Finite elements of voluminal hulls

Summary:

With an aim of supplementing the library of finite elements of plate plans [R3.07.03] currently available in *Code_Aster* (DKT, DST, Q4G…), one proposes to introduce two finite elements of voluminal or three-dimensional hull [bib1]. This modeling COQUE_3D [U3.12.03] allows to carry out structural analyses hull of an unspecified form with a better approximation of the geometry and kinematics.

One will limit oneself to the framework of linear kinematics. One thus remains in small displacements and small deformations. No restriction is made on the type of behavior in plane constraints.

The two elements which are introduced are the quadratic element quadrangle Hétérosis with 9 nodes and its triangular equivalent with 7 nodes. The formulation of the continuous problem is done in Cartesian coordinates, which makes it possible to avoid explicit calculations of the curves. These two elements have as a correspondent the linear element of hull with 3 nodes presented in the document [R3.07.02].

These two new elements are validated on existing CAS-tests of plate, and on three new CAS-tests of hull developed in the documentation of validation and whose principal conclusions are presented briefly here.

This note also presents in appendix how to take into account the anisotropy of materials.

Contents

1 Introduction

One introduces into *Code_Aster* two finite elements of hull voluminal with transverse shearing (the quadrangle with 9 nodes MEC3QU9H and the triangle with 7 nodes MEC3TR7H) in structural analysis hull of an unspecified form. To represent this kind of structures, one used until now with *Code_Aster* elements of plate with plane facets which induced parasitic inflections and too restrictive hulls of revolution on the type of structure [R3.07.02]. The development was carried out for isotropic materials with linear kinematics. They cannot thus be used that within the framework of small displacements and small deformations. This formulation can be wide with anisotropic materials [Appendix 1] and with nonlinear kinematics [R3.07.05].

For the resolution of chained thermomechanical problems, one must use before the finite elements of thermal hull with 7 and 9 nodes described in [R3.11.01].

One develops hereafter the mechanical continuous problem by describing the kinematics of hull of the type Hencky-Mindlin-Naghdi (assumption of the cross-sections or plane) supplemented by a transverse distortion and the thermoelastoplastic law of behavior. Thanks to a parameter of penalization one can pass from a theory with shearing to a theory without shearing. One presents then the selected finite elements which are isoparametric quadratic elements making it possible to have a fine representation of a curved geometry and good estimates of the constraints. The interpolation and the method of integration are also described.

One validates finally the development on some cases of test.

The nonlinear kinematics of these hulls is treated in the reference material [R3.07.05].

2 Formulation

2.1 Geometry of the hull

For the elements of voluminal hull Ω a surface of reference is defined ω, or surfaces average, left (curvilinear coordinates $\xi_1 \xi_2$ for example) and a thickness $h(\xi_1 \xi_2)$ measured according to the normal on the average surface. This thickness must be small compared to other dimensions (extensions, radii of curvature) of the structure to model. Figure [Figure 2.1-a] Ci - below illustrates our matter.

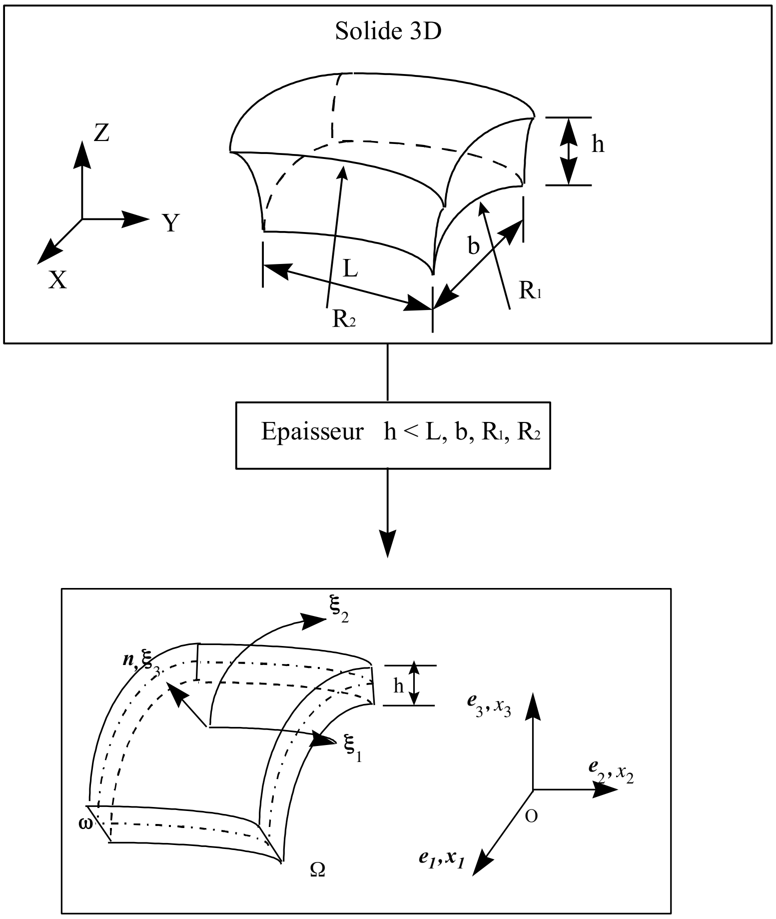


Figure 2.1-a

The position of the points of the hull is given by the curvilinear coordinates $(\xi_1 \xi_2)$ average surface ω and rise ξ_3 compared to this surface. (O, e_k) is the total Cartesian reference mark, associated axes (x_k).

2.1.1 Geometrical description of average surface

Base natural local and bases Cartesian local

That is to say P an unspecified point of the average surface of reference ω, one a:

$$\mathbf{OP}=x_k^0(\xi_1,\xi_2)e_k$$

The vectors are defined a_α natural local base of the tangent plan in P with ω, attached to P by:

$$a_\alpha=\frac{\partial\mathbf{OP}}{\partial\xi_\alpha}=\mathbf{OP}_{,\alpha}$$

and the unit normal is defined n by:

$$n=\frac{a_1\wedge a_2}{\|a_1\wedge a_2\|}$$

ξ_3 is the variable of position in the thickness associated with n.

(a_1,a_2,a_3) constitute the natural base attached to P.

The curvilinear frame of reference $(\xi_1\xi_2)$ not being inevitably orthogonal, the base (a_α) is thus not inevitably orthogonal (and even less orthonormal). An orthonormal local base is thus defined t_k as follows:

$$t_1=\frac{a_1}{\|a_1\|},t_2=n\wedge t_1 t_3=n$$

and one notes (s_1,s_2) the frame of reference associated with (t_1,t_2).

Intrinsic reference mark

The local base t_k will be useful to define the intrinsic reference mark of an element hull by choosing for point P the first top of the element and for vector a_1 a vector coinciding with the projection on the first side on the tangent level at the first top.

Calculation of the tensor of curve

The tensor of curve is related to the variation of the normal on ω. It is defined by its mixed components:

$$n_{,\beta}=-C_\beta^\gamma a_\gamma$$

or by its components covariantes: $C_{\alpha\beta}=-a_\alpha . n_{,\beta}=n . a_{\alpha,\beta}$. This tensor is symmetrical since $a_{\alpha,\beta}=a_{\beta,\alpha}$. Its trace $\mathrm{tr}\, C_{\alpha\beta}$ is the average curve and its determinant the Gaussian curve.

2.1.2 Description of the geometry of the hull

That is to say Q an unspecified point of Ω , volume of the hull thickness h considered constant, one a:

$$\mathbf{OQ}+\mathbf{OP}+\mathbf{PQ}=\mathbf{OP}+\frac{\xi_3}{2}h\,\boldsymbol{n}$$

where $\xi_3\in[-1,1]$.

$\left(\xi_1,\xi_2,\frac{\xi_3}{2}h\right)$ constitute a curvilinear frame of reference of Ω .

One can also write $\mathbf{OQ}$ according to its components (x_k) in the total base (e_k) :

$$\mathbf{OQ}=x_k\,e_k$$

Base natural local, bases orthonormal local and tensor metric

As for P , one defines the natural base of space 3D (g_k) attached to Q by:

$$g_\alpha=\frac{\partial\mathbf{OQ}}{\partial\xi_\alpha}=a_\alpha+\xi_3\frac{h}{2}n_{,\alpha},\ g_3=\frac{g_1\wedge g_2}{\|g_1\wedge g_2\|}=\boldsymbol{n}$$

Like (g_k) is not inevitably orthogonal, one defines an orthonormal local base (T_k) as follows:

$$T_1=\frac{g_1}{\|g_1\|},\ T_2=n\wedge T_1,\ T_3=n$$

and one notes (x_k) the frame of reference associated with (T_k) .

One will call (T_k) the local orthonormal base, and (x_k) coordinates in this local orthonormal base.

By definition, one a:

$$T_k=\frac{\partial\mathbf{OQ}}{\partial\tilde{x}_k}=\frac{\partial x_j}{\partial\tilde{x}_k}e_j=T_k^j e_j$$

with $\frac{\partial x_j}{\partial \tilde{x}_k} = T_k^j$ components of (T_k) in the total base (e_j). (They are also the components of the matrix of passage of (T_k) with (e_j) since the matrix of passage is orthogonal. Thus if $T_k = T_k^j e_j$ one has too $e_k = T_j^k T_j$).

The metric tensor is defined G associated with Q by its components deduced from the scalar products of the vectors of the local orthonormal base:

$$G_{ij} = T_i . T_j$$

This tensor G the identity is worth $\mathbf{Id}$.

2.1.3 Notice

The figures [Figure 2.1.3-a] and [Figure 2.1.3-b] illustrate geometrical magnitudes mentioned above Ci -.

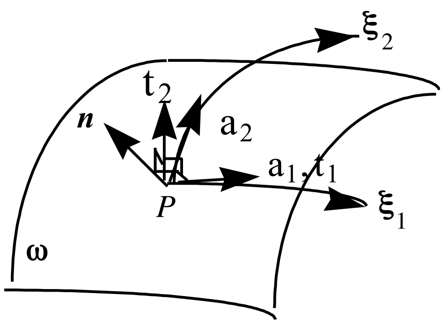


Figure 2.1.3-a

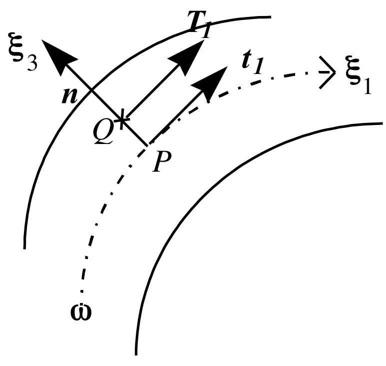


Figure 2.1.3-b

It should be noted that two local orthonormal bases, that associated with average surface (t_k) and the other with the volume of the hull (T_k) are confused only when the curve is worthless. In this case the elements of hull are comparable to elements of plate

2.2 Theory of the plates and the hulls

These elements are based on the theory of the plates and the hulls according to which:

2.2.1 Kinematics

2.2.1.1 Field of displacement

The cross-sections which are the sections perpendicular to average surface remain right; the material points located on a normal at not deformed average surface remain on a line in the deformed configuration. It results from this approach that **the fields of displacement vary linearly in the thickness of the hull**.

If one notes Q' , the position of Q after deformation, one a:

$$\mathbf{OQ}'=\mathbf{OQ}+\mathbf{QQ}'=\mathbf{OQ}+U(Q)$$

where the field of displacement chosen, corresponding to the kinematics of Hencky-Mindlin, is written:

$$U(Q)=u(P)+\frac{\xi_3}{2}h\beta(P) \text{ with } \beta(P).n=0¿$$

where $u(P)$ and $\beta(P)$ are respectively the vector displacement and the vector rotation of P , projection of Q on the average surface of the hull. The fact that $\beta(P).n=0$ indicate that one does not take into account in this kinematics rotations of the hull around his normal.

