Voluminal elements of hulls into nonlinear geometrical

Summary:

We present in this document the theoretical formulation and the digital establishment of a finite element of voluminal hull for analyses into nonlinear geometrical. This approach must make it possible to take into account great displacements and great rotations of mean structures, whose thickness report over characteristic length is lower than $1/10$. One will take care that these rotations remain lower than 2π .

This formulation is based on an approach of continuous medium 3D, degenerated by the introduction of the kinematics of hull in plane constraints into the weak form of balance. The measurement of the deformations which we retain is that of Green-Lagrange, énergétiquement combined with the constraints of Piola-Kirchhoff of second species. The formulation of balance is thus Lagrangian total.

The geometrical entirely nonlinear problem is examined in first. The case of linear buckling is treated like a borderline case of the first approach.

Contents

1 Introduction

The great transformations of hull are characterized by great displacements of average surface and great rotations of initially normal fibres on this surface. The transformation thus is represented exactly, at least in the continuous problem. The derivation of the finite elements objects associated with the linearized system of equations resulting from the principle of virtual work is carried out without any simplifying assumption on displacements or rotations. Moreover, one new diagram of selective digital integration is presented in order to solve the problem of blocking out of membrane and transverse shearing.

The degrees of freedom of rotation retained are the components of the vector of space iterative rotation. Between two iterations, it is the vector of the infinitesimal rotation superimposed on the deformed configuration. This choice led to a tangent matrix of rigidity which is not symmetrical. This is due to the nonvectorial character of great rotations which actually belong to the differential variety $SO(3)$. Rotations must remain lower than 2π because of the choice of update of great rotations established in *Code_Aster*, for which there is not bijection between the vector of full slewing and the orthogonal matrix of rotation.

An important difference compared to the linear analysis is to be announced. The finite elements objects are directly built in the total reference mark; displacements and rotations nodal are measured in the total reference mark.

2 Formulation

In this chapter, we present the various equations controlling the problem of deformation of the hull within the framework of a theory of great transformations.

2.1 Geometry of the elements of voluminal hull

The voluminal hull is represented by volume Ω (together of the points $Q(\xi_3 \neq 0)$) built around average surface ω (together of the points $P(\xi_3 = 0)$). In any point Q of Ω , a local orthonormal reference mark is built $[t_1(\xi_1, \xi_2, \xi_3) : t_2(\xi_1, \xi_2, \xi_3) : n(\xi_1, \xi_2)]$. The vector $n(\xi_1, \xi_2)$ represent the normal on the surface ω .

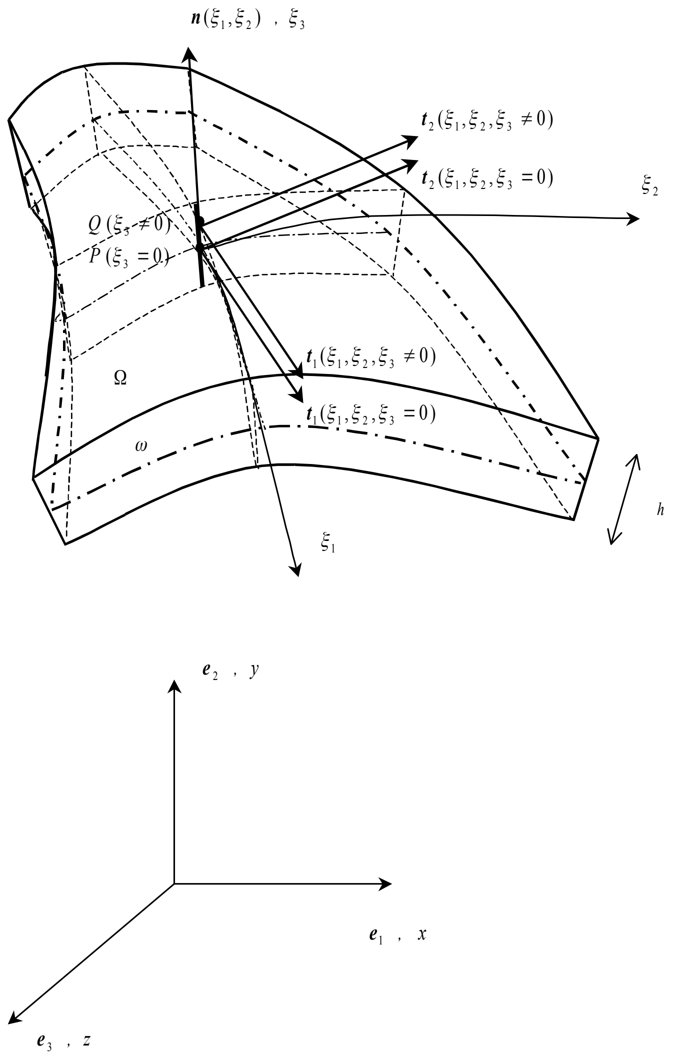

Figure 2.1-a: Voluminal hull. Local reference marks on the configuration of reference

In the initial configuration, the position of an unspecified point Q normal on the average surface can be expressed, according to the position of the revolved center P normal fibre, in the following way:

$$x_Q(\xi_1,\xi_2,\xi_3)=\xi_P(\xi_1,\xi_2)+\xi_3\frac{h}{2}n(\xi_1,\xi_2)$$

2.2 Kinematics of the voluminal hulls

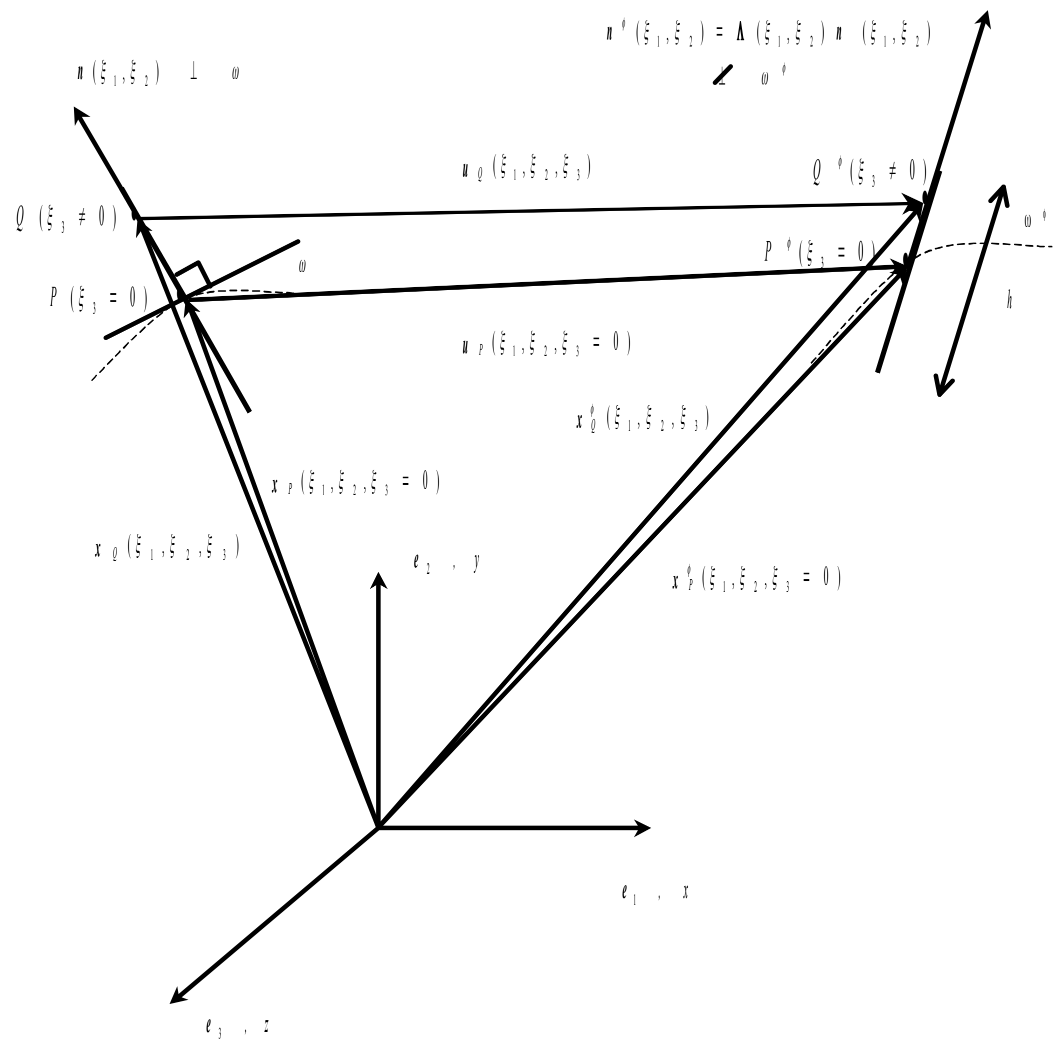

Figure 2.2-a: Voluminal hull.
Great transformations of an initially normal fibre on the average surface

In the deformed configuration, the position of the point Q can also be expressed according to the position of the point P :

$$x_Q^{\varphi}(\xi_1,\xi_2,\xi_3)=x_P^{\varphi}(\xi_1,\xi_2)+\xi_3\frac{h}{2}\boldsymbol{n}^{\boldsymbol{\varphi}}(\xi_1,\xi_2)$$

where $\boldsymbol{n}^{\varphi}$ is the unit vector obtained by great rotation of the normal $\boldsymbol{n}$.

The vector $\boldsymbol{n}^{\varphi}$ is not necessarily normal on the deformed average surface, because of transverse shearing strain. It is connected to the initial normal vector by the relation:

$$\boldsymbol{n}^{\boldsymbol{\varphi}}=\Lambda(\xi_1,\xi_2)\boldsymbol{n}$$

Λ is the orthogonal operator of the great rotation around the vector Θ , of angle θ , undergone by the fibre which was initially normal on the average surface whose expression is given by:

$$\Lambda=\exp[\Theta\times]=\cos\theta[I]+\frac{\sin\theta}{\theta}[\Theta\times]+\frac{1-\cos\theta}{\theta^2}[\Theta\otimes\Theta]$$

where $[\Theta\times]$ is the antisymmetric operator of the vector of full slewing Θ whose matric expression is:

$$[\Theta\times]=\begin{bmatrix}0 & -\Theta_z & \Theta_y\\ \Theta_z & 0 & -\Theta_x\\ -\Theta_y & \Theta_x & 0\end{bmatrix}$$

and $[\Theta\otimes\Theta]$ is the symmetrical operator given by $[\Theta\otimes\Theta]=\Theta\,\Theta^T$.

More details on great rotations and their digital processing can be found in [bib1] or [R5.03.40]. One can also write:

$$t_1^{\varphi}=\Lambda(\xi_1,\xi_2)t_1$$

$$t_2^{\varphi}=\Lambda(\xi_1,\xi_2)t_2$$

One can express the virtual variation of the operator of great rotation in the form:

$$\delta\Lambda=[\delta w\times]\Lambda$$

where $[\delta w\times]$ is the antisymmetric operator of the vector of space virtual rotation δw who is also the rotation part of the functions tests:

$$[\delta\mathbf{w}\times]\mathbf{b}=\delta\mathbf{w}\wedge\mathbf{b}\,\forall\,b\in\mathbb{R}^3$$

Its matric expression is:

$$[\delta w\times]=\begin{bmatrix}0 & -\delta w_z & \delta w_y\\ \delta w_z & 0 & -\delta w_x\\ -\delta w_y & \delta w_x & 0\end{bmatrix}$$

One can also express the iterative variation of the operator of great rotation in the form:

$$\Delta\Lambda=[\Delta w\times]\Lambda$$

where Δw is the vector of space iterative rotation, which is also the rotation part of the solution of the system of linearized equations.

This vector can be connected to the vector of total iterative rotation. There are thus the relations:

$$\Delta w=T(\Theta)\Delta\Theta \text{ and } \delta w=T(\Theta)\delta\Theta$$

where $T(\Theta)$ is the differential operator of rotation, whose expression according to the vector of full slewing is given by:

$$T(\Theta)=\frac{\sin\theta}{\theta}[I]-\frac{1-\cos\theta}{\theta^2}[\Theta\times]+\frac{\theta-\sin\theta}{\theta^3}[\Theta\otimes\Theta]$$

This matrix has the same values and clean vectors that the matrix Λ and the relation checks:

$$T(\Theta)=\Lambda(\Theta)T^T(\Theta)$$

In addition, the iterative variation of the matrix of virtual rotation can be put in the form:

$$\Delta\delta\Lambda=[\delta w\times][\Delta w\times]\Lambda$$

The total displacement of the point Q on fibre can be connected to the displacement of the centre of gravity P :

$$u_Q(\xi_1,\xi_2,\xi_3)=u_P(\xi_1,\xi_2)+\xi_3\frac{h}{2}\left(n^{\varphi}(\xi_1,\xi_2)-n(\xi_1,\xi_2)\right)$$

In order to lead to a system of linearized equations, obtained starting from the weak form of balance, we need to calculate various differential variations of this total displacement. Virtual displacement has as an expression:

$$\delta u_Q(\xi_1,\xi_2,\xi_3)=\delta u_P(\xi_1,\xi_2)+\xi_3\frac{h}{2}\delta w(\xi_1,\xi_2)\wedge n^{\varphi}(\xi_1,\xi_2)\,;\,\delta n=0$$

Iterative displacement has as an expression:

$$\Delta u_Q(\xi_1,\xi_2,x_3)=\Delta u_P(\xi_1,\xi_2)+x_3\frac{h}{2}\Delta w(\xi_1,\xi_2)\wedge n^{\varphi}(\xi_1,\xi_2)\,;\,\Delta n=0$$

The iterative variation of virtual displacement has as an expression:

$$\Delta \delta u_Q(\xi_1, \xi_2, \xi_3) = \xi_3 \frac{h}{2} \delta w(\xi_1, \xi_2) \wedge \left(\Delta w(\xi_1, \xi_2) \wedge n^{\varphi}(\xi_1, \xi_2)\right)$$

Note: The formulation suggested remains limited to rotations lower than 2π . This limit is due to the particular choice of update of the great rotations established in *Code_Aster*. This is due to nonthe bijection between the vector of full slewing and the orthogonal matrix of rotation.

2.3 Law of behavior

We consider a law of linear behavior very-rubber band: the local constraints of Piola - Kirchhoff of second species are proportional to the local deformations of Green - Lagrange:

$$\tilde{S} = D \tilde{E}$$

Hereafter, the symbol $\tilde{}$ indicate the quantities expressed in the orthonormal reference mark $\left[t_1(\xi_1, \xi_2, \xi_3) : t_2(\xi_1, \xi_2, \xi_3) : n(\xi_1, \xi_2)\right]$.

