,\,e_y)$ or room $(t,\,n)$. The second member is:

$$\int_\omega\int_0^{2\pi}\left(F_x\bar{u}_x+F_y\bar{u}_y+M_z\bar{\beta}_s\right)r\,d\theta\,ds\qquad(46)$$

Note:

The specific actions are treated as nodal forces where they are applied, since they work in the degrees of freedom of the finite element.

3.3 Calculation of the strains and the stresses

After resolution, there is the possibility with the operator `CALC_CHAMP` [U4.81.04] to calculate with the nodes the elementary fields according to:

- generalized deformations $E_{\alpha\beta}$ and $K_{\alpha\beta}$: option `DEGE_ELNO`,
- three-dimensional deformations $\varepsilon_{\alpha\beta}$ on average fibre and in skins internal and external (with or without correction of curve): option `EPSI_ELNO`,
- three-dimensional constraints $\sigma_{\alpha\beta}$ on average fibre and in skins internal and external (with or without correction of curve): option `SIGM_ELNO`,
- generalized efforts $N_{\alpha\beta}$ and $M_{\alpha\beta}$ (with or without correction of curve): option `EFGE_ELNO`.

These values with the nodes are obtained by extrapolation starting from the values at the points of GAUSS element, according to the exposed method in [bib4] [R3.06.03]. Lastly, one can have also the values $N_{\alpha\beta}$ and $M_{\alpha\beta}$ at the points of GAUSS of the element: option `SIEF_ELGA`. No postprocessing of constraints or generalized efforts is for the moment available for nonlinear behaviors materials.

4 Conclusion

The finite elements that we propose were selected with a quite particular aim: axisymmetric mean structural analysis with the concern of obtaining a good precision on the membrane and flexional solution while having a simple element of establishment and not too expensive.

The choice of the degrees of freedom allows a good representation of the boundary conditions. Moreover, this displacement formulation and rotation lead to elements of smaller degree: the elements are $P2$ out of membrane and $P2$ in inflection. It appears that they are easy to handle and that their formulation makes it possible to use a structure of pre and post simple processor, significant advantage to carry out rather fine grids (unidimensional) and to display the results easily (on a simple curve). Selected kinematics: formulation of HENCKY-MINDLIN-NAGHDI, in displacements and rotations of average surface makes it possible to utilize the transverse energy of shearing (interesting for the hulls average thickness).

This energy can be affected of a factor of correction k : if one wants to place oneself in theory of REISSNER, it is enough to choose $k=5/6$ instead of 1 (but of course, the arrow W and rotations β in this theory only weighted averages in the thickness are). Moreover, the formulation of hull of LOVE-KIRCHHOFF (for the very mean structures) can be simulated by penalization of the condition of nullity of the transverse distortion, by choosing a factor $k=10^6\times{}^{h}\!/_{L}$, h being the thickness and L a characteristic distance (radius of curvature, enforcement zone of the loads…).

The non-linear behaviors in plane constraints are available for these elements. It is announced however that the constraints generated by the transverse distortion are treated elastically, for want of anything better. Indeed, the taking into account of a transverse shearing constant not no one on the thickness and the determination of the correction associated on rigidity with shearing compared to a model satisfying the boundary conditions are not possible and thus return the use of these elements, when transverse shearing is nonnull, rigorously impossible in plasticity. In any rigour, for nonlinear behaviors, it would thus be necessary to use these elements within the framework of the theory of Coils-Kirchhoff.

Elements corresponding to the machine elements exist in thermics; the thermomechanical chainings are thus available with finite elements of thermal hulls to three nodes described in [R3.11.01] in its axisymmetric version.

In the CAS-test treated, the phenomena of blocking did not appear. The decomposition of the deformation energy will make it possible, where necessary, to integrate in a selective way the terms responsible for blocking, such a modification not having to raise particular difficulties. A more detailed study must of course be carried out on this subject, as for the digital methods to use to avoid this blocking when the thickness becomes low.

The possible developments are:

- anisotropy in order to be able to treat the multi-layer hulls,
- problems of buckling,
- decomposition in Fourier series to study nonaxisymmetric problems of hulls of revolution,
- the taking into account a variable thickness…

5 Bibliography

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6 Description of the versions of the document

Version Aster	Author (S) Organization (S)	Description of the modifications
4	F.VOLDOIRE, C.SEVIN EDF-R&D/AMA	Initial text
5	P.MASSIN, EDF-R&D/AMA	Update