Elements of plate: modelings `DKT`, `DST`, `DKTG` and `Q4G`

Summary:

These modelings of finite elements of plate are intended for calculations in small deformations and small displacements of curved or plane mean structures. They are finite elements plans which do not take into account the geometrical curve of the mean structures, contrary with the elements of hull which are curved: it results from it from the parasitic inflections which can be reduced by using more elements in order to be able to approach the curved geometries correctly. The formulation is thus simplified by it and the reduced number of degrees of freedom. These finite elements are famous as being among most precise for the calculation of displacements and the modal analysis.

For each one of these various modelings several finite elements are available, according to the meshs:

- modeling `DKT`, according to the model of inflection of Coils-Kirchhoff, comprises the finite elements triangular (`DKT`) and quadrangular (`DKQ`), which uses fields under-points, so for example integrating the relation of behavior in the layers constituting the thickness ;
- modeling `DST`, with transverse energy of shearing in elasticity, comprises the finite elements triangular (`DST`) and quadrangular (`DSQ`);
- modeling `DKTG`, according to the model of inflection of Coils-Kirchhoff, comprises the finite elements triangular (`DKTG`) and quadrangular (`DKQG`), dedicated to the “total” relations of behavior, which have only one sleep and one point of integration in the thickness;
- modeling `Q4G` (named too $Q4\gamma$) with transverse energy of shearing in elasticity, but with another interpolation that for modeling `DST`, comprises only the quadrangular finite element (`Q4G`).

Note:

In the document [R3.07.09], one presents modeling `Q4GG`, dedicated to thick plates. This modeling comprises quadrangular finite elements (`Q4G`) theoretical description is made in this document and the triangular elements (`T3G`) .

Contents

1 Introduction

The elements of hulls and plates are particularly used to model mean structures where the relationship between dimensions (characteristic thickness/length) is with more $1/10$. They thus intervene particularly in fields like the civil engineer, the interns of heart REFERENCE MARK, the vibratory analysis, the analysis of buckling of Euler, analysis of the mean elastic multi-layer composite material structures… One limits oneself to the framework of small displacements (even if it possible but is little recommended to use the reactualization 'PETIT_REAC') and of the small deformations.

Contrary to the elements of hull, the elements of plate plans do not make it possible to take into account the geometrical curve of the structure to be represented and induce parasitic inflections. It is thus necessary to use a large number of these elements in order to approach the geometry of the structure correctly, and this, more especially as it is curved. On the other hand, one gains in simplicity of formulation and the number of degrees of freedom is reduced. In addition, formulations "*Discrete Shear*"(DST, DSQ and Q4Γ) or "*Discrete Kirchhoff*"(DKT, DKTG and DKQ, DKQG) kinematics, with or without transverse distortion respectively, allow good performances in terms of displacements and modal analysis.

The meshs support of these finite elements are linear (triangles and quadrangles). The degrees of freedom as of these finite elements are the translations and rotations of the nodes tops. The characteristics that one affects to them are: the thickness, the coefficient of shearing, offsetting,…

The way in which these elements are established in *Code_hasster* as certain aspects of the use are given to [§5] present documentation.

In particular these formulations are classified:

modeling	finite elements	Use
DKT	DKT, DKQ	model of Coils-Kirchhoff, dedicated to the relations of linear behavior and nonlinear with fields under-points of integration in the thickness, cf § 4.9.
DKTG	DKTG, DKQG	model of Coils-Kirchhoff, dedicated to the "total" relations of behavior, which have only one sleep and one point of integration in the thickness
DST	DST, DSQ	model with transverse energy of shearing in elasticity
Q4G	Q4 Γ	model with transverse energy of shearing in elasticity

For sections thin out of material composite multi-layer rubber bands orthotropic, one will defer to [R4.01.01] and [U4.42.03], where one describes how one produces in a stage of preprocessing the homogenized elastic characteristics, out of membrane, inflection, transverse shearing (with modeling DST or Q4G).

2 Formulation

2.1 Geometry of the elements plates [1]

For the elements of plate one defines a surface of reference, or surfaces average, planes (plan x, y for example) and a thickness $h(x, y)$. This thickness must be small compared to other dimensions (extensions, radii of curvature) of the structure to model. [Figure 2.1-a] below our matter illustrates.

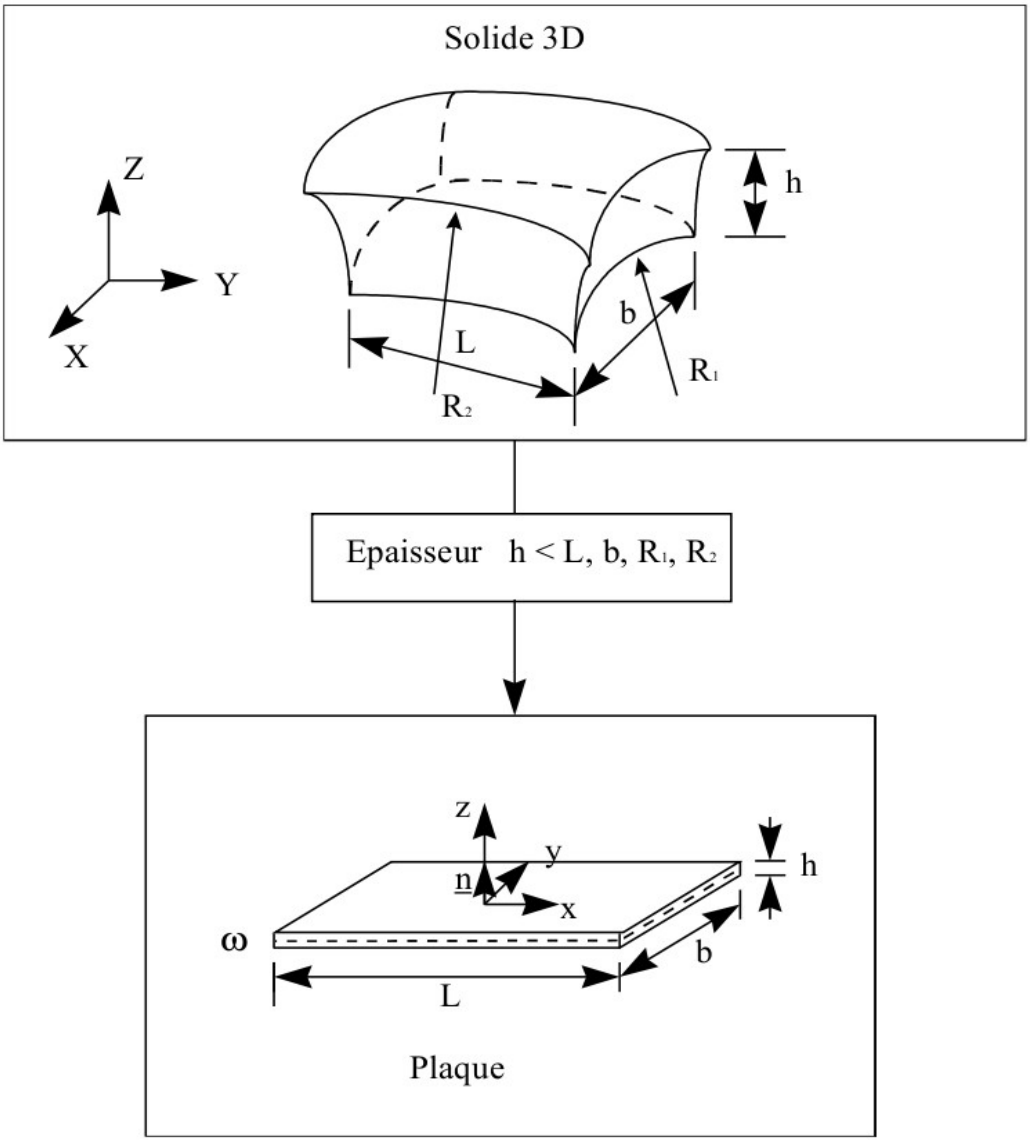


Figure 2.1-a

One attaches to average surface ω a local orthonormal reference mark $Oxyz$ associated with the tangent plan of the structure different from the total reference mark $OXYZ$. The position of the points of the plate is given by the Cartesian coordinates (x, y) average surface and rise z compared to this surface.

2.1.1 Intrinsic reference mark

By taking the local reference mark $Oxyz$ precedent with for origin the first top of the element and for axis Ox the side uniting tops 1 and 2, one defines the reference mark known as intrinsic.

2.2 Theory of the plates

These elements are based on the theory of the plates in small displacements and small deformations.

2.2.1 Kinematics

The cross-sections which are the sections perpendicular to average surface remain right; the material points located on a normal at not deformed average surface remain on a line in the deformed configuration. It results from this approach that **the fields of displacement vary linearly in the thickness of the plate**. If one indicates by u, v, w displacements of a point $q(x, y, z)$ according to x, y and z, there is thus the kinematics of Hencky-Mindlin:

$$\begin{pmatrix} u_x(x,y,z) \\ u_y(x,y,z) \\ u_z(x,y,z) \end{pmatrix} = \begin{pmatrix} u(x,y) \\ v(x,y) \\ w(x,y) \end{pmatrix} + z \begin{pmatrix} \theta_y(x,y) \\ -\theta_x(x,y) \\ 0 \end{pmatrix} = \begin{pmatrix} u(x,y) \\ v(x,y) \\ w(x,y) \end{pmatrix} + z \begin{pmatrix} \beta_x(x,y) \\ \beta_y(x,y) \\ 0 \end{pmatrix} \quad (1)$$

where u, v, w are displacements of average surface and θ_x and θ_y rotations of this surface compared to the two axes x and y respectively. One prefers to introduce two rotations $\beta_x(x,y) = \theta_y(x,y), \beta_y(x,y) = -\theta_x(x,y)$.

The three-dimensional deformations in any point, with kinematics introduced previously, are thus given by:

$$\begin{aligned} &\varepsilon_{xx} = e_{xx} + z\,\kappa_{xx} \\ &\varepsilon_{yy} = e_{yy} + z\,\kappa_{yy} \\ &2\,\varepsilon_{xy} = \gamma_{xy} = 2\,e_{xy} + 2z\kappa_{xy} \\ &2\,\varepsilon_{xz} = \gamma_x \\ &2\,\varepsilon_{yz} = \gamma_y \end{aligned} \quad (2)$$

where e_{xx}, e_{yy} and e_{xy} are the membrane deformations of average surface, γ_x and γ_y deformations associated with transverse shearings, and $\kappa_{xx}, \kappa_{yy}, \kappa_{xy}$ the deformations of inflection (or variations of curve) of average surface, which are written:

$$
\begin{aligned}
e_{xx} &= \frac{\partial u}{\partial x} \\
e_{yy} &= \frac{\partial v}{\partial y} \\
2 e_{xy} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \\
\kappa_{xx} &= \frac{\partial \beta_x}{\partial x} \\
\kappa_{yy} &= \frac{\partial \beta_y}{\partial y} \\
2 \kappa_{xy} &= \frac{\partial \beta_x}{\partial y} + \frac{\partial \beta_y}{\partial x} \\
\gamma_x &= \beta_x + \frac{\partial w}{\partial x} \\
\gamma_y &= \beta_y + \frac{\partial w}{\partial y}
\end{aligned}
\tag{3}
$$

Note:

In the theories of plate the introduction of β_x and β_y allows to symmetrize the formulations of the deformations and, we will see it thereafter, the equilibrium equations. In the theories of hull one uses rather θ_x and θ_y and associated couples M_x and M_y compared to x and y.

2.2.2 Law of behavior

The behavior of the plates is a behavior 3D in "plane constraints". **The transverse constraint σ_{zz} is worthless** because regarded as negligible compared to the other components of the tensor of the constraints (assumption of the plane constraints). The most general law of behavior is written then as follows:

$$
\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{pmatrix} = \mathbf{C}(e,\alpha) \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \\ \gamma_x \\ \gamma_y \end{pmatrix} = \mathbf{C}\mathbf{e} + z\,\mathbf{C}\,\kappa + \mathbf{C}\,\gamma \text{ with } \boldsymbol{e} = \begin{pmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \\ 0 \\ 0 \end{pmatrix}, \kappa = \begin{pmatrix} \kappa_{xx} \\ \kappa_{yy} \\ 2\kappa_{xy} \\ 0 \\ 0 \end{pmatrix} \text{ et } \gamma = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \gamma_x \\ \gamma_y \end{pmatrix}. \tag{4}
$$

where $\mathbf{C}(e,\alpha)$ is the matrix of local tangent rigidity combining forced plane and transverse distortion and α represent the whole of the internal variables when the behavior is nonlinear.

For behaviors where the transverse distortions are uncoupled from the deformations of membrane and inflection, $\mathbf{C}(e,\alpha)$ puts itself in the form:

$$
\mathbf{C} = \begin{pmatrix} \mathbf{H} & 0 \\ 0 & \mathbf{H}_\gamma \end{pmatrix} \tag{5}
$$

where $\mathbf{H}(e,\alpha)$ is a matrix 3×3 and $\mathbf{H}_\gamma(e,\alpha)$ a matrix 2×2. One will remain within the framework of this assumption.

For an isotropic homogeneous linear behavior elastic, one has as follows:

$$\mathbf{C}=\frac{E}{1-\nu^2}\begin{pmatrix}1 & \nu & 0 & 0 & 0\\ \nu & 1 & 0 & 0 & 0\\ 0 & 0 & \frac{1-\nu}{2} & 0 & 0\\ 0 & 0 & 0 & \frac{k(1-\nu)}{2} & 0\\ 0 & 0 & 0 & 0 & \frac{k(1-\nu)}{2}\end{pmatrix} \quad (6)$$

where k is factor of transverse correction of shearing whose significance is given to the following paragraph.

Note:

One does not describe the variation thickness nor that of the transverse deformation e_{zz} that one can however calculate by using the preceding assumption of plane constraints. In addition no restriction is made on the type of behavior that one can represent.

2.2.3 Taking into account of transverse shearing [2]

The taking into account of transverse shearing depends on factors of correction determined a priori by energy equivalences with models 3D, so that rigidity in transverse shearing of the model of plate is nearest possible to that defined by the theory of three-dimensional elasticity. Two theories including the deformation due to the shearing action exist and are presented in [2].

2.2.3.1 The theory known as of Hencky

This theory as that of Coils-Kirchhoff which results from this immediately rests on the kinematics presented to the §2.2.1. The relation of behavior is usual and the factor of correction of shearing is worth $k=1$.

Note:

When one does not take into account the transverse distortions γ_x and γ_y in the theory of Hencky, the model obtained is that of Coils-Kirchhoff (finite elements DKT (G) and DKQ (G)). Two rotations of average surface are then related to displacements of average surface by the following relation:

$$\beta_x=-\frac{\partial w}{\partial x}$$
$$\beta_y=-\frac{\partial w}{\partial y}$$

2.2.3.2 The theory known as of Reissner (DST, DSQ and Q4Γ)

The second theory, known as of Reissner, is developed starting from the constraints. Variation of the membrane stresses (σ_{xx} , σ_{yy} and σ_{xy}) is supposed to be linear in the thickness as in the case of the theory of Hencky where that results from the linearity of the variation of the deformations of membrane with the thickness.

However, whereas one supposes, in the theory of Hencky which the distortion is constant in the thickness and thus stresses shear, which violates the boundary conditions $\sigma_{xz}=\sigma_{yz}=0$ on the faces higher and lower of the plate because of law of behavior stated than the §2.2.2. , one uses within the framework of the theory of Reissner the equilibrium equations to deduce the variation from it from shear stresses in the thickness of the plate, by in particular observing the equilibrium conditions on the faces higher and lower of plate. Energy interns model obtained after resolution of the equations of balance in 3D, for inflection only, with the variation of the plane constraints according to z, reveals, for an elastic material, a relation between the resulting efforts and rotations and the arrow averages. It is in this relation that the factor of correction of shearing appears of $k=5/6$ instead of 1 in the relation which binds the shearing action to the distortion for a homogeneous and isotropic plate. The determination of the factors of correction of shearing for orthotropic plates or laminated plates is described in appendix.

2.2.3.3 Equivalence of the approaches Hencky-Coils-Kirchhoff and Reissner

If one assimilates the slopes of average surface β_x, β_y with the averages of the slopes in the thickness of the plate and the arrow w with the average arrow, the only difference between the theory of Hencky and that of Reissner are the coefficient of transverse correction of shearing in homogeneous elasticity isoptrope of $5/6$ instead of 1. This difference is due to the fact that the starting assumptions are of different nature and especially that the selected variables are not the same ones. Indeed, the arrow on average surface is not equal to the average of the arrows on the thickness of the plate. It is thus normal that relations of behavior which utilize different variables are not identical.

The fact of having to solve on the level finite elements of the problems in displacements rather than of the problems in constraints by interpolation of displacements leads us to use the equivalent approach in displacements of the problem of Reissner formulated in constraints.

2.2.3.4 Remarks

Because of preceding equivalence one presents here only the model in displacement for all the elements. In the facts elements DKT, DKTG, DKQG and DKQ are based on the theory of Hencky-Coils-Kirchhoff and elements DST, DSQ and Q4G are based on the theory of Reissner.

The determination of the factors of correction rests within the framework of another theory, that of Mindlin, on equivalences of Eigen frequency associated with the mode of vibration by transverse shearing. One obtains then $k=\pi^2/12$, value very close to $5/6$ for the DST elements, DSQ and Q4G in the isotropic case.

Within the framework of plasticity the problem of the choice of the coefficient of correction of transverse shearing arises because the equivalent approach in displacements of the problem of Reissner formulated in constraints utilizes the non-linearity of the behavior. One cannot thus deduce some, as it is the case for elastic materials a value of the coefficient of correction of transverse shearing. Plasticity (and other nonlinear behaviors) are thus not developed for these elements.

2.2.3.5 Calculation of the shear stress and the transverse distortion in C*ode_hasster*

The calculation of the shear stress is carried out while considering:

- equilibrium equations in constraint and generalized effort:

$$\begin{array}{l}\partial_z \tau_{xz}=\partial_x \sigma_{xx}+\partial_y \sigma_{xy}\\ \partial_z \tau_{yz}=\partial_y \sigma_{yy}+\partial_x \sigma_{xy}\end{array} \quad ; \quad \begin{array}{l}\partial_x M_{xx}+\partial_y M_{xy}=T_x\\ \partial_y M_{yy}+\partial_x M_{xy}=T_y\end{array} \tag{7}$$

- conditions of free edge $\sigma_{xz}(-h/2)=\sigma_{yz}(h/2)=0$ on the higher and lower faces;

- relations connecting the plane constraints to the derivative of the moments:

$$\begin{array}{l}\partial_x \sigma_{xx}=12/h^3 \cdot z \cdot \partial_x Mxx\\ \partial_x \sigma_{yy}=12/h^3 \cdot z \cdot \partial_y Myy\\ \partial_x \sigma_{xy}=12/h^3 \cdot z \cdot \partial_y Mxy\end{array} \tag{8}$$

Maybe after analytical integration compared to the variable z of $\sigma_{xz}(z), \sigma_{yz}(z)$ and identification of the coefficients of the primitive:

- the expression of the shear stress:

$\sigma_{xz}=T_x\cdot 3/2\mathrm{h}*(1-4z^2/h^2)\quad \sigma_{yz}=T_y\cdot 3/2\mathrm{h}\cdot(1-4z^2/h^2)$

– the expression of the shearing strain within the framework of the theory of Reissner:
$\gamma_x=\tau_{xz}\cdot 2(1+\nu)/Ek\quad \gamma_y=\tau_{yz}\cdot 2(1+\nu)/Ek$ with k the coefficient of correction in shearing

– the expression of the shearing strain within the framework of the theory of Kirchoff:
$\gamma_x=\tau_{xz}\cdot 2(1+\nu)/Ek-\partial_x w\quad \gamma_y=\tau_{yz}\cdot 2(1+\nu)/Ek-\partial_y w$

It is about an approximation which one also finds in [3] (eq 9).

One generalizes the expression of shear stresses by introducing a function $d1iel(z)$ such as:
$\sigma_{xz}=T_x\cdot d1iel(z)\quad \sigma_{yz}=T_y\cdot d1iel(z)$

In the classical cases, one a: $d1iel(z)=3/(2h)\cdot(1-4z^2/h^2)$.

In cases plus generals (in the presence of offsetting for example) $d1iel(z)$ must be modified to take into account the involved phenomenon. To approximate the shear stress correctly, one makes the choice to apply a general quadratic form for $d1iel(z)=a\cdot z^2+b\cdot z+c$ such that the following conditions are observed:

– $\int_{-h/2}^{h/2} d1iel(z)=1$ relation effort slice-constraints
– $d1iel(z=-h/2)=0; d1iel(z=h/2)=0$ condition of free edges

The first condition makes it possible automatically to respect the relations of balance in linear elasticity while the second makes it possible to find correct results on the nonconstrained edges. The extension of these relations to plasticity is not commonplace. However, in plasticity, one chooses to keep a linear elastic description of shear stresses.

The expression of $d1iel(z)$ in the case of offsetting is clarified in Doc. of dedicated reference [R3.07.06].

3 Principle of virtual work

3.1 Work of deformation

The general expression of the work of deformation 3D for a plate is worth:

$$W_{\mathrm{def}}=\int_S \int_{-h/2}^{h/2} (\varepsilon_{xx}\sigma_{xx}+\varepsilon_{yy}\sigma_{yy}+\gamma_{xy}\sigma_{xy}+\gamma_x\sigma_{xz}+\gamma_y\sigma_{yz})\,dV \tag{9}$$

where S is average surface and the position in the thickness of the plate varies between $-h/2$ and $+h/2$.

3.1.1 Expression of the resulting efforts

By adopting the kinematics of the §2.2.1, one identifies the work of the interior efforts:

$$W_{\mathrm{def}}=\int_S (e_{xx}N_{xx}+e_{yy}N_{yy}+2e_{xy}N_{xy}+\kappa_{xx}M_{xx}+\kappa_{yy}M_{yy}+2\kappa_{xy}M_{xy}+\gamma_x T_x+\gamma_y T_y)\,dS$$ where:

$$N=\begin{pmatrix}N_{xx}\\N_{yy}\\N_{xy}\end{pmatrix}=\int_{-h/2}^{+h/2}\begin{pmatrix}\sigma_{xx}\\\sigma_{yy}\\\sigma_{xy}\end{pmatrix}dz\ ;\ M=\begin{pmatrix}M_{xx}\\M_{yy}\\M_{xy}\end{pmatrix}=\int_{-h/2}^{+h/2}\begin{pmatrix}\sigma_{xx}\\\sigma_{yy}\\\sigma_{xy}\end{pmatrix}z\,dz\ ;\ T=\begin{pmatrix}T_x\\T_y\end{pmatrix}=\int_{-h/2}^{+h/2}\begin{pmatrix}\sigma_{xz}\\\sigma_{yz}\end{pmatrix}dz \tag{10}$$

N_{xx}, N_{yy}, N_{xy} are the efforts resulting from membrane (in N/m);
M_{xx}, M_{yy}, M_{xy} are the efforts resulting from inflection or moments (in N);
T_x, T_y are the efforts resulting from shearing or efforts cutting-edges (in N/m).