The matrix of elastic behavior linear in plane constraints is written as follows:

$$D = \begin{bmatrix} \frac{E}{1-\nu^2} & \frac{\nu E}{1-\nu^2} & 0 & 0 & 0 \\ & \frac{E}{1-\nu^2} & 0 & 0 & 0 \\ & & \frac{E}{2(1+\nu)} & 0 & 0 \\ & sym & & \frac{Ek}{2(1+\nu)} & 0 \\ & & & & \frac{Ek}{2(1+\nu)} \end{bmatrix}$$

E being the Young modulus, ν the Poisson's ratio and k the coefficient of transverse correction of shearing.

In the local reference mark, the state of Piola-Kirchhoff stress of second species is plan $\left(\tilde{S}_{nn} = 0\right)$ and can be characterized by a vector with 5 components:

$$\tilde{S} = \begin{pmatrix} \tilde{S}_{t_1 t_1} \\ \tilde{S}_{t_2 t_2} \\ \tilde{S}_{t_1 t_2} \\ \tilde{S}_{t_1 n} \\ \tilde{S}_{t_2 n} \end{pmatrix}$$

The vector of the deformations of Green-Lagrange is also expressed in the local reference mark by a vector with 5 components:

$$\tilde{E}=\begin{vmatrix}\tilde{E}_{t_1t_1}\\ \tilde{E}_{t_2t_2}\\ \tilde{\gamma}_{t_1t_2}\\ \\ \tilde{\gamma}_{t_1n}\\ \tilde{\gamma}_{t_2n}\end{vmatrix}$$

Here, we were unaware of the term $\tilde{E}_{nn}$ who is normal on the average surface and who is not inevitably null. This is a consequence of the assumption of the plane constraints.

2.3.1 Taking into account of transverse shearing

The correction of the transverse shear stress is carried out by extension of energy equivalences given in the case of the small deformations and of small displacements [R3.07.03].

3 Principle of virtual work

The principle of virtual work is the weak formulation of the static balance of the internal forces and the external forces:

$$\delta\pi_{\mathrm{int}}-\delta\pi_{ext}=0$$

The non-linearity of the equilibrium equations leads us to solve the system above in an iterative way by a method of Newton. We carry out thus the exact linearization of the principle of virtual work to each iteration, which leads to the equality:

$$\Delta\delta\pi_{\mathrm{int}}-\Delta\delta\pi_{ext}=\delta\pi_{ext}-\delta\pi_{\mathrm{int}}$$

3.1 Internal virtual work

The virtual work of the internal forces can be written on the initial configuration in the form:

$$\delta\pi_{\mathrm{int}}=\int_{\Omega}\left(\delta\tilde{\mathbf{E}}.\tilde{\mathbf{S}}\right)d\Omega$$

where $\tilde{E}$ and $\tilde{S}$ are the vectors of deformation of Green-Lagrange and Piola-Kirchhoff constraint of second species respectively, expressed in the local reference mark. Indeed, as the state of stress is plan for Piola-Kirchhoff of second species, we use the formulation of the principle of virtual work in the local reference mark. However, to limit the passages of the local reference mark to the total reference mark and vice versa, the vectors of strains and local stresses are not calculated explicitly in the local reference mark but they are obtained by the rotation of their representation in the total reference mark.

3.1.1 Incremental form of internal virtual work

The iterative variation of the work of virtual work interns is written:

$$\Delta\delta\pi_{\mathrm{int}}=\int_{\Omega}\left(\delta\tilde{E}.\Delta\tilde{S}+\Delta\delta\tilde{E}.\tilde{S}\right)d\Omega$$

In this equality, the iterative variation of the vector of local constraints of Piola-Kirchhoff of second species is calculated by the iterative discrete form of the relation of behavior:

$$\Delta \tilde{S} = D \Delta \tilde{E}$$

3.1.2 Passage du locates total with the local reference mark

In tensorial form one passes from the tensor of the total constraints to the tensor of the local constraints 3×3 (see [bib4] p. 111 for the constraints of Cauchy, the same relations applying to the constraints of Piola-Kirchhoff of second species) while using:

$$[\tilde{S}] = P[S]P^T$$

and of the tensor of the local constraints to the tensor of the total constraints by the inversion of the preceding relation:

$$[S] = P^T[\tilde{S}]P$$

In the two preceding expressions, the matrix of passage of the local reference mark to the total reference mark is an orthogonal matrix $P^{-1} = P^T$, and its explicit expression according to the unit vectors of the local orthonormal reference mark is:

$$\mathbf{P}(\xi_1, \xi_2, \xi_3) = \begin{bmatrix} \mathbf{t_1^T}(\xi_1, \xi_2, \xi_3) \\ \mathbf{t_2^T}(\xi_1, \xi_2, \xi_3) \\ \mathbf{n^T}(\xi_1, \xi_2, \xi_3) \end{bmatrix}$$

Within the framework of the conventional notation, one will be able to note:

$$\begin{aligned} t_1(\xi_1, \xi_2, \xi_3) &= \Lambda_0 e_1 \\ t_2(\xi_1, \xi_2, \xi_3) &= \Lambda_0 e_2 \\ t_3(\xi_1, \xi_2, \xi_3) &= n(\xi_1, \xi_2) = \Lambda_0 e_3 \end{aligned}$$

with the orthogonal matrix of passage (initial rotation):

$$\Lambda_0(\xi_1, \xi_2, \xi_3) = \left[t_1(\xi_1, \xi_2, \xi_3) : t_2(\xi_1, \xi_2, \xi_3) : t_3(\xi_1, \xi_2, \xi_3) \right]$$

It will be noticed that:

$$\Lambda_0 = P^T$$

The two relations of rotation of the constraints are also valid for the tensors of the deformations of Green-Lagrange. Nevertheless, a writing which connects the vectors of local and total deformation is necessary. This relation makes it possible to pass from the vector 6×1 total deformations with the vector 6×1 local deformations:

$$\overset{6\times1}{\tilde{E}} = \overset{6\times6}{\bar{H}}\ \overset{6\times1}{E}$$

with the form of the matrix of transformation of the vectors 6×1 of deformation (see [bib2] p. 258):

$$\overset{6\times6}{\bar{H}} = \begin{bmatrix} l_1^2 & m_1^2 & n_1^2 & l_1 m_1 & m_1 n_1 & n_1 l_1 \\ l_2^2 & m_2^2 & n_2^2 & l_2 m_2 & m_2 n_2 & n_2 l_2 \\ l_3^2 & m_3^2 & n_3^2 & l_3 m_3 & m_3 n_3 & n_3 l_3 \\ 2l_1 l_2 & 2m_1 m_2 & 2n_1 n_2 & l_1 m_2 + l_2 m_1 & m_1 n_2 + m_2 n_1 & n_1 l_2 + n_2 l_1 \\ 2l_2 l_3 & 2m_2 m_3 & 2n_2 n_3 & l_2 m_3 + l_3 m_2 & m_2 n_3 + m_3 n_2 & n_2 l_3 + n_3 l_2 \\ 2l_3 l_1 & 2m_3 m_1 & 2n_3 n_1 & l_3 m_1 + l_1 m_3 & m_3 n_1 + m_1 n_3 & n_3 l_1 + n_1 l_3 \end{bmatrix}$$

and components of the unit vectors of the local reference mark:

$$\begin{array}{lll} l_1 = t_1 . e_1 & m_1 = t_1 . e_2 & n_1 = t_1 . e_3 \\ l_2 = t_2 . e_1 & m_2 = t_2 . e_2 & n_2 = t_2 . e_3 \\ l_3 = t_3 . e_1 & m_3 = t_3 . e_2 & n_3 = t_3 . e_3 \end{array}$$

These expressions are general for the curvilinear reference marks. In the Cartesian total reference mark $\left[e_1 : e_2 : e_3\right]$, these components are:

$$\begin{array}{lll} l_1 = t_1(1) & m_1 = t_1(2) & n_1 = t_1(3) \\ l_2 = t_2(1) & m_2 = t_2(2) & n_2 = t_2(3) \\ l_3 = t_3(1) & m_3 = t_3(2) & n_3 = t_3(3) \end{array}$$

We have, actually, need for a writing which connects the vector of local deformation 5×1 and the vector of total deformation 6×1 :

$$\overset{5\times1}{\tilde{E}} = \overset{5\times6}{H}\ \overset{6\times1}{E}$$

For that, one forgets the back-row forward of the expression of $\overset{6\times 6}{H}$ (line associated with S_{nn}):

$$\overset{5\times 6}{H}=\left[\begin{matrix} (t_1(1))^2 & (t_1(2))^2 & (t_1(3))^2 \\ (t_2(1))^2 & (t_2(2))^2 & (t_2(3))^2 \\ 2t_1(1)t_2(1) & 2t_1(2)t_2(2) & 2t_1(3)t_2(3) \\ 2t_2(1)t_3(1) & 2t_2(2)t_3(2) & 2t_2(3)t_3(3) \\ 2t_3(1)t_1(1) & 2t_3(2)t_1(2) & 2t_3(3)t_1(3) \end{matrix}\right.$$
$$\left.\begin{matrix} t_1(1)t_1(2) & t_1(2)t_1(3) & t_1(3)t_1(1) \\ t_2(1)t_2(2) & t_2(2)t_2(3) & t_2(3)t_2(1) \\ t_1(1)t_2(2)+t_2(1)t_1(2) & t_1(2)t_2(3)+t_2(2)t_1(3) & t_1(3)t_2(1)+t_2(3)t_1(1) \\ t_2(1)t_3(2)+t_3(1)t_2(2) & t_2(2)t_3(3)+t_3(2)t_2(3) & t_2(3)t_3(1)+t_3(3)t_2(1) \\ t_3(1)t_1(2)+t_1(1)t_3(2) & t_3(2)t_1(3)+t_1(2)t_3(3) & t_3(3)t_1(1)+t_1(3)t_3(1) \end{matrix}\right]$$

The same preceding relations can be applied for the passage of the vectors of total deformation to the local deformation.

3.1.3 Relation deformation-displacement

The tensor 3×3 total deformations of Green-Lagrange is defined by (see for example [bib2]):

$$[E]=\frac{1}{2}\left(\nabla u+\nabla u^T+\nabla u^T\nabla u\right)$$

with the tensor of the gradient of displacements:

$$\nabla u=\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}\langle u\, v\, w\rangle=\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{bmatrix}$$

The tensor of deformation of Green-Lagrange can be also written:

$$[E]=\frac{1}{2}\left(F^T F-I\right)$$

with F the tensor gradient of the deformations 3×3 who is not symmetrical:

$$F=\nabla x^{\varphi}=I+\nabla u$$

and I the tensor identity:

$$I=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

The vector 6×1 total deformations of Green-Lagrange is ordered as follows (see [bib4] p 117):

$$E=\begin{pmatrix}E_{xx}\\ E_{yy}\\ E_{zz}\\ \gamma_{xy}\\ \gamma_{xz}\\ \gamma_{yz}\end{pmatrix}=\begin{pmatrix}u_{,x}\\ v_{,y}\\ w_{,z}\\ u_{,y}+v_{,x}\\ u_{,z}+w_{,x}\\ v_{,z}+w_{,y}\end{pmatrix}+\begin{pmatrix}\frac{1}{2}\left(u_{,x}{}^2+v_{,x}{}^2+w_{,x}{}^2\right)\\ \frac{1}{2}\left(u_{,y}{}^2+v_{,y}{}^2+w_{,y}{}^2\right)\\ \frac{1}{2}\left(u_{,z}{}^2+v_{,z}{}^2+w_{,z}{}^2\right)\\ u_{,x}u_{,y}+v_{,x}v_{,y}+w_{,x}w_{,y}\\ u_{,x}u_{,z}+v_{,x}v_{,z}+w_{,x}w_{,z}\\ u_{,y}u_{,z}+v_{,y}v_{,z}+w_{,y}w_{,z}\end{pmatrix}$$

It as follows is calculated:

$$E=\left[Q+\frac{1}{2}A(\frac{\partial u}{\partial x})\right]\frac{\partial u}{\partial x}$$

with:

$$Q=\begin{bmatrix}1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0\end{bmatrix}$$

and the vector of the gradient of displacements:

$$\frac{\partial u}{\partial x}=\begin{pmatrix} u_{,x} \\ u_{,y} \\ u_{,z} \\ v_{,x} \\ v_{,y} \\ v_{,z} \\ w_{,x} \\ w_{,y} \\ w_{,z} \end{pmatrix}$$

and the tensor A depending on the gradient of displacements:

$$A\left(\frac{\partial u}{\partial x}\right)=\begin{bmatrix} u_{,x} & 0 & 0 & v_{,x} & 0 & 0 & w_{,x} & 0 & 0 \\ 0 & u_{,y} & 0 & 0 & v_{,y} & 0 & 0 & w_{,y} & 0 \\ 0 & 0 & u_{,z} & 0 & 0 & v_{,z} & 0 & 0 & w_{,z} \\ u_{,y} & u_{,x} & 0 & v_{,y} & v_{,x} & 0 & w_{,y} & w_{,x} & 0 \\ u_{,z} & 0 & u_{,x} & v_{,z} & 0 & v_{,x} & w_{,z} & 0 & w_{,x} \\ 0 & u_{,z} & u_{,y} & 0 & v_{,z} & v_{,y} & 0 & w_{,z} & w_{,y} \end{bmatrix}$$

Virtual variation, noted δ , deformations of Green-Lagrange is obtained by a differential calculus:

$$\delta E=\left[Q+A\left(\frac{\partial u}{\partial x}\right)\right]\frac{\partial \delta u}{\partial x}$$

In this expression and that which follows, we took account of the following property (see [bib4] p 141):

$$\frac{1}{2}A\left(\frac{\partial \delta u}{\partial x}\right)\frac{\partial u}{\partial x}=\frac{1}{2}A\left(\frac{\partial u}{\partial x}\right)\frac{\partial \delta u}{\partial x}$$

Iterative variation Δ is also obtained by a differential calculus:

$$\Delta E=\left[Q+A\left(\frac{\partial u}{\partial x}\right)\right]\frac{\partial \Delta u}{\partial x}$$

The iterative variation of the virtual deformation of Green-Lagrange is put thus in the form:

$$\Delta\delta E=\underbrace{A\left(\frac{\partial \Delta u}{\partial x}\right)\frac{\partial \delta u}{\partial x}}_{\text{terme classique}}+\underbrace{\left[Q+A\left(\frac{\partial u}{\partial x}\right)\right]\frac{\partial \Delta\delta u}{\partial x}}_{\text{terme non classique}}$$

Whereas the first term of this expression is classical for the continuous mediums 3D, the second, which translates the taking into account of great rotations, is less.