Notation:

One notes ˜ quantities expressed in the local Cartesian bases (t_k) or (T_k) for the points P and Q respectively. It results from it that:

- the vector three-dimensional displacement U can be written $U=\tilde{U}_k T_k$ or $U=U_k e_k$, where it is expressed respectively in its local orthonormal base or the total Cartesian base,
- the vector displacement of average surface u can be written $u=\tilde{u}_k t_k$ or $u=u_k e_k$ according to whether it is expressed in its local orthonormal base or the total Cartesian base,
- the vector rotation of average surface is written $b=\tilde{\beta}_\alpha t_\alpha$ in its local orthornormée base. β being the rotation of the normal $\mathbf{n}$ (on average surface), one writes too $\beta=\theta\wedge n$ with θ , vector rotation of average surface, such as $\theta=\tilde{\theta}_\alpha t_\alpha$. The equivalence of the two formulations shows that $\tilde{\beta}_1=\tilde{\theta}_2, \tilde{\beta}_2=-\tilde{\theta}_1$.

2.2.1.2 Expression of the three-dimensional deformations

The tensor of deformation is calculated in the local orthonormal Cartesian base (T_k) . It is defined like the half-difference of the metric tensors associated with the local orthonormal bases after and before deformation. The metric tensor associated with this base in the not-deformed state is simply the identity $\mathbf{Id}$, while the metric tensor of the deformed state is $\overline{G}_{ij}=T_i'.T_j'$ with $T'_k=\frac{\partial\mathbf{OQ}'}{\partial\tilde{x}_k}$.

Components of the tensor of deformation in (T_k) are thus given by:

$$\tilde{\varepsilon}_{\alpha\beta}=\frac{1}{2}\left(\frac{\partial \tilde{U}_{\alpha}}{\partial \tilde{x}_{\beta}}+\frac{\partial \tilde{U}_{\beta}}{\partial \tilde{x}_{\alpha}}\right)$$
$$\tilde{\varepsilon}_{\alpha 3}=\frac{1}{2}\left(\frac{\partial \tilde{U}_{\alpha}}{\partial \tilde{x}_{3}}+\frac{\partial \tilde{U}_{3}}{\partial \tilde{x}_{\alpha}}\right)$$

The equations above are linear relations deformation-displacements. The variables of displacement are the components $\tilde{U}_k$.

Components $\tilde{\varepsilon}_{kl}$ tensor $\tilde{\varepsilon}$ can also express itself according to the components in the total reference mark $\frac{\partial U_p}{\partial x_m}$. Indeed as in the total reference mark $\varepsilon=\varepsilon_{\mathrm{ij}} e^i \times e^j=\varepsilon_{\mathrm{ij}} T_k^i T_l^j T^k \times T^l=\tilde{\varepsilon}_{\mathrm{kl}} T^k \times T^l$ one from of thus deduced immediately that: $\tilde{\varepsilon}_{\mathrm{kl}}=\varepsilon_{\mathrm{ij}} T_k^i T_l^j$. (e^k) and (T^k) are the bases contravariantes associated with (e_k) and (T_k) respectively such as: $e^i . e_j=\delta_{ij}$ and. $T^i . T_j=\delta_{ij}$ Like the bases (e_k) and (T_k) are orthonormal, their associated bases contravariantes are confused with themselves. Thus in the same manner that one had $T_k=T_k^j e_j$ one finds $T^k=T_k^j e^j$.
If one notes $T=T_k^i e_i \times T^k$ then $T \times T : \varepsilon=\tilde{\varepsilon}_{kl} T^k \times T^l=\tilde{\varepsilon}$. For the continuation one indicates by $\tilde{\varepsilon}$ the form of the tensor of the deformations in the local orthonormal reference mark and by ε the form of the same tensor in the total reference mark. The relation of passage of the one with the other is given above in term of tensors.

Note:

Terms T_k^j the terms of curve of the hull contain Ω .

One notes in the relations deformation-displacements that the component $\tilde{\varepsilon}_{33}$ is not determined by kinematics. This is to be associated with the assumption of nullity of the transverse normal constraints $\tilde{\sigma}_{33}=0$ justified by the behavior of the hulls.

In the literature (see for example [bib3]), the modeling of the hulls by the approach based on the curvilinear components $\tilde{u}_k$ displacement reveals explicitly the sizes of curve on the level of the form of the tensor of deformation [bib5]. Like, in general, the geometry of the hull is not known explicitly, one must thus numerically determine the geometrical characteristics which are the vectors a_α, g_α ,… and curves $C_{\alpha\beta}$. With the finite element method it is necessary to derive twice the functions from form (see page 20 of [bib5] and [R3.07.02]) to calculate them $C_{\alpha\beta}$. This can make their calculation vague according to the family of the functions of selected form. The made mistake depends on these last (linear, quadratic, cubic polynomials….) and becomes independent of the refinement of the grid. A formulation utilizing the derivation first of the functions of form (calculation of slopes) does not present this disadvantage. Thus the consequent error with calculations of the terms of curve in a formulation based on the curvilinear approach does not decrease with the refinement of the grid whereas for the formulation described above it becomes small by increasing the number of finite elements. Within sight of the preceding observations, the approach known as curvilinear was not followed.

2.2.2 Law of behavior

The behavior of the hulls is a behavior 3D in “plane constraints”. It binds the components of the constraints and the deformations, in the form of vectors, in the local orthonormal base. The constraint transversal $\tilde{\sigma}_{33}$ worthless because of being regarded as negligible compared to the other components of the tensor of the constraints (assumption of the plane constraints). The most general law of behavior is written then as follows:

$$\begin{pmatrix} \tilde{\sigma}_{11} \\ \tilde{\sigma}_{22} \\ \tilde{\sigma}_{12} \\ \tilde{\sigma}_{13} \\ \tilde{\sigma}_{23} \end{pmatrix} = \tilde{C}(\varepsilon, \mu) \begin{pmatrix} \tilde{\varepsilon}_{11} - \tilde{\varepsilon}_{11}^{\text{th}} \\ \tilde{\varepsilon}_{22} - \tilde{\varepsilon}_{22}^{\text{th}} \\ \tilde{\gamma}_{12} \\ \tilde{\gamma}_1 \\ \tilde{\gamma}_2 \end{pmatrix}$$

where $\tilde{C}(\varepsilon, \mu)$ is the local matrix of behavior in plane constraints and μ represent the whole of the internal variables when the behavior is nonlinear.

For behaviors where the transverse distortions are uncoupled from the deformations of membrane and inflection, $\tilde{C}(\varepsilon, \mu)$ puts itself in the form:

$$\tilde{C} = \begin{pmatrix} \tilde{H} & 0 \\ 0 & \tilde{H}_\gamma \end{pmatrix}$$

where $\tilde{H}(\varepsilon,\mu)$ is a matrix of behavior of membrane-inflection 3×3 and $\tilde{H}_\gamma(\varepsilon,\mu)$ a matrix of transverse behavior of distortion 2×2 . The two phenomena being uncoupled one can also write the behavior in the form:

$$\begin{pmatrix}\tilde{\sigma}_{mf}\\ \tilde{\sigma}_g\end{pmatrix}=\tilde{C}(\varepsilon,\mu)\begin{pmatrix}\tilde{\varepsilon}_{mf}\\ \tilde{\gamma}\end{pmatrix} \text{ with:}$$

$$\tilde{\sigma}_{mf}=\begin{pmatrix}\tilde{\sigma}_{11}\\ \tilde{\sigma}_{22}\\ \tilde{\sigma}_{12}\end{pmatrix}=\tilde{H}(\varepsilon,\mu)\begin{pmatrix}\tilde{\varepsilon}_{11}-\tilde{\varepsilon}_{11}^{\text{th}}\\ \tilde{\varepsilon}_{22}-\tilde{\varepsilon}_{22}^{\text{th}}\\ \tilde{\gamma}_{12}\end{pmatrix}=\tilde{H}(\varepsilon,\mu)\tilde{\varepsilon}_{mf} \text{ and } \tilde{\sigma}_\gamma=\begin{pmatrix}\tilde{\sigma}_{13}\\ \tilde{\sigma}_{23}\end{pmatrix}=\tilde{H}_\gamma(\varepsilon,\mu)\begin{pmatrix}\tilde{\gamma}_1\\ \tilde{\gamma}_2\end{pmatrix}=\tilde{H}_\gamma(\varepsilon,\mu)\tilde{\gamma}$$

One will remain from now on within the framework of this assumption.

For an isotropic homogeneous linear behavior elastic, one has as follows:

$$\tilde{C}=\frac{E}{1-\nu^2}\begin{pmatrix}1 & \nu & 0 & 0 & 0\\ \nu & 1 & 0 & 0 & 0\\ 0 & 0 & \frac{1-\nu}{2} & 0 & 0\\ 0 & 0 & 0 & \frac{k(1-\nu)}{2} & 0\\ 0 & 0 & 0 & 0 & \frac{k(1-\nu)}{2}\end{pmatrix}$$

where k is factor of transverse correction of shearing whose significance is given in the reference material of the elements of plate [R3.07.03], and [bib4] for more details. This coefficient is worth $5/6$ for a theory of the Reissner type and 1 within the framework of the theory of Hencky - Mindlin. Lastly, if one chooses k very large, one brings back oneself to a theory of the type Coils-Kirchhoff. One neutralizes the transverse distortion by penalization of associated energy while taking $k=10^6 h/R$ (h being the thickness of the hull and R its average radius of curvature).

Always in the isotropic case, the two only nonworthless components of $\tilde{\varepsilon}^{th}$ are $\tilde{\varepsilon}_{ii}^{th}$ for $i=1,2$, such as:

$$\tilde{\varepsilon}_{ii}^{th}=\alpha(T-T^{réf})$$

where α is the thermal dilation coefficient and $T-T^{réf}$ the difference in presumedly known temperature.

Note:

One does not describe the variation thickness nor that of the transverse deformation $\tilde{\varepsilon}_{33}$ that one can however calculate by using the preceding assumption of plane constraints. In addition no restriction is made on the type of behavior in plane constraints which one can represent.
Same manner as $T\times T:\varepsilon=\tilde{\varepsilon}$ one can deduce some $(T\times T)_{mf}:\varepsilon=T_{mf}:\varepsilon=\tilde{\varepsilon}_{mf}$ and $(T\times T)_\gamma:\varepsilon=T_\gamma:\varepsilon=\tilde{\varepsilon}_\gamma$, which makes it possible to find $\tilde{\varepsilon}_{mf}$ and $\tilde{\varepsilon}_\gamma$ starting from the tensor of the deformations in the total reference mark.

3 Principle of virtual work

3.1 Work of deformation

In 3D the expression of the work of deformation is written:

$$W_{\text{def}}=\int_S \int_{-h/2}^{h/2} (\tilde{\varepsilon}_{ij}\tilde{\sigma}_{ij})dV=\int_S \int_{-h/2}^{h/2} (\tilde{\varepsilon}_{ij}\tilde{C}_{ijkl}\tilde{\varepsilon}_{kl})dV=\int_S \int_{-h/2}^{h/2} (\varepsilon_{rs}P_i^r P_k^s \tilde{C}_{ijkl}P_k^p P_l^q \varepsilon_{pq})dV$$
$$=\int_S \int_{-h/2}^{h/2} (\varepsilon_{rs}C_{rspq}\varepsilon_{pq})dV=\int_S \int_{-h/2}^{h/2} (\varepsilon_{ij}\sigma_{ij})dV$$

It is checked that this expression is invariant compared to the base in which the tensors are expressed. One chooses for the continuation of this document all to express in the local base (T_k) by knowing that one passes from the local tensor of behavior to the total tensor of behavior by the relation $C_{rspq}=P_i^r P_k^s \tilde{C}_{ijkl}P_k^p P_l^q$.

The general expression of the work of deformation 3D for the element of hull is worth:

$$W_{\text{def}}=\int_S \int_{-h/2}^{h/2} (\tilde{\varepsilon}\,\tilde{\sigma})dV=\int_S \int_{-h/2}^{h/2} (\tilde{\varepsilon}\,\tilde{C}\,\tilde{\varepsilon})dV=\int_S \int_{-h/2}^{h/2} (\tilde{\varepsilon}_{mf}\tilde{H}\,\tilde{\varepsilon}_{mf})dV+\int_S \int_{-h/2}^{h/2} (\tilde{\varepsilon}_{\gamma}\tilde{H}_{\gamma}\tilde{\varepsilon}_{\gamma})dV$$

where S is average surface and the position in the thickness of the hull varies between $-h/2$ and $+h/2$. It appears in the expression of the work of deformation a contribution of deformation out of membrane - inflection and a contribution of transverse shearing strain.