3.1.2 Relation efforts resulting-deformations

The expression of the work of deformation is also written:

$$W_{\mathrm{def}}=\int_s \int_{-h/2}^{h/2} [\varepsilon\,\mathbf{C}(e,\alpha)\,\varepsilon]\,dV=\int_s \int_{-h/2}^{h/2} [\mathbf{eCe}+z\,\mathbf{eC}\kappa+z\,\kappa\,\mathbf{Ce}+z^2\kappa\,\mathbf{C}\kappa+\gamma\,\mathbf{C}\gamma]\,dV \tag{11}$$

where $\mathbf{C}(e,\alpha)$ is the local matrix of behavior.

By using the expression obtained for W_{def} in the preceding paragraph one finds the relation following between the resulting efforts and the deformations:

$$\begin{array}{l}\mathbf{N}=\mathbf{H}_m\mathbf{e}+\mathbf{H}_{\mathrm{mf}}\kappa\\ \mathbf{M}=\mathbf{H}_{\mathrm{mf}}\mathbf{e}+\mathbf{H}_f\kappa\\ \mathbf{T}=\mathbf{H}_{\mathrm{ct}}\gamma\end{array}\quad \text{with}\quad \mathbf{H}_m=\int_{-h/2}^{+h/2}\mathbf{H}\,dz\,,\ \mathbf{H}_{\mathrm{mf}}=\int_{-h/2}^{+h/2}\mathbf{H}\,z\,dz\,,\ \mathbf{H}_f=\int_{-h/2}^{+h/2}\mathbf{H}\,z^2\,dz \tag{12}$$

where:

$$H_{\mathbf{ct}}=\begin{pmatrix} G_{11} & 0 \\ 0 & G_{22} \end{pmatrix} \qquad \mathbf{e}=\begin{pmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{pmatrix}, \kappa=\begin{pmatrix} \kappa_{xx} \\ \kappa_{yy} \\ 2\kappa_{xy} \end{pmatrix}, \gamma=\begin{pmatrix} \gamma_x \\ \gamma_y \end{pmatrix}$$

Matrices $\mathbf{H}_{\boldsymbol{m}}, \mathbf{H}_{\boldsymbol{f}} \text{ et } \mathbf{H}_{\boldsymbol{ct}}$ are the matrices of rigidity out of membrane, inflection and transverse shearing, respectively. The matrix $\mathbf{H}_{\boldsymbol{mf}}$ is a matrix of rigidity of coupling between the membrane and the inflection.

For an isotropic homogeneous elastic behavior of plate these matrices have as an expression:

$$\mathbf{H}_{\boldsymbol{m}}=\frac{Eh}{1-\nu^2}\begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{pmatrix}, \mathbf{H}_{\boldsymbol{f}}=\frac{Eh^3}{12(1-\nu^2)}\begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{pmatrix}, \mathbf{H}_{\mathbf{ct}}=\frac{kEh}{2(1+\nu)}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (13)$$

and $\mathbf{H}_{\boldsymbol{mf}}=0$ because there is material symmetry compared to the plan $z=0$.

For an orthotropic material, the behavior is given in appendix.

3.1.3 Energy interns elastic of plate

Taking into account the preceding remarks, energy interns elastic plate is more usually expressed for this kind of geometry in the following way:

$$\Phi_{\text{int}}=\frac{1}{2}\int_S [\mathbf{e}(\mathbf{H}_{\boldsymbol{m}}\mathbf{e}+\mathbf{H}_{\mathbf{mf}}\kappa)+\kappa(\mathbf{H}_{\mathbf{mf}}\mathbf{e}+\mathbf{H}_{\boldsymbol{f}}\kappa)+\gamma\mathbf{H}_{\mathbf{ct}}\gamma]dS \quad (14)$$

that one can break up in the following way:

$$\Phi_{\text{membrane}}=\frac{1}{2}\int_S \mathbf{e}\,\mathbf{H}_{\boldsymbol{m}}\mathbf{e}\,dS$$ energy of membrane

$$\Phi_{\text{flexion}}=\frac{1}{2}\int_S \kappa\,\mathbf{H}_{\boldsymbol{m}}\kappa\,dS$$ energy of inflection

$$\Phi_{\text{cisaillement}}=\frac{1}{2}\int_S \gamma\,\mathbf{H}_{\boldsymbol{ct}}\gamma\,dS$$ energy of shearing

$$\Phi_{\text{couplage}}=\frac{1}{2}\int_S (\mathbf{e}\,\mathbf{H}_{\mathbf{mf}}\kappa+\kappa\,\mathbf{H}_{\mathbf{mf}}\mathbf{e})\,dS$$ energy of coupling membrane-inflection

3.1.4 Remarks

Relations flexible $\mathbf{H}_{\boldsymbol{m}}$, $\mathbf{H}_{\boldsymbol{f}}$, $\mathbf{H}_{\boldsymbol{mf}}$ with $\mathbf{H}$ and $\mathbf{H}_{\boldsymbol{ct}}$ with $\mathbf{H}_{\boldsymbol{\gamma}}$ are valid whatever the law of behavior: rubber band, with unelastic deformations (thermoelasticity, plasticity,….).

For a plate made up of N orthotropic layers in elasticity, matrices $\mathbf{H}_{\boldsymbol{m}}$, $\mathbf{H}_{\boldsymbol{f}}$, $\mathbf{H}_{\boldsymbol{mf}}$ and $\mathbf{H}_{\boldsymbol{ct}}$ are written:

$$\mathbf{H}_{m}=\sum_{i=1}^{N} h_i \mathbf{H}_i \,,\, \mathbf{H}_{mf}=\sum_{i=1}^{N} h_i \eta_i \mathbf{H}_i \,,\, \mathbf{H}_f=\sum_{i=1}^{N} \frac{1}{3}(z_{i+1}^3-z_i^3)\mathbf{H}_i \,,\, \mathbf{H}_{ct}=\sum_{i=1}^{N} h_i \mathbf{H}_{\gamma i} \tag{15}$$

where: $h_i=z_{i+1}-z_i \,,\, \eta_i=\frac{1}{2}\left(z_{i+1}+z_i\right)$ and $\mathbf{H}_i, \mathbf{H}_{\gamma i}$ the matrices represent $\mathbf{H}$ and $\mathbf{H}_\gamma$ for the layer i.

The homogenisation for multi-layer hulls can lead to matrices of rigidity of membrane and inflection nonproportional of the type:

$$\mathbf{H}_m=\begin{pmatrix} C_{1111} & C_{1122} & 0 \\ C_{1122} & C_{2222} & 0 \\ 0 & 0 & C_{1212} \end{pmatrix}, \mathbf{H}_f=\begin{pmatrix} D_{1111} & D_{1122} & 0 \\ D_{1122} & D_{2222} & 0 \\ 0 & 0 & D_{1212} \end{pmatrix}, \mathbf{H}_{ct}=\begin{pmatrix} G_{11} & 0 \\ 0 & G_{22} \end{pmatrix} \tag{16}$$

for which one cannot find equivalent values of the Young modulus and thickness allowing to find the classical expressions of rigidity, cf. [4].

3.2 Work of the forces and external couples

The work of the forces and couples being exerted on the plate is expressed in the following way:

$$W_{\text{ext}}=\int_S \int_{-h/2}^{+h/2} \mathbf{F}_v . \mathbf{U}\, dV+\int_S \mathbf{F}_s . \mathbf{U}\, dS+\int_C \int_{-h/2}^{+h/2} \mathbf{F}_c . \mathbf{U}\, dz\, ds \tag{17}$$

where $\mathbf{F}_v, \mathbf{F}_s, \mathbf{F}_c$ are the voluminal, surface efforts and of contour being exerted on the plate, respectively. C is the part of the contour of the plate on which efforts of contour $\mathbf{F}_c$ are applied. With the kinematics of the §2.2.1, one determines as follows:

$$\begin{aligned} W_{\text{ext}}&=\int_S (f_x u+f_y v+f_z w+c_x\theta_x+c_y\theta_y)\, dS+\int_C (\phi_x u+\phi_y v+\phi_z w+\chi_x\theta_x+\chi_y\theta_y)\, ds \\ &=\int_S (f_x u+f_y v+f_z w+c_y\beta_x-c_x\beta_y)\, dS+\int_C (\phi_x u+\phi_y v+\phi_z w+\chi_y\beta_x-\chi_x\beta_y)\, ds \end{aligned}$$

•where are present on the plate:

f_x, f_y, f_z : surface forces acting according to x, y and z

$f_i=\int_{-h/2}^{+h/2} \mathbf{F}_v . \mathbf{e}_i dz+\mathbf{F}_s . \mathbf{e}_i$ where $\mathbf{e}_x$ and $\mathbf{e}_y$ are the basic vectors of the tangent plan and $\mathbf{e}_z$ their normal vector.

c_x, c_y : surface couples acting around the axes x and y.

$c_i=\int_{-h/2}^{+h/2} (z\,\mathbf{e}_z \wedge \mathbf{F}_v) . \mathbf{e}_i dz+(\pm\frac{h}{2}\mathbf{e}_z \wedge \mathbf{F}_s) . \mathbf{e}_i$ where $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ are the basic vectors previously definite.

•and where are present on the contour of the plate:

ϕ_x, ϕ_y, ϕ_z : linear forces acting according to x, y and z

$\phi_i = \int_{-h/2}^{+h/2} \mathbf{F}_c . \mathbf{e}_i dz$ where $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ are the basic vectors previously definite.

χ_x, χ_y : linear couples around the axes x and y .

$\chi_i = \int_{-h/2}^{+h/2} (z\, \mathbf{e}_z \wedge \mathbf{F}_c) . \mathbf{e}_i dz$ where $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ are the basic vectors previously definite.

Note:

Moments compared to z are worthless.

3.3 Principle of virtual work

He is written in the following way: $\delta W_{\text{ext}} = \delta W_{\text{def}}$ for all displacements and rotations virtual kinematically acceptable.

3.3.1 Kinematics of Hencky

With this kinematics, it results from them after integration by parts of work of deformation the equilibrium equations static of the following plates:

•For the efforts: $$\begin{aligned} &N_{xx,x} + N_{xy,y} + f_x = 0, \\ &N_{yy,y} + N_{xy,x} + f_y = 0, \\ &T_{x,x} + T_{y,y} + f_z = 0 . \end{aligned}$$

•For the couples: $$\begin{aligned} &M_{xx,x} + M_{xy,y} - T_x + c_y = 0, \\ &M_{yy,y} + M_{xy,x} - T_y - c_x = 0 . \end{aligned}$$

as well as the boundary conditions following on contour C of S :

$$\begin{aligned} &N_{xx} n_x + N_{xy} n_y = \phi_x \\ &N_{yy} n_y + N_{xy} n_x = \phi_y \\ &T_x n_x + T_y n_y = \phi_z , \\ &M_{xx} n_x + M_{xy} n_y = \chi_y \\ &M_{yy} n_y + M_{xy} n_x = -\chi_x \end{aligned} \quad \text{where} \quad \begin{aligned} &u = \bar{u} \\ &v = \bar{v} \\ &w = \bar{w} \\ &\beta_x = \bar{\theta}_y \\ &\beta_y = -\bar{\theta}_x \end{aligned}$$

where n_x and n_y are the cosine directors of the normal with C directed towards the outside of the plate.
and $\bar{u}$ indicate the trace of u on C .

The physical interpretation of these efforts (N , T and M) starting from the preceding equations Ci - is given below:

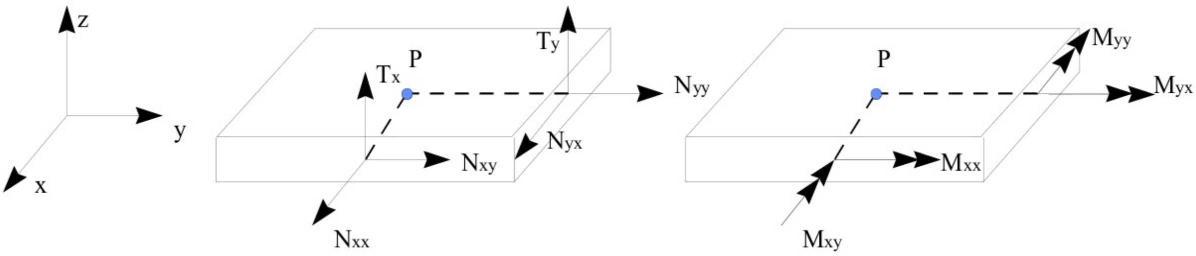


Figure 3.3.1-a: Efforts resulting for an element from plate

Note:

N_{xx}, N_{yy} represent the tractive efforts and N_{xy} shearing plan. M_{xx} and M_{yy} the couples of inflection represent and M_{xy} the torque. T_x and T_y are the shearing forces transverse.

3.3.2 Kinematics of Coils-Kirchhoff

It is pointed out that within the framework of this kinematics, one has the following relation binding the derivative of the arrow to rotations: $\begin{matrix}\beta_x = -\dfrac{\partial w}{\partial x} \\ \beta_y = -\dfrac{\partial w}{\partial y}\end{matrix}$. After a double integration by parts of the work of deformation, one obtains the following equilibrium equations static:

- For the efforts of membrane: $\begin{matrix} N_{xx,x} + N_{xy,y} + f_x = 0, \\ N_{yy,y} + N_{xy,x} + f_y = 0, \end{matrix}$
- For the transverse shearing and bending stresses:

$$\begin{aligned}
&M_{xx,xx} + 2\mathrm{M}_{xy,xy} + M_{yy,yy} + f_z + c_{y,x} - c_{x,y} = 0,\\
&M_{xx,x} + M_{xy,y} - T_x + c_y = 0,\\
&M_{yy,y} + M_{xy,x} - T_y - c_x = 0.
\end{aligned}$$

as well as the boundary conditions on contour C and at the angular points O contour C of S :

$$\begin{aligned}
&N_{xx} n_x + N_{xy} n_y = \phi_x,\\
&N_{yy} n_y + N_{xy} n_x = \phi_y,\\
&T_n + M_{ns,s} = \phi_z - \chi_{n,s},\\
&M_{nn} = \chi_s,\\
&M_{ns}(O+) - M_{ns}(O-) = -[\chi_n(O+) - \chi_n(O-)].
\end{aligned}
\quad \text{where} \quad
\begin{aligned}
&u = \bar{u}\\
&v = \bar{v}\\
&w = \bar{w}\\
&\beta_n = -\bar{w}_{,n} = \bar{\theta}_s
\end{aligned}$$

with
$$\begin{aligned}
&T_n = T_x n_x + T_y n_y,\\
&M_{nn} = M_{xx} n_x^2 + 2\mathrm{M}_{xy} n_x n_y + M_{yy} n_y^2,\\
&M_{ns} = -M_{xx} n_x n_y + M_{xy}(n_x^2 - n_y^2) + M_{yy} n_x n_y.
\end{aligned}$$
.

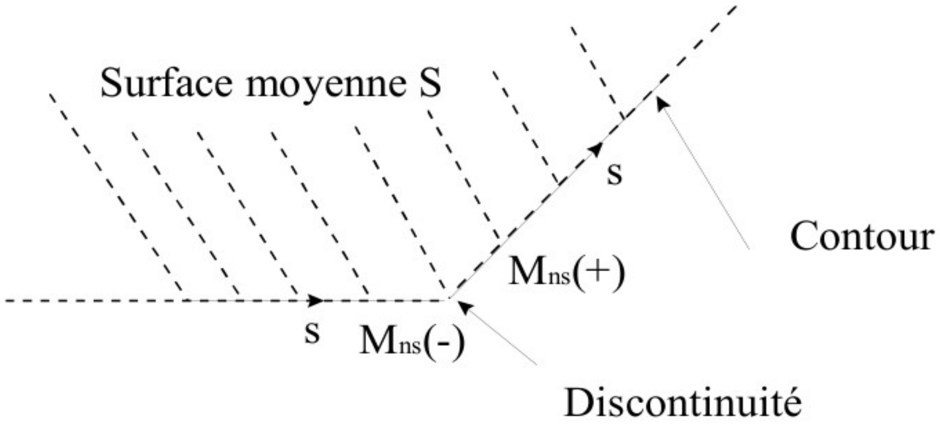


Figure 3.3.2-a: Boundary condition with angular points for an element of plate

Note:

The kinematics of Coils-Kirchhoff implies that on the contour of the plate the transverse shearing force is related to the torque. It is noted that the order of the equilibrium equations of inflection is higher than with the kinematics of Hencky. Thus, to choose the kinematics of Coils - Kirchhoff amounts increasing the degree of the functions of interpolation because one needs a larger regularity for the terms of arrow compared to the terms of membrane because of presence of derived seconds of the arrow in the expression of the work of the deformations. No element of plate of Code_Aster uses this kinematics. One can thus have differences between the results got with the elements of Code_Aster and of the analytical results got by using the kinematics of Coils-Kirchhoff for structures with angular contours.

3.3.3 Principal boundary conditions met [1]

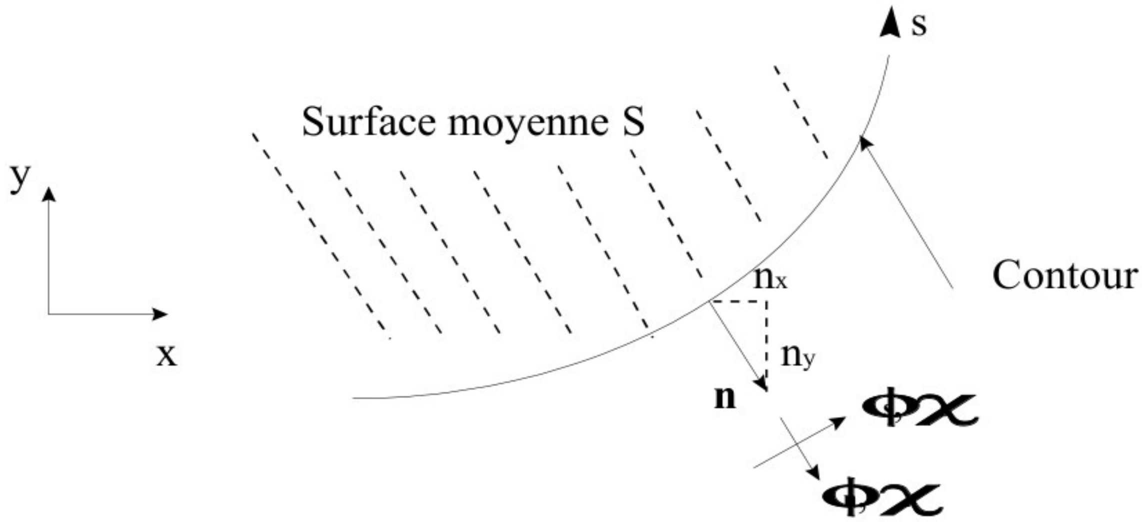


Figure 3.3.3-a: Boundary condition for an element of plate

The boundary conditions frequently met are gathered in the table which follows. They are given for the kinematics of Hencky in the reference mark defined by S and the normal external with the plate:

Embedding	Simple support	Free edge	Symmetry by report with an axis s	Antisymetry compared to an

				axis s
$\bar{u}=0,$ $\bar{v}=0,$ $\bar{w}=0,$ $\bar{\theta}_s=0,$ $\bar{\theta}_n=0.$	$\bar{u}_n=0,$ $\bar{w}=0,$ $\bar{\theta}_n=0.$		$\bar{u}_n=0,$ $\bar{\theta}_s=0.$	$\bar{u}_s=0,$ $\bar{w}=0,$ $\bar{\theta}_n=0.$
	$\phi_s=0,$ $\chi_s=0.$	$\phi_s=0,$ $\phi_n=0,$ $\phi_z=0,$ $\chi_s=0,$ $\chi_n=0$	$\phi_s=0,$ $\phi_z=0,$ $\chi_n=0.$	$\phi_n=0,$ $\chi_s=0.$

with:
$$\begin{aligned}
&u_n=un_x+vn_y\,;u_s=-un_y+vn_x,\\
&\theta_n=\theta_x n_x+\theta_y n_y\,;\theta_s=-\theta_x n_y+\theta_y n_x,\\
&\phi_n=\phi_x n_x^2+2\phi_{xy} n_x n_y+\phi_y n_y^2,\\
&\phi_s=-\phi_x n_x n_y+\phi_{xy}(n_x^2-n_y^2)+\phi_y n_x n_y,\\
&\chi_n=\chi_x n_x^2+2\chi_{xy} n_x n_y+\chi_y n_y^2,\\
&\chi_s=-\chi_x n_x n_y+\chi_{xy}(n_x^2-n_y^2)+\chi_y n_x n_y.
\end{aligned}$$

Note: one has $\begin{aligned}&\beta_s=-\theta_n,\\&\beta_n=\theta_s.\end{aligned}$.

4 Digital discretization of the variational formulation resulting from the principle of virtual work

4.1 Introduction

By exploiting the law of behavior, the virtual work of the interior efforts is written (with $H_{\mathbf{mf}}=0$ until the §4.4, which does not remove anything with the general information following results, but allows to reduce the notations):

$$\delta W_{\text{int}}=\int_S (\delta\,\mathbf{e}\mathbf{H}_{\boldsymbol{m}}\mathbf{e}+\delta\,\kappa\,\mathbf{H}_{\boldsymbol{f}}\,\kappa+\delta\,\gamma\,\mathbf{H}_{\boldsymbol{ct}}\,\gamma)\,dS$$

with: $\mathbf{e}=\begin{pmatrix} u_{,x} \\ v_{,y} \\ u_{,y}+v_{,x}\end{pmatrix},\ \kappa=\begin{pmatrix}\beta_{x,x} \\ \beta_{y,y} \\ \beta_{x,y}+\beta_{y,x}\end{pmatrix},\ \gamma=\begin{pmatrix} w_{,x}+\beta_x \\ w_{,y}+\beta_y\end{pmatrix}.$

It results from it that the elements of plate are elements with five degrees of freedom per node. These degrees of freedom are displacements in the plan of the element u and v , except plan w and two rotations β_x and β_y .