3.1.4 Calculation of the constraints of Cauchy

3.1.4.1 Case general

The tensor 3×3 total constraints of Piola-Kirchhoff of second species is connected to the tensor 3×3 total constraints of Cauchy by the relation:

$$[S]=\det(F)F^{-1}[\sigma]F^{-T}$$

Thus, knowing the state of the constraints of Piola-Kirchhoff of second species, one can calculate the state of the constraints of Cauchy by the relation:

$$[\sigma]=\frac{1}{\det(F)}F[S]F^{T}$$

It should be noted that the state of stresses of Cauchy is not plan, in general, contrary to the state of stresses of Piola-Kirchhoff of second species. In addition, the choice of a local reference mark in which to represent this tensor is not at all obvious. One will show however, in the following paragraph, that within the framework of the small deformations, there exists a local reference mark, easily identifiable, in which the state of stresses of Cauchy is him-also plan.

In the case of completely general laws, a special attention will have to relate to the diagrams of digital integration making it possible to calculate the values of substitution of the gradient F at the points of normal digital integration.

3.1.4.2 Approximation in small deformations

It is pointed out [bib4] that the gradient F can be written thanks to the polar decomposition in two forms:

$$F=\mathbf{RU}=\mathbf{VR}$$

where $R=R^{-T}$ is an orthogonal tensor, and where U and V are positive definite symmetrical matrices of elongation.

Into the geometrical nonlinear field, we can introduce an important simplification into the polar decomposition of the gradient of the deformations if the deformations remain small. This simplification is not introduced into nonlinear calculation but in postprocessing of the constraints.

Elongation at the point Q being minor in front of the great rotation of the section:

$$U\approx V\approx I$$

One can then write:

$$F\approx R=\Lambda$$

where Λ is the tensor of great rotation which transforms the normal n in n^{φ} :

$$\Lambda n = n^{\varphi}$$

Simplification translates the fact that on a section, the transformation is reduced to a great rotation. With this approximation of the gradient of the deformations, one can write:

$$F \approx R = \Lambda$$

and thus, by exploiting the orthogonality of Λ one obtains:

$$F^{-1} \approx \Lambda^{T}$$

and:

$$\det(F) \approx 1 \ .$$

These simplifications lead to the final relation:

$$[\sigma] \approx \Lambda [S] \Lambda^{T}$$

This relation translates the fact that the constraints of Cauchy are quite simply obtained by the great rotation of the constraints of Piola-Kirchhoff of second species.

One can now rewrite the property of plane constraints of the tensor of Piola-Kirchhoff of second species $n.[S]n=0$ in the new form:

$$n.\Lambda^{T}[s]\Lambda n = 0$$

who leads in addition to the property:

$$n^{\varphi}.[\sigma] n^{\varphi} = 0$$

That is to say still:

$$\tilde{\sigma}_{n^{\varphi} n^{\varphi}} = 0$$

Constraints of Cauchy $\left[\sigma\left(\xi_1, \xi_2, \xi_3\right)\right]$ are also plane in the local reference mark $\left[t_1^{\varphi}\left(\xi_1, \xi_2, \xi_3\right) : t_2^{\varphi}\left(\xi_1, \xi_2, \xi_3\right) : n^{\varphi}\left(\xi_1, \xi_2\right)\right]$ obtained by great rotation of the local reference mark on the initial configuration:

$$\left[t_1^{\varphi} : t_2^{\varphi} : n^{\varphi}\right] = \Lambda \left[t_1 : t_2 : n\right]$$

In this reference mark, we can write all the components of the tensor $[s]$ as follows:

$$\begin{bmatrix} \tilde{\sigma}_{t_1^\varphi t_1^\varphi} & \tilde{\sigma}_{t_1^\varphi t_2^\varphi} & \tilde{\sigma}_{t_1^\varphi n^\varphi} \\ \tilde{\sigma}_{t_2^\varphi t_1^\varphi} & \tilde{\sigma}_{t_2^\varphi t_1^\varphi} & \tilde{\sigma}_{t_2^\varphi n^\varphi} \\ \tilde{\sigma}_{t_1^\varphi n^\varphi} & \tilde{\sigma}_{t_1^\varphi n^\varphi} & 0 \end{bmatrix} = \begin{bmatrix} t_1^\varphi.[\sigma]t_1^\varphi & t_1^\varphi.[\sigma]t_2^\varphi & t_1^\varphi.[\sigma]n^\varphi \\ t_2^\varphi.[\sigma]t_1^\varphi & t_2^\varphi.[\sigma]t_2^\varphi & t_2^\varphi.[\sigma]n^\varphi \\ n^\varphi.[\sigma]t_1^\varphi & n^\varphi.[\sigma]t_2^\varphi & n^\varphi.[\sigma]n^\varphi \end{bmatrix}$$

By taking again the relation $[\sigma]\approx\Lambda[S]\Lambda^T$, one obtains:

$$\begin{bmatrix} t_1^\varphi.[\sigma]t_1^\varphi & t_1^\varphi.[\sigma]t_2^\varphi & t_1^\varphi.[\sigma]n^\varphi \\ t_2^\varphi.[\sigma]t_1^\varphi & t_2^\varphi.[\sigma]t_2^\varphi & t_2^\varphi.[\sigma]n^\varphi \\ n^\varphi.[\sigma]t_1^\varphi & n^\varphi.[\sigma]t_2^\varphi & n^\varphi.[\sigma]n^\varphi \end{bmatrix} = \begin{bmatrix} t_1.[S]t_1 & t_1.[S]t_2 & t_1.[S]n \\ t_2.[S]t_1 & t_2.[S]t_2 & t_2.[S]n \\ n.[S]t_1 & n.[S]t_2 & n.[S]n \end{bmatrix}$$

from where the final result:

$$\begin{bmatrix} \tilde{\sigma}_{t_1^\varphi t_1^\varphi} & \tilde{\sigma}_{t_1^\varphi t_2^\varphi} & \tilde{\sigma}_{t_1^\varphi n^\varphi} \\ \tilde{\sigma}_{t_2^\varphi t_1^\varphi} & \tilde{\sigma}_{t_2^\varphi t_1^\varphi} & \tilde{\sigma}_{t_2^\varphi n^\varphi} \\ \tilde{\sigma}_{t_1^\varphi n^\varphi} & \tilde{\sigma}_{t_1^\varphi n^\varphi} & 0 \end{bmatrix} = \begin{bmatrix} \tilde{S}_{t_1 t_1} & \tilde{S}_{t_1 t_2} & \tilde{S}_{t_1 n} \\ \tilde{S}_{t_2 t_2} & \tilde{S}_{t_2 t_2} & \tilde{S}_{t_2 n} \\ \tilde{S}_{t_1 n} & \tilde{S}_{t_2 n} & 0 \end{bmatrix}$$

In so far as the deformation remains small, the components of the tensor of the constraints of Cauchy in the local reference mark attached to the deformed configuration are identical to the components of the tensor of the constraints of Piola-Kirchhoff of second species in the local reference mark attached to the initial configuration.

We take the party in the continuation, to consider only the constraints of Piola-Kirchhoff of second species. We must note that within the framework of a more general constitutive law, one will be able to pass from a stress measurement to another as indicated in the preceding paragraph.

4 Digital discretization of the variational formulation resulting from the principle of virtual work

4.1 Finite elements

The three figures below summarize the finite elements choices concerning the voluminal hulls [R3.07.04]. The selected finite elements are isoparametric quadrangles or triangles. The quadrangle is represented below. One chooses among the elements with quadratic functions of interpolation, the element hétérosis whose displacements are approached by the functions of interpolation of the Sérendip element and rotations by the functions of the element of Lagrange. All the justifications as for these choices are given in [R3.07.04].

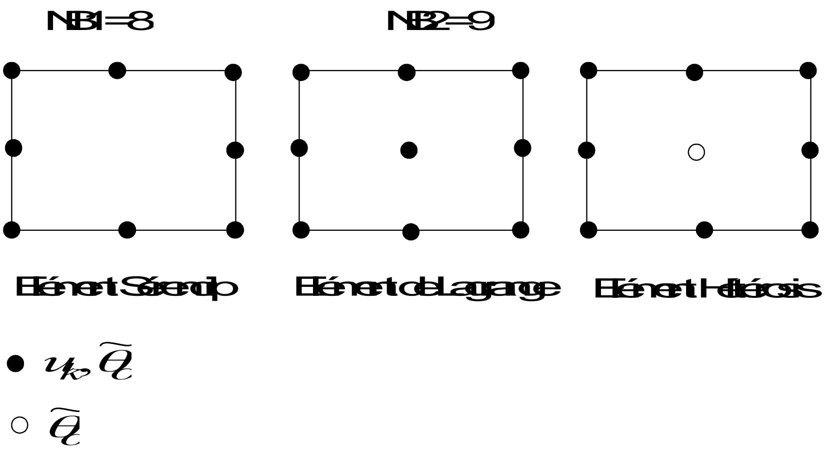

Figure 4.1-a: Families of finite elements for the isoparametric quadrangle

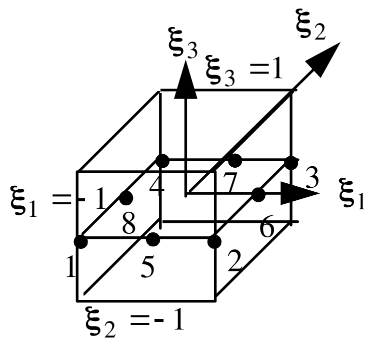

Figure 4.1-b: Voluminal element of reference

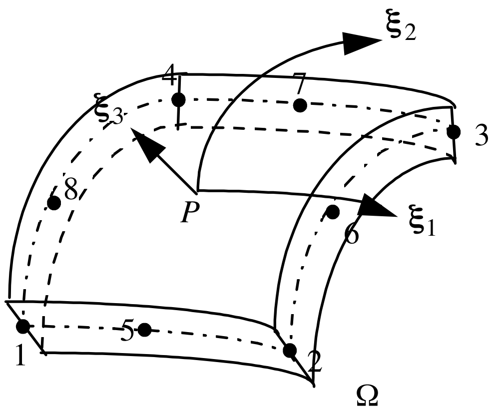

Figure 41-c: Real voluminal element

4.2 Discretization of the field of displacement

With an aim of avoiding the explicit calculation of the curves, which becomes extremely heavy in the case of great rotations, we choose to interpolate the normal on the initial average surface instead of interpolating rotations:

$$n(\xi_1,\xi_2)=\sum_{I=1}^{NB2} N_I^{(2)}(\xi_1,\xi_2)\, n_I$$

where $N_I^{(2)}(\xi_1,\xi_2)$ indicate the function of interpolation to the node I among $NB2$ nodes of Lagrange.

The same interpolations are adopted for the transform of the initial normal:

$$n^{\varphi}(\xi_1,\xi_2)=\sum_{I=1}^{NB2} N_1^{(2)}(\xi_1,\xi_2)\, n_I^{\varphi}$$

The interpolation of the initial position of a point on the average surface of the hull (not P) is given by:

$$x(\xi_1,\xi_2)=\sum_{I=1}^{NB1} N_I^{(1)}(\xi_1,\xi_2)\begin{pmatrix} x\\ y\\ z\end{pmatrix}_I$$

where $N_I^{(1)}(\xi_1,\xi_2)$ indicate the function of interpolation to the node I among $NB1=NB2-1$ nodes of Serendip.

The interpolation of the initial position of an unspecified point of the hull (not Q) can then be written in the form:

$$x(\xi_1,\xi_2,\xi_3)=\sum_{I=1}^{NB1} N_I^{(1)}(\xi_1,\xi_2)\begin{pmatrix}x\\y\\z\end{pmatrix}_I+\xi_3\frac{h}{2}\sum_{I=1}^{NB2} N_I^{(2)}(\xi_1,\xi_2)\begin{pmatrix}n_x\\n_y\\n_z\end{pmatrix}$$

The same interpolations are adopted for the deformed position of an unspecified point of fibre:

$$x^{\varphi}(\xi_1,\xi_2,\xi_3)=\sum_{I=1}^{NB1} N_I^{(1)}(\xi_1,\xi_2)\begin{pmatrix}x^{\varphi}\\y^{\varphi}\\z^{\varphi}\end{pmatrix}_I+\xi_3\frac{h}{2}\sum_{I=1}^{NB2} N_I^{(2)}(\xi_1,\xi_2)\begin{pmatrix}n_x^{\varphi}\\n_y^{\varphi}\\n_z^{\varphi}\end{pmatrix}$$

The interpolations for the positions initial and deformation being the same ones, we can adopt them for the real displacement of an unspecified point of the hull:

$$u(\xi_1,\xi_2,\xi_3)=\sum_{I=1}^{NB1} N_I^{(1)}(\xi_1,\xi_2)\begin{pmatrix}u\\v\\w\end{pmatrix}_I+\xi_3\frac{h}{2}\sum_{I=1}^{NB2} N_I^{(2)}(\xi_1,\xi_2)\left(\begin{pmatrix}n_x^{\varphi}\\n_y^{\varphi}\\n_z^{\varphi}\end{pmatrix}_I-\begin{pmatrix}n_x\\n_y\\n_z\end{pmatrix}_I\right)$$

Thus, the interpolation of virtual displacement becomes:

$$\delta u(\xi_1,\xi_2,\xi_3)=\sum_{I=1}^{NB1} N_I^{(1)}(\xi_1,\xi_2)\begin{pmatrix}\delta u\\\delta v\\\delta w\end{pmatrix}_I-\xi_3\frac{h}{2}\sum_{I=1}^{NB2} N_I^{(2)}(\xi_1,\xi_2)\begin{bmatrix}0&-n_z^{\varphi}&n_y^{\varphi}\\n_z^{\varphi}&0&-n_x^{\varphi}\\-n_y^{\varphi}&n_x^{\varphi}&0\end{bmatrix}_I\begin{pmatrix}\delta w_x\\\delta w_y\\\delta w_z\end{pmatrix}_I$$

In the same way, the interpolation of iterative displacement becomes:

$$\Delta u(\xi_1,\xi_2,\xi_3)=\sum_{I=1}^{NB1} N_I^{(1)}(\xi_1,\xi_2)\begin{pmatrix}\Delta u\\\Delta v\\\Delta w\end{pmatrix}_I-\xi_3\frac{h}{2}\sum_{I=1}^{NB2} N_I^{(2)}(\xi_1,\xi_2)\begin{bmatrix}0&-n_z^{\varphi}&n_y^{\varphi}\\n_z^{\varphi}&0&-n_x^{\varphi}\\-n_y^{\varphi}&n_x^{\varphi}&0\end{bmatrix}_I\begin{pmatrix}\Delta w_x\\\Delta w_y\\\Delta w_z\end{pmatrix}_I$$