3.1.1 Energy interns elastic hull

It is expressed in the following way:

$$\Phi_{\text{int}}=\frac{1}{2}\int_S [\frac{E}{1-\nu^2}(\tilde{\varepsilon}_{11}^2+\tilde{\varepsilon}_{22}^2+2\nu\,\tilde{\varepsilon}_{11}\tilde{\varepsilon}_{22})+G(\tilde{\gamma}_{12}^2+k(\tilde{\gamma}_1^2+\tilde{\gamma}_2^2))]dV$$

where K is the factor of correction in transverse shearing defined in paragraph 2 and $G=\frac{E}{2(1+\nu)}$.

3.1.2 Expression of the resulting efforts

One notes:

$$N=\begin{pmatrix}N_{11}\\N_{22}\\N_{12}\end{pmatrix}=\int_{-h/2}^{+h/2}\begin{pmatrix}\tilde{\sigma}_{11}\\\tilde{\sigma}_{22}\\\tilde{\sigma}_{12}\end{pmatrix}dz\ \ ;\ M=\begin{pmatrix}M_{11}\\M_{22}\\M_{12}\end{pmatrix}=\int_{-h/2}^{+h/2}\begin{pmatrix}\tilde{\sigma}_{11}\\\tilde{\sigma}_{22}\\\tilde{\sigma}_{12}\end{pmatrix}z\,dz\ \ ;\ T=\begin{pmatrix}T_1\\T_2\end{pmatrix}=\int_{-h/2}^{+h/2}\begin{pmatrix}\tilde{\sigma}_{13}\\\tilde{\sigma}_{23}\end{pmatrix}dz\ .$$

N_{11}, N_{22}, N_{12} are the generalized efforts of membrane (in N/m);
M_{11}, M_{22}, M_{12} are the generalized efforts of inflection or moments (in N);
T_1, T_2 , are the generalized efforts of shearing or efforts cutting-edges (in N/m);

The expression of the resulting efforts that one gives here is an approximate expression which does not take account of the curve of the hull (cf p.316 of [bib3]). The mistake made on these efforts is then in h^2/R where $1/R$ is the average curve. When the hull becomes plane, expressions given Ci - above are exact and the significance of the resulting efforts can be found in [R3.07.03]. We will not develop more this aspect in addition documented well in [bib3] because the theory of hull used here does not rest on generalized a deformations formulation/resulting efforts but on three-dimensional/forced a deformations formulation.

3.2 Work of the forces and external couples

The work of the forces being exerted on the voluminal hull is expressed in the following way:

$$W_{\text{ext}}=\int_S\int_{-h/2}^{+h/2}F_v.U\,dV+\int_S F_s.U\,dS+\int_C\int_{-h/2}^{+h/2}F_c.U\,dz\,ds$$

where F_v, F_s, F_c are the voluminal, surface efforts and of contour being exerted on the hull, respectively. C is the part of the contour of the hull on which efforts of contour F_c are applied.

a) Loads given in the total reference mark:

With the kinematics of [§2.2.1], one determines as follows:

$$W_{\text{ext}}=\int_S(f_i u_i+c_i\beta_i)dS+\int_C(\phi_i u_i+\chi_i\beta_i)ds=\int_S(f_i u_i+c_i(\tilde{\theta}_2 t_{1i}-\tilde{\theta}_1 t_{2i}))dS$$
$$+\int_C(\phi_i u_i+\chi_i(\tilde{\theta}_2 t_{1i}-\tilde{\theta}_1 t_{2i}))ds=\int_S(f_i u_i+c_i(\tilde{\theta}_2 t_{1i}-\tilde{\theta}_1 t_{2i}))dS+\int_C(\phi u+\chi\beta)ds$$

- where are present on the hull:

f_1, f_2, f_3 :	surface forces acting along the axes of the total Cartesian reference mark
$f_i=\int_{-h/2}^{+h/2} F_v . e_i dz+F_s . e_i$	where them e_i are the vectors of the total Cartesian base.
c_1, c_2, c_3 :	surface couples acting around the axes of the total reference mark.
$c_i=\int_{-h/2}^{+h/2} zF_v . e_i dz\pm\frac{h}{2} F_s . e_i$	where them e_i are the vectors of the total Cartesian base.

- and where are present on the contour of the hull:

ϕ_1, ϕ_2, ϕ_3 :	linear forces acting along the axes of the total Cartesian reference mark.
$\phi_i=\int_{-h/2}^{+h/2} F_c . e_i dz$	where them e_i are the vectors of the total Cartesian base.
χ_1, χ_2, χ_3 :	linear couples acting around the axes of the total reference mark.
$\chi_i=\int_{-h/2}^{+h/2} zF_c . e_i dz$	where them e_i are the vectors of the total Cartesian base.

Note:

One notes too ϕ and χ linear distributions of force and moment applied to the contour of the finite element.

b) Loads given in the local reference mark:

One has then:

$$W_{\text{ext}}=\int_S \left(\sum_{i=1}^{3} \tilde{f}_\alpha t_{\alpha i} u_i+\tilde{c}_1 \tilde{\beta}_1+c_2 \tilde{\beta}_2\right) dS+\int_C \left(\sum_{i=1}^{3} \tilde{\phi}_\alpha t_{\alpha i} u_i+\chi_1 \tilde{\beta}_1+\chi_2 \tilde{\beta}_2\right) ds=$$
$$\int_S \left(\sum_{i=1}^{3} \tilde{f}_\alpha t_{\alpha i} u_i+\tilde{c}_1 \tilde{\theta}_2-\tilde{c}_2 \tilde{\theta}_1\right) dS+\int_C \left(\sum_{i=1}^{3} \tilde{\phi}_\alpha t_{\alpha i} u_i+\chi_1 \tilde{\theta}_2-\chi_2 \tilde{\theta}_1\right) ds$$

Expressions of $\tilde{f}_1, \tilde{f}_2, \tilde{f}_3$ and $\tilde{c}_1, \tilde{c}_2, \tilde{c}_3$ are the analogues of the expressions obtained for f_1, f_2, f_3 and c_1, c_2, c_3 by replacing them e_i by t_i .

Note:

For the couple c , the contribution $\tilde{c}_3$ associated with n is worthless in theory of hull.

3.3 Work of the inertial forces

Work due to the quantities of acceleration is written:

$$W^{ac}=\int_{\Omega}\rho\,\ddot{\boldsymbol{OQ}}'\cdot\boldsymbol{OQ}'\,dv$$

where ρ is the density.

It is supposed that $\ddot{\mathbf{OQ}}'$, the vector of acceleration of the point Q' is following form:

$$\ddot{\mathbf{OQ}}'=\ddot{U}_k e_k+W\wedge\left[W\wedge x_k^0 e_k\right]$$

where one neglected the forces of Coriolis and the correction of metric in the thickness.

One notes $\ddot{U}_k=\dfrac{d^2U_k}{dt^2}$, and Ω is the uniform vector of rotation of the total reference mark (O,e_k) (compared to a Galilean reference mark which has the same origin O that the total reference mark).

One expresses Ω in the total base (e_k) :

$$\Omega=\Omega_k e_k$$

For virtual displacement $\mathbf{OQ}'$, one a:

$$\mathbf{OQ}'=U_k e_k$$

Work due to the quantities of acceleration becomes then:

$$W^{ac}=\int_{\Omega}\rho\,U_k e_k\left[\ddot{U}_k e_k+\Omega\wedge\left(\Omega\wedge x_k^0 e_k\right)\right]dv=W^{ac}_{mass}+W^{ac}_{cent}$$

with:

$$W^{ac}_{masse}=\int_{\Omega}\rho\,U_k\ddot{U}_k\,dv$$

and:

$$W^{ac}_{cent}=\int_{\Omega}\rho U_k e_k\left[\Omega\wedge\left(\Omega\wedge x_k^0 e_k\right)\right]dv$$

3.4 Principle of virtual work

For a static loading, he is written in the following way: $\delta W_{ext}=\delta W_{def}$ where W_{ext} is the sum of various elementary work, corresponding to the various loadings.

In harmonic dynamics (calculations of clean modes), the principle of virtual work gives: $\delta W_{ext}+\delta W^{ac}_{mass}=0$

4 Digital discretization of the variational formulation resulting from the principle of virtual work

4.1 Introduction

This chapter is devoted to the discretization of the various terms of energy introduced into the preceding chapter. The choice of framework HENCKY-MINDLIN-NAGHDI to describe the kinematics of hull, presented to the paragraph [§2] led to expressions of the deformations where the derivative are limited to order 1, contrary to the model of LOVE-KIRCHHOFF. One can thus use a finite element of a nature limited while ensuring conformity (see p.110 [bib7]).

The degrees of freedom are 3 displacements in the total reference mark and 2 rotations in local reference mark.

The selected elements are isoparametric quadrangles or triangles. The quadrangle is represented below. The quadrangles give the best results (see p.202 [bib8]). The best choice consists in taking for these elements of the quadratic functions of interpolation (see p.224 [bib8]) in order to correctly model the effects of membrane, inflection and shearing. According to the results based on many CAS-tests of the literature, the best alternative is the quadratic isoparametric quadrangle, which makes it possible to have a fine representation of a curved geometry and good estimates of the constraints. One chooses among the elements with quadratic functions the element hétérosis (Q9H) whose displacements are approached by the functions of interpolation of the Sérendip element and rotations by the functions of the element of Lagrange (cf Annexe3). This choice is justified hereafter.

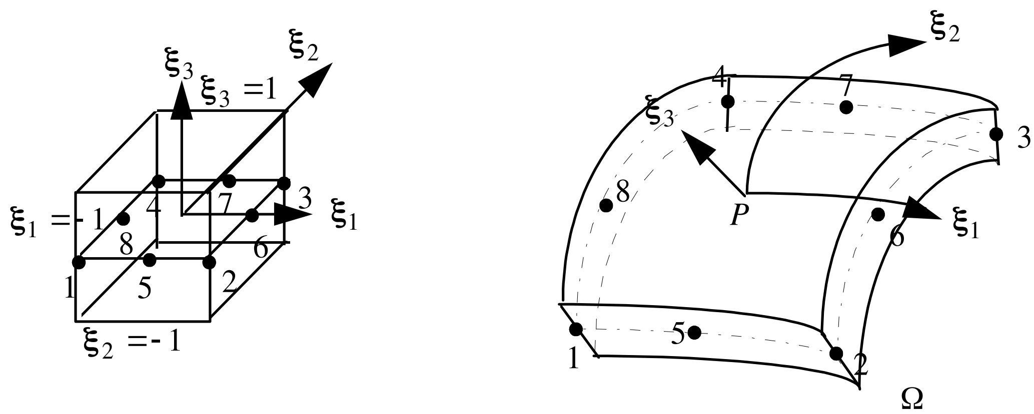


Figure 4.1-a: Representations of the isoparametric quadrangle

The figure [Figure 4.1-b] summarizes the three families of elements previously named.

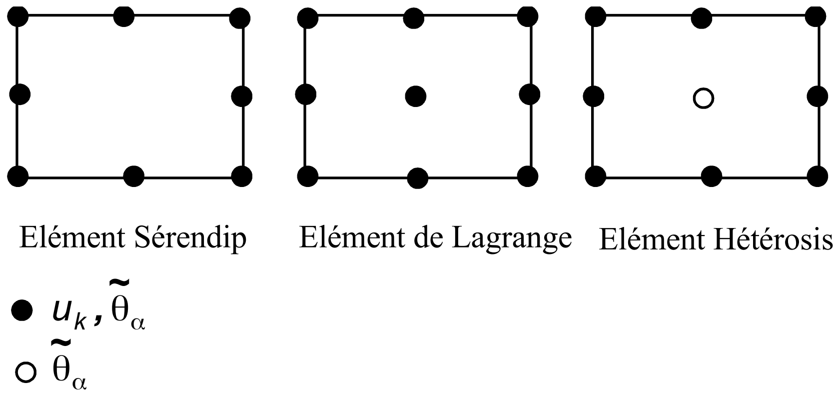


Figure 4.1-b: Families of finite elements for the isoparametric quadrangle

Risks of blocking or locking of membrane or shearing appear when the thickness of the hull becomes small compared to its radius of curvature and that the functions of interpolation are of a too low nature. To solve them a selective digital integration is used [bib6]. For certain types of boundary conditions (embedding) with the Sérendip element locking persists in spite of selective integration. Moreover, for the element of Lagrange, this kind of integration leads to singularities in the matrix of rigidity. The element Hétérosis Q9H with selective integration does not encounter the problems mentioned and seems being more performing for modeling of the very thin hulls (see p.224 [bib8]). It should be noted that this element has a mode of deformation without associated energy if it is used only. This mode disappears when one uses more than two elements [bib7].
For the elements triangle, the element Hétérosis T7H is essential for the same reasons but proves definitely less performing (see paragraph 5 concerning the validation).