The elements DKT, DKTG and DST are triangular elements. Elements DKQ, DKQG, DSQ and $Q4\gamma$ are quadrilateral elements. They are represented below:

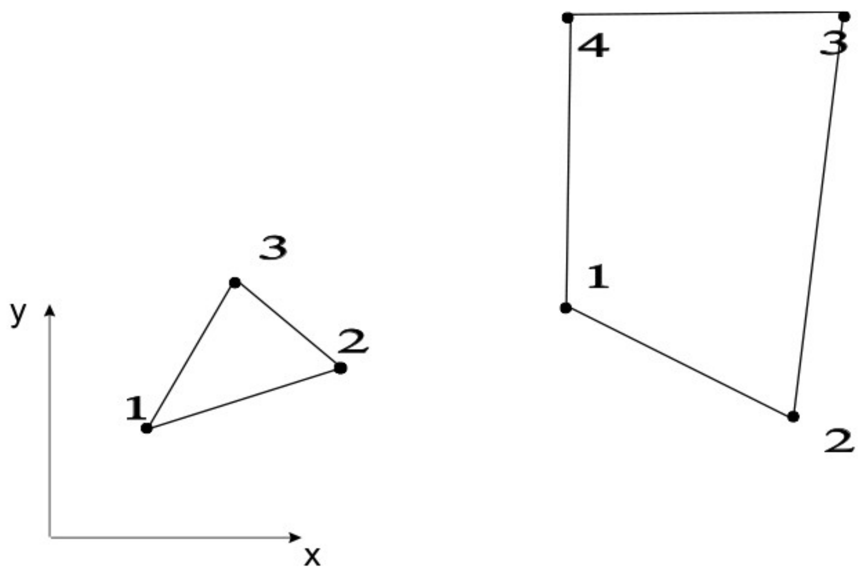


Figure 4.1-a: Real elements

The elements of reference are presented below:

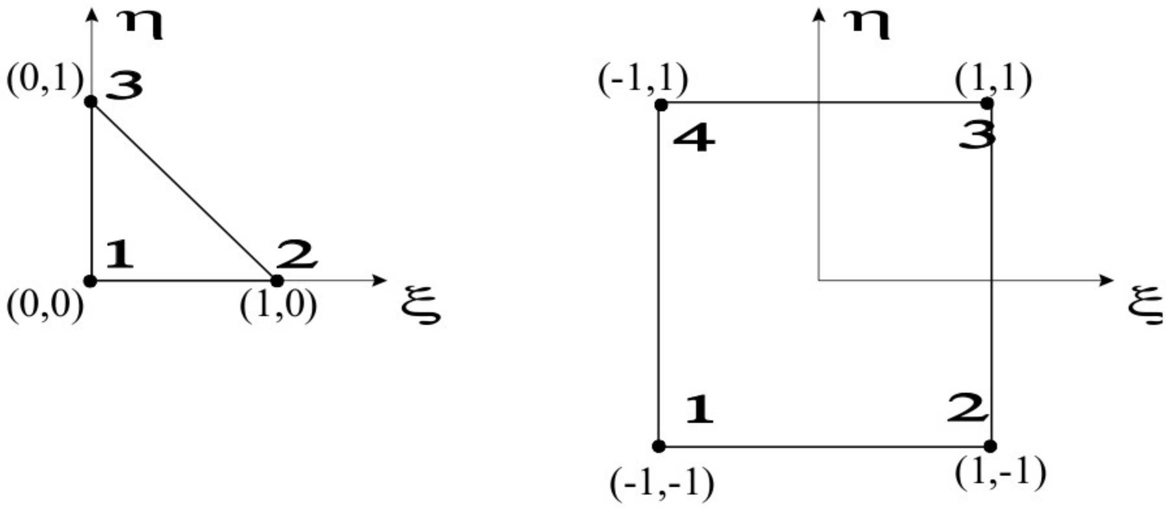


Figure 4.1-b: Elements of reference triangle and quadrangle

One defines the reduced reference mark of the element as the reference mark (ξ,η) element of reference. The local reference mark of the element, in its plan (x,y) is defined by the user. Direction $X1$ this local reference mark is the projection of a direction of reference $\underline{d}$ as regards the element. This direction of reference $\underline{d}$ is chosen by the user who defines it by two nautical angles in the total reference mark. The normal N with the plan of the element ($12\wedge13$ for a triangle numbered 123 and $12\wedge14$ for a numbered quadrangle 1234) fix the second direction. The vector product of the two vectors previously definite $\mathrm{Y1}=\mathrm{N}\wedge\mathrm{X1}$ allows to define the local trihedron in which will be expressed the generalized efforts representing the state of stresses. The user will have to take care that the selected reference axis is not found parallel with the normal of certain elements of plate. By default, direction of reference $\underline{d}$ is the axis X total reference mark of definition of the grid.

The essential difference between elements DKT, DKQ, DKTG, DKQG on the one hand and DST, DSQ, $\mathrm{Q4}\gamma$ in addition, comes owing to the fact that for the first the transverse distortion is worthless, that is to say still $\gamma=0$. The difference enters $\mathrm{Q4}\gamma$ and elements DST and DSQ comes from a choice different of interpolation for the representation of transverse shearing.

4.2 Discretization of the field of displacement

If one discretizes the fields of displacement in the usual way for isoparametric elements i.e.:

$$u=\sum_{i=1}^{N} N_i u_i \,,\, v=\sum_{i=1}^{N} N_i v_i \,,\, w=\sum_{i=1}^{N} N_i w_i \,,\, \beta_x=\sum_{i=1}^{N} N_i \beta_{xi} \,,\, \beta_y=\sum_{i=1}^{N} N_i \beta_{yi} \,,$$

and that one introduces this discretization into the variational formulation of the §4.1 it a blocking in transverse shearing analyzed results from it in [1] who returns the solution in inflection controlled by the effects of transverse shearing, and not by the inflection, when the thickness of the plate becomes small compared to its characteristic dimension.

To cure this disadvantage the variational form presented in introduction is slightly modified so that:

$$\delta W_{\mathrm{int}}=\int_S (\delta\,\mathbf{e}\mathbf{H}_m\mathbf{e}+\delta\,\kappa\,\mathbf{H}_f\,\kappa+\delta\,\bar{\gamma}\,\mathbf{H}_{ct}\,\bar{\gamma})\,dS=\int_S (\delta\,\mathbf{e}\mathbf{H}_m\,\mathbf{e}+\delta\,\kappa\,\mathbf{H}_f\,\kappa+\delta\,\mathbf{T}\mathbf{H}_{ct}^{-1}\,\mathbf{T})\,dS \qquad (18)$$

where $\bar{\gamma}$ are deformations of substitution checking $\bar{\gamma}=\gamma$ in a weak way (integral on the sides of the element) and such as $\mathbf{T}=\mathbf{H}_{\mathbf{ct}}\bar{\gamma}$. One checks thus that on the sides ij element $\int_i^j(\bar{\gamma}_s-\gamma_s)ds=0$ with $\gamma_s=w_{,s}+\beta_s$.

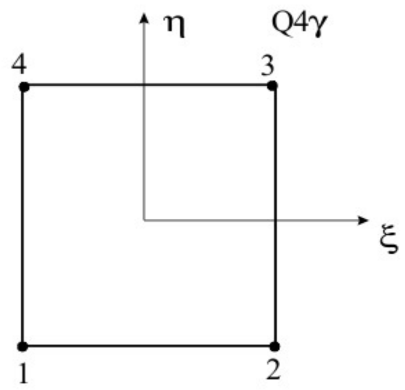
Two approaches are then possible; in the first, that of the element $\mathrm{Q4}\gamma$, one uses the bilinear discretization of the fields of displacement and the fact that $\bar{\gamma}$ is constant on the sides of the element. Relations on the sides ij then allow to express the values of $\bar{\gamma}$ on the sides according to the degrees of freedom of inflection. In the second approach, which is that of the elements of the type DKT, DKTG and DST, one uses the weak formulation of the preceding paragraph which makes it possible to bind the inflection to the shearing forces to deduce the interpolation from it from the terms of inflection.

4.2.1 Q4 approachΓ

It rests on the linear discretization of the fields of displacement presented above:

$$u=\sum_{i=1}^{N}N_i u_i\,,\,v=\sum_{i=1}^{N}N_i v_i\,,\,w=\sum_{i=1}^{N}N_i w_i\,,\,\beta_x=\sum_{i=1}^{N}N_i\beta_{xi}\,,\,\beta_y=\sum_{i=1}^{N}N_i\beta_{yi}\,, \tag{19}$$

where functions N_i are given below.

	$N_i\ (i=1,n)$
	$i=1$ with 4 $N_1(\xi,\eta)=\frac{1}{4}(1-\xi)(1-\eta)$ $N_2(\xi,\eta)=\frac{1}{4}(1+\xi)(1-\eta)$ $N_3(\xi,\eta)=\frac{1}{4}(1+\xi)(1+\eta)$ $N_4(\xi,\eta)=\frac{1}{4}(1-\xi)(1+\eta)$

Functions N_i for the elements $\mathrm{Q4}\gamma$

Note:

One notes too $N_i(\xi,\eta)=\frac{1}{4}(1+\xi_i\xi)(1+\eta_i\eta)$ *with* $(\xi_1,\xi_2,\xi_3,\xi_4)=(-1,1,1,-1)$ *and* $(\eta_1,\eta_2,\eta_3,\eta_4)=(-1,-1,1,1)$.

4.2.2 Approach DKT, DKQ, DKTG, DKQG, DST, DSQ

Like $T_x = M_{xx,x} + M_{xy,y}$ et $T_y = M_{yy,y} + M_{xy,x}$ and $\mathbf{M} = \mathbf{H}_f \kappa$ one from of deduced that $\bar{\gamma}$ is defined according to the derivative second of β_x and β_y via two equilibrium equations internal and law of behavior in inflection. Discretization retained for β_x and β_y, such as β_s is quadratic on the sides and β_n linear, then utilizes of the functions of incomplete quadratic forms in the form:

$$\beta_x = \sum_{k=1}^{N} N_k \beta_{xk} + \sum_{k=N+1}^{2N} P_{xk} \alpha_k \, , \beta_y = \sum_{k=1}^{N} N_k \beta_{yk} + \sum_{k=N+1}^{2N} P_{yk} \alpha_k \quad \text{with } P_{xk} = P_k C_k \,\text{et}\, P_{yk} = P_k S_k$$

where C_k and S_k are the cosine and directing sines on the side ij which the node belongs k defined by: $C_k = x_{ji} / L_k = (x_j - x_i) / L_k \, ; S_k = y_{ij} / L_k = (y_j - y_i) / L_k \, ; \, L_k = \left(x_{ji}^2 + y_{ji}^2 \right)^{1/2}$.

Note:

To introduce the preceding discretization amounts adding like degrees of freedom to the element of rotations α_k in the middle of the sides k element. Indeed, rotations β_s and β_n such as:

$$\begin{pmatrix} \beta_s \\ \beta_n \end{pmatrix} = \begin{pmatrix} C & S \\ S & -C \end{pmatrix} \begin{pmatrix} \beta_x \\ \beta_y \end{pmatrix}$$

are quadratic for β_s and linear for β_n with:

$$\beta_s = (1 - s') \beta_{si} + s' \beta_{sj} + 4 s' (1 - s') \alpha_k \, ; \beta_n = (1 - s') \beta_{ni} + s' \beta_{nj} \quad \textit{where } 0 \le s' = s / L_k \le 1 \; .$$

One observes thus that: $\beta_{sk} = \beta_s (s' = \frac{1}{2}) = \frac{1}{2} (\beta_{si} + \beta_{sj}) + \alpha_k$. It is the relation $\int_i^j (\bar{\gamma}_s - \gamma_s) ds = 0$ with $\gamma_s = w_{,s} + \beta_s$ who will allow to eliminate the additional degrees of freedom and to express them according to displacements and of nodal rotations.

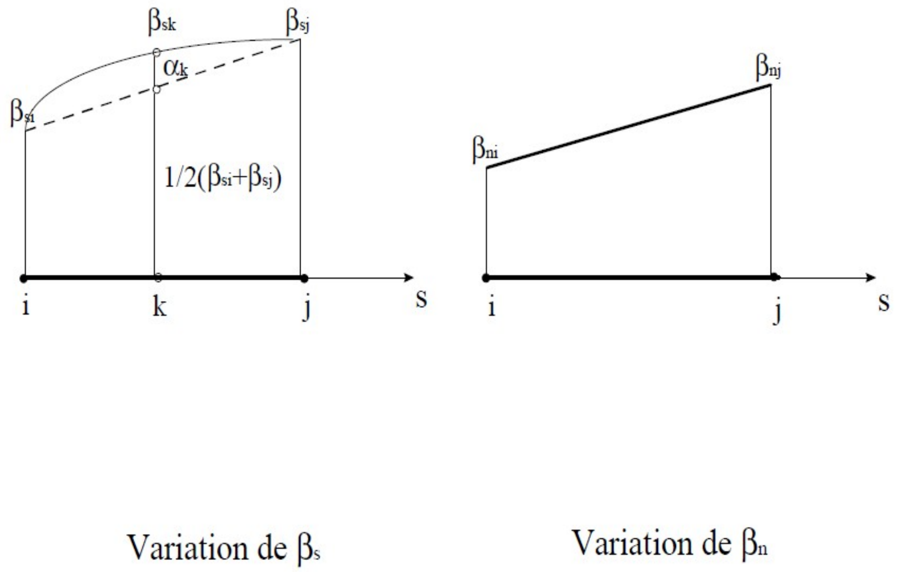
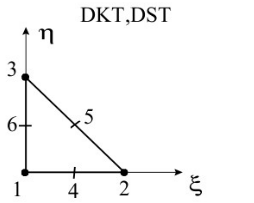
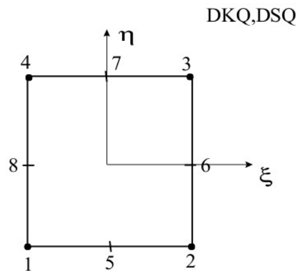


Figure 4.2.2-a: Variations of β_s and β_n .

	N_i ($i=1,n$)	P_i ($i=n+1,2n$)
	$i=1$ with 3 $N_1(x,\eta)=\lambda=1-\xi-\eta$ $N_2(x,\eta)=\xi$ $N_3(x,\eta)=\eta$	$i=4$ with 6 $P_4(\xi,\eta)=4\,\xi\,\lambda$ $P_5(\xi,\eta)=4\,\xi\,\eta$ $P_6(\xi,\eta)=4\,\eta\,\lambda$
	$i=1$ with 4 $N_1(\xi,\eta)=\frac{1}{4}(1-\xi)(1-\eta)$ $N_2(\xi,\eta)=\frac{1}{4}(1+\xi)(1-\eta)$ $N_3(\xi,\eta)=\frac{1}{4}(1+\xi)(1+\eta)$ $N_4(\xi,\eta)=\frac{1}{4}(1-\xi)(1+\eta)$	$i=5$ with 8 $P_5(\xi,\eta)=\frac{1}{2}(1-\xi^2)(1-\eta)$ $P_6(\xi,\eta)=\frac{1}{2}(1-\eta^2)(1+\xi)$ $P_7(\xi,\eta)=\frac{1}{2}(1-\xi^2)(1+\eta)$ $P_8(\xi,\eta)=\frac{1}{2}(1-\eta^2)(1+\xi)$

Functions N_i and P_i for elements DKT, DST, DKTG, DKQG, DKQ, DSQ.

4.3 Discretization of the field of deformation

The matrix jacobienne $\mathbf{J}(\xi,\eta)$ is:

$$\mathbf{J}=\begin{pmatrix} x_{,\xi} & y_{,\xi} \\ x_{,\eta} & y_{,\eta} \end{pmatrix}=\begin{pmatrix} \sum_{i=1}^{N} N_{i,\xi}\,x_i & \sum_{i=1}^{N} N_{i,\xi}\,y_i \\ \sum_{i=1}^{N} N_{i,\eta}\,x_i & \sum_{i=1}^{N} N_{i,\eta}\,y_i \end{pmatrix}=\begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} \tag{20}$$

Moreover:

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}=j\begin{pmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{pmatrix} \text{avec } \mathbf{j}=\begin{pmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{pmatrix}=\mathbf{J}^{-1}=\frac{1}{J}\begin{pmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{pmatrix} \text{où } J=\det\mathbf{J}=J_{11}J_{22}-J_{12}J_{21} \tag{21}$$

It is pointed out that the field of displacement is discretized by:

$$\begin{pmatrix} u \\ v \end{pmatrix}=\sum_{k=1}^{N} N_k(\xi,\eta)\begin{pmatrix} u^k \\ v^k \end{pmatrix} \text{ and } \begin{pmatrix} w \\ \beta_x \\ \beta_y \end{pmatrix}=\sum_{k=1}^{N} N_k(\xi,\eta)\begin{pmatrix} w_k \\ \beta_{xk} \\ \beta_{yk} \end{pmatrix}+[\sum_{k=N+1}^{2N}\begin{pmatrix} 0 \\ P_{xk}(\xi,\eta) \\ P_{yk}(\xi,\eta) \end{pmatrix}\alpha_k] \tag{22}$$

the term between hooks being present for the elements of type DKT, DKTG, DST, but not for the elements $\mathrm{Q4}\gamma$.

Note:

> *In Code_hasster, the Jacobienne matrix makes it possible to pass from the element of reference to the local reference mark of the element (and not the total reference mark) because it is easier to work in this reference mark.*

4.3.1 Discretization of the membrane field of deformation:

$$
\begin{aligned}
e_{xx}&=u_{,x}=\sum_{k=1}^{N} N_{k,x}(\xi,\eta)u^k=\sum_{k=1}^{N}(j_{11}N_{k,\xi}+j_{12}N_{k,\eta})u^k,\\
e_{yy}&=v_{,y}=\sum_{k=1}^{N} N_{k,y}(\xi,\eta)v^k=\sum_{k=1}^{N}(j_{21}N_{k,\xi}+j_{22}N_{k,\eta})v^k,\\
2e_{xy}&=u_{,x}+v_{,y}=\sum_{k=1}^{N} N_{k,y}(\xi,\eta)u^k+N_{k,x}(\xi,\eta)v^k\\
&=\sum_{k=1}^{N}(j_{21}N_{k,\xi}+j_{22}N_{k,\eta})u^k+(j_{11}N_{k,\xi}+j_{12}N_{k,\eta})v^k
\end{aligned}
\tag{23}
$$

Maybe in matric form:

$\begin{pmatrix} e_{xx}\\ e_{yy}\\ 2e_{xy}\end{pmatrix}=\sum_{k=1}^{N}\mathbf{B_{mk}}\mathbf{U_k}$ where $\mathbf{U}_{\boldsymbol{k}}=\begin{pmatrix}u_k\\ v_k\end{pmatrix}$ is the field of membrane displacement to the node k

and:

$$
B_{\mathbf{mk}}=\begin{pmatrix} j_{11}N_{k,\xi}+j_{12}N_{k,\eta} & 0\\ 0 & j_{21}N_{k,\xi}+j_{22}N_{k,\eta}\\ j_{21}N_{k,\xi}+j_{22}N_{k,\eta} & j_{11}N_{k,\xi}+j_{12}N_{k,\eta}\end{pmatrix}
$$

The matrix of passage of the membrane deformations to the field of displacement $U_m=\begin{pmatrix}u_1\\ v_1\\ \dots\\ u_N\\ v_N\end{pmatrix}$ in the plan of the element is written as follows: $\mathbf{B}_{\boldsymbol{m}[3\times 2\mathbf{N}]}=\left(\mathbf{B}_{\mathbf{m1}}\dots\mathbf{B}_{\mathbf{mN}}\right)$.

4.3.2 Discretization of the transverse distortion

4.3.2.1 For the finite elements Q4Γ

The field linearly is discretized $\bar{\gamma}$ constant by side so that:

$$\bar{\gamma}=\boldsymbol{j}\,\bar{\gamma}^{ref}$$

$\bar{\gamma}^{ref}=\begin{pmatrix}\bar{\gamma}_\xi\\ \bar{\gamma}_\eta\end{pmatrix}=\begin{pmatrix}\dfrac{1-\eta}{2}\gamma_\xi^{12}+\dfrac{1+\eta}{2}\gamma_\xi^{34}\\ \dfrac{1-\xi}{2}\gamma_\eta^{23}+\dfrac{1-\xi}{2}\gamma_\eta^{41}\end{pmatrix}$ where $\bar{\gamma}^{ref}$ is the transverse field of distortion in the element of reference.

By using the relations (cf then. 4.2): $\begin{matrix} \int_{-1}^{+1} (\bar{\gamma}_{\xi} - (w_{,\xi} + \beta_{\xi}))\, d\xi = 0\,; \\ \int_{-1}^{+1} (\bar{\gamma}_{\eta} - (w_{,\eta} + \beta_{\eta}))\, d\eta = 0 \end{matrix}$ for $\begin{matrix} \xi = \pm 1 \\ \eta = \pm 1 \end{matrix}$,

it is established that: $\begin{matrix} \gamma_{\xi}^{ij} = \frac{1}{2}(w_j - w_i + \beta_{\xi i} + \beta_{\xi j})\,; \\ \gamma_{\eta}^{kp} = \frac{1}{2}(w_p - w_k + \beta_{\eta p} + \beta_{\eta k})\,; \end{matrix}$ for $(ij) \in (12,34)$ and $(kp) \in (23,41)$.

By deferring the two results above in the expression of $\bar{\gamma}^{loc}$, one from of deduced that:

$\bar{\gamma}^{ref} = \begin{pmatrix} \bar{\gamma}_{\xi} \\ \bar{\gamma}_{\eta} \end{pmatrix} = \begin{pmatrix} \mathbf{B}_{\xi}^{ref} u_{\xi}^{ref} \\ \mathbf{B}_{\eta}^{ref} u_{\eta}^{ref} \end{pmatrix} = \mathbf{B}^{ref} u^{ref}$ where $u^{ref} = \begin{pmatrix} w_1 \\ \beta_{\xi 1} \\ \beta_{\eta 1} \\ \vdots \\ w_N \\ \beta_{\xi N} \\ \beta_{\eta N} \end{pmatrix}$ and $\mathbf{B}^{ref} = (\mathbf{B}_1, \ldots, \mathbf{B}_N)$ with

$\mathbf{B}_k = \begin{pmatrix} N_{k,\xi} & \xi_k N_{k,\xi} & 0 \\ N_{k,\eta} & 0 & \eta_k N_{k,\eta} \end{pmatrix}$, $N = 4$, $k \in [1, N]$, ξ_k, η_k are defined with 4.2.1.

It is now necessary to express the rotations given here in the element of reference according to rotations in the local reference mark.

Like $\begin{pmatrix} \beta_{\xi k} \\ \beta_{\eta k} \end{pmatrix} = \boldsymbol{J}_k \begin{pmatrix} \beta_{xk} \\ \beta_{yk} \end{pmatrix} = \begin{pmatrix} J_{11k} & J_{12k} \\ J_{21k} & J_{22k} \end{pmatrix} \begin{pmatrix} \beta_{xk} \\ \beta_{yk} \end{pmatrix}$ one from of deduced that $\bar{\gamma}^{ref} = \mathbf{B}_{loc} u_{loc}$ where

$u_{loc} = \begin{pmatrix} w_1 \\ \beta_{x1} \\ \beta_{y1} \\ \vdots \\ w_N \\ \beta_{xN} \\ \beta_{yN} \end{pmatrix}$ and $\mathbf{B}_{loc} = (\mathbf{B}_{loc\,1}, \ldots, \mathbf{B}_{loc\,N})$ with $\mathbf{B}_{loc\,k} = \begin{pmatrix} N_{k,\xi} & \xi_k N_{k,\xi} J_{11k} & \xi_k N_{k,\xi} J_{12k} \\ N_{k,\eta} & \eta_k N_{k,\eta} J_{21k} & \eta_k N_{k,\eta} J_{22k} \end{pmatrix}$. It

will be noticed that the Jacobienne matrix $\boldsymbol{J}_k$ is expressed in each point of the element.

Finally: $\bar{\gamma} = \begin{pmatrix} \bar{\gamma}_x \\ \bar{\gamma}_y \end{pmatrix} = \begin{pmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{pmatrix} \bar{\gamma}^{ref} = \mathbf{B}_c u_{loc}$ with $\mathbf{B}_{c[2 \times 3N]} = \mathbf{j} \mathbf{B}_{loc}$.