Moreover, the interpolation of the iterative variation of virtual displacement is:

$$\Delta\delta u(\xi_1,\xi_2,\xi_3)=\xi_3\frac{h}{2}\sum_{I=1}^{NB2} N_I^{(2)}(\xi_1,\xi_2)\left(\delta w\wedge(\Delta w\wedge n^{\varphi})\right)_I$$

4.3 Discretization of the gradient of displacement

4.3.1 Gradient of total displacement

The vector of the gradient of real displacement can be connected to the isoparametric gradient of real displacement by the following relation:

$$\frac{\partial u}{\partial x}=\tilde{J}^{-1}\frac{\partial u}{\partial \xi}$$

The isoparametric gradient of displacement is organized as follows:

$$\frac{\partial u}{\partial \xi}=\begin{vmatrix} u,_{\xi_1} \\ u,_{\xi_2} \\ u,_{\xi_3} \\ v,_{\xi_1} \\ v,_{\xi_2} \\ v,_{\xi_3} \\ w,_{\xi_1} \\ w,_{\xi_2} \\ w,_{\xi_3} \end{vmatrix}$$

The matrix jacobienne generalized 9×9 $\tilde{J}^{-1}$ can be expressed according to the matrix jacobienne of the isoparametric transformation 3×3 as follows:

$$\tilde{J}^{-1}=\begin{bmatrix} J^{-1} & 0 & 0 \\ 0 & J^{-1} & 0 \\ 0 & 0 & J^{-1} \end{bmatrix}$$

The isoparametric gradient of real displacement can be calculated as follows:

$$\frac{\partial u}{\partial \xi}=\left[\frac{\partial N}{\partial \xi}\right]_1 p^e$$

with the first matrix of the derivative of the functions of form:

$$\left[\ldots \begin{bmatrix} N^{(1)}_{I,\xi_1} & 0 & 0 \\ N^{(1)}_{I,\xi_2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & N^{(1)}_{I,\xi_1} & 0 \\ 0 & N^{(1)}_{I,\xi_2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & N^{(1)}_{I,\xi_1} \\ 0 & 0 & N^{(1)}_{I,x_2} \\ 0 & 0 & 0 \end{bmatrix} \frac{h}{2} \begin{bmatrix} \xi_3 N^{(2)}_{I,\xi_1} & 0 & 0 \\ \xi_3 N^{(2)}_{I,\xi_2} & 0 & 0 \\ N^{(2)}_{I} & 0 & 0 \\ 0 & \xi_3 N^{(2)}_{I,\xi_1} & 0 \\ 0 & \xi_3 N^{(2)}_{I,\xi_2} & 0 \\ 0 & N^{(2)}_{I} & 0 \\ 0 & 0 & \xi_3 N^{(2)}_{I,\xi_1} \\ 0 & 0 & \xi_3 N^{(2)}_{I,\xi_2} \\ 0 & 0 & N^{(2)}_{I} \end{bmatrix}\right] \ldots I=1, NB1$$

$$\frac{h}{2} \begin{bmatrix} \xi_3 N^{(2)}_{NB2,\xi_1} & 0 & 0 \\ \xi_3 N^{(2)}_{NB2,\xi_2} & 0 & 0 \\ N^{(2)}_{NB2} & 0 & 0 \\ 0 & \xi_3 N^{(2)}_{NB2,\xi_1} & 0 \\ 0 & \xi_3 N^{(2)}_{NB2,\xi_2} & 0 \\ 0 & N^{(2)}_{NB2} & 0 \\ 0 & 0 & \xi_3 N^{(2)}_{NB2,x_1} \\ 0 & 0 & \xi_3 N^{(2)}_{NB2,\xi_2} \\ 0 & 0 & N^{(2)}_{NB2} \end{bmatrix}$$

$$\left[\frac{\partial N}{\partial \xi}\right]_1 = ¿[¿][¿][]¿$$

and the vector of “generalized nodal real displacement”:

$$p^e = \begin{pmatrix} \vdots \\ \begin{pmatrix} u \\ v \\ w \\ n_x^{\varphi} - n_x \\ n_y^{\varphi} - n_y \\ n_z^{\varphi} - n_z \end{pmatrix}_I \\ \vdots \\ I = 1, NB\,1 \\ \vdots \\ \begin{pmatrix} n_x^{\varphi} - n_x \\ n_y^{\varphi} - n_y \\ n_z^{\varphi} - n_z \end{pmatrix}_{NB\,2} \end{pmatrix}$$

Finally, one will be able to write the gradient of real displacement in the form:

$$\frac{\partial u}{\partial x} = \tilde{J}^{-1} \left[\frac{\partial N}{\partial \xi} \right]_1 p^e$$

4.3.2 Gradient of virtual displacement

While proceeding in a way similar to the gradient of real displacement, one can connect the two gradients of virtual displacement:

$$\frac{\partial \delta u}{\partial x} = \tilde{J}^{-1} \frac{\partial \delta u}{\partial \xi}$$

The isoparametric gradient of virtual displacement can be calculated as follows:

$$\frac{\partial \delta u}{\partial \xi} = \left[\frac{\partial N}{\partial \xi} \right]_2 \delta u^e$$

with the second matrix of the derivative of the functions of form:

$$\cdots\left[\begin{bmatrix} N^{(1)}_{I,\xi_1} & 0 & 0 \\ N^{(1)}_{I,\xi_2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & N^{(1)}_{I,\xi_1} & 0 \\ 0 & N^{(1)}_{I,\xi_2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & N^{(1)}_{I,\xi_1} \\ 0 & 0 & N^{(1)}_{I,\xi_2} \\ 0 & 0 & 0 \end{bmatrix} \frac{h}{2}\begin{bmatrix} 0 & \xi_3 N^{(2)}_{I,\xi_1} n^{\varphi}_z & -\xi_3 N^{(2)}_{I,\xi_1} n^{\varphi}_y \\ 0 & \xi_3 N^{(2)}_{I,\xi_2} n^{\varphi}_z & -\xi_3 N^{(2)}_{I,\xi_2} n^{\varphi}_y \\ 0 & N^{(2)}_1 n^{\varphi}_z & -N^{(2)}_1 n^{\varphi}_y \\ -\xi_3 N^{(2)}_{I,\xi_1} n^{\varphi}_z & 0 & \xi_3 N^{(2)}_{I,\xi_1} n^{\varphi}_x \\ -\xi_3 N^{(2)}_{I,\xi_2} n^{\varphi}_z & 0 & \xi_3 N^{(2)}_{I,\xi_2} n^{\varphi}_x \\ -N^{(2)}_1 n^{\varphi}_z & 0 & N^{(2)}_1 n^{\varphi}_x \\ \xi_3 N^{(2)}_{I,\xi_1} n^{\varphi}_y & -\xi_3 N^{(2)}_{I,\xi_1} n^{\varphi}_x & 0 \\ \xi_3 N^{(2)}_{I,\xi_2} n^{\varphi}_y & -\xi_3 N^{(2)}_{I,\xi_2} n^{\varphi}_x & 0 \\ N^{(2)}_1 n^{\varphi}_y & -N^{(2)}_1 n^{\varphi}_x & 0 \end{bmatrix}\right]\cdots I=1, NB1$$

$$|\frac{h}{2}\begin{bmatrix} 0 & \xi_3 N^{(2)}_{NB2,\xi_1} n^{\varphi}_z & -\xi_3 N^{(2)}_{NB2,\xi_1} n^{\varphi}_y \\ 0 & \xi_3 N^{(2)}_{NB2,\xi_2} n^{\varphi}_z & -\xi_3 N^{(2)}_{NB2,\xi_2} n^{\varphi}_y \\ 0 & N^{(2)}_{NB2} n^{\varphi}_z & -N^{(2)}_{NB2} n^{\varphi}_y \\ -\xi_3 N^{(2)}_{NB2,\xi_1} n^{\varphi}_z & 0 & \xi_3 N^{(2)}_{NB2,\xi_1} n^{\varphi}_x \\ -\xi_3 N^{(2)}_{NB2,\xi_2} n^{\varphi}_z & 0 & \xi_3 N^{(2)}_{NB2,\xi_2} n^{\varphi}_x \\ -N^{(2)}_{NB2} n^{\varphi}_z & 0 & N^{(2)}_{NB2} n^{\varphi}_x \\ \xi_3 N^{(2)}_{NB2,\xi_1} n^{\varphi}_y & -\xi_3 N^{(2)}_{NB2,\xi_1} n^{\varphi}_x & 0 \\ \xi_3 N^{(2)}_{NB2,\xi_2} n^{\varphi}_y & -\xi_3 N^{(2)}_{NB2,\xi_2} n^{\varphi}_x & 0 \\ N^{(2)}_{NB2} n^{\varphi}_y & -N^{(2)}_{NB2} n^{\varphi}_x & 0 \end{bmatrix}$$

$$\left[\frac{\partial N}{\partial \xi}\right]_2 = ¿[¿][¿][]$$

and the vector of the virtual nodal variables:

$$\delta u^e = \begin{pmatrix} \vdots \\ \begin{pmatrix} \delta u \\ \delta v \\ \delta w \\ \delta w_x \\ \delta w_y \\ \delta w_z \end{pmatrix}_I \\ \vdots \\ I=1, NB1 \\ \vdots \\ \begin{pmatrix} \delta w_x \\ \delta w_y \\ \delta w_z \end{pmatrix}_{NB2} \end{pmatrix}$$

Finally, one will be able to write the gradient of virtual displacement in the form:

$$\frac{\partial \delta u}{\partial x} = \tilde{J}^{-1} \left[\frac{\partial N}{\partial \xi} \right]_2 \delta u^e$$

4.3.3 Gradient of iterative displacement

The approach here is similar to virtual calculation. It is enough to replace δ by Δ :

$$\frac{\partial \Delta u}{\partial x} = \tilde{J}^{-1} \left[\frac{\partial N}{\partial \xi} \right]_2 \Delta u^e$$

with the vector of the iterative nodal variables:

$$\Delta u^e = \begin{pmatrix} \vdots \\ \begin{pmatrix} \Delta u \\ \Delta v \\ \Delta w \\ \Delta w_x \\ \Delta w_y \\ \Delta w_z \end{pmatrix}_I \\ \vdots \\ I=1, NB1 \\ \vdots \\ \begin{pmatrix} \Delta w_x \\ \Delta w_y \\ \Delta w_z \end{pmatrix}_{NB2} \end{pmatrix}$$

4.3.4 Gradient of the iterative variation of virtual displacement

$$\frac{\partial \Delta \delta u}{\partial x} = \tilde{J}^{-1} \left[\frac{\partial N}{\partial \xi} \right]_3 q^e$$

with:

$$\left[\frac{\partial N}{\partial \xi} \right]_3 = \left[\cdots \frac{h}{2} \begin{bmatrix} x_3 N^{(2)}_{I,\xi_1} & 0 & 0 \\ x_3 N^{(2)}_{I,\xi_2} & 0 & 0 \\ N^{(2)}_I & 0 & 0 \\ 0 & \xi_3 N^{(2)}_{I,\xi_1} & 0 \\ 0 & \xi_3 N^{(2)}_{I,\xi_2} & 0 \\ 0 & N^{(2)}_I & 0 \\ 0 & 0 & \xi_3 N^{(2)}_{I,\xi_1} \\ 0 & 0 & \xi_3 N^{(2)}_{I,\xi_2} \\ 0 & 0 & N^{(2)}_I \end{bmatrix} \cdots I = 1, NB2 \right]$$

and the vector of the iterative variation “nodal virtual displacement” generalized:

$$q^e = \begin{pmatrix} . \\ . \\ . \\ \left(\delta w \wedge \left(\Delta w \wedge n^{\varphi} \right) \right)_I \\ . \\ . \\ . \\ I = 1, NB2 \end{pmatrix}$$

4.4 Discretization of the variational formulation resulting from the principle of virtual work

We take again the iterative variation (between two iterations) of internal virtual work:

$$\Delta\,\delta\,\pi_{\text{int}}=\int_{\Omega}\left(\delta\,\tilde{E}\,.\,\Delta\,\tilde{S}+\Delta\,\delta\,\tilde{E}\,.\,\tilde{S}\right)d\,\Omega$$

and iterative variation of the vector of local constraints of Piola-Kirchhoff of second:

$$\Delta\,\tilde{S}=D\Delta\,\tilde{E}$$

Then, the linearized form of the principle of virtual work of the §3 can be written for the finite element described above in the following matric form:

$$\delta\,u^e\,.\,K^{e^T}\Delta\,u^e=\delta\,u^e.\left(\lambda\,f^e-r^e\right)$$

where $\delta\,u^e$ is the nodal vector of the functions tests. One from of deduced the system from equations:

$$K^{e^T}\Delta\,u^e=\lambda\,f^e-r^e$$

where:

K^{e^T}	is the tangent matrix of rigidity
$\Delta\,u^e$	is the elementary vector of the solution of the linearized system of equations (nodal vector between two iterations)
λ	is the external level of load
f^e	is the nodal vector of the external forces (associate with $\lambda=1$)
r^e	is the nodal vector of the internal forces

4.4.1 Vector of the internal forces

It is a vector $\left(6\times Nb1+3\right)\times 1$ entirely expressed in the total reference mark and which must be evaluated with each iteration by the relation:

$$r^e=\int_{\Omega_\xi}\tilde{B}_2^T\,\tilde{S}\,\det J\,d\,\xi_1 d\,\xi_2 d\,\xi_3$$

with the vector of the local constraints Piola-Kirchhoff of second species:

$$\tilde{S}=D\,\tilde{E}$$

It is pointed out that the symbol **|** indicates an object expressed in the local reference mark.