One decides to carry out all calculations of discretization in the total Cartesian base.

4.2 Discretization of the geometrical terms

Coordinates x_k^0 of a point P average surface ω are interpolated by the functions of form in the following way:

$$x_k^0 = \sum_{i=1}^{Nb1} N_i^{(1)} x_{ik}^0$$

where the number $Nb1$ and functions of form $N_i^{(1)}$ depend on the type of element chosen, and x_{ik}^0 are the coordinates with the node i element.

The vectors covariants a_α (attaches at the point P) are then given by:

$$a_\alpha = \sum_{i=1}^{Nb1} \frac{\partial N_i^{(1)}}{\partial \xi_\alpha} x_{ik}^0 e_k$$

The calculation of the vectors is avoided T_k because components T_k^j the sizes of curve contain whose calculation is often vague as it was shown in the paragraph [§2.2.1].

In order to avoid the presence of the terms of curve, one writes:

$$n=\sum_{i=1}^{Nb\,1} N_i^{(1)} n_i$$

where n_i is the normal vector with the nodes of the element.

4.3 Discretization of the field of displacement

One adopts the following writing for displacement at the point Q :

$$U=\sum_{i=1}^{Nb\,1} N_i^{(1)} u_{ik} e_k+\frac{\xi_3}{2}\sum_{i=1}^{Nb\,2} N_i^{(2)} h_i\left(\tilde{\theta}_{i2} t_{i1}-\tilde{\theta}_{i1} t_{i2}\right)$$

where them t_α are evaluated with the nodes, and where it is observed that the functions of interpolation $N_i^{(2)}$ and their number $Nb\,2$ for rotations $\tilde{\theta}_\alpha$ are a priori different from those used for displacements u_k .

By expressing them $t_{i\alpha}$ according to their components in the total Cartesian base, one obtains:

$$U=\sum_{i=1}^{Nb\,1} N_i^{(1)} u_{ik} e_k+\frac{\xi_3}{2}\sum_{i=1}^{Nb\,2} N_i^{(2)} h_i\left(\tilde{\theta}_{i2} t_{i1k}+\tilde{\theta}_{i1} t_{i2k}\right) e_k$$

One calculates then the various elementary terms, in order to obtain the complete discretized formulation. In the continuation one uses the convention of summation of Einstein, while having with the spirit which the number of interpolations is $Nb\,1$ for x_k^0, n, u_k , and $Nb\,2$ for $\tilde{\theta}_\alpha$, t_α .

4.3.1 Element Hétérosis Q9H

With this element, the number of interpolations for the geometry $\left(x_k^0, n\right)$ and displacements u_k is $Nb1=8$ (nodes tops and mediums on the sides), while the number of interpolations for t_α and rotations $\tilde{\theta}_\alpha$ is $Nb2=9$ (nodes tops and mediums on the sides + barycentre). The number of degrees of freedom total of the element is thus $Nddle=3\times 8+2\times 9=42$.

Functions of interpolation $N_i^{(1)}$ and $N_i^{(2)}$ respectively for the geometry and displacements, and for rotations, can be found for example in [bib2] and are quoted in appendix 2.

The elementary vector of displacement can be put in the following form:

$$\tilde{q}^e=\left(u_{11}, u_{12}, u_{13}, \tilde{\theta}_{11}, \tilde{\theta}_{12}, \ldots, u_{i1}, u_{i2}, u_{i3}, \tilde{\theta}_{i1}, \tilde{\theta}_{i2}, \ldots, \tilde{\theta}_{91}, \tilde{\theta}_{92}\right) i=1,8$$

4.3.2 Element triangle T7H

With this element $Nb1=6$ (nodes tops and mediums on the sides) and $Nb2=7$ (nodes tops and mediums on the sides + barycentre). The number of degrees of freedom total of the element is $Nddle=3\times 6+2\times 7=32$.

6 functions of interpolation $N_i^{(1)}$ who are classical can be found in [bib2] and are quoted in appendix 4. On the other hand 7 $N_i^{(2)}$ are much less and their expressions are given in Appendix 3.

The elementary vector of displacement can be put in the following form:

$$\tilde{q}^e=\left(u_{11},u_{12},u_{13},\tilde{\theta}_{11},\tilde{\theta}_{12},\ldots,u_{i1},u_{i2},u_{i3},\tilde{\theta}_{i1},\tilde{\theta}_{i2},\ldots,\tilde{\theta}_{71},\tilde{\theta}_{72}\right) i=1,6$$

4.3.3 Notice

One notices on the level of the elementary vector $\tilde{q}^e$ the presence of terms associated with the local base and the total base.

4.4 Discretization of the field of deformation

The field of deformation is expressed like the symmetrized gradient of the field of displacement:

$$\varepsilon=S\nabla U=\frac{1}{2}(\nabla U+\nabla U^T)$$

Like:

$$U(x)=N\left[\xi(x)\right]\tilde{q}^e$$

one thus has:

$$\nabla U=\nabla N(\xi)\frac{\partial\xi}{\partial x}\tilde{q}^e$$

where N gather the functions of form $N_i^{(1)}$ and $N_i^{(2)}$ and matrices of passage $t_{i\alpha k}$, $\frac{\partial\xi}{\partial x}$ is the reverse of the jacobien J and $\tilde{q}^e$ is the vector of the degrees of freedom to the nodes (translations u_k and rotations $\tilde{\theta}_\alpha$).

Taking into account these relations and of $\tilde{\varepsilon}=T\times T\,\varepsilon$, one obtains the components of the tensor of deformation in the local reference mark:

$$\tilde{\varepsilon}=\tilde{B}\,\tilde{q}^e$$

where $\tilde{B}$ is the matrix of interpolation of $\tilde{\varepsilon}$, such as:

$$\tilde{B}=\boldsymbol{T}\times\boldsymbol{T}\,S\,J^{-1}\nabla N\left(\xi\right)$$

Note:

If one takes again the expression of $U(x)=\sum_{i=1}^{Nb1} N_i^{(1)} u_{ik} e_k + \frac{\xi_3}{2} \sum_{i=1}^{Nb2} N_i^{(2)} h_i \left(\tilde{\theta}_{i2} t_{i1k} + \tilde{\theta}_{i1} t_{i2k}\right) e_k = U_t(x)+U_r(x)$ *it is noticed that the terms of membrane are contained in the first part* $U_t(x)$ *of* $U(x)$ *and that the terms of inflection are contained in the second part* $U_r(x)$ *of* $U(x)$ *. The terms of transverse shearing come from the two contributions. One obtains as follows:* $\tilde{\varepsilon}_m = \tilde{B}_m \tilde{q}^e$, $\tilde{\varepsilon}_f = \tilde{B}_f \tilde{q}^e$, $\tilde{\varepsilon}_\gamma = \tilde{B}_\gamma \tilde{q}^e$ *where*

$$\tilde{B}_m = T_{mf} \mathbf{S} \mathbf{J}^{-1} \nabla N_1(x)$$
$$\tilde{B}_f = T_{mf} \mathbf{S} \mathbf{J}^{-1} \nabla [\xi_3 \frac{h}{2} N_2(\xi)]$$
$$\tilde{B}_\gamma = T_\gamma \mathbf{S} \mathbf{J}^{-1} \nabla N(\xi)$$

by simple decomposition of the expression $\tilde{\varepsilon} = \tilde{B} \tilde{q}^e$ *. One calls membrane part of the deformation projection on the membrane-inflection part of the local field of deformation of the symmetrized gradient of the translations in the total reference mark. One calls inflection part of the deformation projection on the membrane-inflection part of the local field of deformation of the symmetrized gradient of rotations in the total reference mark. One calls transverse distortion projection on the shearing part of the local field of deformation of the symmetrized gradient of total displacement.*

4.5 Matrix of rigidity

The principle of virtual work is written in the following way: $\delta W_{ext} = \delta W_{def}$ that is to say still $\delta U^T \mathrm{KU} = \delta U^T F$ in matric form where K is the matrix of rigidity coming from the assembly in the total reference mark of the whole of the elementary matrices of rigidity. At the elementary level the discretization of the work of deformation is written with the preceding notations:

$$\delta W_{def}^{el} = \delta \tilde{q}^{e^t} \int_{-1}^{1} \int_{A_r} \tilde{B}^t \tilde{C} \tilde{B} \det J \, d\xi_1 d\xi_2 d\xi_3 \tilde{q}^e = \delta \tilde{q}^{e^t} \tilde{K}^e \tilde{q}^e$$

where A_r is the area of reference of the element.

4.5.1 Decomposition of the elementary matrices

This matrix of rigidity understands three contributions due to the deformations of membrane, inflection and transverse distortion. One has as follows: $\tilde{K}^e=\tilde{K}^e_m+\tilde{K}^e_f+\tilde{K}^e_\gamma$ with:

$$\tilde{K}^e_m=\int_{-1}^{1}\int_{A_r}\tilde{B}^t_m H\,\tilde{B}_m \det J\, d\xi_1\, d\xi_2 d\xi_3\,;$$

$$\tilde{K}^e_f=\int_{-1}^{1}\int_{A_r}\tilde{B}^t_f H\,\tilde{B}_f \det J\, d\xi_1\, d\xi_2 d\xi_3\,;$$

$$\tilde{K}^e_\gamma=\int_{-1}^{1}\int_{A_r}\tilde{B}^t_\gamma H_\gamma\,\tilde{B}_\gamma \det J\, d\xi_1\, d\xi_2 d\xi_3\,.$$

4.5.2 Assembly of the elementary matrices

The principle of virtual work for the whole of the elements is written:

$\delta W_{def}=\sum_{e=1}^{\mathrm{nb}\,elem}\delta W^e_{def}=\delta \boldsymbol{U}^{\boldsymbol{T}}\mathbf{KU}$ where U is the whole of the degrees of freedom of the discretized structure and K comes from the assembly of the elementary matrices.

4.5.2.1 Degrees of freedom

The process of assembly of the elementary matrices implies that all the degrees of freedom are expressed in the total reference mark. In the total reference mark, the degrees of freedom are three displacements compared to the three axes of the total Cartesian reference mark and three rotations compared to these three axes. One thus uses, for the degrees of freedom of rotation, of the matrices of passage of the orthonormal local reference mark t_α with the total reference mark for each element.

4.5.2.2 Fictitious rotations

Rotation compared to the normal with the hull is not a true degree of freedom. To ensure compatibility between the passage of the local reference mark the total reference mark, one thus adds a degree of additional freedom local of rotation to the hull which is that corresponding to rotation compared to the normal on the average surface of the element. This implies an expansion of the blocks of dimension $(5,5)$ matrix of local rigidity into cubes blocks of dimension $(6,6)$ by adding a line and a column corresponding to this rotation. These additional lines and these columns are a priori worthless. One then carries out the passage of the matrix of local rigidity extended to the matrix of total rigidity.
In the preceding transformation, one was satisfied to add rotations compared to the normals on the surface of the elements without modifying the deformation energy. The contribution to the energy brought by these additional degrees of freedom is indeed worthless and no rigidity is associated for them.
The matrix of total rigidity thus obtained presents the risk however to be noninvertible. To avoid this nuisance it is allowed to allot a small rigidity to these additional degrees of freedom on the level of the matrix of widened local rigidity. Practically, one chooses it between 10^{-6} and 10^{-3} time the diagonal minor term of the matrix of rigidity of local rotation. The user can choose this multiplicative coefficient `COEF_RIGI_DRZ` itself in `AFFE_CARA_ELEM` ; by default it is worth 10^{-5}.