4.3.2.2 For the finite elements of type DKT, DST, DKTG

With regard to the transverse distortions one deduces from $T_x=M_{xx,x}+M_{xy,y}$ et $T_y=M_{yy,y}+M_{xy,x}$ with $\mathbf{M}=\mathbf{H}_f\kappa$ that $\mathbf{T}=\bar{\mathbf{H}}_f\beta_{,xx}$ where:

$\beta_{,xx}^T=\begin{pmatrix}\beta_{x,xx} & \beta_{x,yy} & \beta_{x,xy} & \beta_{y,xx} & \beta_{y,yy} & \beta_{y,xy}\end{pmatrix}$ and

$\bar{\mathbf{H}}_f=\begin{pmatrix} H_{11} & H_{33} & 2\mathrm{H}_{13} & H_{13} & H_{23} & H_{12}+H_{33} \\ H_{13} & H_{23} & H_{12}+H_{33} & H_{33} & H_{22} & 2\mathrm{H}_{23}\end{pmatrix}$ where them $\mathbf{H}_{\mathbf{ij}}$ are the terms (i,j) of $\mathbf{H}_f$.

$$\beta_{x,xx}=\sum_{k=1}^{N} N_{k,xx}(\xi,\eta)\beta_{xk}+\sum_{k=N+1}^{2N} P_{xk,xx}(\xi,\eta)\alpha_k=\sum_{k=1}^{N} N_{k,xx}(\xi,\eta)\beta_{xk}+\sum_{k=N+1}^{N}(j_{11}^2 P_{xk,\xi\xi}+2 j_{11} j_{12} P_{xk,\xi\eta}+j_{12}^2 P_{xk,\eta\eta})\alpha_k,$$

$$\beta_{x,yy}=\sum_{k=1}^{N} N_{k,yy}(\xi,\eta)\beta_{xk}+\sum_{k=N+1}^{2N} P_{xk,yy}(\xi,\eta)\alpha_k=\sum_{k=1}^{N} N_{k,yy}(\xi,\eta)\beta_{xk}+\sum_{k=N+1}^{N}(j_{21}^2 P_{xk,\xi\xi}+2 j_{21} j_{22} P_{xk,\xi\eta}+j_{22}^2 P_{xk,\eta\eta})\alpha_k,$$

$$\beta_{x,xy}=\sum_{k=1}^{N} N_{k,xy}(\xi,\eta)\beta_{xk}+\sum_{k=N+1}^{2N} P_{xk,xy}(\xi,\eta)\alpha_k$$

$$=\sum_{k=1}^{N} N_{k,xy}(\xi,\eta)\beta_{xk}+\sum_{k=N+1}^{N}(j_{11} j_{21} P_{xk,\xi\xi}+[j_{11} j_{22}+j_{12} j_{21}]P_{xk,\xi\eta}+j_{12} j_{22} P_{xk,\eta\eta})\alpha_k,$$

$$\beta_{y,xx}=\sum_{k=1}^{N} N_{k,xx}(\xi,\eta)\beta_{yk}+\sum_{k=N+1}^{2N} P_{yk,xx}(\xi,\eta)\alpha_k=\sum_{k=1}^{N} N_{k,xx}(\xi,\eta)\beta_{yk}+\sum_{k=N+1}^{N}(j_{11}^2 P_{yk,\xi\xi}+2 j_{11} j_{12} P_{yk,\xi\eta}+j_{12}^2 P_{yk,\eta\eta})\alpha_k$$

$$\beta_{y,yy}=\sum_{k=1}^{N} N_{k,yy}(\xi,\eta)\beta_{yk}+\sum_{k=N+1}^{2N} P_{yk,yy}(\xi,\eta)\alpha_k=\sum_{k=1}^{N} N_{k,yy}(\xi,\eta)\beta_{yk}+\sum_{k=N+1}^{N}(j_{21}^2 P_{yk,\xi\xi}+2 j_{21} j_{22} P_{yk,\xi\eta}+j_{22}^2 P_{yk,\eta\eta})\alpha_k,$$

$$\beta_{y,xy}=\sum_{k=1}^{N} N_{k,xy}(\xi,\eta)\beta_{yk}+\sum_{k=N+1}^{2N} P_{yk,xy}(\xi,\eta)\alpha_k$$

$$=\sum_{k=1}^{N} N_{k,xy}(\xi,\eta)\beta_{yk}+\sum_{k=N+1}^{N}(j_{11} j_{21} P_{yk,\xi\xi}+[j_{11} j_{22}+j_{12} j_{21}]P_{yk,\xi\eta}+j_{12} j_{22} P_{yk,\eta\eta})\alpha_k$$

where P_{xk} , P_{yk} and α_k are defined into 4.2.2.

that is to say still in matric form:

$$\mathbf{T}=\bar{\mathbf{H}}_f\begin{pmatrix}\beta_{x,xx}\\ \beta_{x,yy}\\ \beta_{x,xy}\\ \beta_{y,xx}\\ \beta_{y,yy}\\ \beta_{y,xy}\end{pmatrix}=$$

$$\bar{\mathbf{H}}_f\sum_{k=1}^{N}\begin{pmatrix}0 & j_{11}^2 N_{k,\xi\xi}+2j_{11}j_{12}N_{k,\xi\eta}+j_{12}^2N_{k,\eta\eta} & 0\\ 0 & j_{21}^2 N_{k,\xi\xi}+2j_{21}j_{22}N_{k,\xi\eta}+j_{22}^2N_{k,\eta\eta} & 0\\ 0 & j_{11}j_{21}N_{k,\xi\xi}+[j_{11}j_{22}+j_{12}j_{21}]N_{k,\xi\eta}+j_{12}j_{22}N_{k,\eta\eta} & 0\\ 0 & 0 & j_{11}^2 N_{k,\xi\xi}+2j_{11}j_{12}N_{k,\xi\eta}+j_{12}^2N_{k,\eta\eta}\\ 0 & 0 & j_{21}^2 N_{k,\xi\xi}+2j_{21}j_{22}N_{k,\xi\eta}+j_{22}^2N_{k,\eta\eta}\\ 0 & 0 & j_{11}j_{21}N_{k,\xi\xi}+[j_{11}j_{22}+j_{12}j_{21}]N_{k,\xi\eta}+j_{12}j_{22}N_{k,\eta\eta}\end{pmatrix}\begin{pmatrix}w_k\\ \beta_{xk}\\ \beta_{yk}\end{pmatrix}$$

$$+\bar{\mathbf{H}}_f\sum_{k=1}^{2N}\alpha_k\begin{pmatrix}C_k(j_{11}^2P_{k,\xi\xi}+2j_{11}j_{12}P_{k,\xi\eta}+j_{12}^2P_{k,\eta\eta})\\ C_k(j_{21}^2P_{k,\xi\xi}+2j_{21}j_{22}P_{k,\xi\eta}+j_{22}^2P_{k,\eta\eta})\\ C_k(j_{11}j_{22}P_{k,\xi\xi}+[j_{11}j_{22}+j_{12}j_{21}]P_{k,\xi\eta}+j_{12}j_{22}P_{k,\eta\eta})\\ S_k(j_{11}^2P_{k,\xi\xi}+2j_{11}j_{12}P_{k,\xi\eta}+j_{12}^2P_{k,\eta\eta})\\ S_k(j_{21}^2P_{k,\xi\xi}+2j_{21}j_{22}P_{k,\xi\eta}+j_{22}^2P_{k,\eta\eta})\\ S_k(j_{11}j_{22}P_{k,\xi\xi}+[j_{11}j_{22}+j_{12}j_{21}]P_{k,\xi\eta}+j_{12}j_{22}P_{k,\eta\eta})\end{pmatrix}=$$

$$\bar{\mathbf{H}}_f\sum_{k=1}^{N}\mathbf{P}_{f\beta k}\begin{pmatrix}w_k\\ \beta_{xk}\\ \beta_{yk}\end{pmatrix}+\sum_{k=N+1}^{2N}\bar{\mathbf{H}}_f\mathbf{T}_2\begin{pmatrix}C_kP_{k,\xi\xi}\\ C_kP_{k,\eta\eta}\\ C_kP_{k,\xi\eta}\\ S_kP_{k,\xi\xi}\\ S_kP_{k,\eta\eta}\\ S_kP_{k,\xi\eta}\end{pmatrix}\alpha_k=\bar{\mathbf{H}}_f\sum_{k=1}^{N}\mathbf{P}_{f\beta k}\mathbf{U}_{f\beta k}+\bar{\mathbf{H}}_f\mathbf{T}_2\sum_{k=N+1}^{2N}\mathbf{T}_{ck}\alpha_k=\bar{\mathbf{H}}_f\mathbf{P}_{f\beta}\mathbf{U}_{f\beta}+\bar{\mathbf{H}}_f\mathbf{T}_2\mathbf{T}_\alpha\alpha$$

where $\mathbf{T}_\alpha=(\mathbf{T}_{c(N+1)}\dots\mathbf{T}_{c2N})$ and $\mathbf{T}_2=\begin{pmatrix}\mathbf{t}_2 & 0\\ 0 & \mathbf{t}_2\end{pmatrix}$ with $\mathbf{t}_2=\begin{pmatrix}j_{11}^2 & j_{12}^2 & 2j_{11}j_{12}\\ j_{21}^2 & j_{22}^2 & 2j_{21}j_{22}\\ j_{11}j_{21} & j_{12}j_{22} & j_{11}j_{22}+j_{12}j_{21}\end{pmatrix}$.

We use the relation then $\int_i^j(\bar{\gamma}_s-\gamma_s)ds=0$ with $\gamma_s=w_{,s}+\beta_s$ for each side ij of the element which makes it possible to obtain them α_k since she is still written:

$$w_j-w_i+\frac{L_k}{2}(C_k\beta_{xi}+S_k\beta_{yi}+C_k\beta_{xj}+S_k\beta_{yj})+\frac{2}{3}L_k\alpha_k=L_k\bar{\gamma}_{sk}$$ where:

$$\bar{\gamma}_{sk}=(C_k\quad S_k)\bar{\gamma}=(C_k\quad S_k)\mathbf{H}_{ct}^{-1}\mathbf{T}=(C_k\quad S_k)\mathbf{H}_{ct}^{-1}[\bar{\mathbf{B}}_{c\beta}\mathbf{U}_{f\beta}+\bar{\mathbf{B}}_{c\alpha}\alpha]$$

where C_k, S_k and L_k are defined into 4.2.2.

Note:

Terms $\bar{\mathbf{B}}_{c\alpha}$ and $\bar{\mathbf{B}}_{c\beta}$ correspond to the integration of the term $\bar{\gamma}_s$ on each side ij element. One evaluates the integral by using two points of Gauss of X-coordinates $\pm 1/\sqrt{3}$ and of weight $1/2$ in the element of reference $[-1,+1]$. Thus the term $\bar{\mathbf{B}}_{c\alpha}$ and $\bar{\mathbf{B}}_{c\beta}$ can they be written:

$$\bar{\mathbf{B}}_{c\alpha}=\frac{1}{2}\left[\bar{H}_f T_2(PG_1)T_\alpha(PG_1)+\bar{H}_f T_2(PG_2)T_\alpha(PG_2)\right]$$ *and*

$$\bar{\mathbf{B}}_{c\beta}=\frac{1}{2}\left[\bar{H}_f P_{f\beta}(PG_1)+\bar{H}_f P_{f\beta}(PG_2)\right] .$$

The relation above is still written in matric form: $\mathbf{A}_\alpha \alpha=\mathbf{A}_w \mathbf{U}_{f\beta}$

with: $$\mathbf{A}_\alpha=\frac{2}{3}\begin{pmatrix} L_{N+1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & L_{2N} \end{pmatrix}-\begin{pmatrix} L_{N+1}C_{N+1} & L_{N+1}S_{N+1} \\ \vdots & \vdots \\ L_{2N}C_{2N} & L_{2N}S_{2N} \end{pmatrix}\mathbf{H}_{ct}^{-1}\bar{\mathbf{B}}_{c\alpha}$$

and:

$$A_w=-\frac{1}{2}\begin{pmatrix}
-2 & L_{N+1}C_{N+1} & L_{N+1}S_{N+1} & 2 & L_{N+1}C_{N+1} & L_{N+1}S_{N+1} & 0 & 0 & 0 \\
0 & 0 & 0 & -2 & L_{k+1}C_{k+1} & L_{k+1}S_{k+1} & 2 & L_{k+1}C_{k+1} & L_{k+1}S_{k+1} \\
0 & 0 & 0 & 0 & 0 & 0 & -2 & L_{2N-1}C_{2N-1} & L_{2N-1}S_{2N-1} \\
2 & L_{2N}C_{2N} & L_{2N}S_{2N} & 0 & 0 & \cdots & \cdots & 0 & 0 \\
\cdots & 0 & 0 & 0 & & & & & \\
\cdots & 0 & 0 & 0 & & & & & \\
\cdots & 2 & L_{2N-1}C_{2N-1} & L_{2N-1}S_{2N-1} & & & & & \\
\cdots & -2 & L_{2N}C_{2N} & L_{2N}S_{2N} & & & & &
\end{pmatrix}$$

$$+\begin{pmatrix} L_{N+1}C_{N+1} & L_{N+1}S_{N+1} \\ \vdots & \vdots \\ L_{2N}C_{2N} & L_{2N}S_{2N} \end{pmatrix}\mathbf{H}_{ct}^{-1}\bar{\mathbf{B}}_{c\beta}$$

Thus $\alpha=\mathbf{A}_\beta \mathbf{U}_{f\beta} \text{avec} \mathbf{A}_\beta=\mathbf{A}_\alpha^{-1}\mathbf{A}_w$, which implies $\mathbf{T}=[\bar{\mathbf{B}}_{c\beta}+\bar{\mathbf{B}}_{c\alpha}\mathbf{A}_\beta]\mathbf{U}_{f\beta}$.

Note:

For the DST elements, this expression is simplified a little since $\bar{\mathbf{B}}_{c\beta}=0$ because of linearity of the functions of form N_k $(k=1,2,3)$.
This expression is simpler for elements DKT, DKTG and DKQ since they are without transverse distortion, i.e. $\bar{\gamma}$ = 0, which implies $\mathbf{A}_\alpha=\begin{pmatrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \end{pmatrix}$ *and*

¿

$$A_w=-\frac{3}{4}\begin{pmatrix} -2/L_{N+1} & C_{N+1} & S_{N+1} & 2/L_{N+1} & C_{N+1} & S_{N+1} & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & -2/L_{k+1} & C_{k+1} & S_{k+1} & 2/L_{k+1} & C_{k+1} & S_{k+1} & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & -2/L_{2N-1} & C_{2N-1} & S_{2N-1} & \cdots \\ 2/L_{2N} & C_{2N} & S_{2N} & 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots \\ \cdots\ 0 & 0 & 0 \\ \cdots\ 0 & 0 & 0 \\ \cdots\ 2/L_{2N-1} & C_{2N-1} & S_{2N-1} \\ \cdots\ -2/L_{2N} & C_{2N} & S_{2N} \end{pmatrix}$$

It is also noticed that for elements DKT, DKTG the expression of the shearing actions is calculated starting from balance and not starting from the behavior (on the basis of the behavior one would find a value zero of the efforts cutting-edges what would not make it possible to carry out balance!). It results from it according to the §3.1.1 from nonworthless stresses shear transverse in the thickness from the plate that one is in formulation DKT or DST.

4.3.3 Discretization of the field of deformation of inflection:

4.3.3.1 For the Q4 elementsΓ

The relation binding the deformations of inflection to the field of displacement of inflection is written:

$$
\begin{aligned}
&k_{xx}=\beta_{x,x}=j_{11}\beta_{x,\xi}+j_{12}\beta_{x,\eta}=j_{11}\sum_{k=1}^{N}N_{k,\xi}\beta_{xk}+j_{12}\sum_{k=1}^{N}N_{k,\eta}\beta_{xk},\\
&\kappa_{yy}=\beta_{y,y}=j_{21}\beta_{y,\xi}+j_{22}\beta_{y,\eta}=j_{21}\sum_{k=1}^{N}N_{k,\xi}\beta_{yk}+j_{22}\sum_{k=1}^{N}N_{k,\eta}\beta_{yk},\\
&2\kappa_{xy}=\beta_{y,x}+\beta_{x,y}=j_{11}\beta_{y,\xi}+j_{12}\beta_{y,\eta}+j_{21}\beta_{x,\xi}+j_{22}\beta_{x,\eta}=j_{21}\sum_{k=1}^{N}N_{k,\xi}\beta_{xk}+j_{22}\sum_{k=1}^{N}N_{k,\eta}\beta_{xk}\\
&+j_{11}\sum_{k=1}^{N}N_{k,\xi}\beta_{yk}+j_{12}\sum_{k=1}^{N}N_{k,\eta}\beta_{yk}.
\end{aligned}
\tag{24}
$$

That is to say still in matric form:

$\begin{pmatrix}\kappa_{xx}\\ \kappa_{yy}\\ 2\kappa_{xy}\end{pmatrix}=\sum_{k=1}^{N}\mathbf{B}_{fk}\mathbf{U}_{fk}$ where $\mathbf{U}_{fk}=\begin{pmatrix}w_k\\ \beta_{xk}\\ \beta_{yk}\end{pmatrix}$ represent the field of displacement of inflection to the node k,

with:

$$
\mathbf{B}_{fk}=\begin{pmatrix}0 & j_{11}N_{k,\xi}+j_{12}N_{k,\eta} & 0\\ 0 & 0 & j_{21}N_{k,\xi}+j_{22}N_{k,\eta}\\ 0 & j_{21}N_{k,\xi}+j_{22}N_{k,\eta} & j_{11}N_{k,\xi}+j_{12}N_{k,\eta}\end{pmatrix}.
$$

The matrix of passage of the field of displacement of inflection $\mathbf{U}_f=\begin{pmatrix}w_1\\ \beta_{x1}\\ \beta_{y1}\\ \vdots\\ w_N\\ \beta_{xN}\\ \beta_{yN}\end{pmatrix}$ with the deformations of inflection is written then: $\mathbf{B}_{f[3\times 3n]}=(\mathbf{B}_{f1},\cdots,\mathbf{B}_{fN})$.

4.3.3.2 For the finite elements of type DKT, DKTG, DST:

The relation binding the deformations of inflection to the field of displacement of inflection is written:

$$\kappa_{xx}=\beta_{x,x}=j_{11}\beta_{x,\xi}+j_{12}\beta_{x,\eta}=j_{11}\Big(\sum_{k=1}^{N}N_{k,\xi}\beta_{xk}+\sum_{k=N+1}^{2N}P_{xk,\xi}\alpha_k\Big)+j_{12}\Big(\sum_{k=1}^{N}N_{k,\eta}\beta_{xk}+\sum_{k=N+1}^{2N}P_{xk,\eta}\alpha_k\Big),$$

$$\kappa_{yy}=\beta_{y,y}=j_{21}\beta_{y,\xi}+j_{22}\beta_{y,\eta}=j_{21}\Big(\sum_{k=1}^{N}N_{k,\xi}\beta_{yk}+\sum_{k=N+1}^{2N}P_{yk,\xi}\alpha_k\Big)+j_{22}\Big(\sum_{k=1}^{N}N_{k,\eta}\beta_{yk}+\sum_{k=N+1}^{2N}P_{yk,\eta}\alpha_k\Big),$$

$$2\kappa_{xy}=\beta_{y,x}+\beta_{x,y}=j_{11}\beta_{y,\xi}+j_{12}\beta_{y,\eta}+j_{21}\beta_{x,\xi}+j_{22}\beta_{x,\eta}=$$

$$j_{21}\Big(\sum_{k=1}^{N}N_{k,\xi}\beta_{xk}+\sum_{k=N+1}^{2N}P_{xk,\xi}\alpha_k\Big)+j_{22}\Big(\sum_{k=1}^{N}N_{k,\eta}\beta_{xk}+\sum_{k=N+1}^{2N}P_{xk,\eta}\alpha_k\Big)+j_{11}\Big(\sum_{k=1}^{N}N_{k,\xi}\beta_{yk}+\sum_{k=N+1}^{2N}P_{yk,\xi}\alpha_k\Big)$$

$$+j_{12}\Big(\sum_{k=1}^{N}N_{k,\eta}\beta_{yk}+\sum_{k=N+1}^{2N}P_{yk,\eta}\alpha_k\Big).$$

For elements DKT, DKTG, DKQ:

In matric form the preceding relation is also written by introducing the relation $\alpha=\mathbf{A}_{\beta}\mathbf{U}_{f\beta}$:

$$\begin{pmatrix}\kappa_{xx}\\ \kappa_{yy}\\ 2\kappa_{xy}\end{pmatrix}=\begin{pmatrix}j_{11}\mathbf{B}_{\beta x\xi}+j_{12}\mathbf{B}_{\beta x\eta}\\ j_{21}\mathbf{B}_{\beta y\xi}+j_{22}\mathbf{B}_{\beta y\eta}\\ j_{11}\mathbf{B}_{\beta y\xi}+j_{12}\mathbf{B}_{\beta y\eta}+j_{21}\mathbf{B}_{\beta x\xi}+j_{22}\mathbf{B}_{\beta x\eta}\end{pmatrix}\mathbf{U}_f=\mathbf{B}_{f[3\times 3N]}\mathbf{U}_f \quad \text{where} \quad \mathbf{U}_f=\begin{pmatrix}w_1\\ \beta_{x1}\\ \beta_{y1}\\ \vdots\\ w_N\\ \beta_{xN}\\ \beta_{yN}\end{pmatrix}$$

represent the field of displacement in inflection for the element with:

$$B_{\beta x\xi}=\Big(\frac{6P_{N+1,\xi}C_{N+1}}{4L_{N+1}}-\frac{6P_{2N,\xi}C_{2N}}{4L_{2N}},N_{1,x}-\frac{3}{4}(P_{N+1,\xi}C_{N+1}^2+P_{2N,\xi}C_{2N}^2),$$

$$-\frac{3}{4}(P_{N+1,\xi}C_{N+1}S_{N+1}+P_{2N,\xi}C_{2N}S_{2N}),L,$$

$$\frac{6P_{N+k,\xi}C_{N+k}}{4L_{N+k}}-\frac{6P_{N+k-1,\xi}C_{N+k-1}}{4L_{N+k-1}},N_{k,\xi}-\frac{3}{4}(P_{N+k,\xi}C_{N+k}^2+P_{N+k-1,x}C_{N+k-1}^2),$$

$$-\frac{3}{4}(P_{N+k,\xi}C_{N+k}S_{N+k}+P_{N+k-1,\xi}C_{N+k-1}S_{N+k-1}),L$$

$$(k=2,..,N)\Big)$$

$$\mathbf{B}_{\beta x\eta}=(\frac{6P_{N+1,\eta}C_{N+1}}{4L_{N+1}}-\frac{6P_{2N,\eta}C_{2N}}{4L_{2N}},N_{1,\eta}-\frac{3}{4}(P_{N+1,\eta}C_{N+1}^2+P_{2N,\eta}C_{2N}^2),$$
$$-\frac{3}{4}(P_{N+1,\eta}C_{N+1}S_{N+1}+P_{2N,\eta}C_{2N}S_{2N}),\cdots,$$
$$\frac{6P_{N+k,\eta}C_{N+k}}{4L_{N+k}}-\frac{6P_{N+k-1,\eta}C_{N+k-1}}{4L_{N+k-1}},N_{k,\eta}-\frac{3}{4}(P_{N+k,\eta}C_{N+k}^2+P_{N+k-1,h}C_{N+k-1}^2),$$
$$-\frac{3}{4}(P_{N+k,\eta}C_{N+k}S_{N+k}+P_{N+k-1,\eta}C_{N+k-1}S_{N+k-1}),\cdots$$
$$(k=2,..,N))$$

$$\mathbf{B}_{\beta y\xi}=(\frac{6P_{N+1,\xi}S_{N+1}}{4L_{N+1}}-\frac{6P_{2N,\xi}S_{2N}}{4L_{2N}},-\frac{3}{4}(P_{N+1,\xi}C_{N+1}S_{N+1}+P_{2N,\xi}C_{2N}S_{2N}),$$
$$N_{1,\xi}-\frac{3}{4}(P_{N+1,\xi}S_{N+1}^2+P_{2N,\xi}S_{2N}^2),\cdots,$$
$$\frac{6P_{N+k,\xi}S_{N+k}}{4L_{N+k}}-\frac{6P_{N+k-1,\xi}S_{N+k-1}}{4L_{N+k-1}},-\frac{3}{4}(P_{N+k,\xi}C_{N+k}S_{N+k}+P_{N+k-1,\xi}C_{N+k-1}S_{N+k-1}),$$
$$N_{k,\xi}-\frac{3}{4}(P_{N+k,\xi}S_{N+k}^2+P_{N+k-1,\xi}S_{N+k-1}^2),\cdots$$
$$(k=2,..,N))$$

$$\mathbf{B}_{\beta y\eta}=(\frac{6P_{N+1,\eta}S_{N+1}}{4L_{N+1}}-\frac{6P_{2N,\eta}S_{2N}}{4L_{2N}},-\frac{3}{4}(P_{N+1,\eta}C_{N+1}S_{N+1}+P_{2N,\eta}C_{2N}S_{2N}),$$
$$N_{1,\eta}-\frac{3}{4}(P_{N+1,\eta}S_{N+1}^2+P_{2N,\eta}S_{2N}^2),\cdots,$$
$$\frac{6P_{N+k,\eta}S_{N+k}}{4L_{N+k}}-\frac{6P_{N+k-1,\eta}S_{N+k-1}}{4L_{N+k-1}},-\frac{3}{4}(P_{N+k,\eta}C_{N+k}S_{N+k}+P_{N+k-1,\eta}C_{N+k-1}S_{N+k-1}),$$
$$N_{k,\eta}-\frac{3}{4}(P_{N+k,\eta}S_{N+k}^2+P_{N+k-1,\eta}S_{N+k-1}^2),\cdots$$
$$(k=2,..,N))$$