The local deformations of Green-Lagrange are updatings to each iteration:

$$\tilde{E}=\tilde{B}_1\,p^e$$

where the operator of the total deflections (first operator of the deformations) is written:

$$\tilde{B}_1=H\left[Q+\frac{1}{2}\,A\left(\frac{\partial\,u}{\partial\,x}\right)\right]\tilde{J}^{-1}\left[\frac{\partial\,N}{\partial\,\xi}\right]_1$$

with the gradient of real displacement:

$$\frac{\partial u}{\partial x}=\tilde{J}^{-1}\left[\frac{\partial N}{\partial \xi}\right]_1 p^e$$

The operator of the virtual deformations (second operator of the deformations):

$$\tilde{B}_2=H\left(Q+A\left(\frac{\partial u}{\partial x}\right)\right)\tilde{J}^{-1}\left[\frac{\partial N}{\partial \xi}\right]_2$$

is highlighted by the relations:

$$\delta\,\tilde{E}=\tilde{B}_2\,\delta\,u^e$$

$$\Delta\,\tilde{E}=\tilde{B}_2\Delta\,u^e$$

4.4.2 Tangent matrix of rigidity

The tangent matrix of rigidity which is of size $(6\times NB1+3)\times(6\times NB1+3)$ express yourself also entirely in the total reference mark. One must be able to evaluate it with each iteration if it is wanted that the convergence of the method of Newton is quadratic. In a classical way into nonlinear geometrical, it takes the shape:

$$K_T^e=K_m^e+K_g^e$$

where the first part represents the material part:

$$K_m^e=\int_{\Omega_\xi}\tilde{B}_2^T D\,\tilde{B}_2 \det J\,d\,\xi_1\,d\,\xi_2\,d\,\xi_3$$

and the second part represents the geometrical part, it even made up of two parts:

$$K_g^e=K_g^{e\,\text{classique}}+K_g^{e\,\text{non classique}}$$

with the classical part of the geometrical part (see [bib4] p. 141):

$$K_g^{e\,\text{classique}}=\int_{\Omega_\xi}\left[\tilde{J}^{-1}\left[\frac{\partial N}{\partial \xi}\right]_2\right]^T\bar{S}\left[\tilde{J}^{-1}\left[\frac{\partial N}{\partial \xi}\right]_2\right]\det J\,d\,\xi_1\,d\,\xi_2\,d\,\xi_3$$

where $\bar{S}$ the generalized tensor of the constraints expressed in the total reference mark is written:

$$\overset{9\times 9}{\bar{S}}=\begin{bmatrix}\overset{3\times 3}{[S]} & 0 & 0\\ 0 & [S] & 0\\ 0 & 0 & [S]\end{bmatrix}$$

The nonclassical part of the geometrical part represents terms uncoupled from rotation nonsymmetrical which have as a form:

$$\overset{3\times 3}{K_{g\,\text{non classique}}^e}(I\,,I)=[z_I\times][n_I^j\times]$$

where n_I^φ is the transform of the initial normal to the node I and z_I a vector 3×1 with the node $I=1,NB2$ nodal vector $3\times(NB2\times 1)\,Z_I$

$$Z_I=\begin{pmatrix}\vdots\\ z_I\\ \vdots\\ I=1, NB2\end{pmatrix}$$

The nodal vector Z_I is similar to a vector of internal force and its expression is:

$$Z_I=\int_{\Omega_\xi}\tilde{B}_3^T\tilde{S}\det J\,d\,\xi_1\,d\,\xi_2\,d\,\xi_3$$

with the operator of the iterative variation of the virtual deformations (third operator of the deformations):

$$\tilde{B}_3=H\left[Q+A\left(\frac{\partial u}{\partial x}\right)\right]\tilde{J}^{-1}\left[\frac{\partial N}{\partial \xi}\right]_3$$

who is highlighted by the relation:

$$\int_{\Omega_\xi}\tilde{B}_3^T\tilde{S}\det J\,d\,\xi_1\,d\,\xi_2\,d\,\xi_3=\int_\Omega\Delta\,\delta\,\tilde{E}_{\text{non classique}}\,.\,\tilde{S}\,d\,\Omega$$

4.4.3 Diagrams of integration

The integration of the terms of rigidity in the thickness of the hull is identical to the method used in geometrical linear analysis [R3.07.04] for nonlinear behaviors. The initial thickness is divided into N identical layers thicknesses. There are three points of integration per layer. The points of integration are located in higher skin of layer, in the middle of the layer and in lower skin of layer. A layer in the thickness of the hull appears sufficient in most case.

In order to be able stage with the problem of blocking out of membrane of the curved hulls and to solve the problem of blocking in transverse shearing, it is necessary to modify the diagram of integration on average surface. If the technique is completely known in linear analysis, it is it less in geometrical nonlinear analysis.

The procedure is presented in the form of a generalization of the separation of the effects of membrane, inflection and transverse shearing if the deformations of Green-Lagrange are used:

$$\tilde{E}=\begin{pmatrix}\tilde{E}_m\\ \tilde{E}_S\end{pmatrix}$$

where $\tilde{E}_m=\begin{pmatrix}\tilde{E}_{t_1t_1}\\ \tilde{E}_{t_2t_2}\\ \tilde{\gamma}_{t_1t_2}\end{pmatrix}$ represent the deformation of membrane-inflection and $\tilde{E}_S=\begin{pmatrix}\tilde{\gamma}_{t_1n}\\ \tilde{\gamma}_{t_2n}\end{pmatrix}$ transverse shearing strain.

During the digital evaluation of the deformations at the points of normal digital integration of Gauss (9 points for the quadrilateral and 7 points for the triangle), one uses the relation $\tilde{E}=\tilde{B}_1\,p^e$. The modification is introduced on the level of the first operator of the deformations:

$$\tilde{B}_1 = \begin{bmatrix} \tilde{B}_{mf_1^{\text{substitution}}} \\ \tilde{B}_{s_1^{\text{substitution}}} \end{bmatrix}$$

$\tilde{B}_{\mathbf{mf}_1^{\text{substitution}}}$ and $\tilde{B}_{s_1^{\text{substitution}}}$ are the first operators of the deformations of substitution of membrane-inflection and transverse shearing, respectively.

During the calculation of the nodal vector of the internal forces and material part of the tangent matrix of rigidity, the modification is introduced in a way similar to the level of the second operator of the deformations:

$$\tilde{B}_2 = \begin{bmatrix} \tilde{B}_{mf_2^{\text{su}}} \\ \tilde{B}_{s_2^{\text{su}}} \end{bmatrix}$$

4.4.3.1 Operators of deformations of substitution

In what follows the points of digital integration normal and reduced by Gauss, on average surface, are with the number of $\mathrm{NPGSN}=9$ and $\mathrm{NPGSR}=4$, respectively, for the quadrilateral element, and $\mathrm{NPGSN}=7$ and $\mathrm{NPGSR}=3$, respectively, for the triangular element.

Membrane-inflection part

At the point $INTSN$ among $NPGSN$ points of normal digital integration of Gauss of average surface, one will calculate:

$$\tilde{\mathbf{B}}_{\mathrm{mf1}}^{\mathrm{substitution}}(\mathrm{INTSN})=\tilde{\mathbf{B}}_{\mathrm{mf1}}^{\overset{\mathrm{normal}}{\mathrm{complet}}}(INTSN)-\tilde{\mathbf{B}}_{\mathrm{mf1}}^{\overset{\mathrm{normal}}{\mathrm{incomplet}}}(\mathrm{INTSN})+\sum_{\mathrm{INTSR}=1,\mathrm{NPGSR}} N_I(\mathrm{INTSN})\,\tilde{\mathbf{B}}_{\mathrm{mf1}}^{\overset{\mathrm{r\acute{e}duit}}{\mathrm{incomplet}}}(\mathrm{INTSR})$$

where $INTSR$ is a point among $NPGSR$ points of digital integration reduced by Gauss of average surface.

In the expression above, $\tilde{\mathbf{B}}_{\mathrm{mf1}}^{\overset{\mathrm{normal}}{\mathrm{complet}}}$ represent the first three lines of $\tilde{B}_1$ calculated at the points of normal digital integration by considering the complete matrix $\left[\frac{\partial N}{\partial\xi}\right]_1$. The operator $\tilde{\mathbf{B}}_{\mathrm{mf1}}^{\overset{\mathrm{normal}}{\mathrm{incomplet}}}$ represent the first three lines of $\tilde{B}_1$ calculated at the points of normal digital integration by considering a matrix $\left[\frac{\partial N}{\partial\xi}\right]_1$ incomplete where the columns of rotation are cancelled:

$$\left[\frac{\partial N}{\partial\xi}\right]_1^{inc}=\left[\cdots\begin{bmatrix} N_{I,\xi_1}^{(1)} & 0 & 0 & 0 & 0 & 0\\ N_{I,\xi_2}^{(1)} & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & N_{I,\xi_1}^{(1)} & 0 & 0 & 0 & 0\\ 0 & N_{I,\xi_2}^{(1)} & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & N_{I,\xi_1}^{(1)} & 0 & 0 & 0\\ 0 & 0 & N_{I,\xi_2}^{(1)} & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}\cdots I=1,NB1\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\\0&0&0\\0&0&0\\0&0&0\\0&0&0\\0&0&0\\0&0&0\end{bmatrix}\right]$$

$\tilde{\mathbf{B}}_{\mathrm{mf1}}^{\overset{\mathrm{r\acute{e}duit}}{\mathrm{incomplet}}}(INSTR)$ the first three lines represent of $\tilde{B}_1$ calculated at the points of reduced digital integration with the matrix $\left[\frac{\partial N}{\partial\xi}\right]_1^{inc}$ incomplete above definite. They are thus stored to be extrapolated at each point of normal digital integration.

Transverse shearing part

For the transverse shearing part, one will calculate:

$$\tilde{\mathbf{B}}_{S1}^{\mathrm{substitution}}(INTSN)=\sum_{INTSR=1,NPGSR} N_I(INTSN)\,\tilde{\mathbf{B}}_{S1}^{\overset{\mathrm{r\acute{e}duit}}{\mathrm{complet}}}(INTSR)$$

where $\tilde{\mathbf{B}}_{S1}^{\substack{\text{réduit}\\\text{complet}}}(INTSR)$ represent the two last lines of $\tilde{B}_1$ calculated at the points of reduced digital integration with a matrix $\left[\frac{\partial N}{\partial \xi}\right]_1$ complete. They are also stored to be extrapolated at each point of normal digital integration.

4.4.3.2 Substitution of the geometrical part of the tangent matrix of rigidity

The nonclassical part of the tangent matrix of rigidity $K_g^{e\,\text{non classique}}$ is numerically integrated into the points of normal integration of Gauss. No operation of substitution is necessary. For the classical part of the tangent matrix of rigidity, we use substitution:

$$\mathbf{K}_{\mathbf{g}}^{\mathbf{e}\,\substack{\text{substitution}\\\text{classique}}}=\mathbf{K}_{\mathbf{g}}^{\mathbf{e}\,\substack{\text{normale}\\\text{complet}\\\text{membrane}\\\text{flexion}\\\text{classique}}}-\mathbf{K}_{\mathbf{g}}^{\mathbf{e}\,\substack{\text{normale}\\\text{incomplet}\\\text{membrane}\\\text{flexion}\\\text{classique}}}+\mathbf{K}_{\mathbf{g}}^{\mathbf{e}\,\substack{\text{réduit}\\\text{incomplet}\\\text{membrane}\\\text{flexion}\\\text{classique}}}+\mathbf{K}_{\mathbf{g}}^{\mathbf{e}\,\substack{\text{réduit}\\\text{complet}\\\text{cisaillement}\\\text{transverse}\\\text{classique}}}$$

where:

$\mathbf{K}_{\mathbf{g}}^{\mathbf{e}\,\substack{\text{normale}\\\text{complet}\\\text{membrane}\\\text{flexion}\\\text{classique}}}$ is numerically integrated on the points of normal integration with a matrix $\left[\frac{\partial N}{\partial \xi}\right]_2$ supplements, and the local constraints of membrane inflection only;

$\mathbf{K}_{\mathbf{g}}^{\mathbf{e}\,\substack{\text{normale}\\\text{incomplet}\\\text{membrane}\\\text{flexion}\\\text{classique}}}$ is numerically integrated on the points of normal integration with a matrix $\left[\frac{\partial N}{\partial \xi}\right]_2$ incomplete, and local constraints of membrane inflection only;

$\mathbf{K}_{\mathbf{g}}^{\mathbf{e}\,\substack{\text{réduit}\\\text{incomplet}\\\text{membrane}\\\text{flexion}\\\text{classique}}}$ is numerically summoned on the points of integration reduced with a matrix $\left[\frac{\partial N}{\partial \xi}\right]_2$ incomplete, and integrated local constraints of membrane inflection only;

$\mathbf{K}_{\mathbf{g}}^{\mathbf{e}\,\substack{\text{normale}\\\text{complet}\\\text{cisaillement}\\\text{transverse}\\\text{classique}}}$ is numerically summoned on the points of integration reduced with a matrix $\left[\frac{\partial N}{\partial \xi}\right]_2$ supplements, and the integrated local constraints of transverse shearing only;

To be able to calculate the two last tangent matrices in the preceding equation, we carry out the digital integration of the local constraints on $NPGSN$ points of normal integration:

$$\tilde{S}(INTSR)=\int_{\Omega_\xi}\sum_{INTSN=1,NPGSN} N_I(INTSR)\,\tilde{S}(INTSN)\det J\, d\xi_1\, d\xi_2\, d\xi_3$$

This equation contains the terms of weight of the points of Gauss.

5 Rigidity around the transform of the normal

5.1 Singularity of the tangent matrix of rigidity

Although the finite elements objects of the hull are expressed directly in the total reference mark $[e_1 : e_2 : e_3]$ (the degrees of freedom are displacements and rotations in the total reference mark), the tangent matrix of rigidity presents a singularity compared to the component of rotation around the transform of the normal in each node:

$$\left(\underset{\delta w \Delta w}{K^e_T} \; n^{\varphi} = 0 \right)_{I=1,NB2}$$

Contributions $\left(\Delta w \cdot n^{\varphi} \right)_{I=1,NB2}$ are worthless.

In the preceding equation, this matrix represents the rigidity of rotation in the total reference mark. Its structure is full:

$$\left[\underset{\delta w \Delta w}{K^e_T} \right]_I = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{12} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix}_I$$

it is a nonsymmetrical matrix.

This singularity is a direct consequence of the kinematics of hull. It is due to the vector product appearing in linearized displacements (virtual and incremental). Thus displacement between two iterations is given by:

$$\Delta u_Q(\xi_1, \xi_2, \xi_3) = \Delta u_p(\xi_1, \xi_2) + \xi_3 \frac{h}{2} \Delta w(\xi_1, \xi_2) \wedge n^{\varphi}(\xi_1, \xi_2)$$

It is noticed that the contribution $\xi_3 \frac{h}{2} \Delta w(\xi_1, \xi_2) \wedge n^{\varphi}(\xi_1, \xi_2)$ is perpendicular to n^{φ}. One interprets this singularity in the following way: the rotation of an initially normal fibre on the average surface does not lead to an elongation of this one, and consequently does not induce deformation.

5.2 Principle of virtual work for the terms associated with rotation around the normal

We propose to define the full slewing around the transform of the normal in the hull like the projection of the vector of full slewing on the transform of the normal:

$$\theta_{n^{\varphi}} = \Theta . n^{\varphi}$$

It is pointed out that the vector of rotation Θ is an invariant of the matrix of rotation $\Lambda=\exp[\Theta\times]$

$$\Lambda\,\Theta=\Theta$$

The vector of rotation is a clean vector of the matrix of rotation associated with the identity eigenvalue. So the first relation is rewritten:

$$\begin{aligned}\theta_{n^\varphi}&=(\Lambda\,\Theta).(\Lambda\,n)\\&=\Theta.\,n\\&=\theta_n\end{aligned}$$

This relation translates an important result:

The projection of the vector of full slewing on the transform of the normal is equal to the projection of the vector of full slewing on the initial normal

In discrete form, one defines a deformation energy associated with this rotation:

$$\pi_{n^\varphi}=\frac{1}{2}k\sum_{I=1,NB2}\left(\theta^2_{n^\varphi}\right)_I$$

where k is a rigidity of torsion of which the determination of the value will be discussed further. It is supposed that this rigidity remains constant and undergoes neither virtual variation nor incremental variation.