4.6 Matrix of mass

The virtual work of the effects of inertia can be expressed in the form:

$$\delta W_{mass}^{ac}=\int_{\Omega}\rho\ddot{U}(Q).\delta U(Q)\,d\Omega$$

It is supposed that the deformations and displacements remain sufficiently small so that the normal on the average surface of the hull remains unchanged.
With these assumptions, we can write the field of virtual displacement:

$$\delta U(Q)(\xi_1,\xi_2,\xi_3)=\delta u(P)(\xi_1,\xi_2)+\xi_3\frac{h}{2}\delta\theta(\xi_1,\xi_2)\wedge n(\xi_1,\xi_2)$$

and the field of acceleration:

$$\ddot{U}(Q)(\xi_1,\xi_2,\xi_3)=\ddot{u}(P)(\xi_1,\xi_2)+\xi_3\frac{h}{2}\ddot{\theta}(\xi_1,\xi_2)\wedge n(\xi_1,\xi_2)$$

In this expression, we neglected the gyroscopic terms.

4.6.1 Discretization of displacement for the matrix of mass

At the point Q , one takes as interpolation of the field of displacement:

$$\delta U(Q)(\xi_1,\xi_2,\xi_3)=\sum_{I=1}^{Nb1}N_I^1(\xi_1,\xi_2)\begin{pmatrix}\delta u_{I1}\\ \delta u_{I2}\\ \delta u_{I3}\end{pmatrix}-\xi_3\frac{h}{2}\sum_{I=1}^{Nb2}N_I^2(\xi_1,\xi_2)\begin{bmatrix}0 & -n_{I3} & n_{I2}\\ n_{I3} & 0 & -n_{I1}\\ -n_{I2} & n_{I1} & 0\end{bmatrix}\begin{pmatrix}\delta\theta_{I1}\\ \delta\theta_{I2}\\ \delta\theta_{I3}\end{pmatrix}$$

For the field of acceleration, the interpolation is written:

$$\ddot{U}(Q)(\xi_1,\xi_2,\xi_3)=\sum_{I=1}^{Nb1}N_I^1(\xi_1,\xi_2)\begin{pmatrix}\ddot{u}_{I1}\\ \ddot{u}_{I2}\\ \ddot{u}_{I3}\end{pmatrix}-\xi_3\frac{h}{2}\sum_{I=1}^{Nb2}N_I^2(\xi_1,\xi_2)\begin{bmatrix}0 & -n_{I3} & n_{I2}\\ n_{I3} & 0 & -n_{I1}\\ -n_{I2} & n_{I1} & 0\end{bmatrix}\begin{pmatrix}\ddot{\theta}_{I1}\\ \ddot{\theta}_{I2}\\ \ddot{\theta}_{I3}\end{pmatrix}$$

We rewrite the two preceding equations in the matric form:

$$\delta U(Q)(\xi_1,\xi_2,\xi_3)=N\,\delta u^e$$
$$\ddot{U}(Q)(\xi_1,\xi_2,\xi_3)=N\,\ddot{u}^e$$

where N is the matrix of interpolation, whose expression is:

$$\mathbf{N}=\left[\left[N_I^1\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}-\xi_3\frac{h}{2}N_I^2\begin{pmatrix}0 & -n_{I3} & n_{I2}\\ n_{I3} & 0 & -n_{I1}\\ -n_{I2} & n_{I1} & 0\end{pmatrix}\right]_{I=1,Nb1}\quad -\xi_3\frac{h}{2}N_{Nb2}^2\begin{pmatrix}0 & -n_{Nb23} & n_{Nb22}\\ n_{Nb23} & 0 & -n_{Nb21}\\ -n_{Nb22} & n_{Nb21} & 0\end{pmatrix}\right]$$

The vector u^e is the elementary nodal vector of displacements in the total reference mark which is put in the following form:

4.6.2 Elementary matrix of mass

With the preceding notations, the virtual work of the effects of inertia is put in the following matric form:

$$\delta W_{mass}^{inertie} = \delta u^{eT} M^e \ddot{u}^e$$

with M^e the matrix of coherent mass which can be expressed in the form:

$$M^e = \int_{\Omega e} \rho N^T N \det(J(\xi_3)) d\xi_1 d\xi_2 d\xi_3$$

It is important to note that because of the curve, a coupling of the terms of translation with those of rotation is possible (indeed, $\det(J(\xi_3))$ is not constant in the thickness).

4.6.3 Assembly of the elementary matrices of mass

The assembly of the matrices of mass follows same logic as that of the matrices of rigidity. The degrees of freedom are the same ones and one finds the treatment specific to normal rotations on the surface of the hull. Although the matrix of coherent mass is built in the total reference mark, it remains singular compared to the rotation of the normal in each node. We need to supply this matrix on the basis of the variational form:

$$\delta W_n^{inertie} = \sum_{I=1}^{Nb2} m_e \delta\theta_I (n_I \times n_I) \ddot{\theta}_I$$

where m_e is selected constant by element and calculated according to the formula:

$$m_e = C m_{max}$$

$m_{\max}$ being the major term due to rotations (in the local reference mark of the element) on the diagonal of the matrix M^e . It is thus to note that with this intention it was necessary to bring back the contribution of the rotations initially expressed in the total reference mark of the element, in the local reference mark of the element by change of reference mark.

For modal calculations utilizing at the same time the calculation of the matrix of rigidity and that of the matrix of mass, it is necessary to take a mass on the being worth degree of normal rotation on the surface of the hull C time the diagonal minor term of the matrix of mass for the terms of rotation in the local reference mark, where C is worth between 10^{-6} and 10^{-3}. One chooses to confuse the values of this coefficient with those of COEF_RIGI_DRZ for the equivalent operation on the matrix of rigidity. By default C is worth thus 10^{-5}. That makes it possible to inhibit, during a modal analysis, the modes being able to appear on the additional degree of freedom of rotation around the normal on the surface of the hull.

4.7 Digital integration for elasticity

4.7.1 Surface integration

For the element Hétérosis Q9H the inflection part of the matrix of stiffness is integrated classically with 9 points of Gauss while the parts membrane and shearing are obtained by integration reduced with 4 points of Gauss.

For element T7H, by analogy with Q9H, the matrix of stiffness is obtained with 7 points of integration of Hammer for the inflection part and 3 points of integration of Hammer for the parts shearing and membrane.

Cordonnées of the points	**Weight** ω_t
$\xi_1=1/3\,;\eta_1=1/3$	$9/80$
$\xi_2=a\,;\eta_2=a$ $a=\dfrac{6+\sqrt{15}}{21}$	$A=\dfrac{155+\sqrt{15}}{2400}$
$\xi_3=1-2\mathrm{a}\,;\eta_3=a$	A
$\xi_4=a\,;\eta_4=1-2\mathrm{a}$	A
$\xi_5=b\,;\eta_5=b$ $b=4/7-a$	$31/240-A$
$\xi_6=1-2\mathrm{b}\,;\eta_6=b$	$31/240-A$
$\xi_7=b\,;\eta_7=1-2\mathrm{b}$	$31/240-A$

$$\int_0^1\int_0^{1-\xi} y(\xi,\eta)\,d\eta\,d\xi=\sum_{i=1}^{n}\omega_i\,y(\xi_i,\eta_i)$$

Normal digital formulas of integration on triangle T7H (Hammer)

Cordonnées of the points	**Weight** ω_t
$\xi_1=-a\,;\eta_1=-a$ $a=-0.774596669241483$	$25/81$
$\xi_2=0.\,;\eta_2=-a$	$40/81$
$\xi_3=a\,;\eta_3=-a$	$25/81$
$\xi_4=a\,;\eta_4=0$	$40/81$
$\xi_5=a\,;\eta_5=a$	$25/81$
$\xi_6=0\,;\eta_6=a$	$40/81$
$\xi_7=-a\,;\eta_7=a$	$25/81$
$\xi_8=-a\,;\eta_8=0$	$40/81$
$\xi_9=0\,;\eta_9=0$	$64/81$

$$\int_{-1}^1\int_{-1}^{1} y(\xi,\eta)\,d\eta\,d\xi=\sum_{i=1}^{n}\omega_i\,y(\xi_i,\eta_i)$$

Normal digital formulas of integration 3×3 on quadrangle Q9H (Gauss)

It is noticed that the order of the points of Gauss of the preceding formula is not the same one as for the isoparametric elements. The first 8 points are described here while turning in the direct direction.

The principle of reduced integration consists in evaluating the membrane and shearing strains at the points of reduced integration and extrapolating them at the points of classical integration. This amounts supposing that these deformations are bilinear on element Q9H and linear on the T7H. The functions of form chosen to make this extrapolation are the bilinear of the quadrangle with 4 nodes for Q9H and linear classical functions of form of the triangle with 3 nodes for the T7H being worth 1 at the points of reduced integration.

For more details on the principle of reduced or selective integration, one can refer to [bib6].

Cordonnées of the points	Weight ω_t
$\xi_1=1/6\,;\eta_1=1/6$	1/6
$\xi_2=2/3\,;\eta_2=1/6$	1/6
$\xi_3=1/6\,;\eta_3=2/3$	1/6

$$\int_0^1 \int_0^{1-\xi} y(\xi,\eta)\,d\eta\,d\xi=\sum_{i=1}^{n} \omega_i\, y(\xi_i,\eta_i)$$

Digital formulas of integration reduced on triangle T7H (Hammer)

For the elements quadrangle an integration of Gauss 2×2 is used.

Cordonnées of the points	Weight ω_t
$\xi_1=1/\sqrt{3}\,;\eta_1=1/\sqrt{3}$	1
$\xi_2=1/\sqrt{3}\,;\eta_2=-1/\sqrt{3}$	1
$\xi_3=-1/\sqrt{3}\,;\eta_3=1/\sqrt{3}$	1
$\xi_3=-1/\sqrt{3}\,;\eta_3=-1/\sqrt{3}$	1

$$\int_{-1}^{1} \int_{-1}^{1} y(\xi,\eta)\,d\eta\,d\xi= \sum_{i=1}^{n} \omega_i\, y(\xi_i,\eta_i)$$

Reduced digital formulas of integration 2×2 on quadrangle Q9H (Gauss)

4.7.2 Integration in the thickness

Integration in the thickness is made with three points for the two elements.

Cordonnées of the points	Weight ω_t
$\xi_1=-1$	1/3
$\xi_2=0$	4/3
$\xi_3=+1$	1/3

$$\int_{-1}^{1} y(\xi)\,d\xi= \sum_{i=1}^{n} \omega_i\, y(\xi_i)$$

Digital formula of integration in the thickness in elasticity

4.8 Digital integration for plasticity

The principle of surface integration remains the same one as in elasticity, but the initial thickness is divided into N identical layers thicknesses. There are three points of integration per layer. The points of integration are located in higher skin of layer, in the middle of the layer and in lower skin of layer. For N layers, the number of points of integration is of $2N+1$. One advises to use from 3 to 5 layers in the thickness for a number of points of integration being worth 7.9 and 11 respectively.

For rigidity, one calculates for each layer, in plane constraints, the contribution to the matrices of rigidity of membrane, inflection and transverse distortion. These contributions are added and assembled to obtain the matrix of total tangent rigidity.

For each layer, the state of the constraints is calculated $(\sigma_{11,}\sigma_{22,}\sigma_{12})$ and the whole of the internal variables, in the middle of the layer and in skins higher and lower of layer, starting from the local plastic behavior and of the local field of deformation $(\varepsilon_{11,}\varepsilon_{22,}\varepsilon_{12})$. The positioning of the points of integration enables us to have the rightest estimates, because not extrapolated, in skins lower and higher of layer, where it is known that the constraints are likely to be maximum. The plastic behavior does not understand for the moment the terms of transverse shearing which are treated in an elastic way, because transverse shearing is uncoupled from the membrane behavior in plane constraints.