For elements DST, DSQ:

The relation binding the deformations of inflection to the field of displacement in inflection is also written in matric form:

$\begin{pmatrix}\kappa_{xx}\\ \kappa_{yy}\\ 2\kappa_{xy}\end{pmatrix}=\sum_{k=1}^{N}\mathbf{B}_{f\beta k}\mathbf{U}_{f\beta k}+\sum_{k=N+1}^{2N}\mathbf{B}_{f\alpha k}\mathbf{U}_{f\alpha k}$ where $\mathbf{U}_{f\beta k}=\begin{pmatrix}w_k\\ \beta_{xk}\\ \beta_{yk}\end{pmatrix}$ and $\mathbf{U}_{f\alpha k}=\alpha_k$ represent the field of displacement of inflection to the node K, so that:

$$\mathbf{B}_{f\beta k}=\begin{pmatrix}0 & j_{11}N_{k,\xi}+j_{12}N_{k,\eta} & 0\\ 0 & 0 & j_{21}N_{k,\xi}+j_{22}N_{k,\eta}\\ 0 & j_{21}N_{k,\xi}+j_{22}N_{k,\eta} & j_{11}N_{k,\xi}+j_{12}N_{k,\eta}\end{pmatrix}$$ and

$$\mathbf{B}_{f\alpha k}=\begin{pmatrix}j_{11}P_{xk,\xi}+j_{12}P_{xk,\eta}\\ j_{21}P_{yk,\xi}+j_{22}P_{yk,\eta}\\ j_{11}P_{yk,\xi}+j_{12}P_{yk,\eta}+j_{21}P_{xk,\xi}+j_{22}P_{xk,\eta}\end{pmatrix}.$$

The matrix of passage of the field of displacement of inflection $\mathbf{U}_f=(\mathbf{U}_{f\beta},\alpha)$ with $\mathbf{U}_{f\beta}=\begin{pmatrix}w_1\\ \beta_{x1}\\ \beta_{y1}\\ \vdots\\ w_N\\ \beta_{xN}\\ \beta_{yN}\end{pmatrix}$

and $\alpha=\begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_N\end{pmatrix}$ with the deformations of inflection is written then:

$$\mathbf{B}_{f[3\times 4N]}=(\mathbf{B}_{f\beta 1},\cdots,\mathbf{B}_{f\beta N},\mathbf{B}_{f\alpha(N+1)},\cdots,\mathbf{B}_{f\alpha 2N})=(\mathbf{B}_{f\beta[3\times 3N]},\mathbf{B}_{f\alpha[3\times N]}).$$

4.4 Matrix of rigidity

The principle of virtual work is written in the following way: $\delta W_{\text{ext}}=\delta W_{\text{int}}$ that is to say still in elasticity $\delta\mathbf{U}^T\mathbf{K}\mathbf{U}=\mathbf{F}\delta\mathbf{U}$ in matric form where $\mathbf{K}$ is the matrix of rigidity coming from the assembly in the total reference mark of the whole of the elementary matrices of rigidity.

4.4.1 Elementary matrix of rigidity for the Q4 elementsΓ

$$\delta W_{\text{int}}^e=\int_e[\delta e(H_m e+H_{mf}\kappa)+\delta\kappa(H_{mf}e+H_f\kappa)+\delta\bar{\gamma}H_{ct}\bar{\gamma}]dS=$$
$$\int_e(\delta U_m^T B_m^T H_m B_m U_m+\delta U_m^T B_m^T H_{mf}B_f U_f+\delta U_f^T B_f^T H_{mf}B_m U_m+\delta U_f^T B_f^T H_f B_f U_f$$
$$+\delta U_f^T B_c^T H_{ct}B_c U_f)dS=$$
$$\delta U_m^T(\int_e B_m^T H_m B_m dS)U_m+\delta U_f^T(\int_e B_f^T H_f B_f dS)U_f+dU_m^T(\int_e B_m^T H_{mf}B_f dS)U_f$$
$$+\delta U_f^T(\int_e B_f^T H_{mf}B_m dS)U_m$$
$$+\delta U_f^T(\int_e B_c^T H_{ct}B_c dS)U_f=\delta U_m^T K_m U_m+\delta U_f^T K_f U_f+\delta U_m^T K_{mf}U_f+\delta U_f^T K_{fm}U_m+\delta U_f^T K_c U_f$$

with $K_{mf}=K_{fm}^T$.

This is still written: $\delta W_{\text{int}}^e=\left(\delta U_m, \delta U_f\right) K \begin{pmatrix} U_m \\ U_f \end{pmatrix}$ where

$K_{[5\mathrm{N}\times 5\mathrm{N}]}=\begin{pmatrix} K_{m[2\mathrm{N}\times 2\mathrm{N}]} & K_{mf[2\mathrm{N}\times 3\mathrm{N}]} \\ K_{mf[3\mathrm{N}\times 2\mathrm{N}]}^T & K_{f[3\mathrm{N}\times 3\mathrm{N}]}+K_{c[3\mathrm{N}\times 3\mathrm{N}]} \end{pmatrix}$ is the matrix of rigidity of the element.

4.4.2 Elementary matrix of rigidity for elements DKT, DKTG, DKQ

Since the relation $\bar{\gamma}=0$ is satisfied, one can write:

$$
\begin{aligned}
&\delta W_{\text{int}}^e=\int_e \delta e(H_m e+H_{mf} k)+\delta \kappa(H_{mf} e+H_f k) dS= \\
&\int_e (\delta U_m^T B_m^T H_m B_m U_m+\delta U_m^T B_m^T H_{mf} B_f U_f+\delta U_f^T B_f^T H_{mf} B_m U_m+\delta U_f^T B_f^T H_f B_f U_f) dS= \\
&\delta U_m^T(\int_e B_m^T H_m B_m dS) U_m+\delta U_f^T(\int_e B_f^T H_f B_f dS) U_f+\delta U_m^T(\int_e B_m^T H_{mf} B_f dS) U_f \\
&+\delta U_f^T(\int_e B_f^T H_{mf} B_m dS) U_m=\delta U_m^T K_m U_m+\delta U_f^T K_f U_f+\delta U_m^T K_{mf} U_f+\delta U_f^T K_{fm} U_m
\end{aligned} \tag{25}
$$

with $K_{mf}=K_{fm}^T$.

This is still written: $\delta W_{\text{int}}^e=\left(\delta U_m, \delta U_f\right) K \begin{pmatrix} U_m \\ U_f \end{pmatrix}$

where $K_{[5N\times 5N]}=\begin{pmatrix} K_{m[2N\times 2N]} & K_{mf[2N\times 3N]} \\ K_{mf[3N\times 2N]}^T & K_{f[3N\times 3N]} \end{pmatrix}$ is the matrix of rigidity of the element.

4.4.3 Elementary matrix of rigidity for elements DST, DSQ

$$
\begin{aligned}
&\delta W_{\text{int}}^e=\int_e \delta e(H_m e+H_{mf}\kappa)+\delta\kappa(H_{mf} e+H_f\kappa)+\delta\,\mathrm{T}\mathrm{H}_{ct}^{-1} T\, dS= \\
&\int_e (\delta U_m^T B_m^T H_m B_m U_m+\delta U_m^T B_m^T H_{mf} B_f U_f+\delta U_f^T B_f^T H_{mf} B_m U_m+\delta U_f^T B_f^T H_f B_f U_f \\
&+\delta U_{f\beta^T} B_{c\beta}^T H_{ct}^{-1} B_{c\beta} U_{f\beta}+\delta U_{f\beta^T} B_{c\beta}^T H_{ct}^{-1} B_{c\alpha}\alpha+\delta\alpha^T B_{c\alpha}^T H_{ct}^{-1} B_{c\beta} U_{f\beta}+\delta\alpha^T B_{c\alpha}^T H_{ct}^{-1} B_{c\alpha}\alpha) dS= \\
&\delta U_m^T(\int_e B_m^T H_m B_m dS) U_m+\delta U_f^T(\int_e B_f^T H_f B_f dS) U_f+\delta U_m^T(\int_e B_m^T H_{mf} B_f dS) U_f+\delta U_f^T(\int_e B_f^T H_{mf} B_m dS) U_m \\
&+\delta U_{f\beta}^T(\int_e B_{f\beta}^T H_{ct}^{-1} B_{c\beta} dS) U_{f\beta}+\delta U_{f\beta}^T(\int_e B_{f\beta}^T H_{ct}^{-1} B_{c\alpha} dS)\alpha+\delta\alpha^T(\int_e B_{c\alpha}^T H_{ct}^{-1} B_{c\beta} dS) U_{f\beta}+\delta\alpha^T(\int_e B_{c\alpha}^T H_{ct}^{-1} B_{c\alpha} dS)\alpha= \\
&\delta U_m^T K_m U_m+dU_f^T K_f U_f+\delta U_m^T K_{mf} U_f+\delta U_f^T K_{fm} U_m+\delta U_{f\beta}^T K_{\beta\beta} U_{f\beta}+\delta U_{f\beta}^T K_{\beta\alpha}\alpha+\delta\alpha^T K_{\alpha\beta} U_{f\beta}+\delta\alpha^T K_{c\alpha}\alpha
\end{aligned}
$$

It is also known that $\mathbf{U}_f=(\mathbf{U}_{f\beta},\alpha)$ from where it results that:

$$K_f=\begin{pmatrix} K_{f11} & K_{f12} \\ K_{f12}^T & K_{22} \end{pmatrix} \text{ with: } \begin{aligned} K_{f11}&=\int_s B_{f\beta}^T H_f B_{f\beta}\, dS\,; \\ K_{f12}&=\int_s B_{f\beta}^T H_f B_{f\alpha}\, dS\,; \\ K_{f22}&=\int_s B_{f\alpha}^T H_f B_{f\alpha}\, dS\,. \end{aligned}$$

$$K_{mf}=\begin{pmatrix} K_{mf11} & K_{mf12} \end{pmatrix} \text{ with: } \begin{aligned} K_{mf11}&=\int_s B_m^T H_{mf} B_{f\beta}\, dS\,; \\ K_{mf12}&=\int_s B_m^T H_{mf} B_{f\alpha}\, dS\,. \end{aligned}$$

$$K_{fm}=K_{mf}^T\,.$$

Using the fact that $\alpha=A_\beta U_{f\beta}$ one from of deduced that:

$$\delta W_{\text{int}}=\delta U_m^T K_m U_m+\delta U_{f\beta}^T K_f^{'} U_{f\beta}+\delta U_m^T K_{mf}^{'} U_{f\beta}+\delta U_{f\beta}^T K_{fm}^{'} U_m \text{ where:}$$

$$\begin{aligned} K_f^{'}&=K_{f11}+K_{\beta\beta}+A_\beta^T(K_{f22}+K_{c\alpha})A_\beta+(K_{f12}+K_{\beta\alpha})A_\beta+A_\beta^T(K_{f12}^T+K_{\beta\alpha}^T) \\ K_{mf}^{'}&=K_{mf11}+K_{mf12}A_\beta \end{aligned}\,.$$

This is still written: $\delta W_{\text{int}}^e=\begin{pmatrix}\delta U_m, \delta U_{f\beta}\end{pmatrix} K \begin{pmatrix} U_m \\ U_{f\beta}\end{pmatrix}$ where $K_{[5\text{N}\times5\text{N}]}=\begin{pmatrix} K_{m[2\text{N}\times2\text{N}]} & K'_{mf[2\text{N}\times3\text{N}]} \\ K'^T_{mf[3\text{N}\times2\text{N}]} & K'_{f[3\text{N}\times3\text{N}]}\end{pmatrix}$ is the elementary matrix of rigidity for an element of plate.

4.4.4 Assembly of the elementary matrices

The principle of virtual work for the whole of the elements is written:

$$\delta W_{\text{int}}=\sum_{e=1}^{\text{nb}\,elem} \delta W_{\text{int}}^e=\delta \mathbf{U}^{\boldsymbol{T}}\mathbf{K}\mathbf{U} \tag{26}$$

where $\mathbf{U}$ is the whole of the degrees of freedom of the discretized structure and K comes from the assembly of the elementary matrices.

4.4.4.1 Degrees of freedom

The process of assembly of the elementary matrices implies that all the degrees of freedom are expressed in the total reference mark. In the total reference mark, the degrees of freedom are three displacements compared to the three axes of the total Cartesian reference mark and three rotations compared to these three axes. One thus uses matrices of passage of the local reference mark to the total reference mark for each element. However it was seen previously that the degrees of freedom of the elements of plate are two displacements in the plan of the plate, displacement except plan and two rotations. These rotations not being exactly rotations compared to the axes of the plate since $\beta_x(x,y)=\theta_y(x,y), \beta_y(x,y)=-\theta_x(x,y)$ it is necessary to take account of it with the level of the assembly to reveal the good degrees of freedom θ_{xi}, θ_{yi} .

4.4.4.2 Fictitious rotations

Case general:

Rotation compared to the normal with the plate is regarded as not being a degree of freedom. To ensure compatibility between the passage of the local reference mark the total reference mark, one thus adds a degree of additional freedom local of rotation to the plate which is that corresponding to rotation compared to the normal with the plan of the element. This implies an expansion of the blocks of dimension $(5,5)$ matrix of local rigidity into cubes blocks of dimension $(6,6)$ by adding a line and a column corresponding to this rotation. These additional lines and these columns are a priori worthless. One then carries out the passage of the matrix of local rigidity extended to the matrix of total rigidity.

In the preceding transformation, one was satisfied to add rotations compared to the normals with the plan of the elements without modifying the deformation energy. The contribution to the energy brought by these additional degrees of freedom is indeed worthless and no rigidity is associated for them.

The matrix of total rigidity thus obtained presents the risk however to be noninvertible.
To avoid this nuisance, it is allowed to allot a small rigidity to these additional degrees of freedom on the level of the matrix of widened local rigidity. Practically, one chooses it between 10^{-6} and 10^{-3} time the diagonal minor term of the matrix of rigidity of local inflection. The user can choose this multiplicative coefficient `COEF_RIGI_DRZ` itself in `AFFE_CARA_ELEM` ; by default it is worth 10^{-5} .

Typical case of the DKT:
It is possible to associate a physical direction with ddl DRZ often called “drilling rotation” in reference to a tendency of the plate to put itself in torsion. In this case, the writing of kinematics associated with this ddl is:

$$\theta_Z=\frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right).$$

With this kinematics, one associates a dual quantity τ who is equivalent to the torque of the plate to obtain θ_Z . The difficulty to integrate this new kinematics in the classical writing of the DKT it is:
- To adopt a variational framework allowing to introduce a not fictitious but real rigidity related to the rotation of the plate,
- To introduce a discretization of θ_Z and τ .

One notes by:

- $\nabla^Z U=\frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$ the operator differential who allows knowing the values of membrane displacement to calculate the kinematics of “drilling rotation”.
- γ is a reality strictly positive.

Kinematics is reinforced ($\nabla^Z U-\theta_Z=0$) by one method of Lagrangian increased.

$$\delta W^e_{\mathrm{int}}=\int_{-h/2}^{+h/2}\int_e \delta e\left(H_m e+H_{mf} k\right)+\delta\kappa\left(H_{mf} e+H_f k\right)dS+\int_{-h/2}^{+h/2}\int_e\left(\left(\nabla^Z U-\theta_Z\right)-\frac{1}{2\gamma}\tau\right)\tau^* d\Omega$$

There is a second condition which is $\tau=2\gamma\left(\nabla^Z U-\theta_Z\right)$ sufficient small to guarantee a weak reinforcement of kinematics around DRZ. With this new variational writing, the classical framework of study of the DKT is enriched. In particular one can expect a kinematic answer different with the models from classical DKT which does not integrate physical rigidity around the normal. It is seen well that one reveals all the same a penalization of the physical kinematic condition (τ sufficient small). It is currently a weakness of the method: its dependence according to γ .

One interpolates from now on θ_Z at the points of Gauss thanks to the nodal values of θ_Z element by using the functions of linear form N . In the same way the differential operator is calculated $\nabla^Z U$ grace with the value nodal of u,v element by using the linear functions of form N and incomplete polynomials P . After discretization, the following matric form is obtained:

$$\begin{pmatrix} K_{[6N\times 6N]} & L_\tau \\ L_\tau^T & \frac{1}{\gamma^{-1}} T \end{pmatrix} \begin{pmatrix} U \\ \tau \end{pmatrix} = \begin{pmatrix} F^{\text{int}} \\ 0 \end{pmatrix}$$

Lastly, as one does not want to reveal a degree of freedom on the ddl additional τ one will proceed by a method of condensation staitque which gives finally:

$$\left(K_{[6N\times 6N]} - L_\tau \frac{1}{\gamma^{-1}} T L_\tau^T\right) U = F^{\text{int}}$$

One enriched the kinematic framework with an impact on the matrix by total rigidity by the system.

4.5 Matrix of mass

The terms of the matrix of mass are obtained after discretization of the following variational formulation:

$$\delta W_{mass}^{ac} = \int_{-h/2}^{+h/2} \int_S \rho\, \ddot{\mathbf{u}}\, \delta \mathbf{u}\, dz dS = \int_S \rho_m (\ddot{u}\delta u + \ddot{v}\delta v + \ddot{w}\delta w) + \rho_{mf} (\ddot{u}\delta\beta_x + \ddot{v}\delta\beta_y + \ddot{\beta}_x \delta u + \ddot{\beta}_y \delta v)$$
$$+ \rho_f (\ddot{\beta}_x \delta\beta_x + \ddot{\beta}_y \delta\beta_y) dS$$

with $\rho_m = \int_{-h/2}^{+h/2} \rho\, dz$, $\rho_{mf} = \int_{-h/2}^{+h/2} \rho\, z dz$, et $\rho_f = \int_{-h/2}^{+h/2} \rho\, z^2 \mathrm{dz}$.

Note:

If the plate is homogeneous or symmetrical compared to $z=0$ then $\rho_{mf}=0$. One considers in the continuation of the talk that it is always the case.