The existence of the potential is supposed:

$$\pi_{n^j}=\frac{1}{2}k\sum_{I=1,NB2}\left((\Theta.\,n^\varphi)(\Theta.\,n^\varphi)\right)_I$$

that one can rewrite in a more elegant form:

$$\pi_{n^\varphi}=\frac{1}{2}k\sum_{I=1,NB2}\left(\Theta[n^\varphi\otimes n^\varphi]\Theta\right)_I$$

By exploiting the property of orthogonality $\Lambda^{-1}=\Lambda^T$ matrix of rotation:

$$n^\varphi\otimes n^\varphi=n^\varphi n^{\varphi^T}=(\Lambda\,n)(\Lambda\,n)^T=\mathbf{nn}^{\boldsymbol{T}}=n\otimes n$$

This property will be exploited in the double linearization of the potential energy.

One rewrites the potential in the form:

$$\pi_{n^\varphi}=\frac{1}{2}k\sum_{I=1,NB2}\left(\Theta[n\otimes n]\Theta\right)_I$$

First linearization of π_{n^φ}, allows to obtain the virtual variation:

$$\begin{aligned}\delta\pi_{n^\varphi}&=\frac{1}{2}k\sum_{I=1,NB2}\left(\delta\Theta[n\otimes n]\Theta+\Theta[n\otimes n]\delta\Theta\right)_I\\&=k\sum_{I=1,NB2}\left(\delta\Theta[n\otimes n]\Theta\right)_I\end{aligned}$$

It is necessary to express this form according to the function tests of rotations retained in the variational form.

$$\delta\Theta = T^{-1}(\Theta)\,\delta w$$

with the form of the matrix reverses of the differential operator of rotation:

$$T^{-1}(\Theta) = \frac{\frac{\theta}{2}}{\tan\frac{\theta}{2}}[I] - \frac{1}{2}[\Theta\times] + \frac{1}{\theta^2}\left[1 - \frac{\frac{\theta}{2}}{\tan\frac{\theta}{2}}\right][\Theta\otimes\Theta]$$

From where the final form of the virtual work which makes it possible to deduce the vector from the interior forces:

$$\delta\pi_{n^\varphi} = k \sum_{I=1,NB2} \left(\delta w\, T^{-T}(\Theta)[n\otimes n]\Theta\right)_I$$

One carries out the second linearization of $\delta\pi_{n^\varphi}$:

$$\Delta\delta\pi_{n^\varphi} = k \sum_{I=1,NB2} \left(\delta w.\left(T^{-T}(\Theta)[n\otimes n]\Delta\Theta + \Delta T^{-T}(\Theta)[n\otimes n]\Theta\right)\right)_I$$

with the particular choice of the degrees of freedom of rotation $\Delta\delta w = 0$, and owing to the fact that the initial normal “does not move” during the iterations $\Delta n = 0$.

The expression of the tangent operator who gives rise to the terms corresponding to the degrees of freedom of rotation around the transform of the normal of tangent matrix is the following one:

$$\Delta\delta\pi_{n^\varphi} = k \sum_{I=1,NB2} \left(\delta w.\left(T^{-T}(\Theta)[n\otimes n]T^{-1}(\Theta)\right)\Delta w\right)_I + k \sum_{I=1,NB2} \left(\delta w.\left(\Delta T^{-T}(\Theta)[n\otimes n]\Theta\right)\right)_I$$

In this relation, the last term is a differential term due to the nonlinear relation between the parameters of rotation. Its linearization is heavy to carry out and its contribution will be neglected in the expression of the tangent operator.

With the property: $n^\varphi\otimes n^\varphi = n\otimes n$, we give the final expression:

$$\Delta\delta\pi_{n^\varphi} \equiv k \sum_{I=1,NB2} \left(\delta w.\left(T^{-T}(\Theta)[n^\varphi\otimes n^\varphi]T^{-1}(\Theta)\right)\Delta w\right)_I$$

The contribution of the operator is noted $T^{-1}(\Theta)\neq[I]$ in the tangent matrix of rigidity.

5.3 Notice

The potential energy brought by the terms of rotation around the transform of the normal is nonworthless even for a rigid rotation movement. This energy does not correspond to a deformation. For this reason it must be nonsignificant. The value by default of COEF_RIGI_DRZ must guarantee that.

5.4 Borderline case of the analysis geometrically linear

In the case of the analysis geometrically linear, the initial configuration and the deformed configuration are confused what leads us to confuse the initial normal n with its transform n^φ :

$$n^{\varphi} \approx n$$

Rotations become small in this case and the operator of great rotation becomes:

$$\Lambda = \exp[\Theta \times] \approx [I] + [\Theta \times]$$

The differential operator of rotations becomes:

$$T(\Theta) \approx [I]$$

and the parameters of rotations become confused:

$$\Delta w \approx \Delta \Theta \quad \text{and} \quad \delta w \approx \delta \Theta$$

All these approximations introduced into virtual work lead to its simplification:

$$\delta \pi_{n^{\varphi}} \approx k \sum_{k=1, NB2} (\delta \Theta [n \otimes n] \Theta)_I$$

The same approximations introduced into the differential part of virtual work also lead to its simplification:

$$\Delta \delta \pi_{n^{\varphi}} \equiv k \sum_{k=1, NB2} \left(\delta \Theta [n \otimes n] \Delta \Theta \right)_I$$

The two last equations are those of the analysis geometrically linear. They show that the contributions in the vector of the interior forces and the tangent matrix of rigidity recover the borderline case well of [R3.07.04].

5.5 Determination of the coefficient K

The coefficient k is calculated with each iteration of each step of time. With each iteration of Newton of each step of time, one builds with $NB2$ nodes the matrix of passage

$$\bar{\Lambda}_I = \left[t_1^{\varphi} : t_2^{\varphi} : n^{\varphi} \right]_I ;\ I = 1, NB2$$

who allows to pass from the vector δw_I , vector of iterative rotation to the node expressed in the total reference mark $\left[e_1 : e_2 : e_3 \right]$ with the vector $\delta \tilde{w}_I$ expressed in local reference mark $\left[t_1^{\varphi} : t_2^{\varphi} : n^{\varphi} \right]_I$:

$$\delta \tilde{w}_I = \bar{\Lambda}_I \delta w_I ;\ I = 1, NB2$$

One can build in each node, the matrix $\underset{\delta w \Delta w}{\left[K_T^e \right]_I}$ of size 3×3

$$\underset{\tilde{\delta} w \tilde{\Delta} w}{[\ K_T^e\ ]_I} = \bar{\Lambda}_I{}^T \underset{\delta w \Delta w}{[\ K_T^e\ ]_I} \bar{\Lambda}_I$$

This matrix represents the rigidity of rotation in the local reference mark. Its structure, nonsymmetrical, is:

$$\underset{\tilde{\delta} w \tilde{\Delta} w}{[\ K_T^e\ ]_I} = \begin{bmatrix} k_{t_1^{\varphi} t_1^{\varphi}} & k_{t_1^{\varphi} t_2^{\varphi}} & 0 \\ k_{t_2^{\varphi} t_1^{\varphi}} & k_{t_2^{\varphi} t_2^{\varphi}} & 0 \\ k_{n^{\varphi} t_1^{\varphi}} & k_{n^{\varphi} t_2^{\varphi}} & 0 \end{bmatrix}_I$$

The coefficient k is then calculated according to the relation:

$$k = COEF\_RIGI\_DRZ \times KMIN$$

where $COEF\_RIGI\_DRZ$ is a coefficient introduced like a mechanical characteristic of hull by the user. In classical linear analysis of the hulls or plates, this coefficient is selected enters 0.001 and 0.000001 . By default it is worth 0.00001 . In the case of great rotations calculated with great steps of load, one advises to use the value 0.001 .

$KMIN$ is the minimum of the nonworthless terms of rotation on the diagonal of $\tilde{K}_T^e$.

$$KMIN = \underset{I=1, NB2}{MIN} \left(k_{t_1^\varphi t_1^\varphi}, k_{t_2^\varphi t_2^\varphi} \right)_I$$

Note:

It would be undoubtedly more rigorous to calculate k with the first iteration of the first step of time and to store this value like invariant information during the iterations and the steps of following times. The experiment shows that this way of proceeding is often not optimal insofar as the values of the coefficients of the matrices of rigidity can evolve in an important way during a calculation in great displacements. The value of k determined initially can then become too small and the matrix rigidity to end up being singular.

6 Linear buckling

Linear buckling is presented in the form of a typical case of the geometrical nonlinear problem. It is based on the assumption of a linear dependence of the fields of displacements, strains and stresses compared to the level of load, and this, before the critical load is reached.

In a total Lagrangian formulation, one recalls that linearized balance can be written in the variational form:

$$\Delta\delta\pi_{\mathrm{int}}-\Delta\delta\pi_{ext}=\delta\pi_{ext}-\delta\pi_{\mathrm{int}}$$

maybe in matric form after discretization:

$$\delta u\cdot K_T\Delta u=\delta u\cdot(\lambda f-r)$$

where dependence of the tangent matrix of rigidity K_T is nonlinear compared to the vector of generalized nodal displacement $p=\underset{e=1,Nel}{U} p^e$.

If we suppose the linear dependence of displacement compared to the level of load, one can write:

$$u=\lambda u_0$$

where u_0 is the solution obtained following a linear analysis for $\lambda=1$ by:

$$K_0u_0=f$$

where K_0 is the tangent matrix of initial rigidity. One can then develop the tangent matrix of rigidity in a linear way compared to the level of load:

$$\underset{e=1,Nel}{U} K_T^e=K_0^e+\lambda\left(K_u^e+K_\sigma^e\right)+\dots$$

where K_u^e is the matrix of initial displacements depending on p_0^e , traditionally neglected in *Code_Aster,* and K_σ^e the matrix of the initial constraints depending on the total tensor of the constraints of Piola-Kirchhoff of second species $[S_0]$ and of the local vector $\tilde{S}_0$. These constraints are voluntarily confused with the constraints of Cauchy. They are obtained by a postprocessing of the linear analysis.

For the rotation part of p^e , the assumption of linearity of the deformations according to the level of load results in the equality of:

$$n_I^\varphi=n_I$$

who leads us to confuse the initial normals n_I with their transforms n_I^φ .

The matrix of the initial constraints K_σ^e represent the constant part in λ geometrical part of the tangent matrix of rigidity. It is evaluated at the point p_0^e and $\lambda=1$ with a transform of the normal replaced by the initial normal:

$$K_\sigma^e=K_\sigma^{e\,\text{classique}}+K_\sigma^{e\,\text{non classique}}$$

with the classical part of the geometrical part (see [bib4] volume 1 p. 141):

$$K_{\sigma}^{e_{classique}}=\int_{\Omega_{\zeta}}\left[\tilde{J}^{-1}\left[\frac{\partial N}{\partial \sigma}\right]_2\right]^T \bar{S}\left[\tilde{J}^{-1}\left[\frac{\partial N}{\partial \xi}\right]_2\right]\det J\, d\xi_1\, d\xi_2\, d\xi_3$$

where the second matrix of the derivative of the functions of form becomes:

$$\cdots\left[\begin{bmatrix} N^{(1)}_{I,\xi_1} & 0 & 0 \\ N^{(1)}_{I,\xi_2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & N^{(1)}_{I,\xi_1} & 0 \\ 0 & N^{(1)}_{I,\xi_2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & N^{(1)}_{I,\xi_1} \\ 0 & 0 & N^{(1)}_{I,\xi_2} \\ 0 & 0 & 0 \end{bmatrix} \frac{h}{2}\begin{bmatrix} 0 & \xi_3 N^{(2)}_{I,\xi_1} n_z & -\xi_3 N^{(2)}_{I,\xi_1} n_y \\ 0 & \xi_3 N^{(2)}_{I,\xi_2} n_z & -\xi_3 N^{(2)}_{I,\xi_2} n_y \\ 0 & N^{(2)}_I n_z & -N^{(2)}_I n_y \\ -\xi_3 N^{(2)}_{I,\xi_1} n_z & 0 & \xi_3 N^{(2)}_{I,\xi_1} n_x \\ -\xi_3 N^{(2)}_{I,\xi_2} n_z & 0 & \xi_3 N^{(2)}_{I,\xi_2} n_x \\ -N^{(2)}_I n_z & 0 & N^{(2)}_I n_x \\ \xi_3 N^{(2)}_{I,\xi_1} n_y & -\xi_3 N^{(2)}_{I,\xi_1} n_x & 0 \\ \xi_3 N^{(2)}_{I,\xi_2} n_y & -\xi_3 N^{(2)}_{I,\xi_2} n_x & 0 \\ N^{(2)}_I n_y & -N^{(2)}_I n_x & 0 \end{bmatrix}\right]\cdots I=1,NB1$$

$$|\frac{h}{2}\begin{bmatrix} 0 & \xi_3 N^{(2)}_{NB2,\xi_1} n_z & -\xi_3 N^{(2)}_{NB2,\xi_1} n_y \\ 0 & \xi_3 N^{(2)}_{NB2,\xi_2} n_z & -\xi_3 N^{(2)}_{NB2,\xi_2} n_y \\ 0 & N^{(2)}_{NB2} n_z & -N^{(2)}_1 n_y \\ -\xi_3 N^{(2)}_{NB2,\xi_1} n_z & 0 & \xi_3 N^{(2)}_{NB2,\xi_1} n_x \\ -\xi_3 N^{(2)}_{NB2,\xi_2} n_z & 0 & \xi_3 N^{(2)}_{NB2,\xi_2} n_x \\ -N^{(2)}_{NB2} n_z & 0 & N^{(2)}_{NB2} n_x \\ \xi_3 N^{(2)}_{NB2,\xi_1} n_y & -\xi_3 N^{(2)}_{NB2,\xi_1} n_x & 0 \\ \xi_3 N^{(2)}_{NB2,\xi_2} n_y & -\xi_3 N^{(2)}_{NB2,\xi_2} n_x & 0 \\ N^{(2)}_{NB2} n_y & -N^{(2)}_{NB2} n_x & 0 \end{bmatrix}$$

$$\left[\frac{\partial N}{\partial \xi}\right]_2 = ¿[¿][¿][\,]$$

and the generalized tensor of the total constraints:

$$\overset{9\times 9}{\bar{S}}=\begin{bmatrix} [\overset{3\times 3}{S}] & 0 & 0 \\ 0 & [S] & 0 \\ 0 & 0 & [S] \end{bmatrix}$$

The nonclassical part of the geometrical part represents the terms uncoupled from rotation, nonsymmetrical. Since the current algorithm of resolution of the problem to the eigenvalues [R5.06.01] $\left[K_0+\lambda\left(K_u+K_\sigma\right)\right]F=0$ (λ being the level of critical load) considers only symmetrical matrices, one makes symmetrical, while dividing by two the sum with its transposed, the matrix:

$$K_\sigma^{e\,\text{non classique}}(I,I)=\frac{1}{2}\left(\left[z_I\times\right]\left[n_I\times\right]\right)$$

where n_I is the normal with the node I and z_I a vector 3×3 with the node $I=1,NB2$ nodal vector $3\times(3\times NB2)Z_I$:

$$Z_I=\begin{pmatrix} . \\ . \\ . \\ z_I \\ . \\ . \\ . \\ I=1,NB2 \end{pmatrix}$$

The nodal vector Z_I is similar to a vector of internal force and its expression is:

$$Z_I=\int_{\Omega_z}\tilde{B}_3^T\,\tilde{S}\det J\,d\xi_1\,d\xi_2\,d\xi_3$$

with the operator of the iterative variation of the virtual deformations (third operator of the deformations):

$$\tilde{B}_3=\mathbf{HQ}\ \tilde{\boldsymbol{J}}^{-1}\left[\frac{\partial N}{\partial\xi}\right]_3$$

who is highlighted by the relation:

$$\int_{\Omega_z}\tilde{B}_3^T\,\tilde{S}\det J\,d\xi_1\,d\xi_2\,d\xi_3=\int_\Omega\Delta\,\delta\,\tilde{E}_{\text{non classique}}\cdot\tilde{S}\,d\Omega$$

Note:

For digital integration in the thickness of the various terms of rigidity, we just like retain a diagram of Gauss at two points in elasticity for the geometrical linear hulls [R3.07.04].