Cordonnées of the points	Weight ω_t
$\xi_1=-1$	1/3
$\xi_2=0$	4/3
$\xi_3=+1$	1/3
$\int_{-1}^{1} y(\xi)\,d\xi=$	$\sum_{i=1}^{n}\omega_i\,y(\xi_i)$

Digital formula of integration for a layer in the thickness in plasticity

Note:

One already mentioned with [§2.2.2] that the value of the coefficient of correction in transverse shearing for the elements of plate and hull was obtained by identification of elastic complementary energies after resolution of balance 3D. This method is not usable any more in elastoplasticity and the choice of the coefficient of correction in transverse shearing is posed then. The transverse terms of shearing are thus not affected by plasticity and are treated elastically, for want of anything better. If one places oneself in theory of Coils-Kirchhoff for a value of this coefficient of $10^6 h/R$ *(* h *being the thickness of the hull and* R *its average radius of curvature) the transverse terms of shearing become negligible and the approach is more rigorous.*

4.9 Discretization of elementary work for the loadings

4.9.1 Elementary discretization of the work of the forces and external couples being exerted on average surface

According to the paragraph [§3.2], one recalls that one has for these efforts and couples:

$$\delta W_{ext}=\int_S (f\,\delta u+c\,\delta\beta)\,dS$$

where S is the average surface of the hull.
For the first term of this expression one has as follows:

4.9.1.1 Loads given in the total reference mark

$$\delta W_{ext}=\int_{A_r}[F_k N_i^1 \delta u_{ik}+c_k N_j^2(\delta \tilde{\theta}_{j2} t_{j1k}-\delta \tilde{\theta}_{j1} t_{j2k})]\frac{2}{h}\det J° \, d\xi_1 \, d\xi_2$$

with $\det J°=\det J(\xi_3=0)$

4.9.1.2 Loads given in the local reference mark

$$\delta W_{ext}=\int_{A_r}[F_\alpha N_j^1 t_{j\alpha k} N_i^1 \delta u_{ik}+c_\alpha t_{\alpha k} N_j^2(\delta \tilde{\theta}_{j2} t_{j1k}-\delta \tilde{\theta}_{j1} t_{j2k})]\frac{2}{h}\det J° \, d\xi_1 \, d\xi_2$$

4.9.2 Elementary discretization of the work of the forces and external couples being exerted on contour

According to the paragraph [§3.2], one recalls that one has for these efforts and couples:

$$\delta W_{ext}=\int_C(\phi \delta u+\chi \delta \beta)ds$$

where C is the average contour of the hull. ϕ and χ linear distributions of force and moment applied to the contour of the hull in the total reference mark.

The discretization gives then: $\delta W_{ext}=\int_C[\phi_k N_i^1 \delta u_{ik}+\chi_k N_j^2(\delta \tilde{\theta}_{j2} t_{j1k}-\delta \tilde{\theta}_{j1} t_{j2k})]ds$

4.9.3 Discretization of the term of gravity

One has for this term:

$$\delta W_{\text{pes}}=\int_{\Omega_e}\rho g \delta U(Q)dV=\int_{\Omega_e}\rho g_k \delta U_k(Q)dV=\int_{\Omega_e}\rho g_k[N_i^1 \delta u_{ik}+\frac{\xi_3}{2}N_j^2(\delta \tilde{\theta}_{j2} t_{j1k}-\delta \tilde{\theta}_{j1} t_{j2k})]dV$$

That is to say: $\delta W_{pes}=\int_{\Omega_e}\rho g_k N_i^1 \delta u_{ik} \, dV$ by supposing negligible the second term of expression Ci above.

4.9.4 Discretization of the term of pressure

It is supposed that the pressure p is applied to average surface ω hull. One has then:

$$\delta W_{pres}=\int_{A_r} ep \, n \, \delta u(P)dS=\int_{A_r} ep(a_1 \wedge a_2)\delta u(P)d\xi_1 d\xi_2$$

$$dW_{pres}=\int_{A_r} ep \, n \, \delta u(P)dS=\int_{A_r} ep(a_1 \wedge a_2)\delta u(P)d\xi_1 d\xi_2$$

where $e=\pm 1$ according to whether p is applied in internal or external skin.

Like $a_\alpha = a_{\alpha k} e_k$, this is still written: $\delta W_{pres} = \int_{A_r} epN_i^1 \delta u_{ik} \nu_k d\xi_1 d\xi_2$

where $\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} J^\circ_{12} J^\circ_{23} - J^\circ_{13} J^\circ_{22} \\ J^\circ_{13} J^\circ_{21} - J^\circ_{11} J^\circ_{23} \\ J^\circ_{11} J^\circ_{22} - J^\circ_{12} J^\circ_{21} \end{pmatrix}, J^\circ_{ij} = J_{ij}(\xi_3 = 0)$.

4.9.5 Discretization of the terms of centrifugal inertia

One adds with the expression of the field of accelerations of the paragraph [§4.6] the term corresponding to the centrifugal accelerative forces if the total reference mark (O, e_k) is in uniform rotation Ω compared to a Galilean reference mark which has the same origin O that the total reference mark. The expression of the field of accelerations becomes as follows:

$$\ddot{U}(Q)(\xi_1, \xi_2, \xi_3) = \ddot{u}(P)(\xi_1, \xi_2) + \xi_3 \frac{h}{2} \ddot{\theta}(\xi_1, x_2) \wedge n(\xi_1, \xi_2) + \Omega \wedge [\Omega \wedge \mathbf{OP}]$$

where one neglected the forces of Coriolis and the correction of metric in the thickness.

One expresses Ω in the total base (e_k): $\Omega = \Omega_k e_k$.

By taking again the expression of: $dW^{inertie} = \int_\Omega \rho \ddot{\mathbf{U}}(Q).\mathbf{dU}(Q) d\Omega$, one identifies the contribution of the terms of centrifugal inertia: $dW^{inertie}_{cent} = \int_{\Omega_e} \rho \delta u_k e_k [\Omega \wedge (\Omega \wedge x^0_k e_k)] dV$ by neglecting the terms of rotation in virtual displacement. The terms of mass are unchanged compared to [§4.6].

Like one a:

$$\Omega \wedge x^0_k e_k = \Omega_p e_p \wedge x^0_k e_k = \Omega_p x^0_k e_{qpk} e_q$$

where e_{qpk} is the permutation of Lévi-Strauss.

One also writes:

$$\Omega \wedge (\Omega \wedge x^0_k e_k) = e_{qpk} e_{srq} \Omega_r \Omega_p x^0_k e_k$$

From where it results from it that:

$$W^{inertie}_{cent} = \int_{-1}^{1} \int_{Ar} \rho \delta u_{is} N^{(1)}_i e_{qpk} e_{srq} \Omega_r \Omega_p x^0_{\mathrm{jk}} N^{(1)}_j \det J \, dx_1 dx_2 dx_3$$

4.9.6 Taking into account of the loadings of thermal dilation

One treats only the case where thermoelastic characteristics E , ν , α depend only on the average temperature $\bar{T}$ in the thickness. Moreover, the material is thermoelastic isotropic homogeneous in the thickness.

The variational formulation of work due to thermal dilations is written:

The temperature is represented by the model of thermics to three fields according to [R3.11.01]:

$$T(\xi_\alpha,\xi_3)=T^m(\xi_\alpha).P_1(\xi_3)+T^s(\xi_\alpha).P_2(\xi_3)+T^i(\xi_\alpha).P_3(\xi_3) \ ,$$

with: $P_j(\xi_3)$: three polynomials of LAGRANGE in the thickness: $]-1,+1[$:

$$P_1(\xi_3)=1-(\xi_3)^2 \ ; \ P_2(\xi_3)=\frac{\xi_3}{2}(1+\xi_3) \ ; \ P_3(\xi_3)=-\frac{\xi_3}{2}(1-\xi_3) \ ;$$

WITH to leave the representation of the temperature above, one obtains:

- the average temperature in the thickness:
$$\bar{T}(\xi_\alpha)=\frac{1}{2}\int_{-1}^{+1}T(\xi_\alpha,\xi_3)\,d\xi_3=\frac{1}{6}\left(4\mathrm{T}^m(\xi_\alpha)+T^s(\xi_\alpha)+T^i(\xi_\alpha)\right) \ ;$$
- the average variation in temperature in the thickness:
$$\hat{T}(\xi_\alpha)=3\int_{-1}^{+1}T(\xi_\alpha,\xi_3)\xi_3\,d\xi_3=T^s(\xi_\alpha)-T^i(\xi_\alpha) \ ;$$

Thus the temperature can be written in the following way:

$$T(\xi_\alpha,\xi_3)=\bar{T}(\xi_\alpha)+\hat{T}(\xi_\alpha).\xi_3/2+\tilde{T}(\xi_\alpha,\xi_3) \text{ such as:}$$
$$\int_{-1}^{+1}\tilde{T}(\xi_\alpha,\xi_3)=0;\int_{-1}^{+1}\xi_3\tilde{T}(\xi_\alpha,\xi_3)=0 \ .$$

If the temperature is indeed closely connected in the thickness one has, $\tilde{T}=0$.

It is necessary to evaluate the three-dimensional thermal stresses, in each point of integration in the thickness. These constraints of thermal origin withdrawn from the usual mechanical constraints are calculated at the points of integration in the thickness by:

$$\tilde{\sigma}^{ther}_{\beta\gamma}=\frac{\alpha.E}{1-\nu^2}\left(\bar{T}-T^{réf}+\hat{T}.\xi_3/2\right)\delta_{\beta\gamma}$$

4.9.7 Assembly

The variational formulation of the work of the efforts external for the unit of the elements is written then:

$\delta W_{\mathbf{ext}}=\sum_{e=1}^{\mathrm{nb}\,elem}\delta W^e_{\mathbf{ext}}=\delta U^T F$ where U is the whole of the degrees of freedom of the discretized structure and F comes from the assembly of the vectors forces elementary.

As for the matrices of rigidity, the process of assembly of the vectors forces elementary implies that all the degrees of freedom are expressed in the total reference mark. In the total reference mark, the degrees of freedom are three displacements compared to the three axes of the total Cartesian reference mark and three rotations compared to these three axes. One thus uses matrices of passage of the local reference mark to the total reference mark for rotations of each element.

Note:

The external efforts can also be defined in the reference mark user. One then uses a matrix of passage of the reference mark user towards the local reference mark of the element to have the expression of these efforts in the local reference mark of the element and to deduce the vector from it elementary corresponding room forces. For the assembly one passes then from the local reference mark of the element to the total reference mark.

5 Validation

To judge relevance of thick the hull formulation, the few examples of application according to relate to as well linear statics as the calculation of clean modes. Three new cases tests relative to the two finite elements described in the preceding parts were integrated in *Code_Aster*. They come to enrich the CAS-tests by the elements of plate already present in the environment from *Code_Aster.* Most of these CAS-tests were indexed in [bib10].

The three new CAS-tests, two in statics plus one in dynamics, are classical examples of validation drawn from [bib3]. The reference solutions, analytical or digital, resulting from [bib3] are compared with the digital results given by *Code_Aster*. For more information on these CAS-tests, one will refer to the documentation of validation indicated in reference.

5.1 Case test in linear statics

5.1.1 Static case test n° 1

The first case test is that of a cylindrical panel subjected to its own weight [V3.03.107].

This test makes it possible to highlight effects of membrane more important than those of inflection. It makes it possible to measure the performance of the elements hulls compared to elements DKT or DKQ whose interpolation out of membrane is linear.

5.1.2 Case static test n° 2

The second case test is that of a helicoid hull subjected to two concentrated types of loading [V3.03.108].

The helicoid shape of the hull makes it possible to study the geometrical representation of the finite elements. The concentrated loadings can be:

- in the plan: the influence due to the effects of membrane is then not important and the behavior dominating is that due to the inflection,
- except plan: the effects of membrane affect the behavior of the hull.

5.2 Case test in dynamics

This case test is a simplified model of paddle of compressor, which is in fact a cylindrical panel [V2.03.102].

This test highlights the performances of the elements in dynamic behavior by the data of the frequencies and the clean modes.

The frequencies and clean modes of the paddle are experimental values which are used as results of reference.