4.5.1 Matrix of elementary mass classical

4.5.1.1 Q4 elementΓ

The discretization of displacement for this finite element is:

$$\mathbf{u} = \sum_{k=1}^{N} N_k \left. \begin{pmatrix} u_k \\ v_k \\ w_k \\ \beta_{xk} \\ \beta_{yk} \end{pmatrix} \right\} k = 1, \cdots, N \qquad (27)$$

The matrix of mass, in the base where the degrees of freedom are gathered according to the directions of translation and rotation, has then as an expression:

$$\mathbf{M}=\begin{pmatrix} \mathbf{M}_m & 0 & 0 & \mathbf{M}_{mf} & 0 \\ 0 & \mathbf{M}_m & 0 & 0 & \mathbf{M}_{mf} \\ 0 & 0 & \mathbf{M}_m & 0 & 0 \\ \mathbf{M}_{mf}^T & 0 & 0 & \mathbf{M}_f & 0 \\ 0 & \mathbf{M}_{mf}^T & 0 & 0 & \mathbf{M}_f \end{pmatrix} \tag{28}$$

with:

$$\mathbf{M}_m=\int_S \rho_m \mathbf{N}^T \mathbf{N}\, dS,\quad \mathbf{M}_{mf}=\int_S \rho_{mf} \mathbf{N}^T \mathbf{N}\, \mathrm{dS} \text{ and } \mathbf{M}_f=\int_S \rho_f \mathbf{N}^T \mathbf{N}\, dS \text{ and } \mathbf{N}=(N_1 \cdots N_k) \tag{29}$$

4.5.1.2 Elements of the type DKT, DST

Like $\begin{pmatrix} w \\ \beta_x \\ \beta_y \end{pmatrix}=\sum_{k=1}^{N} N_k(\xi,\eta)\begin{pmatrix} w_k \\ \beta_{xk} \\ \beta_{yk} \end{pmatrix}+\sum_{k=N+1}^{2N}\begin{pmatrix} 0 \\ P_{xk}(\xi,\eta) \\ P_{yk}(\xi,\eta) \end{pmatrix}\alpha_k$ where $\alpha=\mathbf{A}_\beta \mathbf{U}_{f\beta}$ one from of deduced that:

$$\begin{pmatrix} w \\ \beta_x \\ \beta_y \end{pmatrix}=\sum_{k=1}^{N}\begin{pmatrix} N_k(\xi,\eta) & 0 & 0 \\ N_{kxw}(\xi,\eta) & N_{kxx}(\xi,\eta) & N_{kxy}(\xi,\eta) \\ N_{kyw}(\xi,\eta) & N_{kyx}(\xi,\eta) & N_{kyy}(\xi,\eta) \end{pmatrix}\begin{pmatrix} w_k \\ \beta_{xk} \\ \beta_{yk} \end{pmatrix} \tag{30}$$

The membrane part of the elementary matrix of mass is the same one as for Q4Γ with $k=3$ instead of $k=4$ in $\mathbf{N}$. The inflection part is composed of the blocks kp (k ième line and p ième column) following:

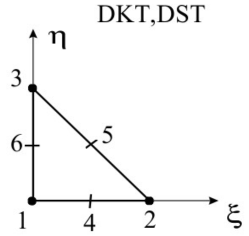
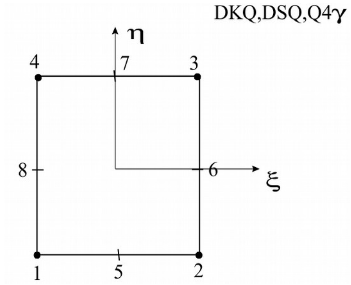
$$\rho_f\begin{pmatrix} N_{kxw}N_{pxw}+N_{kyw}N_{pyw}+\rho_m N_k N_p/\rho_f & N_{kxw}N_{pxx}+N_{kyw}N_{pyx} & N_{kxw}N_{pxy}+N_{kyw}N_{pyy} \\ N_{kxx}N_{pxw}+N_{kyx}N_{pyw} & N_{kxx}N_{pxx}+N_{kyx}N_{pyx} & N_{kxx}N_{pxy}+N_{kyx}N_{pyy} \\ N_{kxy}N_{pxw}+N_{kyy}N_{pyw} & N_{kxy}N_{pxx}+N_{kyy}N_{pyx} & N_{kxy}N_{pxy}+N_{kyy}N_{pyy} \end{pmatrix}$$

4.5.2 Elementary matrix of improved mass

As the arrow of a flexbeam can be represented by a linear approximation with difficulty, one can enrich the functions by form for the terms of inflection. This approach is used in *Code_hasster* for the elements of type DKT, DST and Q4Γ where the functions of form used in the calculation of the matrix of mass of inflection are of order three. The interpolation for w is written as follows:

$$w=\sum_{k=1}^{N} N_{(k-1)N+1}(\xi,\eta)w_k+N_{(k-1)N+2}(\xi,\eta)w_{,\xi k}+N_{(k-1)N+3}(\xi,\eta)w_{,\eta k}$$

where the functions of form are given for the triangle and the quadrangle in the following table:

	DKT,DST  	
Interpolation for w	$\lambda=1-\xi-\eta$ $i=1$ with 9 $N_1(\xi,\eta)=3\lambda^2-2\lambda^3+2\xi\eta\lambda$ $N_2(\xi,\eta)=\lambda^2\xi+\xi\eta\lambda/2$ $N_3(\xi,\eta)=\lambda^2\eta+\xi\eta\lambda/2$ $N_4(\xi,\eta)=3\xi^2-2\xi^3+2\xi\eta\lambda$ $N_5(\xi,\eta)=\xi^2(-1+\xi)-\xi\eta\lambda$ $N_6(\xi,\eta)=\xi^2\eta+\xi\eta\lambda/2$ $N_7(\xi,\eta)=3\eta^2-2\eta^3+2\xi\eta\lambda$ $N_8(\xi,\eta)=\eta^2\xi+\xi\eta\lambda/2$ $N_9(\xi,\eta)=\eta^2(-1+\eta)-\xi\eta\lambda$	$i=1$ with 12 $N_1(\xi,\eta)=\frac{1}{8}(1-\xi)(1-\eta)(2-\xi^2-\eta^2-\xi-\eta)$ $N_2(\xi,\eta)=\frac{1}{8}(1-\xi)(1-\eta)(1-\xi^2)$ $N_3(\xi,\eta)=\frac{1}{8}(1-\xi)(1-\eta)(1-\eta^2)$ $N_4(\xi,\eta)=\frac{1}{8}(1+\xi)(1-\eta)(2-\xi^2-\eta^2+\xi-\eta)$ $N_5(\xi,\eta)=-\frac{1}{8}(1+\xi)(1-\eta)(1-\xi^2)$ $N_6(\xi,\eta)=\frac{1}{8}(1+\xi)(1-\eta)(1-\eta^2)$ $N_7(\xi,\eta)=\frac{1}{8}(1+\xi)(1+\eta)(2-\xi^2-\eta^2+\xi+\eta)$ $N_8(\xi,\eta)=-\frac{1}{8}(1+\xi)(1+\eta)(1-\xi^2)$ $N_9(\xi,\eta)=-\frac{1}{8}(1+\xi)(1+\eta)(1-\eta^2)$ $N_{10}(\xi,\eta)=\frac{1}{8}(1-\xi)(1+\eta)(2-\xi^2-\eta^2-\xi+\eta)$ $N_{11}(\xi,\eta)=\frac{1}{8}(1-\xi)(1+\eta)(1-\xi^2)$ $N_{12}(\xi,\eta)=\frac{1}{8}(1-\xi)(1+\eta)(1-\eta^2)$

Functions of interpolation for the arrow of the elements of type DKT, DST, DKTG and Q4G, in dynamics and modal.

4.5.2.1 Elements of type DKT

It is known that in the approximation of one Coils-Kirchhoff has $\beta_x = -w_{,x}$ and $\beta_y = -w_{,y}$ in any point of the element.

Because of discretization stated above one a:

$$w = \sum_{k=1}^{N} N_{(k-1)N+1}(\xi,\eta) w_k + (J_{11} N_{(k-1)N+2}(\xi,\eta) + J_{21} N_{(k-1)N+3}(\xi,\eta)) w_{,xk} + (J_{12} N_{(k-1)N+2}(\xi,\eta)$$
$$+ J_{22} N_{(k-1)N+3}(\xi,\eta)) w_{,yk}$$

since: $\begin{pmatrix} w_{,\xi k} \\ w_{,\eta k} \end{pmatrix} = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} \begin{pmatrix} w_{,xk} \\ w_{,yk} \end{pmatrix}$.

This is still written:

$$w = \sum_{k=1}^{N} N'_{(k-1)N+1}(\xi,\eta) w_k + N'_{(k-1)N+2}(\xi,\eta) \beta_{xk} + N'_{(k-1)N+3}(\xi,\eta) \beta_{yk}$$

where: $$\begin{aligned} N'_{(k-1)N+1}(\xi,\eta) &= N_{(k-1)N+1}(\xi,\eta) \\ N'_{(k-1)N+2}(\xi,\eta) &= -J_{11} N_{(k-1)N+2}(\xi,\eta) - J_{21} N_{(k-1)N+3}(\xi,\eta) \\ N'_{(k-1)N+3}(\xi,\eta) &= -J_{12} N_{(k-1)N+2}(\xi,\eta) - J_{22} N_{(k-1)N+3}(\xi,\eta) \end{aligned}$$.

By not taking account of the effects of inertia, the matrix of mass has the following form thus:

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_m & 0 & 0 \\ 0 & \mathbf{M}_m & 0 \\ 0 & 0 & \mathbf{M}_f \end{bmatrix} \text{ where } \mathbf{M}_f = \int_S \rho_m \mathbf{N}' \mathbf{N}' dS . \quad (31)$$

4.5.2.2 Finite elements of the DST type

It is known that for these elements one has $\beta_x = \gamma_x - w_{,x}$ and $\beta_y = \gamma_y - w_{,y}$ where the distortion γ is constant on the element.

Like:

$$w = \sum_{k=1}^{N} N_{(k-1)N+1}(\xi,\eta) w_k + (J_{11} N_{(k-1)N+2}(\xi,\eta) + J_{21} N_{(k-1)N+3}(\xi,\eta)) w_{,xk} + (J_{12} N_{(k-1)N+2}(\xi,\eta)$$
$$+ J_{22} N_{(k-1)N+3}(\xi,\eta)) w_{,yk}$$

one can also write:

$$w=\sum_{k=1}^{N} N'_{(k-1)N+1}(\xi,\eta)w_k+N'_{(k-1)N+2}(\xi,\eta)\beta_{xk}+N'_{(k-1)N+3}(\xi,\eta)\beta_{yk}$$
$$+(J_{11}\bar{\gamma}_x+J_{12}\bar{\gamma}_y)SN_{(k-1)N+2}(\xi,\eta)+(J_{21}\bar{\gamma}_x+J_{22}\bar{\gamma}_y)SN_{(k-1)N+3}(\xi,\eta)$$

where:
$$N'_{(k-1)N+1}(\xi,\eta)=N_{(k-1)N+1}(\xi,\eta)$$
$$N'_{(k-1)N+2}(\xi,\eta)=-J_{11}N_{(k-1)N+2}(\xi,\eta)-J_{21}N_{(k-1)N+3}(\xi,\eta)\ ,$$
$$N'_{(k-1)N+3}(\xi,\eta)=-J_{12}N_{(k-1)N+2}(\xi,\eta)-J_{22}N_{(k-1)N+3}(\xi,\eta)$$

$$\sum N_{(k-1)N+1}(\xi,\eta)=\sum_{k=1}^{N} N_{(k-1)N+1}(\xi,\eta)$$
$$\sum N_{(k-1)N+2}(\xi,\eta)=\sum_{k=1}^{N} N_{(k-1)N+2}(\xi,\eta)$$
$$\sum N_{(k-1)N+3}(\xi,\eta)=\sum_{k=1}^{N} N_{(k-1)N+3}(\xi,\eta)$$

and
$$\begin{pmatrix}\bar{\gamma}_x\\ \bar{\gamma}_y\end{pmatrix}=\mathbf{H}_{ct}^{-1}[\mathbf{B}_{c\beta}+\mathbf{B}_{c\alpha}\mathbf{A}_{\beta}]\begin{pmatrix}w_1\\ \beta_{x1}\\ \beta_{y1}\\ \vdots\\ w_N\\ \beta_{xN}\\ \beta_{yN}\end{pmatrix}=\mathbf{T}_{\gamma w}\begin{pmatrix}w_1\\ \beta_{x1}\\ \beta_{y1}\\ \vdots\\ w_N\\ \beta_{xN}\\ \beta_{yN}\end{pmatrix}.$$

One obtains the interpolation then for w :

$$w=\sum_{k=1}^{N} N''_{(k-1)N+1}(\xi,\eta)w_k+N''_{(k-1)N+2}(\xi,\eta)\beta_{xk}+N''_{(k-1)N+3}(\xi,\eta)\beta_{yk}$$

where:
$$\begin{aligned}
&N''_{(k-1)N+1}(\xi,\eta)=N'_{(k-1)N+1}(\xi,\eta)+\\
&(J_{11}T_{\gamma w}(1,(k-1)N+1)+J_{12}T_{\gamma w}(2,(k-1)N+1))\sum N_{(j-1)N+2}(\xi,\eta)+\\
&(J_{21}T_{\gamma w}(1,(k-1)N+1)+J_{22}T_{\gamma w}(2,(k-1)N+1))\sum N_{(j-1)N+3}(\xi,\eta)\\
&N''_{(k-1)N+2}(\xi,\eta)=N'_{(k-1)N+2}(\xi,\eta)+\\
&(J_{11}T_{\gamma w}(1,(k-1)N+2)+J_{12}T_{\gamma w}(2,(k-1)N+2))\sum N_{(j-1)N+2}(\xi,\eta)+\\
&(J_{21}T_{\gamma w}(1,(k-1)N+2)+J_{22}T_{\gamma w}(2,(k-1)N+2))\sum N_{(j-1)N+3}(\xi,\eta)\\
&N''_{(k-1)N+3}(\xi,\eta)=N'_{(k-1)N+3}(\xi,\eta)+\\
&(J_{11}T_{\gamma w}(1,(k-1)N+3)+J_{12}T_{\gamma w}(2,(k-1)N+3))\sum N_{(j-1)N+2}(\xi,\eta)+\\
&(J_{21}T_{\gamma w}(1,(k-1)N+3)+J_{22}T_{\gamma w}(2,(k-1)N+3))\sum N_{(j-1)N+3}(\xi,\eta)
\end{aligned}$$

By not taking account of the effects of inertia, the matrix of mass has the following form thus:

$$\mathbf{M}=\begin{pmatrix} \mathbf{M}_m & 0 & 0 \\ 0 & \mathbf{M}_m & 0 \\ 0 & 0 & \mathbf{M}_f \end{pmatrix} \quad \text{where} \quad \mathbf{M}_f=\int_S \rho_m \mathbf{N}'' \mathbf{N}'' dS . \tag{32}$$

4.5.2.3 Elements of the Q4 typeΓ

One proceeds in the same way that for the elements of the DST type but with:

$$\begin{pmatrix} \bar{\gamma}_x \\ \bar{\gamma}_y \end{pmatrix}=\mathbf{B}_c \begin{pmatrix} w_1 \\ \beta_{x1} \\ \beta_{y1} \\ M \\ w_N \\ \beta_{xN} \\ \beta_{yN} \end{pmatrix}$$ where $\mathbf{B}_c$ is the matrix established with the § 4.3.2.1.

4.5.2.4 Notice

One neglects in the form of the elementary matrix of mass the terms of inertia of rotation $\int_S \rho_f (\ddot{\beta}_x \delta \beta_x + \ddot{\beta}_y \delta \beta_y) dS$ because the latter are negligible [5] compared to the others. Indeed a multiplicative factor of $h^2/12$ the dregs with the other terms and they become negligible for a thickness report over characteristic length lower than $1/20$.

4.5.3 Assembly of the elementary matrices of mass

The assembly of the matrices of mass follows same logic as that of the matrices of rigidity. The degrees of freedom are the same ones and one finds the treatment specific to normal rotations with the plan of the plate. For modal calculations utilizing at the same time the calculation of the matrix of rigidity and that of the matrix of mass, it is necessary to take a rigidity or a mass on the degree of normal rotation to the plan of the plate of 10^3 to 10^6 time smaller than the diagonal minor term of the matrix of rigidity or mass for the terms of inflection. That makes it possible to inhibit the modes being able to appear on the additional degree of freedom of rotation around the normal with the plan of the plate. By default, one takes a rigidity or a mass on the degree of normal rotation to the plan of plate 10^5 time smaller than the diagonal minor term of the matrix of rigidity or mass for the terms of inflection

4.5.4 Matrix of “lumpée” mass

The use of a matrix of “lumpée” mass has two advantages: it is simpler to implement numerically and it allows a better convergence. However the results are less good than with the classical diagram (consistent matrix) for which the error is minimal [6] . The matrix of lumpée mass is recommended in theory only for transitory calculations using an explicit diagram of temporal integration, which is used almost exclusively in fast dynamics. Contrary to calculations with the implicit schemes where with each increment (and each iteration Newton in non-linear case) one assembles and opposite a matrix, given like a linear combination between the matrix of rigidity, the matrix of mass and the matrix of damping, in explicit calculations only the matrix of mass is assembled and reversed. Therefore, by using a matrix of diagonal mass profit in term of time CPU of resolution as well as the profit in term of the storage of the matrices are enormous compared to the use of a matrix masses consistent.

Code_Aster not being a code specialized in fast dynamics, this advantage of the diagonal matrix is not exploited. The option of the matrix of lumpée mass is thus only one choice of modeling, allowing possible comparisons other computer codes.

One presents here two methods for diagonaliser a matrix of coherent mass. It will be also shown in what the choice between the two methods is conditioned by the choice on the coherent matrix between the classic (§4.5.1) and improved (§4.5.2).

The technique undoubtedly simplest to obtain a lumpée matrix is to retain the diagonal value for each degree of freedom as the sum of the elements of the line of the coherent matrix. Moreover, it is pointed out that the most important property of a matrix of lumpée mass is that it makes it possible to represent a movement of the rigid body correctly. It is satisfied in the following way by the method of "summation per line", put in equations:

$$M_C^{\alpha}=\sum_{\beta} M^{\alpha\beta} .$$

Unfortunately the summation of the lines does not guarantee only all the terms M_C^{α} are positive. The negative terms appear in particular by using the matrix of improved coherent mass (mentioned in §4.5.2). This reason and for most elements in Code_Aster, one chose another approach, which had in Hinton (see [6]), where diagonal terms corresponding to the directions x , y , z , $M_H^{\alpha x}$, $M_H^{\alpha y}$ and $M_H^{\alpha z}$, are calculated like:

$$M_H^{\alpha x}=\frac{\int_V \rho\, dV}{\sum_{\beta} M_x^{\beta\beta}} M_x^{\alpha\alpha} \qquad M_H^{\alpha y}=\frac{\int_V \rho\, dV}{\sum_{\beta} M_y^{\beta\beta}} M_y^{\alpha\alpha} \qquad M_H^{\alpha z}=\frac{\int_V \rho\, dV}{\sum_{\beta} M_z^{\beta\beta}} M_z^{\alpha\alpha} \tag{33}$$

where indices α correspond to the numbers of nodes. Although the method of Hinton is generally more robust, it is unsuited to the elements plates and hull, since [eq. 33] does not make it possible to include the terms of inertia, the terms $M_H^{\alpha\cdot}$ defined in [eq. 33] having inevitably the units of mass and never of inertia.

Consequently, for the elements treated here one modifies the calculation of the matrix of consistent mass being used with calculation of the matrix as lumpée mass. One adopts the classical method described in §4.5.1 for the matrix of coherent mass, then the approach of "summation per line" for the lumping. The option impacted of *Code_hasster* is `MASS_MECA_EXPLI` and only for the elements `DKT` and `DKTG` . For the others one does not have the matrix of lumpée mass.

4.5.5 Modification of the terms of inertia

The matrix of lumpée mass described in §4.5.4 is not very effective for a calculation in explicit dynamics, where the step of time of stability is strongly penalized by a bad conditioning of the matrix of mass. Terms corresponding to rotations, i.e. the terms of inertia are the principal culprits, since much smaller than the terms of translation, i.e. displacements. For this reason, one proposed in [7] a method to modify the problematic terms while avoiding degrading the quality of the solution. Although old and not completely rigorous approach suggested in [7] is largely referred by the literature of the field and was not really prone to remarkable improvements.

One focuses oneself on the terms due to the inflection, θ_x , θ_y and $w(=u_z)$, terms corresponding to the membrane being obtained in a way classical and also applied to the elements 2D. The matrix of mass M defined §4.5 becomes:

$$M^{\alpha\beta}=m^{\alpha\beta}\begin{pmatrix}\frac{h^3}{12} & 0 & 0\\ 0 & \frac{h^3}{12} & 0\\ 0 & 0 & h\end{pmatrix} \tag{34}$$

where h is the thickness of the plate and $m^{\alpha\beta}$ defined like:

$$m^{\alpha\beta}=\int_S \rho N^\alpha N^\beta dS \tag{35}$$

It is noted that [eq. 34] and [eq. 35] are equivalent to [eq. 2.3-1] for the kinematics of plates. In [7] one proposes to build the matrix lumpée starting from [éq. 2.5-2] by using the squaring of Lobatto, whose alternatives are the trapezoidal diagram and the diagram of Simpson, where the points of integration coincide with the nodes. The construction of the matrix of mass is done through one finite element of beam, linear and with two nodes, by using the trapezoidal diagram, leading to:

$$M_0^{\text{pout}}=\frac{1}{2}\rho LA\begin{pmatrix}\frac{I}{A} & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & \frac{I}{A} & 0\\ 0 & 0 & 0 & 1\end{pmatrix} \tag{36}$$

for which the vector of degrees of freedom is written like $\begin{pmatrix}\theta_1 & w_1 & \theta_2 & w_2\end{pmatrix}^T$. A , L and I are the surface of the section, the length and the moment of inertia of the element beam, respectively. The use of the matrix [éq. 36] seeming too restrictive compared to the stability condition, one proposes in [7] rather:

$$M^{\text{pout}}=\frac{1}{2}\rho LA\begin{pmatrix}\alpha & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & \alpha & 0\\ 0 & 0 & 0 & 1\end{pmatrix} \tag{37}$$

where the parameter α is introduced so that its adjustment can maximize the step of time of stability. According to [7] its optimal value would be $\alpha=\frac{1}{8}L^2$. By directly applying these results by analogy to the plates, one replaces the matrix of [éq. 34] by:

$$M^{\alpha\beta}=m^{\alpha\beta}\begin{pmatrix}\frac{hA^e}{8} & 0 & 0\\ 0 & \frac{hA^e}{8} & 0\\ 0 & 0 & h\end{pmatrix} \tag{38}$$

where A^e is the surface of the element considered. In the passage of the beam to the plate one supposed a certain equivalence between the length of the element beam and the surface of the element plates, so that $A^e \approx L^2$. It is pointed out that the approach suggested in [7] is not rigorous from the geometrical point of view and which it focuses on the maximization of the step of stability. In the established version, one makes sure of the desired effect of the modification of [éq. 34] with [éq. 37] while using:

$$M^{\alpha\beta}=m^{\alpha\beta}\begin{pmatrix} \max\left(\frac{h^3}{12},\frac{hA^e}{8}\right) & 0 & 0 \\ 0 & \max\left(\frac{h^3}{12},\frac{hA^e}{8}\right) & 0 \\ 0 & 0 & h \end{pmatrix} \quad (39)$$

because [éq 37] is not interesting that for the grids coarse, a priori more current, while [éq. 34] becomes favorable for very fine grids.

4.6 Linear buckling

Linear buckling is presented in the form of a typical case of the geometrical nonlinear problem. It is based on the assumption of a linear dependence of the fields of displacements, strains and stresses compared to the level of load.

4.6.1 Field of deformation

From the assumption of Kirchhoff, the components of the tensor of the deformations of Lagrange Green are related to the components of displacement in the plan of the plate in the following way:

$$\varepsilon=\begin{pmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_{xy}\end{pmatrix}=\begin{pmatrix}u_{,x}\\ v_{,y}\\ (u_{,y}+v_{,x})\end{pmatrix}+\begin{pmatrix}\frac{1}{2}\beta_x^2\\ \frac{1}{2}\beta_y^2\\ (\beta_x\beta_x)\end{pmatrix}+z\begin{pmatrix}\beta_{x,x}\\ \beta_{y,y}\\ (\beta_{x,y}+\beta_{y,x})\end{pmatrix} \quad (40)$$

That one can express in the following form:

$$\varepsilon=e_L+e_{NL}+z\,\kappa$$

with

$<e_L>=<u_{,x}\ v_{,y}\ (u_{,y}+v_{,x})>$ linear deformations of membrane

$<e_{NL}>=<\frac{1}{2}\beta_x^2\ \ \frac{1}{2}\beta_y^2\ \ \beta_x\beta_y>$ non-linear deformations of membrane

$<\kappa>=<\beta_{x,x}\ \ \beta_{y,y}\ \ \beta_{x,y}\beta_{y,x}>$ linear deformations of inflection

4.6.2 Geometrical matrix of rigidity [K$_G$]

From the second variation of the internal energy of deformation and non-linear deformations, one obtains the matrix $[K_G]$ following:

$$\int_S <\delta w_{,x}\,\delta w_{,x}> \begin{bmatrix} N_{xx} & N_{xy} \\ N_{xy} & N_{yy} \end{bmatrix} \begin{Bmatrix} \delta w_{,x} \\ \delta w_{,y} \end{Bmatrix} dA_e = <\delta u_n^f>[K_G]\{\delta u_n^f\} \quad (41)$$

That one can write in the following form:

$$<\delta u_n^f>[K_G]\{\delta u_n^f\} = <\delta u_n^f> \int_S [B_{NL}][N][B_{NL}]^T dS \{\delta u_n^f\}$$

With

$[N]=\begin{bmatrix} N_{xx} & N_{xy} \\ N_{xy} & N_{yy} \end{bmatrix}$ normal efforts

$<u_n^f>$ degrees of freedom of inflection

$[B_{NL}]$ the matrix connecting the quadratic deformations to the degrees of freedom

Note:

Linear buckling is available only for the elements `DKT` and `DKTG` with meshs `TRIA3` and `QUAD4` .