7 Establishment of the elements of hull in *Code_Aster*

7.1 Description

These elements (of names MEC3QU9H and MEC3TR7H) are pressed on meshs QUAD9 and TRIA7 who are of geometry curves [R3.07.04].

7.2 Use

These elements are used in the following way:

AFFE_MODELE (MODELING: 'COQUE_3D'…) for the triangle and the quadrangle.

One calls on the routine INI080 for standard calculations of digital integration.

```
AFFE_CARA_ELEM ( HULL: (THICKNESS: 'EP'
                  ANGL_REP: (' '' ')
                 COEF_RIGI_DRZ: 'CTOR')
```

to introduce the characteristics of hull.

ELAS: (E: NAKED Young: ν ALPHA: α . RHO: ρ .)
For a homogeneous isotropic thermoelastic behavior in the thickness one uses the keyword ELAS in DEFI_MATERIAU where the coefficients are defined E, Young modulus, ν, Poisson's ratio, α, thermal dilation coefficient and ρ density.

AFFE_CHAR_MECA (DDL_IMPO: (
DX:. DY:. DZ:. DRX:. DRY MARTINI:. DRZ:. degree of freedom of plate in the total reference mark.

FORCE_COQUE: (FX:. FY:. FZ:. MX:. MY:. MZ:.)
They is the surface efforts on elements of plate. These efforts can be given in the total reference mark or the reference mark user defined by ANGL_REP.

FORCE_NODALE: (FX:. FY:. FZ:. MX:. MY:. MZ:.)
They is the efforts of hull in the total reference mark.

7.3 Calculation in geometrical nonlinear "elasticity"

Calculation imposes the following instructions user:

```
BEHAVIOR: (RELATION     : 'elas'
                           COQUE_NCOU: 1 (or more)
                           DEFORMATION    : 'GROT_GDEP')
```

Digital integration in the thickness is based on an approach multi-layer with 3 points of integration per layer. It is the approach currently used in nonlinear material [R3.07.04]. Options of postprocessing SIGM_ELNO constraints and VARI_ELNO internal variables (here worthless) are by default activated with the convergence of each filed step.

7.4 Establishment

Options FULL_MECA, RIGI_MECA_TANG, and RAPH_MECA are already active in the elementary catalogues mec3qu9h.cata and mec3tr7h.cata for material non-linearity. They direct calculation towards /fort/te0414.f, then towards /fort/vdxnlr.f to calculate and store, inter alia, the tangent matrix of symmetrical rigidity in the address corresponding to the mode MMATUUR PMATUUR.

For the geometrical nonlinear analysis, the calculation of the tangent matrix of rigidity is directed towards the new routine VDGNLR. This matrix is not symmetrical and must be stored in the address corresponding to the mode MMATUNS PMATUNS.

One defines the two local modes at the same time symmetrical and nonsymmetrical, at exit of the elementary catalogues. The tangent matrix of nonsymmetrical rigidity into nonlinear geometrical is stored with the address reserved for a nonsymmetrical matrix. On the other hand, if it is about nonmaterial linearity in small deformations, all the tangent matrix of rigidity is stored with the address corresponding to the nonsymmetrical mode. The lower triangular part is duplicated starting from the higher triangular part. Material nonlinear calculation in small deformations thus proceeds also in nonsymmetrical.

7.4.1 Modification of TE0414

Calculation is directed towards /fort/vdgnlr.f when the type of behavior BEHAVIOR is checked, i.e. when the problem is nonlinear geometrical.

7.4.2 Addition of a routine VDGNLR

According to the option, the routine /fort/vdgnlr.f has as a role of :

Options: FULL_MECA and RAPH_MECA :

To calculate the 6 components of the state of the local constraints of Cauchy (confused with the state of the constraints of Piola-Kirchhoff of second species) at the points of normal digital integration and the nodal vector of the internal forces. They are stored in the local modes ECONTPG PCONTPR and MVECTUR PVECTUR respectively. In additional remark, one will note that SIEF_ELGA/SIGM_ELGA after the resolution is of type PK2 while SIEF_ELNO/SIGM_ELNO in postprocessing is of type Cauchy.

Options: FULL_MECA and RIGI_MECA_TANG :

To calculate and store the tangent matrix of nonsymmetrical rigidity in the mode MMATUNS PMATUNS.

7.5 Calculation in linear buckling

The option RIGI_MECA_GE, inactive until now, is activated in the elementary catalogues mec3qu9h.cata and mec3tr7h.cata.

The new one TE0402 is dedicated to the calculation of the matrix of geometrical rigidity due to the initial constraints for the buckling of Euler. One recovers the states plans of the local constraints of Cauchy (component S_{nn} worthless) at the points of normal digital integration of Gauss. These states of stresses must be obtained by postprocessing with the option of calculation SIEF_ELGA following a linear analysis (mode ECONTPG PCONTRR).

In analysis of buckling of Euler, constraints $[\sigma]$ of Cauchy can be confused with the constraints $[S]$ of Piola-Kirchhoff of second species. Therefore we will keep the notation $[S]$.

The matrix of rigidity of the initial constraints can be broken up into a symmetrical classical part and a nonsymmetrical nonclassical part. First is calculated according to the total tensor of the constraints 3×3 , contrary to the second which, it, is calculated according to the vector of the local constraints 5×1 .

Since the current algorithm of resolution of the problem to the eigenvalues [R5.06.01] considers only symmetrical matrices, we force the symmetry of the nonclassical part of the geometrical matrix before storing the higher triangular part of all the matrix in the mode MMATUUR PMATUUR.

8 Conclusion

The formulation that we have just described applies to the curved mean structural analyses in great displacements, whose thickness report over characteristic length is lower than $1/10$. It comes in direct object from the finite elements described in the preceding reference material [R3.07.04] and used within the framework from small displacements and deformations. They are pressed on the same meshs quadrangle and triangle.

Their formulation rests on an approach of continuous medium 3D into which one introduces a kinematics of hull of the Hencky-Mindlin-Naghdi type, in plane constraints, in the weak formulation of balance. The measurement of the deformations retained is that of Green-Lagrange, énergétiquement combined with the constraints of Piola-Kirchhoff of second species. The formulation of balance is thus Lagrangian total. The transverse distortion is treated same manner as in [R3.07.04]. Rotations must remain lower than 2π because of the choice of update of great rotations established in *Code_Aster* for which there is not bijection between the vector of full slewing and the orthogonal matrix of rotation.

Because of singularity of the tangent matrix of rigidity compared to the component of rotation around the transform of the normal, one defines a fictitious deformation energy associated with this rotation. With this rotation, one associates a rigidity of constant torsion. The interior efforts associated with this potential energy are taken into account. This potential energy, nonworthless, does not correspond to a physical deformation. One thus needs that it remains negligible, which the user can control by imposing a value of COEF_RIGI_DRZ being worth 10^{-3} with 10^{-5} .

For the postprocessing of the constraints, one limits oneself to the framework of the small deformations. One then could prove the identity between the tensor of the constraints of Piola-Kirchhoff observed in the initial geometry and the tensor of the constraints of Cauchy in the deformed geometry. Moreover, the state of the constraints being plan for the tensor of Piola-Kirchhoff, one finds this property for the state of stresses of Cauchy. It should be noted that in contexts plus generals, this property is not preserved.

Linear buckling is treated like a typical case of the geometrical nonlinear problem. It rests on the assumption of a linear dependence of the fields of displacements, strains and stresses compared to the level of load, before the critical load. It results from it that one can linearly develop the tangent matrix of rigidity compared to the level of load. One then finds the part geometrical of the matrices of nonlinear the geometrical general obtained by identifying the deformation of the normal on the average surface and the initial normal, which is coherent with the linearity of the deformations according to level of load.

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10 Description of the versions of the document

Version Aster	Author (S) Organization (S)	Description of the modifications
5	P.Massin, Mr. AL MIKDAD EDF-R&D/MMN	Initial text
7.4	X. Desroches	Update: minor modifications

Annexe 1: Flow chart of calculation in linear buckling

Local reference marks with $NB2$ nodes $\left[t_1 : t_2 : n\right]_I$

Buckle on the points of normal digital integration of Gauss

- recovery of the vector of the local constraints $\tilde{S}=\begin{pmatrix}\tilde{S}_{t_1 t_1}\\ \tilde{S}_{t_2 t_2}\\ \tilde{\tau}_{t_1 t_2}\\ \\ \tilde{\tau}_{t_1 n}\\ \tilde{\tau}_{t_2 n}\end{pmatrix}=\begin{pmatrix}\tilde{S}_{t_1 t_1}\\ \tilde{S}_{t_2 t_2}\\ \sqrt{2}\,\tilde{S}_{t_1 t_2}\\ \\ \sqrt{2}\,\tilde{S}_{t_1 n}\\ \sqrt{2}\,\tilde{S}_{t_2 n}\end{pmatrix}$

  starting from the 6 components tensors stored in the mode PCONTRR $\begin{pmatrix}\tilde{S}_{t_1 t_1}\\ \tilde{S}_{t_2 t_2}\\ 0\\ \tilde{S}_{t_1 t_2}\\ \tilde{S}_{t_1 n}\\ \tilde{S}_{t_2 n}\end{pmatrix}$

- formation of the symmetrical tensor 3×3 local constraints $[\tilde{S}]$

- construction of the matrix of transformation $\mathbf{P}(\xi_1, \xi_2, \xi_3)=\begin{bmatrix}\mathbf{t_1^T}(\xi_1, \xi_2, \xi_3)\\ \mathbf{t_2^T}(\xi_1, \xi_2, \xi_3)\\ \mathbf{t_3^T}(\xi_1, \xi_2, \xi_3)\end{bmatrix}$ where

  $\mathbf{t_3}(\xi_1, \xi_2, \xi_3)=n_1(\xi_2)$

- calculation of the symmetrical tensor 3×3 total constraints $[S]=P^T[\tilde{S}]P$

- for the nonclassical term, calculation of $\mathbf{HQ}=\begin{bmatrix}\overset{3\times9}{[HSFM]}\\ \hline \overset{2\times9}{[HSS]}\end{bmatrix}$

$$
\mathbf{HQ}=
\begin{bmatrix}
(t_1(1))^2 & (t_1(2))^2 & (t_1(3))^2 & t_1(1)t_1(2) & t_1(2)t_1(3) & t_1(3)t_1(1) \\
(t_2(1))^2 & (t_2(2))^2 & (t_2(3))^2 & t_2(1)t_2(2) & t_2(2)t_2(3) & t_2(3)t_2(1) \\
(t_3(1))^2 & (t_3(2))^2 & (t_3(3))^2 & t_3(1)t_3(2) & t_3(2)t_3(3) & t_3(3)t_3(1) \\
2t_2(1)t_3(1) & 2t_2(2)t_3(2) & 2t_2(3)t_3(3) & t_2(1)t_3(2)+t_3(1)t_2(2) & t_2(2)t_3(3)+t_3(2)t_2(3) & t_2(3)t_3(1)+t_3(3)t_3(1 \\
2t_3(1)t_1(1) & 2t_3(2)t_1(2) & 2t_3(3)t_1(3) & t_3(1)t_1(2)+t_1(1)t_3(2) & t_3(2)t_1(3)+t_1(2)t_3(3) & t_3(3)t_1(1)+t_1(3)t_3(1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0
\end{bmatrix}
$$

- calculation of the opposite matrix jacobienne J^{-1} and of the determinant $\det J$
- calculation of $\tilde{J}^{-1}\left[\dfrac{\partial N}{\partial \xi}\right]_3$ with:

$$
\tilde{J}^{-1}=\begin{bmatrix} J^{-1} & 0 & 0 \\ 0 & J^{-1} & 0 \\ 0 & 0 & J^{-1} \end{bmatrix};\left[\frac{\partial N}{\partial \xi}\right]_3=\left[\cdots\frac{h}{2}\begin{bmatrix}
\xi_3 N^{(2)}_{I,\xi_1} & 0 & 0 \\
\xi_3 N^{(2)}_{I,\xi_2} & 0 & 0 \\
N^{(2)}_I & 0 & 0 \\
0 & \xi_3 N^{(2)}_{I,\xi_1} & 0 \\
0 & \xi_3 N^{(2)}_{I,\xi_2} & 0 \\
0 & N^{(2)}_I & 0 \\
0 & 0 & \xi_3 N^{(2)}_{I,\xi_1} \\
0 & 0 & \xi_3 N^{(2)}_{I,\xi_2} \\
0 & 0 & N^{(2)}_I
\end{bmatrix}\cdots I=1,NB2\right]
$$