6 Thermomechanical chaining

6.1 Description

For the resolution of chained thermomechanical problems, one must use for the thermal calculation of the finite elements of thermal hull [R3.11.01] whose field of temperature is recovered like input datum of *Code_Aster* for mechanical calculation. It is necessary thus that there is compatibility between the thermal field given by the thermal hulls and that recovered by the mechanical hulls. This last is defined by the knowledge of the 3 fields TEMP_SUP, TEMP_MIL and TEMP_INF given in skins lower, medium and higher of hull.

The table below indicates compatibilities between the elements of mechanical hull and thermal hull.

Modeling THERMICS	Mesh	Finite element	to use with	Mesh	Finite element	Modeling MECHANICS
HULL	QUAD9	THCOQU9	///////////	QUAD9	MEC3QU9H	COQUE_3D
HULL	TRIA7	THCOTR7	///////////	TRIA7	MEC3TR7H	COQUE_3D

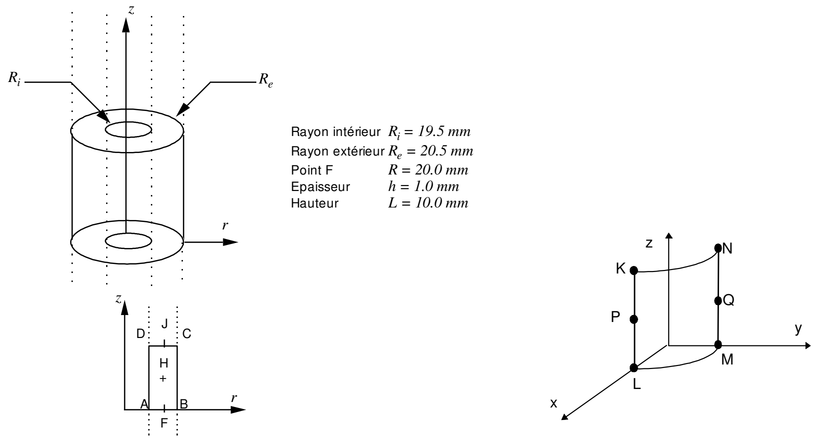
Note:

- *The nodes of the thermal elements of hulls and mechanical hulls must correspond. The grids for thermics and mechanics will thus have the same number and the same type of meshs.*
- *The elements of thermal hulls surface are treated like elements plans by projection of the initial geometry on the level defined by the first 3 tops.*

The thermomechanical chaining is also possible if one knows by experimental measurements the variation of the field of temperature in the thickness of the structure or certain parts of the structure. In this case one works with a map of temperature defined a priori; the field of temperature is not given any more by the three values TEMP_INF, TEMP_MIL and TEMP_SUP thermal calculation obtained by EVOL_THER. It can be much richer and contain an arbitrary number of points of discretization in the thickness of the hull. The operator DEFI_NAPPE allows to create such profiles of temperatures starting from the abundant data by the user. These profiles are affected by the order CREA_CHAMP (cf CAS-test HSNS100B). It will be noted that it is not necessary for mechanical calculation that the number of points of integration in the thickness is equal to the number of points of discretization of the field of temperature in the thickness. The field of temperature is automatically interpolated at the points of integration in the thickness of the elements of hulls.

6.2 CAS-test

The cas-tests for the thermomechanical chaining between thermal elements of hulls and mechanical elements of hulls are the HPLA100C (elements MEC3QU9H) and HPLA100D (elements MEC3TR7H). It is about a heavy thermoelastic hollow roll in uniform rotation [V7.01.100] subjected to a phenomenon of thermal dilation where the fields of temperature are calculated with THER_LINEAIRE by a stationary calculation.



Thermal dilation is worth: $T(\rho) - T_{ref}(\rho) = 0.5(T_s + T_i) + 2.(T_s + T_i)(r - R)/h$

with:

- $T_s = 0.5\,°C, T_i = -0.5\,°C, T_{ref} = 0.\,°C$
- $T_s = 0.1\,°C, T_i = 0.1\,°C, T_{ref} = 0.\,°C$

One tests the constraints, the efforts and bending moments in L and M. The results of reference are analytical. One obtains very good performances whatever the type of element considered.

7 Establishment of the elements of hull in *Code_Aster*

7.1 Description

These elements (of names MEC3TR7H and MEC3QU9H) are pressed on meshs TRIA7 and QUAD9 curves. These elements are not exact with the nodes and it is necessary to net with several elements to get correct results.

7.2 Introduced use and developments

These elements are used in the following way:

```
MY = CREA_MAILLAGE (    GRID: MAILINI
                        MODI_MAILLE : (OPTION: 'QUAD8_9'
                        ALL: 'YES')…)
```

One calls on a routine MODI_MAILLE of modification of the grid to pass from the elements quadrangles to 8 nodes to the elements quadrangles to 9 nodes or many elements triangles to 6 nodes to the elements triangles to 7 nodes.

AFFE_MODELE (MODELING: 'COQUE_3D'…) for the triangle and the quadrangle

One calls on the routine INI080 for the position of the points of Hammer and Gauss on the surface of the hull and the weights corresponding.

```
AFFE_CARA_ELEM (HULL: (THICKNESS: 'EP'
              ANGL_REP: ( α , β )
              COEF_RIGI_DRZ: 'CTOR')
```

To make postprocessings (forced, generalized efforts,…) in a reference mark chosen by the user who is not the local reference mark of the element, one defines the direction $X1$ reference mark user like the projection of a direction of reference $\underline{d}$ on surface ω element. This direction of reference $\underline{d}$ is chosen by the user who defines it by two nautical angles in the total reference mark. The normal N at the surface of the element the second direction fixes at the point of observation concerned. The vector product of the two vectors previously definite $Y1 = N \wedge X1$ allows to define the local trihedron in which will be expressed the generalized efforts representing the state of stresses. The user will have to take care that the selected reference axis is not found parallel with the normal of certain elements of hull. By default, direction of reference $\underline{d}$ is the axis X total reference mark of definition of the grid.
The value CTOR corresponds to the coefficient which the user can introduce for the treatment of the terms of rigidity and mass according to normal rotation on the surface of the hull. This coefficient must be sufficiently small not to disturb the energy assessment of the element and not too small so that the matrices of rigidity and mass are invertible. A value of 10^{-5} by default is put.

ELAS : (E: NAKED Young: ν ALPHA: α . RHO: ρ .)
For a homogeneous isotropic thermoelastic behavior in the thickness one uses the keyword ELAS in DEFI_MATERIAU where the coefficients are defined E , Young modulus, ν Poisson's ratio, α thermal dilation coefficient and RHO density˜

AFFE_CHAR_MECA (DDL_IMPO: (
DX:. DY:. DZ:. DRX:. DRY MARTINI:. DRZ:. DDL of hull in the total reference mark.
FORCE_COQUE: (FX:. FY:. FZ:. MX:. MY:. MZ:.). They is the surface efforts on elements of hull. These efforts can be given in the total reference mark or the reference mark user defined by ANGL_REP.

FORCE_NODALE: (FX:. FY:. FZ:. MX:. MY:. MZ:.). They is the efforts of hull in the total reference mark.

7.3 Calculation in linear elasticity

The matrix of rigidity and the matrix of mass (respectively options RIGI_MECA and MASS_MECA) are integrated numerically in TE0401 and TE0406, respectively. Calculation takes account owing to the fact that the terms corresponding to the degrees of freedom of rotation of hull are expressed in the local reference mark of the element. A matrix of passage makes it possible to pass from the local degrees of freedom to the total degrees of freedom.

Elementary calculations (CALC_CHAMP) currently available correspond to the options:

- EPSI_ELNO and SIGM_ELNO who provide the strains and the stresses to the nodes in the reference mark user of the element in lower skin, to semi thickness and in higher skin of hull. One stores these values in the following way: 6 components of strain or stresses,
- EPXX EPYY EPZZ EPXY EPXZ EPYZ or SIXX SIYY SIZZ SIXY SIXZ SIYZ,
- EFGE_ELNO : who gives the efforts generalize by element with the nodes starting from displacements: NXX, NYY, NXY, MXX, MYY, MXY, QX, QY.
- SIEF_ELGA : who gives the constraints by element to the points of Gauss in the local reference mark of the element starting from displacements: SIXX, SIYY, SIZZ, SIXY, SIXZ, SIYZ.
- EPOT_ELEM : who gives the elastic energy of deformation per element starting from displacements.
- ECIN_ELEM : who gives the kinetic energy by element.

Finally it TE0416 calculate also the option FORC_NODA of calculation of the nodal forces for the operator CALC_CHAMP.

7.4 Plastic design

The matrix of rigidity is also integrated numerically, by layers, in TE0414. One calls on the option of calculation STAT_NON_LINE in which one defines in the level of the nonlinear behavior the number of layers to be used for digital integration. All laws of plane constraints available in *Code_Aster* can be used.

```
STAT_NON_LINE (….
 BEHAVIOR: (RELATION: ‘‘
COQUE_NCOU: ‘MANY LAYERS’)
….)
```

Currently available elementary calculations correspond to the options:

- EPSI_ELNO who provides the deformations by element to the nodes in the reference mark user starting from displacements, in lower skin, with semi thickness and in higher skin of hull.
- SIGM_ELNO who allows to obtain the stress field in the thickness by element with the nodes for all the under-points (all the layers and for all the positions: in lower skin, in the medium or in higher skin of layer). These values are given in the reference mark user.
- EFGE_ELNO who allows to obtain the efforts generalized by element with the nodes in the reference mark user.
- VARI_ELNO who calculates the field of internal variables and the constraints by element with the nodes for all the layers, in the local reference mark of the element.

8 Conclusion

The finite elements of hull curves which we describe here are used in the curved mean structural analyses whose thickness report over characteristic length is lower than 1/10. Two finite elements of voluminal hull being pressed on quadrangular and triangular meshs were introduced into *Code_Aster*. They were selected with a quite particular aim: to be able to represent a complete behavior of curved structures whereas until now one could use only elements with plane facets which induced parasitic inflections and required to refine the grids.

It is elements for which the strains and the stresses in the plan of the element vary linearly with the thickness of the hull. Selected kinematics is a kinematic hull of the Hencky-Mindlin-Naghdi type making it possible to utilize the transverse energy of shearing. The distortion associated with transverse shearing is constant in the thickness of the element. Variable correction on the coefficient k of transverse shearing a flexibility in use offers making it possible to pass from the theory of HENCKY-MINDLIN-NAGHDI for $k=1$, with that of REISSNER for $k=5/6$ and with that of LOVE_KIRCHHOFF (for very mean structures) if one chooses a value of k equalize with $10^6 \times h/L$, h being the thickness and L a characteristic distance (average radius of curvature, enforcement zone of the loads….). As in this last case, one uses a method of penalization to make small the terms of shearing transverse, one can, if one takes a value of k too much important, to make singular the digital system. In this case, it is necessary to decrease the value of k.

The value by default of k is of $5/6$. It is generally used when the structure to be netted has a thickness report over characteristic length understood enters $1/20$ and $1/10$. For lower thicknesses where the transverse distortion becomes low one can want to use a value of $k=10^6 \times h/L$ (to be able to make comparisons with elements of plate DKT for example). When the transverse distortion is nonworthless, the elements of hull do not satisfy the equilibrium conditions 3D and the boundary conditions on nullity with stresses shear transverse on the faces higher and lower of hull, compatible with a constant transverse distortion in the thickness of the hull. It results from it thus that on the level of the behavior a coefficient of 5/6 for a homogeneous hull corrects the usual relation between the constraints and the transverse distortion in order to ensure the equality between energies of shearing of the model 3D and the model of hull constant distortion. In this case, the arrow $\tilde{u}_3$ has as an interpretation average transverse displacement in the thickness of the hull and not the displacement of the average surface of the hull.

For structures low thickness in order to avoid the phenomena of blocking, one uses under - integration reduced for the parts membrane and shearing of the matrix of rigidity. The choice on the finite elements was made on the elements quadrangle Hétérosis Q9H and triangle T7H. Indeed, among the finite elements with quadratic functions of interpolation, the performance of the element Hétérosis Q9H is known. It is in particular higher than that of the elements Sérendip Q9S or the elements of Lagrange Q9. This performance rests however on the selective integration of the element with reduced integration of the terms of membrane and shearing on the one hand, and normal integration of the terms of inflection on the other hand. By analogy with Q9H, one took the finite element T7H like triangular element of form. However, as far as possible, one will use the Q9H rather than the T7H which is definitely less performing.