4.7 Digital integration for elasticity

For the triangular elements `DKT`, `DKTG` and `DST` the matrices of rigidity are obtained exactly with three points of integration of Hammer:

Cordonnées of the points	Weight ω_l
$\xi_1=1/6\,;\eta_1=1/6$	1/6
$\xi_2=2/3\,;\eta_2=1/6$	1/6
$\xi_3=1/6\,;\eta_3=2/3$	1/6
$\int_0^1 \int_0^{1-\xi} y(\xi,\eta)\,d\eta\,d\xi =$	$\sum_{i=1}^{n} \omega_i\, y(\xi_i,\eta_i)$

Formulas of digital integration on a triangle (Hammer)

For the elements quadrangles `DKQ`, `DKQG` and `DSQ` an integration of Gauss 2x2 is used for the matrices of rigidity.

Cordonnées of the points	Weight ω_l
$\xi_1=1/\sqrt{3}\,;\eta_1=1/\sqrt{3}$	1
$\xi_2=1/\sqrt{3}\,;\eta_2=-1/\sqrt{3}$	1
$\xi_3=-1/\sqrt{3}\,;\eta_3=1/\sqrt{3}$	1
$\xi_3=-1/\sqrt{3}\,;\eta_3=-1/\sqrt{3}$	1
$\int_0^1 \int_0^{1-\xi} y(\xi,\eta)\,d\eta\,d\xi =$	$\sum_{i=1}^{n} \omega_i\, y(\xi_i,\eta_i)$

Formulas of digital integration on a quadrangle (Gauss)

4.8 Digital integration for the matrix of mass

For the triangular elements `DKT`, `DKTG` and `DST` the matrices of mass are obtained with four points of integration.

For the elements quadrangles `DKQ, DKQG and DSQ` the matrices of mass are obtained with an integration of Gauss 3x3.

4.9 Digital integration for plasticity and other nonlinear laws

Integration on the surface of the element is supplemented by an integration in the thickness of the behavior since:

$\mathbf{H}_{\boldsymbol{m}}=\int_{-h/2}^{+h/2}\mathbf{H}\,\mathrm{dz}\,,\mathbf{H}_{\boldsymbol{mf}}=\int_{-h/2}^{+h/2}\mathbf{H}\,\mathrm{zdz}\,,\mathbf{H}_{\boldsymbol{f}}=\int_{-h/2}^{+h/2}\mathbf{H}\,z^2\mathrm{dz}$ where $\mathbf{H}$ is the matrix of plastic behavior local (or other nonlinear laws).

The initial thickness is divided into N layers of thicknesses identical and there are three points of integration per layer (except for the elements `DKTG` and `DKQG` who have only one sleep and a point of integration in the layer). The points of integration are located in higher skin of layer, in the middle of the layer and in lower skin of layer. For N layers, the number of points of integration is of $2N+1$.

One advises to use from 3 to 5 layers in the thickness for a number of points of integration being worth 7.9 and 11 respectively.

For rigidity, one calculates for each layer, in plane constraints, the contribution to the matrices of rigidity of membrane, inflection and coupling membrane-inflection. These contributions are added and assembled to obtain the matrix of total tangent rigidity. For each layer, the state of the constraints is calculated $(\sigma_{xx},\sigma_{yy},\sigma_{xy})$ and the whole of the internal variables, in the middle of the layer and in skins higher and lower of layer, starting from the local plastic behavior and of the local field of deformation $(\varepsilon_{xx},\varepsilon_{yy},\varepsilon_{xy})$. The positioning of the points of integration enables us to have the rightest estimates, because not extrapolated, in skins lower and higher of layer, where it is known that the constraints are likely to be maximum.

Cordonnées of the points	Weight ω_l
$\xi_1=-1$	1/3
$\xi_2=0$	4/3
$\xi_3=+1$	1/3
$\int_{-1}^{1} y(\xi)\,d\,\xi=$	$\sum_{i=1}^{n}\omega_i\,y(\xi_i,\eta_i)$

Digital formula of integration for a layer in the thickness

Note:

> *One already mentioned with the §2.2.3 that the value of the coefficient of correction in transverse shearing for the elements DST, DSQ and Q4Γ was obtained by identification of elastic complementary energies after resolution of balance 3D. This method is not usable any more in elastoplasticity and the choice of the coefficient of correction in transverse shearing is posed then. Plasticity is thus not developed for these elements.*

4.10 Discretization of external work

The variational formulation of work external for the elements of plate is written:

$$\delta W_{ext}=\int_S (f_x\delta u+f_y\delta v+f_z\delta w+m_x\delta\beta_x+m_y\delta\beta_y)dS+\int_C (f_x\delta u+f_y\delta v+f_z\delta w+m_x\delta\beta_x+m_y\delta\beta_y)ds$$

By taking account of a linear discretization of displacements, one can write for an element:

$$\begin{aligned}
\delta W^e_{ext}=&\sum_{k=1}^{N}\int_S (f_x N_k(\xi,\eta)du_k+f_y N_k(\xi,\eta)\delta v_k+f_z N_k(\xi,\eta)\delta w_k\\
&+m_x N_k(\xi,\eta)\delta\beta_{xk}+m_y N_k(\xi,\eta)\delta\beta_{yk})dS\\
&+\int_C (\phi_x N_k(\xi,\eta)\delta u_k+\phi_y N_k(\xi,\eta)\delta v_k+\phi_z N_k(\xi,\eta)\delta w_k\\
&+\mu_x N_k(\xi,\eta)\delta\beta_{xk}+\mu_y N_k(\xi,\eta)\delta\beta_{yk})ds\\
=&\sum_{k=1}^{N}(\int_S f_x N_k(\xi,\eta)dS+\int_C \phi_x N_k(\xi,\eta)ds\vdots\ \int_S f_y N_k(\xi,\eta)dS+\int_C \phi_y N_k(\xi,\eta)ds\vdots\\
&\int_S f_z N_k(\xi,\eta)dS+\int_C \phi_z N_k(\xi,\eta)ds\vdots\ \int_S m_x N_k(\xi,\eta)dS+\int_C \mu_x N_k(\xi,\eta)ds\vdots\\
&\int_S m_y N_k(\xi,\eta)dS+\int_C \mu_y N_k(\xi,\eta)ds\vdots)\delta\mathbf{U}^e_{\boldsymbol{k}}\\
=&\sum_{k=1}^{N}\mathbf{F}^e_{\boldsymbol{k}}\delta\mathbf{U}^e_{\boldsymbol{k}}=\mathbf{F}^e\delta\mathbf{U}^e
\end{aligned} \tag{42}$$

The variational formulation of the work of the efforts external for the unit of the elements is written then:

$\delta W_{\mathbf{ext}}=\sum_{e=1}^{\mathrm{nb}elem}\delta W^e_{\mathbf{ext}}=\mathbf{F}\delta\mathbf{U}=\delta\mathbf{U}^{\boldsymbol{T}}\mathbf{F}^{\boldsymbol{T}}$ where $\mathbf{U}$ is the whole of the degrees of freedom of the discretized structure and $\mathbf{F}$ comes from the assembly of the vectors forces elementary.

As for the matrices of rigidity, the process of assembly of the vectors forces elementary implies that all the degrees of freedom are expressed in the total reference mark. In the total reference mark, the degrees of freedom are three displacements compared to the three axes of the total Cartesian reference mark and three rotations compared to these three axes. One thus uses matrices of passage of the local reference mark to the total reference mark for each element.

Note:

The external efforts can also be defined in the reference mark user. One then uses a matrix of passage of the reference mark user towards the local reference mark of the element to have the expression of these efforts in the local reference mark of the element and to deduce the vector from it elementary corresponding room forces. For the assembly one passes then from the local reference mark of the element to the total reference mark.

4.11 Taking into account of the thermal loadings

4.11.1 Thermoelasticity of the plates

The temperature is represented by the model of thermics to three fields according to [R3.11.01]:

$$T(x_\gamma,x_3)=T^m(x_\gamma).P_1(x_3)+T^s(x_\gamma).P_2(x_3)+T^i(x_\gamma).P_3(x_3) \tag{43}$$

with: $T^m(x_\gamma)$: the temperature on the average layer
$T^s(x_\gamma)$: the temperature on the higher skin
$T^i(x_\gamma)$: the temperature on the lower skin

$P_j(x_3)$: three polynomials of Lagrange in the thickness: $]-h/2,+h/2[$:

$$P_1(x_3)=1-(2x_3/h)^2\ ;\ P_2(x_3)=\frac{x_3}{h}(1+2x_3/h)\ ;\ P_3(x_3)=-\frac{x_3}{h}(1-2x_3/h) \tag{44}$$

WITH to leave the representation of the temperature above, one obtains:

•the average temperature in the thickness:

$$\bar{T}(x_\gamma)=\frac{1}{h}\int_{-h/2}^{+h/2}T(x_\gamma,x_3)\,dx_3=\frac{1}{6}\left(4T^m(x_\gamma)+T^s(x_\gamma)+T^i(x_\gamma)\right)\ ;$$

•the average variation in temperature in the thickness:

$$\hat{T}(x_\gamma)=\frac{12}{h^2}\int_{-h/2}^{+h/2}T(x_\gamma,x_3)\,x_3dx_3=T^s(x_\gamma)-T^i(x_\gamma)\ ;$$

Thus the temperature can be written in the following way:

$T(x_\gamma,x_3)=\bar{T}(x_\gamma)+\hat{T}(x_\gamma).x_3/h+\tilde{T}(x_\gamma,x_3)$ such as:

$$\int_{-h/2}^{h/2}\tilde{T}(x_\gamma,x_3)=0\,;\int_{-h/2}^{h/2}x_3\,\tilde{T}(x_\gamma,x_3)=0\ .$$

If the temperature is indeed closely connected in the thickness one has, $\tilde{T}=0$.

Code_Aster draft three different thermoelastic situations, where thermoelastic characteristics E , ν , α depend only on the average temperature $\bar{T}$ in the thickness:

- the case where the material is thermoelastic isotropic homogeneous in the thickness;
- the case where the plate models an orthotropic grid (concrete reinforcing steels);
- the case where the behavior of the plate is deduced from a thermoelastic homogenisation, cf. [4] .

For the elements of plate in thermoelasticity, the heating effects are taken into account via generalized efforts, membrane and inflection. Thus, in the case of a homogeneous plate, knowing the dilation coefficient α , the generalized thermal efforts are defined starting from the plane constraints in the thickness by:

$$\begin{aligned}
N_{\beta\gamma}^{ther}&=\int_{-h/2}^{+h/2}C_{\beta\gamma\eta\zeta}\,e_{\eta\zeta}^{ther}\,dx_3=\int_{-h/2}^{+h/2}\alpha C_{\beta\gamma\eta\zeta}\left(T-T^{réf}\right)\delta_{\eta\zeta}\,dx_3\\
M_{\beta\gamma}^{ther}&=\int_{-h/2}^{+h/2}x_3\,C_{\beta\gamma\eta\zeta}\,e_{\eta\zeta}^{ther}\,dx_3=\int_{-h/2}^{+h/2}\alpha\,x_3\,C_{\beta\gamma\eta\zeta}\left(T-T^{réf}\right)\delta_{\eta\zeta}\,dx_3\\
V_{\beta}^{ther}&=0
\end{aligned} \tag{45}$$

Maybe in the homogeneous isotropic thermoelastic case in the thickness:

$$N_{\beta\gamma}^{ther}=\alpha . C_{\beta\gamma\eta\zeta} . h . \left(\bar{T}-T^{réf}\right)\delta_{\eta\zeta}=\alpha . \frac{Eh}{1-\nu} . \left(\bar{T}-T^{réf}\right)\delta_{\beta\gamma} ;$$
$$M_{\beta\gamma}^{ther}=\alpha . C_{\beta\gamma\eta\zeta} . \frac{h^2}{12}\hat{T}=\alpha . \frac{Eh^2}{12(1-\nu)} . \hat{T}\delta_{\beta\gamma} ; V_{\beta}^{ther}=0 . \quad (46)$$

The thermal constraints of origin withdrawn from the usual mechanical constraints are calculated in three positions (sup., moy. and inf.) in the thickness:

$$\sigma_{\beta\gamma}^{ther}=\frac{\alpha . E}{1-\nu}\left(\bar{T}-T^{réf}+\hat{T} . x_3/h\right)\delta_{\beta\gamma} \quad (47)$$

In the case deduced from the thermoelastic homogenisation, cf. [4] , the generalized thermal efforts are defined by the general relation, starting from the “correctors” of membrane $c^{\beta\gamma}$, those of inflection $x^{\beta\gamma}$, and that of dilation $\mathbf{u}^{dil}$, like averages on representative ground volume (cell z):

$$N_{\beta\gamma}^{ther}=\langle\langle C_{\beta\gamma\eta\zeta} . \alpha . \left(\bar{T}-T^{réf}+\hat{T}\left(x_\gamma\right) . z_3/h+\tilde{T}\left(x_\gamma , x_3\right)\right)\delta_{\eta\zeta}\rangle\rangle_Z+\langle\langle e_{ij}\left(c^{\beta\gamma}\right)C_{ijkl} . e_{kl}\left(u^{dil}\right)\rangle\rangle_Z ;$$
$$M_{\beta\gamma}^{ther}=\langle\langle z_3 . C_{\beta\gamma\eta\zeta} . \alpha . \left(\bar{T}-T^{réf}+\hat{T}\left(x_\gamma\right) . z_3/h+\tilde{T}\left(x_\gamma , x_3\right)\right)\delta_{\eta\zeta}\rangle\rangle_Z+\langle\langle e_{ij}\left(x^{\beta\gamma}\right)C_{ijkl} . e_{kl}\left(u^{dil}\right)\rangle\rangle_Z ;$$
$$V_{\beta}^{ther}=0$$

In this case when one limits oneself to the orthotropic situations without coupling inflection-membrane, one neglects the role of $\tilde{T}\left(x_\gamma , x_3\right)$ on the corrector $\mathbf{u}^{dil}$, and it is thus found that the thermal efforts which appear to the second member have as an expression:

$$N_{\beta\gamma}^{ther}=\alpha . H_{\beta\gamma\eta\zeta}^{m} . \left(\bar{T}-T^{réf}\right)\delta_{hz} ;$$
$$M_{\beta\gamma}^{ther}=\alpha . H_{\beta\gamma\eta\zeta}^{f} . \hat{T}\delta_{\eta\zeta} ; V_{\beta}^{ther}=0 \quad (48)$$

One cannot however go back to the complete three-dimensional constraints: it would be necessary to know the “correctors” within the basic cell having been used with the determination of the coefficients as homogenized behavior.

In the thermoelastoplastic situations, or for the hulls (elements of the family `COQUE_3D`), it is necessary to evaluate the three-dimensional constraints, of which thermal stresses, in each point of integration in the thickness.

Note:

To go back to the complete three-dimensional constraints is not immediate for the multi-layer hulls (laminated) because it is necessary to know layer by layer the state of stress; in elasticity, this one results from the state of deformation and the behavior on the level of each layer.

4.11.2 Thermomechanical chaining

For the resolution of chained thermomechanical problems, one must use for the thermal calculation of the finite elements of thermal hull [R3.11.01] whose field of temperature is recovered like input datum of *Code_hasster* for mechanical calculation. It is necessary thus that there is compatibility between the thermal field given by the thermal hulls and that recovered by the mechanical plates. This last is defined by the knowledge of the 3 fields `TEMP_SUP`, `TEMP_MIL` and `TEMP_INF` given in skins lower, medium and higher of hull.

The table below indicates compatibilities between the elements of plate and the elements of thermal hull:

Modeling THERMICS	Mesh	Finite element	to use with	Mesh	Finite element	Modeling MECHANICS
HULL	QUAD4	THCOQU4		QUAD4	MEDKQU4 MEDKQG4 MEDSQU4 MEQ4QU4	DKT DKTG DST Q4G
HULL	TRIA3	THCOTR3		TRIA3	MEDKTR3 MEDKTG3 MEDSTR3	DKT DKTG DST

Note:

The nodes of the thermal elements of hulls and mechanical plates must correspond. The grids will be identical.
The elements of thermal hulls surface are treated like elements plans by projection of the initial geometry on the level defined by the first 3 tops.

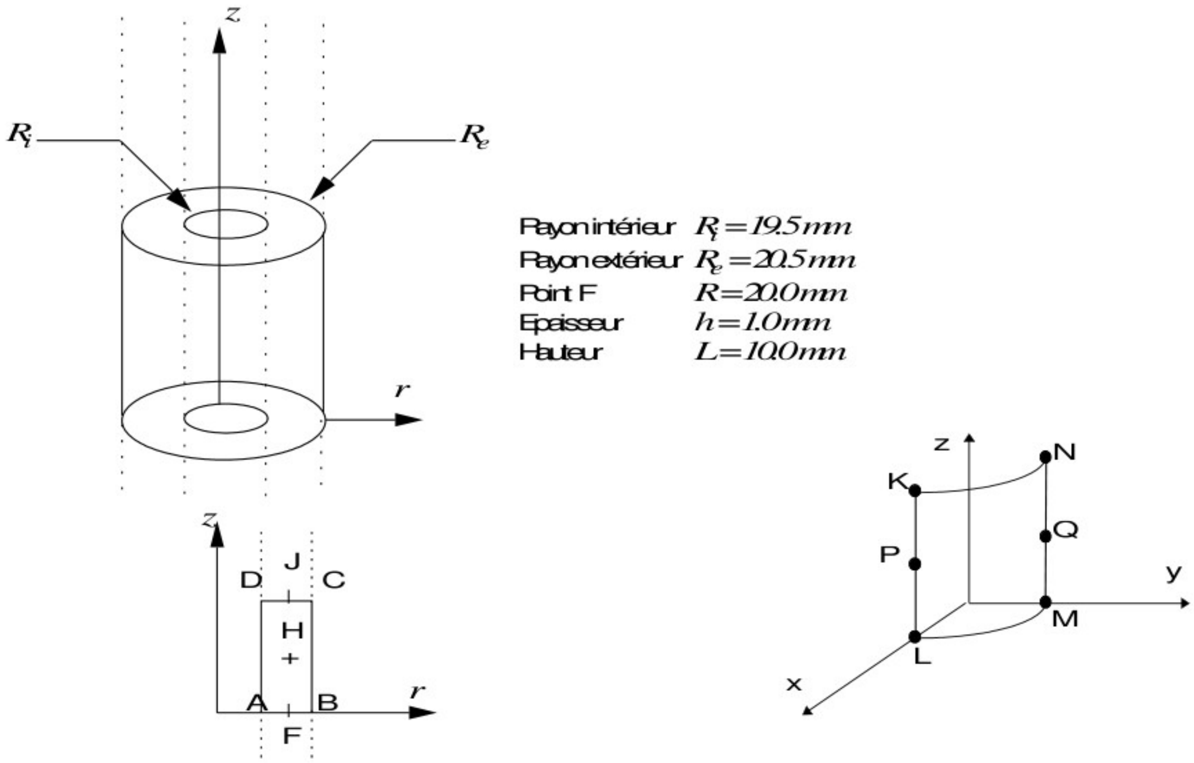
The thermomechanical chaining with definite multi-layer materials via the order DEFI_COMPOSITE [U4.23.03] is not available in *Code_hasster* for the moment.

The thermomechanical chaining is also possible if one knows by experimental measurements the variation of the field of temperature in the thickness of the structure or certain parts of the structure. In this case one works with a map of temperature defined a priori; the field of temperature is not given any more by the three values TEMP_INF, TEMP_MIL and TEMP_SUP thermal calculation obtained by EVOL_THER. It can be much richer and contain an arbitrary number of points of discretization in the thickness of the hull. The operator DEFI_NAPPE allows to create such profiles of temperatures starting from the abundant data by the user. These profiles are affected by the order CREA_CHAMP (cf the CAS-test hpla100e). It will be noted that it is not necessary for mechanical calculation that the number of points of integration in the thickness is equal to the number of points of discretization of the field of temperature in the thickness. The field of temperature is automatically interpolated at the points of integration in the thickness of the elements of plates or hulls by the order CREA_RESU operation PREP_VRC2.

For the elements DKTG on the other hand, which does not have under-points in the thickness, one should not use PREP_VRC2. Three values TEMP_INF, TEMP_MIL and TEMP_SUP are assigned to variables of the same order name, recoverable directly in the programs.

4.11.3 CAS-test

The cas-tests for the thermomechanical chaining between thermal elements of hulls and elements of plate are the hpla100e (elements DKT) and hpla100f (elements DKQ). It is about a heavy thermoelastic hollow roll in uniform rotation [V7.01.100] subjected to a phenomenon of thermal dilation where the fields of temperature are calculated with THER_LINEAIRE by a stationary calculation.



Thermal dilation is worth: $T(\rho)-T_{réf}(\rho)=0.5(T_s+T_i)+2.(T_s+T_i)(r-R)/h$

with:

- $T_s=0.5\,°C$, $T_i=-0.5\,°C$, $T_{réf}=0.\,°C$
- $T_s=0.1\,°C$, $T_i=0.1\,°C$, $T_{réf}=0.\,°C$

One tests the constraints, the efforts and bending moments in L and M. The results of reference are analytical. One obtains very good performances whatever the type of element considered.

5 Establishment of the elements of plate in *Code_Aster*

5.1 Description

These elements (of names MEDKTR3, MEDSTR3, MEDKQU4, MEDSQU4, MEDKTG3, MEDKQG4 and MEQ4QU4) are pressed on meshs TRIA3 and QUAD4 plane. These elements are not exact with the nodes and it is necessary to net with several elements to get correct results.

5.2 Introduced use and developments

These elements are used in the following way:

- AFFE_MODELE (MODELING = 'DKT'.) for the triangle and the quadrangle of the type DKT
- AFFE_MODELE (MODELING = 'DST'.) for the triangle and the quadrangle of the DST type
- AFFE_MODELE (MODELING = 'DKTG'.) for the triangle and the quadrangle of the type DKTG
- AFFE_MODELE (MODELING = 'Q4G'.) for the quadrangle of the type $Q4\Gamma$

One calls on the routine INI079 for the position of the points of Hammer and Gauss on the surface of the plate and the weights corresponding.

```
•AFFE_CARA_ELEM (COQUE=_F (EPAISSEUR=' EP'
                 ANGL_REP = (' α '' β ')
                 COEF_RIGI_DRZ = 'CTOR')
```

To make postprocessings (forced, generalized efforts,…) in a reference mark chosen by the user who is not the local reference mark of the element, one gives a direction of reference D defined by two nautical angles in the total reference mark. The projection of this direction of reference as regards the plate fixes a direction $X1$ of reference. The normal with the plan into fixed one second and the vector product of the two vectors previously definite make it possible to define the local trihedron in which the generalized efforts and the constraints will be expressed. The user will have to take care that the selected reference axis is not found parallel with the normal of certain elements of plate of the model. By default this direction of reference is the axis X total reference mark of definition of the grid.

The value CTOR corresponds to the coefficient which the user can introduce for the treatment of the terms of rigidity and mass according to normal rotation with the plan of the plate. This coefficient must be sufficiently small not to disturb the energy assessment of the element and not too small so that the matrices of rigidity and mass are invertible. A value of 10^{-5} by default is put.

```
•ELAS  =_F (E =YOUNG NAKED = naked ALPHA = alpha . RHO = rho .)
```

for a homogeneous isotropic thermoelastic behavior in the thickness one uses the keyword ELAS in DEFI_MATERIAU where the coefficients are defined E Young modulus, ν Poisson's ratio, α thermal dilation coefficient and RHO density;

```
•ELAS_ORTH (_FO) =_F (
    E_L =ygl. E_T =ygt. G_LT =glt. G_TZ =gtz. NU_LT =nult.
    ALPHA_L =alphal. ALPHA_T =alphat.)
```

for an orthotropic thermoelastic behavior whose axes of orthotropism are L , T and z with isotropy of axis L (fibres in the direction L coated by a matrix, for example) the seven independent coefficients should be given ygl, longitudinal Young modulus, ygt, transverse Young modulus, glt, modulus of rigidity in the plan LT, gtz, modulus of rigidity in the plan TZ nult, Poisson's ratio in the plan LT and dilation coefficients thermal alphal and alphat for longitudinal and transverse thermal dilation, respectively.

The orthotropic elastic behavior available is only associated with the keyword DEFI_COMPOSITE who allows to define a multi-layer composite hull.

For only one orthotropic material, one will thus use DEFI_COMPOSITE with only one sleep. If one wishes to use ELAS_ORTH for transverse shearing, it is necessarily necessary to employ DST modeling. If one uses modelings DKT, or DKTG, the transverse energy of shearing is not taken into account.

```
•ELAS_COQUE (_FO) =F (
    MEMB_L =C1111. MEMB_LT =C1122. MEMB_T =C2222. MEMB_G_LT =C1212.
    FLEX_L =D1111. FLEX_LT =D1122. FLEX_T =D2222. FLEX_G_LT =D1212.
    CISA_L =G11…. CISA_T =G22…. ALPHA =alpha. RHO =rho.)
```

This behavior was added in DEFI_MATERIAU to take into account matrices of rigidity nonproportional out of membrane and inflection, obtained by homogenisation of a multi-layer material. The coefficients of the matrices of rigidity are then introduced with the hand by the user into the reference mark user defined by the keyword ANGL_REP. The thickness given in AFFE_CARA_ELEM is only used with the density defined by RHO. alpha is thermal dilation. If one wishes to use ELAS_COQUE for transverse shearing, it is necessarily necessary to employ DST modeling. If modeling DKT is used, transverse shearing is not taken into account.

```
•DEFI_COMPOSITE _F (LAYER = THICKNESS: 'EP'
                   MATER = 'material'
                   ORIENTATION = (theta))
```

This keyword (cf [R4.01.01] and [U4.42.03]) makes it possible to define a multi-layer composite hull on the basis of the sub-base towards the roadbase starting from its characteristics sleep by layer, thickness, type of material constitutive and orientation of fibres compared to a reference axis. The type of constitutive material is produced by the operator DEFI_MATERIAU under the keyword ELAS_ORTH. theta is the angle of the first direction of orthotropism (longitudinal direction or direction of fibres) in the tangent plan with the element compared to the first direction of the reference mark of reference defined by ANGL_REP. By default theta is null, if not it must be provided in degrees and must be understood enters -90^{o} and $+90^{o}$.

•AFFE_CHAR_MECA (DDL_IMPO =_F (
DX =. DY =. DZ =. DRX =. DRY MARTINI =. DRZ =. degree of freedom of plate in the total reference mark.
FORCE_COQUE =_F (FX =. FY =. FZ =. MX =. MY =. MZ =.) They is the surface efforts (membrane and inflection) on elements of plate. These efforts can be given in the total reference mark or the reference mark user defined by ANGL_REP.

•FORCE_NODALE =_F (FX =. FY =. FZ =. MX =. MY =. MZ =.) They is the efforts of hull in the total reference mark.

•

5.3 Calculation in linear elasticity

The matrix of rigidity and the matrix of mass (respectively options RIGI_MECA and MASS_MECA) are integrated numerically. It is not checked if the mesh is plane or not. Calculation takes account owing to the fact that the terms corresponding to the degrees of freedom of plate are expressed in the reference mark room of the element. A matrix of passage makes it possible to pass from the local degrees of freedom to the total degrees of freedom.

Elementary calculations (CALC_CHAMP) currently available correspond to the options:

- •EPSI_ELNO and SIGM_ELNO who provide the strains and the stresses to the nodes in the reference mark user of the element in lower skin, to semi thickness and in higher skin of plate, the position being specified by the user. One stores these values in the following way: 6 components of strain or stresses:
- •EPXX EPYY EPZZ EPXY EPXZ EPYZ or SIXX SIYY SIZZ SIXY SIXZ SIYZ
- •DEGE_ELNO : who gives the deformations generalized by 'element to the nodes starting from displacements in the reference mark user: EXX, EYY, EXY, KXX, KYY, KXY, GAX, GAY.
- •EFGE_ELNO : who gives the efforts generalize by element with the nodes starting from displacements: NXX, NYY, NXY, MXX, MYY, MXY, QX, QY.
- •SIEF_ELGA : who gives the efforts generalize by element at the points of Gauss starting from displacements: NXX, NYY, NXY, MXX, MYY, MXY, QX, QY.
- •EPOT_ELEM : who gives the elastic energy of deformation per element starting from displacements.
- •ECIN_ELEM : who gives the kinetic energy by element.

Finally one calculates also the option FORC_NODA of calculation of the nodal forces for the operator CALC_CHAMP.

5.4 Calculation in linear buckling

The option RIGI_MECA_GE being activated in the catalogue of the element, it is possible to carry out a classical calculation of buckling of Euler after assembly of the matrices of elastic and geometrical rigidity.

5.5 Another nonlinear behavior or plastic design

The matrix of rigidity is there too integrated numerically. One calls on the option of calculation STAT_NON_LINE in which one defines in the level of the nonlinear behavior the number of layers to be used for digital integration.

For modelings DKT, all the laws of plane constraints available in *Code_Aster* can be used.

For modelings DST and Q4G, only linear elasticity is usable.

For modeling DKTG, the only laws of behavior used are total laws (since there is only one point of integration in the thickness), connecting the deformations generalized to the generalized constraints. These laws are, in version 9.4: GLRC_DM and GLRC_DAMAGE, like their coupling with elastoplastic laws out of membrane (KIT_DDI).

Currently available elementary calculations correspond to the options:

- •EPSI_ELNO who provides the deformations by element to the nodes in the reference mark user starting from displacements, in lower skin, with semi thickness and in higher skin of plate.
- •SIGM_ELNO who allows to obtain the stress field in the thickness by element with the nodes for all the under-points (all the layers and all the positions: in lower skin, in the medium and in higher skin of layer).
- •EFGE_ELNO who allows to obtain the efforts generalized by element with the nodes in the reference mark user.
- •VARI_ELNO who calculates the field of internal variables and the constraints by element with the nodes for all the layers, in the local reference mark of the element.

6 Conclusion

The finite elements of plate plans which we describe here are used in the mean structural analyses, in small displacements and deformations, whose thickness report over characteristic length is lower than $1/10$. As these elements are plans, they do not take into account the curve of the structures, and it is necessary to refine the grids if this one would be important.

It is elements for which the strains and the stresses in the plan of the element vary linearly with the thickness of the plate. Moreover, the distortion associated with transverse shearing is constant in the thickness of the element. Two families of finite elements of plate exist: elements DKT, DKQ (or DKTG, DKQG) for which the transverse distortion is worthless and finite elements with energy of shearing transverse DST, DSQ and Q4G (or Q4Γ) for which it remains constant and nonworthless in the thickness. One advises to use the second type of elements when the structure studied has a thickness report over characteristic length understood enters $1/20$ and $1/10$ and first in the remainder of the cases. When the transverse distortion is nonworthless, the elements of DST plate, DSQ and Q4G do not satisfy the equilibrium conditions 3D and the boundary conditions on nullity with stresses shear transverse on the faces higher and lower of plate, compatible with a constant transverse distortion in the thickness of the plate. It results from it thus that on the level from the elastic behavior a coefficient from $5/6$ for a homogeneous plate corrects the usual relation between the constraints and the distortion transverses in order to ensure the equality between energies of shearing of the model 3D and the model of plate constant distortion. In this case, the arrow w as an interpretation average transverse displacement in the thickness of the plate has.

The nonlinear behaviors in plane constraints are available for the elements of plate DKT and DKQ only. Indeed the rigorous taking into account of a transverse shearing constant not no one on the thickness and the determination of the correction associated on rigidity with shearing compared to a model satisfying the equilibrium conditions and the boundary conditions are not possible and thus return the use of the DST elements, DSQ and rigorously impossible Q4G in plasticity.

For the elements of family DKTG, only of the total relations of behavior (membrane relations moment-curves and efforts – elongations) are available.

Elements corresponding to the machine elements exist in thermics; the thermomechanical chainings are thus available except, for the moment, laminated materials.

Annexe 1 : Orthotropic plates

For an orthotropic material like that represented on the figure 6-1, made up for example of fibres of direction L coated with a matrix, whose axes of orthotropism are L, T and Z with isotropy of axis L, the expression for the matrices $\mathbf{H}$ and $\mathbf{H}_{\gamma}$ in the reference mark of orthotropism previously definite becomes:

$$\mathbf{H}_L=\begin{pmatrix} H_{LL} & H_{LT} & 0 \\ H_{LT} & H_{TT} & 0 \\ 0 & 0 & G_{LT} \end{pmatrix} \quad \text{and} \quad \mathbf{H}_{L\gamma}=\begin{pmatrix} G_{LZ} & 0 \\ 0 & G_{TZ} \end{pmatrix} \tag{49}$$

with $\begin{array}{l} H_{LL}=\dfrac{E_L}{1-\nu_{LT}\nu_{TL}}\,;H_{TT}=\dfrac{E_T}{1-\nu_{LT}\nu_{TL}} \\ H_{LT}=\dfrac{E_T\nu_{LT}}{1-\nu_{LT}\nu_{TL}}=\dfrac{E_L\nu_{TL}}{1-\nu_{LT}\nu_{TL}} \end{array}$ and $\begin{array}{l} G_{LZ}=\dfrac{E_L}{2(1+\nu_{LZ})} \\ G_{TZ}=\dfrac{E_T}{2(1+\nu_{TZ})} \end{array}$.

The knowledge of the five independent coefficients E_L, E_T, G_{LT}, G_{TZ} and ν_{LT} is sufficient to determine the coefficients of the matrices $\mathbf{H}$ and $\mathbf{H}_{\gamma}$ since:

$$\nu_{TL}=\frac{E_T\nu_{LT}}{E_L} \quad \text{and} \quad G_{LZ}=G_{LT}\,.$$

If one indicates by θ the angle enters the reference mark of orthotropism and the main axis of the reference mark defined by the user by means of `ANGL_REP` it is established that:

$$\mathbf{H}=\mathbf{T}_1^T\mathbf{H}_L\mathbf{T}_1 \quad \text{and} \quad \mathbf{H}_{\gamma}=\mathbf{T}_2^T\mathbf{H}_{L\gamma}\mathbf{T}_2 \tag{50}$$

with: $\mathbf{T}_1=\begin{pmatrix} C^2 & S^2 & CS \\ S^2 & C^2 & -CS \\ -2CS & 2CS & C^2-S^2 \end{pmatrix}$ and $\mathbf{T}_2=\begin{pmatrix} C & S \\ -S & C \end{pmatrix}$ where $C=\cos\theta, S=\sin\theta$ and $\theta=(x, L)$ as indicated on the figure below.

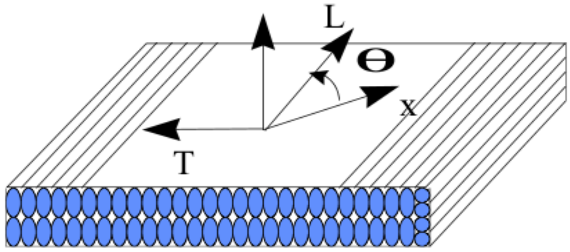


Figure 6-1: Composite plate

In the case of forced initial of thermal origin, we have moreover:

$$\sigma_{th}=-\mathbf{T}_1^T\mathbf{H}_L\begin{pmatrix} \alpha_L\Delta T \\ \alpha_T\Delta T \\ 0 \end{pmatrix} \tag{51}$$

where α_L and α_T are the dilation coefficients thermal in the directions L and T and ΔT temperature variation.

Annexe 2 : Factors of transverse correction of shearing for orthotropic or laminated plates

The matrix $\mathbf{H}_{\mathbf{ct}}$ is defined so that the surface density of transverse energy of shearing obtained in the case of the three-dimensional distribution of the constraints resulting from the resolution of balance is equal to that of the model of plate based on the assumptions of Reissner, for a behavior in pure bending. One must thus find $\mathbf{H}_{\mathbf{ct}}$ such as:

$$\frac{1}{2}\int_{-h/2}^{+h/2} \tau\,\mathbf{H}_{\gamma}^{-1}\tau = \frac{1}{2}\,\mathbf{T}\mathbf{H}_{\mathbf{ct}}^{-1}\mathbf{T} = \frac{1}{2}\,\gamma\,\mathbf{H}_{\mathbf{ct}}\,\gamma \quad \text{with} \quad \tau = \begin{pmatrix}\sigma_{xz}\\ \sigma_{yz}\end{pmatrix} \quad \text{and} \quad \mathbf{T} = \int_{-h/2}^{+h/2} \tau\, dz = \mathbf{H}_{\mathbf{ct}}\,\gamma \qquad (52)$$

To obtain $\mathbf{H}_{\mathbf{ct}}$ one uses the distribution of τ according to z obtained starting from the resolution of the equilibrium equations 3D without external couples:

$$\sigma_{xz} = -\int_{-h/2}^{z} (\sigma_{xx,x} + \sigma_{xy,y})\, d\zeta \,;\, s_{yz} = -\int_{-h/2}^{z} (\sigma_{xy,x} + \sigma_{yy,y})\, d\zeta \quad \text{with} \quad \sigma_{xz} = \sigma_{yz} = 0 \quad \text{for} \quad z = \pm h/2\,.$$

If there is no coupling membrane inflection (symmetry compared to $z=0$), constraints in the plan of the element σ_{xx} , σ_{yy} and σ_{xy} in the case of have as an expression a behavior of pure inflection:

$$\sigma = z\,\mathbf{A}(z)\,\mathbf{M} \quad \text{with} \quad \mathbf{A}(z) = \mathbf{H}(z)\,\mathbf{H}_f^{-1}\,.$$

If $\mathbf{H}(z)$ and $\mathbf{H}_f$ do not depend on x and y one can determine $\mathbf{H}_{\mathbf{ct}}$. Indeed:

$$\tau(z) = \mathbf{D_1}(z)\,\mathbf{T} + \mathbf{D_2}(z)\,\lambda \quad \text{where} \quad \mathbf{T} = \begin{pmatrix}T_x\\ T_y\end{pmatrix} = \begin{pmatrix}M_{xx,x} + M_{xy,y}\\ M_{xy,x} + M_{yy,y}\end{pmatrix} \quad \text{and} \quad \lambda = \begin{pmatrix}M_{xx,x} - M_{xy,y}\\ M_{xy,x} - M_{yy,y}\\ M_{yy,x}\\ M_{xx,y}\end{pmatrix}$$

like:

$$\mathbf{D_1} = -\int_{-h/2}^{z} \frac{\zeta}{2}\begin{pmatrix}A_{11}+A_{33} & A_{13}+A_{32}\\ A_{31}+A_{23} & A_{22}+A_{33}\end{pmatrix} d\zeta\,,$$

$$\mathbf{D_2} = -\int_{-h/2}^{z} \frac{\zeta}{2}\begin{pmatrix}A_{11}-A_{33} & A_{13}-A_{32} & 2A_{12} & 2A_{31}\\ A_{31}-A_{23} & A_{33}-A_{22} & 2A_{32} & 2A_{21}\end{pmatrix} d\zeta\,.$$

It results from it that $\frac{1}{2}\int_{-h/2}^{+h/2} \tau\,\mathbf{H}_{\gamma}^{-1}\tau = \frac{1}{2}\begin{pmatrix}\mathbf{T}\\ \lambda\end{pmatrix}\begin{pmatrix}C_{11} & C_{12}\\ C_{12}^{T} & C_{22}\end{pmatrix}\begin{pmatrix}\mathbf{T}\\ \lambda\end{pmatrix}$ with:

$$\mathbf{C_{11}} = \int_{-h/2}^{+h/2} \mathbf{D_1}^{T}\,\mathbf{H}_{\gamma}^{-1}\,\mathbf{D_1}\, dz\,;$$

$$\mathbf{C_{12}} = \int_{-h/2}^{+h/2} \mathbf{D_1}^{T}\,\mathbf{H}_{\gamma}^{-1}\,\mathbf{D_2}\, dz\,;$$

$$\mathbf{C_{22}} = \int_{-h/2}^{+h/2} \mathbf{D_2}^{T}\,\mathbf{H}_{\gamma}^{-1}\,\mathbf{D_2}\, dz$$

As in addition $\frac{1}{2}\int_{-h/2}^{+h/2}\tau\,\mathbf{H}_{\boldsymbol{\gamma}}^{-1}\tau=\frac{1}{2}\mathbf{T}\mathbf{H}_{\mathbf{ct}}^{-1}\mathbf{T}$ one proposes to take $\mathbf{H}_{\mathbf{ct}}=\mathbf{C}_{\mathbf{11}}^{-1}$ to satisfy the two equations as well as possible whatever T and λ.

While comparing $\mathbf{H}_{\mathbf{ct}}$ thus calculated with $\bar{\mathbf{H}}_{\mathbf{ct}}=\int_{-h/2}^{+h/2}\mathbf{H}_{\boldsymbol{\gamma}}\,dz$ one reveals the coefficients of transverse correction of shearing following

$$k_1=H_{ct}^{11}/\bar{H}_{ct}^{11},\qquad k_{12}=H_{ct}^{12}/\bar{H}_{ct}^{12}\quad\text{and}\quad k_2=H_{ct}^{22}/\bar{H}_{ct}^{22} \tag{53}$$

For a homogeneous, isotropic or anisotropic plate, one finds as follows: $\mathbf{H}_{\mathbf{ct}}=kh\,\mathbf{H}_{\boldsymbol{\gamma}}$ with $k=5/6$.

Note:

This method is valid only when the composite plate is symmetrical compared to $z=0$.

• For a multi-layer material, one establishes that:

$$\begin{aligned}\mathbf{C}_{\mathbf{11}}=&\sum_{i=1}^{N}\frac{h_i}{4}\Big(\sum_{p=1}^{i-1}h_p\eta_p\mathbf{A}_{\boldsymbol{p}}^{\boldsymbol{T}}-\frac{1}{2}z_i^2\mathbf{A}_{\boldsymbol{i}}^{\boldsymbol{T}}\Big)\mathbf{H}_{\boldsymbol{\gamma}}^{-1}\Big(\sum_{p=1}^{i-1}h_p\eta_p\mathbf{A}_{\boldsymbol{p}}-\frac{1}{2}z_i^2\mathbf{A}_{\boldsymbol{i}}\Big)+\\&\frac{1}{24}(z_{i+1}^3-z_i^3)\Big[\mathbf{A}_{\boldsymbol{i}}^{\boldsymbol{T}}\mathbf{H}_{\boldsymbol{\gamma}}^{-1}\Big(\sum_{p=1}^{i-1}h_p\eta_p\mathbf{A}_{\boldsymbol{p}}-\frac{1}{2}z_i^2\mathbf{A}_{\boldsymbol{i}}\Big)+\Big(\sum_{p=1}^{i-1}h_p\eta_p\mathbf{A}_{\boldsymbol{p}}^{\boldsymbol{T}}-\frac{1}{2}z_i^2\mathbf{A}_{\boldsymbol{i}}^{\boldsymbol{T}}\Big)\mathbf{H}_{\boldsymbol{\gamma}}^{-1}\mathbf{A}_{\boldsymbol{i}}\Big]\\&+\frac{1}{80}(z_{i+1}^5-z_i^5)\mathbf{A}_{\boldsymbol{i}}^{\boldsymbol{T}}\mathbf{H}_{\boldsymbol{\gamma}}^{-1}\mathbf{A}_{\boldsymbol{i}}\end{aligned} \tag{54}$$

where: $h_i=z_{i+1}-z_i$, $\eta_i=\frac{1}{2}\left(z_{i+1}+z_i\right)$ and $\mathbf{A}_{\boldsymbol{i}}$ represent the matrix $\begin{pmatrix}A_{11}+A_{33} & A_{13}+A_{32}\\ A_{31}+A_{23} & A_{22}+A_{33}\end{pmatrix}$ for the layer i

.

• Validity of the choice $\mathbf{H}_{\mathbf{ct}}=\mathbf{C}_{\mathbf{11}}^{-1}$ can be examined a posteriori when one has an estimate of the solution (fields of displacements and plane constraints, in particular). One can then estimate the difference between the two estimates on energy. A approach of calculation in two stages for the multi-layer plates and hulls (with $\mathbf{H}_{\mathbf{ct}}$ diagonal and two coefficients k_1 and k_2) was developed besides by Noor and Burton [8] and [8] .

• In the case of an isotropic or anisotropic homogeneous plate, the equality between two energies is satisfied in a strict sense since $\mathbf{D}_{\mathbf{2}}=0$. The choice makes above is then valid and no examination a posteriori is necessary.

7 Bibliography

[1] **Batoz, J. - L. and Dhatt, G.**, (**1992**). *Modeling of the structures by finite elements: beams and plates*. Hermes.

[2] **Bui, D.** (**1992**). *Shearing in the plates and the hulls: modeling and calculation*, EDF R & D - Note HI 71/7784, Internal report.

[3] **Taylor, Mr.; Vasiliev, V. and Dillard, D.** (**1997**). *One the preblem of shear locking in finite element based one shear deformable punt theory*, International Newspaper of Solids and structures 34, pp. 859-875.

[4] **Voldoire, F.** (**1997**). *Modeling by thermal and thermoelastic homogenisation of thin mechanical components*, EDF R & D - CR MMN/97/091, Internal report.

[5] **Ren, J.** (**1986**). *With new theory of laminated punt*, Composite Science and Technology 26, pp. 225-239.

[6] **Hughes, T.**, (**1987**). *The finite element method*. Prentice Hall.

[7] **Hughes, J.; Cohen, Mr. and Haroun, Mr.** (**1978**). *Reduced and selective integration techniques in the finite element analysis of punts*, Nuclear Engineering and Design 46, pp. 203-222.

[8] **Noor, A. and Burton, W.** (**1989**). *Composite Assessment of shear deformation theories for multilayered punts*, ASME - Applied Mechanics Review 42, pp. 1-13.

[9] **Rock'n'roll, T. and Hinton, E.** (**1976**). *With finite transverse element method for the free vibration of punts allowing for shear deformation*, Computers and Structures 6, pp. 37-44.

[10] **Hinton, E.; Rock'n'roll, T. and Zienkiewicz, O.** (**1976**). *With note one farmhouse lumping and related processes in the finite element method*, Earthquake Engineering and Structural Dynamics 4, pp. 245-249.

8 Description of the versions of the document

Index document	Version Aster	Author (S) Organization (S)	Description of the modifications
With	5	P. MASSIN EDF DER MN	Initial text.
B	9.4	X. DESROCHES, D.MARKOVIC, EDF R & D AMA	Addition of DKTG, and the matrices of lumpées masses.
C	13.2	D. KUDAWOO, EDF R & D /AMA	Addition of the explanations of the method of calculating of the shear stress in the thickness.
D	14.1	F.VOLDOIRE, EDF/DR&D/ERMES	Introductory paragraph on the various finite elements of this family. Some corrections.