The non-linear behaviors in plane constraints are available for these elements. It is announced however that the constraints generated by the transverse distortion are treated elastically, for want of anything better. Indeed the rigorous taking into account of a transverse shearing constant not no one on the thickness and the determination of the correction associated on rigidity with shearing compared to a model satisfying the equilibrium conditions and the boundary conditions are not possible and thus return the use of these elements, when transverse shearing is nonnull, rigorously impossible in plasticity. Rigorously, for nonlinear behaviors, it would thus be necessary to use these elements within the framework of the theory of Coils-Kirchhoff.

Elements corresponding to the machine elements exist in thermics; the mechanical chainings thermo - are thus available with finite elements of thermal hulls to 7 and 9 nodes. Extensions of the preceding formulation presented in appendix allow also the taking into account of the anisotropy of materials and kinematic non-linearity. This second extension is operational in *Code_Aster* and is the object of a reference material [R3.07.05].

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10 Description of the versions of the document

Version Aster	Author (S) Organization (S)	Description of the modifications
5	P.Massin, A.Laulusa EDF-R&D/MMN	Initial text
7.4	X. Desroches	Update: minor modifications

Annexe 1 Extension to anisotropic materials not programmed

It is considered that the hull consists of an orthotropic material, axes of orthotropism $\tilde{\tilde{x}}_k$ associated with the base k_k . The law of behavior in these axes is written:

$$\underset{(6\times1)}{\tilde{\tilde{\varepsilon}}} = \underset{(6\times6)}{\tilde{\tilde{S}}_k} \underset{(6\times1)}{\tilde{\tilde{\sigma}}}$$

where $\tilde{S}$ is the matrix of flexibility of the component k .

Are $\tilde{\varepsilon}$ and $\tilde{\sigma}$, tensors of strain and stresses in the axes $\tilde{x}_k$, one a:

$$\tilde{\sigma} = {}^t Q \tilde{\tilde{\sigma}} Q$$
$$\tilde{\varepsilon} = {}^t Q \tilde{\tilde{\varepsilon}} Q$$

where $Q = \left[T_1, T_2, T_3\right]_{/k_k}$ ($Q_{ij} = T_i . k_j$) is the matrix of the cosine directors of T_k in the base k_k .

In vectorial form, one a:

$$\tilde{\sigma} = \tilde{T} \tilde{\tilde{\sigma}}$$
$$\tilde{\varepsilon} = \tilde{T} \tilde{\tilde{\varepsilon}}$$

where components of $\tilde{T}$ are defined according to those of Q .

Conversely, one a:

$$\tilde{\tilde{\sigma}} = \tilde{T}^{-1} \tilde{\sigma}$$
$$\tilde{\tilde{\varepsilon}} = \tilde{T}^{-1} \tilde{\varepsilon}$$

therefore, one obtains:

$$\tilde{\varepsilon} = \tilde{T} \tilde{\tilde{S}}_k \tilde{T}^{-1} \tilde{\sigma}$$

that one writes:

$$\tilde{\varepsilon} = \tilde{S}_k \tilde{\sigma}$$

To be coherent with the assumption of plane constraint $\tilde{\sigma}_{33} = 0$, one writes:

$$\underset{(5\times1)}{\tilde{\varepsilon}_r} = \underset{(5\times5)}{\tilde{S}_{kr}} \underset{(5\times1)}{\tilde{\sigma}_r}$$

with the symbol R like tiny room, which gives:

$$\tilde{\sigma}_r=\tilde{C}_k\tilde{\varepsilon}_r,\tilde{C}_k=\tilde{S}_{kr}^{-1}$$

that one récrit by omitting the symbol r ,

$$\tilde{\sigma}=\tilde{C}_k\tilde{\varepsilon}$$

The elastic deformation energy W^{el} is:

$$W^{el}=\frac{1}{2}{}^t\tilde{q}^e\int_{-1}^{1}\int_{Ar}{}^t\tilde{B}\tilde{C}_k\tilde{B}\det J\,d\xi_1 d\xi_2 d\xi_3\tilde{q}^e$$

If the hull consists of Nc layers, each layer being regarded as a component k , then:

$$W^{el}=\frac{1}{2}{}^t\tilde{q}^e\sum_{k=1}^{Nc}\int_{2e_k^-/h}^{2e_k^+/h}\int_{Ar}{}^t\tilde{B}\tilde{C}_k\tilde{~}B\det J\,d\xi_1 d\xi_2 d\xi_3\tilde{~}q^e$$

where e_k^- and e_k^+ are the X-coordinates of the limits lower and higher of the layer k of thickness $e_k=e_k^+-e_k^-$, with $e_1^-=-h/2$ and $e_{Nc}^+=h/2$.

While posing:

$$\xi_3=\frac{e_k}{h}\bar{\xi}_3+\frac{e_k^++e_k^-}{h},\bar{\xi}_3\in[-1,1]$$

one a:

$$W^{el}=\frac{1}{2}{}^t\tilde{q}^e\sum_{k=1}^{Nc}\frac{e_k}{h}\int_{-1}^{1}\int_{Ar}{}^t\tilde{B}\tilde{C}_k\tilde{B}\det J(\xi_1,\xi_2,\bar{\xi}_3)\,d\xi_1 d\xi_2 d\bar{\xi}_3\tilde{q}^e$$

In the same way, for work due to thermal dilations W^{th} , one a:

$$\tilde{\tilde{\varepsilon}}_{th}^k=(\alpha_1^k T,\alpha_2^k T,\alpha_3^k T,0,0,0)$$

where them α_i^k are the dilation coefficients thermal of the layer k in the axes of orthotropism ($\tilde{\tilde{\xi}}_k$).

With the relation:

$$\tilde{\varepsilon}_{th}^k=\tilde{T}\tilde{\tilde{\varepsilon}}_{th}^k$$

one obtains:

$$W^{th}=-{}^t\tilde{q}^e\int_{-1}^{1}\int_{Ar}{}^t\tilde{B}(-\tilde{C}_k\tilde{\varepsilon}_{th}^k)\det J\,d\xi_1 d\xi_2 d\xi_3$$

That is to say:

$$W^{th}={}^t\tilde{q}^e\sum_{h=1}^{Nc}\frac{e_k}{h}\int_{-1}^{1}\int_{Ar}{}^t\tilde{B}\tilde{C}_k\tilde{\varepsilon}_{th}^k\det J\,d\xi_1 d\xi_2 d\bar{\xi}_3$$

Annexe 2 Functions of form for element Q9H

These functions are given on page 174 of [bib8].

A2.1 Functions of form for the translations

8 functions of the shape of incomplete Lagrange of the element quadrangle Q9H [A2.2-a Figure] for the interpolation of displacements u_k are:

- $N_i^{(1)}(\xi_1,\xi_2)=\frac{1}{4}(-1+\xi_{1i}\xi_1+\xi_{2i}x_2)(1+\xi_{1i}\xi_1)(1+x_{2i}\xi_2)\,i=1,2,3,4$
- $N_i^{(1)}(\xi_1,\xi_2)=\frac{1}{2}(1-x_1^2)(1+\xi_{2i}\xi_2)\,i=5,7$
- $N_i^{(1)}(\xi_1,\xi_2)=\frac{1}{2}(1-x_2^2)(1+\xi_{1i}\xi_1)\,i=6,8$

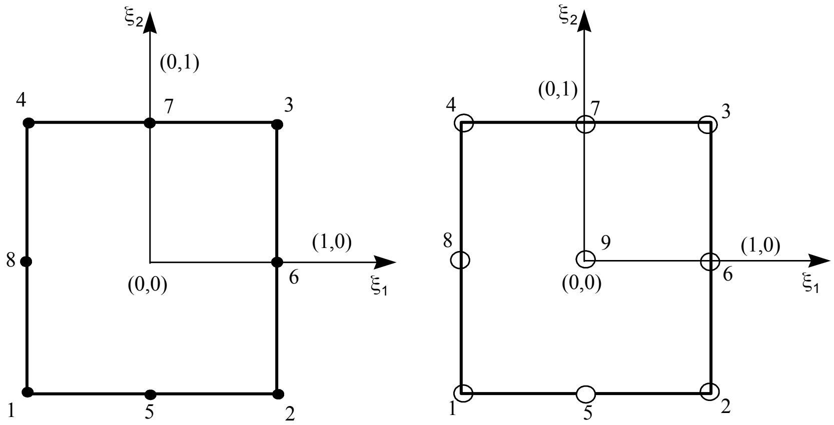
with: $\xi_{1i}=\text{-}1\mathrm{i}=1,8,4;\quad \xi_{1i}=0\mathrm{i}=5,7;\quad \xi_{1i}=+1\mathrm{i}=2,6,3.$ and $\xi_{2i}=\text{-}1\mathrm{i}=1,5,2;\quad \xi_{2i}=0\mathrm{i}=6,8;\quad \xi_{2i}=+1\mathrm{i}=3,7,4.$.

A2.2 Functions of form for rotations

9 functions of the shape of Lagrange of the element quadrangle Q9H [A2.2-a Figure] for the interpolation of rotations $\tilde{\theta}_\alpha$ are:

$N_i^{(2)}(x_1,\xi_2)=N_i(\xi_1)N_i(\xi_2)$ where $N_i(\xi_P)=\underset{r'i}{P}\frac{\xi_{\mathrm{Pr}}-\xi_P}{\xi_{\mathrm{Pr}}-\xi_{\mathrm{Pi}}}$ for $p=1,2$ and where r described all two node aligned with the node i in the direction ξ_P .

One a: $x_{1i}=\text{-}1\mathrm{i}=1,8,4;\quad x_{1i}=0\mathrm{i}=5,7;\quad x_{1i}=+1\mathrm{i}=2,6,3.$ and $x_{2i}=\text{-}1\mathrm{i}=1,5,2;\quad x_{2i}=0\mathrm{i}=6,8;\quad x_{2i}=+1\mathrm{i}=3,7,4.$.



A2.2-a figure: Degrees of freedom for the translations and rotations of the element quadrangle Q9H

Annexe 3 Functions of form for element T7H

A3.1 Functions of form for the translations

6 functions of form of triangular element T7H [A3.2-a Figure] for the interpolation of displacements u_k are given on page 175 of [bib8]:

- $N_1^{(1)}(x_1, x_2)=\lambda(2\lambda-1)$
- $N_2^{(1)}(x_1, x_2)=x_1(2x_1-1)$
- $N_3^{(1)}(x_1, x_2)=x_2(2x_2-1)$
- $N_4^{(1)}(x_1, x_2)=4x_1\lambda$
- $N_5^{(1)}(x_1, x_2)=4x_1 x_2$
- $N_6^{(1)}(x_1, x_2)=4\lambda x_2$

where: $\lambda=1-x_1-x_2$

A3.2 Functions of form for rotations

7 functions of form of triangular element T7H [A3.2-a Figure] for the interpolation of rotations $\tilde{q}_\alpha$ are:

- $N_1^{(2)}(x_1, x_2)=(1-x_1-x_2)[2(1-x_1-x_2)-1]+\frac{1}{9}N_7^{(2)}$
- $N_2^{(2)}(x_1, x_2)=x_1(2x_1-1)+\frac{1}{9}N_7^{(2)}$
- $N_3^{(2)}(x_1, x_2)=x_2(2x_2-1)+\frac{1}{9}N_7^{(2)}$
- $N_4^{(2)}(x_1, x_2)=4x_1(1-x_1-x_2)-\frac{4}{9}N_7^{(2)}$
- $N_5^{(2)}(x_1, x_2)=4x_1 x_2-\frac{4}{9}N_7^{(2)}$
- $N_6^{(2)}(x_1, x_2)=4x_2(1-x_1-x_2)-\frac{4}{9}N_7^{(2)}$

with:

- $N_7^{(2)}(x_1, x_2)=27x_1 x_2(1-x_1-x_2)$

A3.2-a figure: Degrees of freedom for the translations and rotations of the element triangle T7H