- calculation of the third operator of the deformations $\tilde{B}_3=\boldsymbol{HQ}\,\tilde{\boldsymbol{J}}^{-1}\left[\dfrac{\partial N}{\partial \xi}\right]_3$
- calculation and digital integration $Z_I=\int_{\Omega_\zeta}\tilde{B}_3^T\,\tilde{S}\det J\,d\xi_1\,d\xi_2\,d\xi_3$
- calculation of the generalized tensor of the constraints **total**

$$
\overset{9\times 9}{\tilde{S}}=\begin{bmatrix} \overset{3\times 3}{[S]} & 0 & 0 \\ 0 & [S] & 0 \\ 0 & 0 & [S] \end{bmatrix}
$$

- calculation of $\tilde{J}^{-1}\left[\dfrac{\partial N}{\partial \xi}\right]_2$ avec :

$$\cdots\left[\begin{bmatrix} N^{(1)}_{I,\xi_1} & 0 & 0 \\ N^{(1)}_{I,\xi_2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & N^{(1)}_{I,\xi_1} & 0 \\ 0 & N^{(1)}_{I,\xi_2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & N^{(1)}_{I,\xi_1} \\ 0 & 0 & N^{(1)}_{I,\xi_2} \\ 0 & 0 & 0 \end{bmatrix} \frac{h}{2}\begin{bmatrix} 0 & \xi_3 N^{(2)}_{I,\xi_1} n_z & -\xi_3 N^{(2)}_{I,\xi_1} n_y \\ 0 & \xi_3 N^{(2)}_{I,\xi_2} n_z & -\xi_3 N^{(2)}_{I,\xi_2} n_y \\ 0 & N^{(2)}_1 n_z & -N^{(2)}_1 n_y \\ -x_3 N^{(2)}_{I,\xi_1} n_z & 0 & \xi_3 N^{(2)}_{I,\xi_1} n_x \\ -\xi_3 N^{(2)}_{I,\xi_2} n_z & 0 & \xi_3 N^{(2)}_{I,\xi_2} n_x \\ -N^{(2)}_1 n_z & 0 & N^{(2)}_1 n_x \\ \xi_3 N^{(2)}_{I,\xi_1} n_y & -\xi_3 N^{(2)}_{I,\xi_1} n_x & 0 \\ \xi_3 N^{(2)}_{I,\xi_2} n_y & -\xi_3 N^{(2)}_{I,\xi_2} n_x & 0 \\ N^{(2)}_1 n_y & -N^{(2)}_1 n_x & 0 \end{bmatrix}\right]\cdots I=1,NB1$$

$$|\frac{h}{2}\begin{bmatrix} 0 & \xi_3 N^{(2)}_{NB2,\xi_1} n_z & -\xi_3 N^{(2)}_{NB2,\xi_1} n_y \\ 0 & \xi_3 N^{(2)}_{NB2,\xi_2} n_z & -\xi_3 N^{(2)}_{NB2,\xi_2} n_y \\ 0 & N^{(2)}_{NB2} n_z & -N^{(2)}_1 n_y \\ -\xi_3 N^{(2)}_{NB2,\xi_1} n_z & 0 & \xi_3 N^{(2)}_{NB2,\xi_1} n_x \\ -\xi_3 N^{(2)}_{NB2,\xi_2} n_z & 0 & \xi_3 N^{(2)}_{NB2,\xi_2} n_x \\ -N^{(2)}_{NB2} n_z & 0 & N^{(2)}_{NB2} n_x \\ \xi_3 N^{(2)}_{NB2,\xi_1} n_y & -\xi_3 N^{(2)}_{NB2,\xi_1} n_x & 0 \\ \xi_3 N^{(2)}_{NB2,\xi_2} n_y & -\xi_3 N^{(2)}_{NB2,\xi_2} n_x & 0 \\ N^{(2)}_{NB2} n_y & -N^{(2)}_{NB2} n_x & 0 \end{bmatrix}$$

$$\left[\frac{\partial N}{\partial \xi}\right]_2 = ¿[¿][¿][\,]$$

- calculation and digital integration of the matrix of geometrical rigidity classical

$$K^{e\,\text{classique}}_{\sigma} = \int_{\Omega_\zeta}\left[\tilde{J}^{-1}\left[\frac{\partial N}{\partial \xi}\right]_2\right]^T \bar{S}\left[\tilde{J}^{-1}\left[\frac{\partial N}{\partial \xi}\right]_2\right]\det J\, d\xi_1\, d\xi_2\, d\xi_3$$

Fine buckles on the points of integration

Buckle on all the nodes of Lagrange with distinction of the super node

- calculation of $[z_I \times]$ knowing that $Z_I = \begin{Bmatrix} . \\ . \\ . \\ z_I \\ . \\ . \\ . \\ I=1, NB2 \end{Bmatrix}$
- calculation of the vector normal n_I and of its antisymmetric matrix $[n_I \times]$
- calculation of the nonclassical matrix of geometrical rigidity

$$\overset{3\times 3}{K^e_{\sigma\,\text{non classique}}}(I, I) = [z_I \times][n_I \times] \; I=1, NB2$$

- addition of $\overset{3\times 3}{K^e_{\sigma\,\text{non classique}}}(I, I)$ with distinction of the super node

Fine buckles on the nodes

Storage of the higher triangular part of K^e_σ

END

Annexe 2: Flow chart of geometrical nonlinear calculation

Local reference marks with the NB2 nodes $[t_1 : t_2 : n]_I$

Beginning Buckle JN on the NB2 nodes

IF JN $\leq$ **NB1**

- recovery of the vector of total displacement already updated by MAJOUR:

  $u_I(ii) = ZR(IDEPLP + IDEPLM - 1 + 6\times(JN-1) + ii)\,; ii=1,3\,; (JN=1, NB1)$
- recovery of the vector of full slewing already updated by MAJOUR

  $\theta_I(ii) = ZR(IDEPLP - 1 + 6\times(JN-1) + ii + 3)\,; ii=1,3\,; (JN=1, NB1)$

  ELSE JN
- recovery of the vector of full slewing

  $\theta_I(ii) = ZR(IDEPLP - 1 + 6\times NB1 + ii)\,; ii=1,3\,; (JN=NB2)$

END IF JN

- calculation of the matrix of rotation $\Lambda_I = \exp[\theta_I \times]$
- transform of the normal $n_I^{\varphi} = \Lambda_I n_I$

End Buckle on $NB2$ nodes

calculation of $$p^e=\begin{pmatrix} \vdots \\ \begin{pmatrix} u \\ v \\ w \\ n_x^{\varphi}-n_x \\ n_y^{\varphi}-n_y \\ n_z^{\varphi}-n_z \end{pmatrix}_I \\ \vdots \\ I=1, NB1 \\ \vdots \\ \begin{pmatrix} n_x^{\varphi}-n_x \\ n_y^{\varphi}-n_y \\ n_z^{\varphi}-n_z \end{pmatrix}_{NB2} \end{pmatrix}$$

Beginning Buckle INTSR on the points of normal reduced integration of Gauss

- construction of part of the operators $\tilde{B}_1, \tilde{B}_2$
  with the J = 1, INTSR points of integrations to be able to extrapolate them

End Buckle INTSR on the points of normal reduced integration of Gauss

Beginning Buckle INTSN on the points of normal digital integration of Gauss

- construction of the matrix of transformation:
  $$\mathbf{P}(\xi_1,\xi_2,\xi_3)=\begin{bmatrix} \mathbf{t}_1^{\mathbf{T}}(\xi_1,\xi_2,\xi_3) \\ \mathbf{t}_2^{\mathbf{T}}(\xi_1,\xi_2,\xi_3) \\ \mathbf{t}_3^{\mathbf{T}}(\xi_1,\xi_2,\xi_3) \end{bmatrix}$$
  where $t_3(\xi_1,\xi_2,\xi_3)=n(\xi_1,\xi_2)$
- calculation of the opposite matrix jacobienne J^{-1} and of the determinant $\det J$
- calculation of $\tilde{J}^{-1}=\begin{bmatrix} J^{-1} & 0 & 0 \\ 0 & J^{-1} & 0 \\ 0 & 0 & J^{-1} \end{bmatrix}$
- calculation of the second matrix of the derivative of the functions of form $\left[\dfrac{\partial N}{\partial \xi}\right]_1$

$$\dots \left[\begin{bmatrix} N^{(1)}_{I,\xi_1} & 0 & 0 \\ N^{(1)}_{I,\xi_2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & N^{(1)}_{I,\xi_1} & 0 \\ 0 & N^{(1)}_{I,\xi_2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & N^{(1)}_{I,\xi_1} \\ 0 & 0 & N^{(1)}_{I,\xi_2} \\ 0 & 0 & 0 \end{bmatrix} \frac{h}{2} \begin{bmatrix} \xi_3 N^{(2)}_{I,\xi_1} & 0 & 0 \\ \xi_3 N^{(2)}_{I,\xi_2} & 0 & 0 \\ N^{(2)}_{I} & 0 & 0 \\ 0 & \xi_3 N^{(2)}_{I,\xi_1} & 0 \\ 0 & \xi_3 N^{(2)}_{I,\xi_2} & 0 \\ 0 & N^{(2)}_{I} & 0 \\ 0 & 0 & \xi_3 N^{(2)}_{I,\xi_1} \\ 0 & 0 & \xi_3 N^{(2)}_{I,\xi_2} \\ 0 & 0 & N^{(2)}_{I} \end{bmatrix} \right] \dots I=1, NB1$$

$$| \frac{h}{2} \begin{bmatrix} \xi_3 N^{(2)}_{NB2,\xi_1} & 0 & 0 \\ \xi_3 N^{(2)}_{NB2,\xi_2} & 0 & 0 \\ N^{(2)}_{NB2} & 0 & 0 \\ 0 & \xi_3 N^{(2)}_{NB2,\xi_1} & 0 \\ 0 & \xi_3 N^{(2)}_{NB2,\xi_2} & 0 \\ 0 & N^{(2)}_{NB2} & 0 \\ 0 & 0 & \xi_3 N^{(2)}_{NB2,\xi_1} \\ 0 & 0 & \xi_3 N^{(2)}_{NB2,\xi_2} \\ 0 & 0 & N^{(2)}_{NB2} \end{bmatrix}$$

$$\left[\frac{\partial N}{\partial \xi} \right]_1 = ¿ [¿][¿][\,]$$

- calculation of $\frac{\partial u}{\partial x} = \tilde{J}^{-1} \left[\frac{\partial N}{\partial \xi} \right]_1 p^e$

- calculation of $A\left(\frac{\partial u}{\partial x}\right) = \begin{bmatrix} u_{,x} & 0 & 0 & v_{,x} & 0 & 0 & w_{,x} & 0 & 0 \\ 0 & u_{,y} & 0 & 0 & v_{,y} & 0 & 0 & w_{,y} & 0 \\ 0 & 0 & u_{,z} & 0 & 0 & v_{,z} & 0 & 0 & w_{,z} \\ u_{,y} & u_{,x} & 0 & v_{,y} & v_{,x} & 0 & w_{,y} & w_{,x} & 0 \\ u_{,z} & 0 & u_{,x} & v_{,z} & 0 & v_{,x} & w_{,z} & 0 & w_{,x} \\ 0 & u_{,z} & u_{,y} & 0 & v_{,z} & v_{,y} & 0 & w_{,z} & w_{,y} \end{bmatrix}$

- calculation of $H\left[Q + \frac{1}{2} A\left(\frac{\partial u}{\partial x}\right) \right]$

$H=$

$$\begin{bmatrix} (t_1(1))^2 & (t_1(2))^2 & (t_1(3))^2 & t_1(1)t_1(2) & t_1(2)t_1(3) & t_1(3)t_1(1) \\ (t_2(1))^2 & (t_2(2))^2 & (t_2(3))^2 & t_2(1)t_2(2) & t_2(2)t_2(3) & t_2(3)t_2(1) \\ (t_3(1))^2 & (t_3(2))^2 & (t_3(3))^2 & t_3(1)t_3(2) & t_3(2)t_3(3) & t_3(3)t_3(1) \\ 2\mathrm{t}_2(1)t_3(1) & 2\mathrm{t}_2(2)t_3(2) & 2\mathrm{t}_2(3)t_3(3) & t_2(1)t_3(2)+t_3(1)t_2(2) & t_2(2)t_3(3)+t_3(2)t_2(3) & t_2(3)t_3(1)+t_3(3)t_3(1) \\ 2\mathrm{t}_3(1)t_1(1) & 2\mathrm{t}_3(2)t_1(2) & 2\mathrm{t}_3(3)t_1(3) & t_3(1)t_1(2)+t_1(1)t_3(2) & t_3(2)t_1(3)+t_1(2)t_3(3) & t_3(3)t_1(1)+t_1(3)t_3(1) \end{bmatrix}$$

- calculation of the first operator of the deformations

$$\tilde{B}_1=H\left[Q+\frac{1}{2}A\left(\frac{\partial u}{\partial x}\right)\right]\tilde{J}^{-1}\left[\frac{\partial N}{\partial \xi}\right]_1$$

- calculation of the vector of the local deformations $\tilde{E}=\tilde{B}_1 p^e$
- calculation of the second matrix of the derivative of the functions of form $\left[\frac{\partial N}{\partial \xi}\right]_2$

$$\dots\left[\begin{bmatrix} N^{(1)}_{I,\xi_1} & 0 & 0 \\ N^{(1)}_{I,\xi_2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & N^{(1)}_{I,\xi_1} & 0 \\ 0 & N^{(1)}_{I,\xi_2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & N^{(1)}_{I,\xi_1} \\ 0 & 0 & N^{(1)}_{I,\xi_2} \\ 0 & 0 & 0 \end{bmatrix}\frac{h}{2}\begin{bmatrix} 0 & \xi_3 N^{(2)}_{I,\xi_1} n^{\varphi}_z & -\xi_3 N^{(2)}_{I,\xi_1} n^{\varphi}_y \\ 0 & \xi_3 N^{(2)}_{I,\xi_2} n^{\varphi}_z & -\xi_3 N^{(2)}_{I,\xi_2} n^{\varphi}_y \\ 0 & N^{(2)}_1 n^{\varphi}_z & -N^{(2)}_1 n^{\varphi}_y \\ -\xi_3 N^{(2)}_{I,\xi_1} n^{\varphi}_z & 0 & \xi_3 N^{(2)}_{I,\xi_1} n^{\varphi}_x \\ -\xi_3 N^{(2)}_{I,\xi_2} n^{\varphi}_z & 0 & \xi_3 N^{(2)}_{I,\xi_2} n^{\varphi}_x \\ -N^{(2)}_1 n^{\varphi}_z & 0 & N^{(2)}_1 n^{\varphi}_x \\ \xi_3 N^{(2)}_{I,\xi_1} n^{\varphi}_y & -\xi_3 N^{(2)}_{I,\xi_1} n^{\varphi}_x & 0 \\ \xi_3 N^{(2)}_{I,\xi_2} n^{\varphi}_y & -\xi_3 N^{(2)}_{I,\xi_2} n^{\varphi}_x & 0 \\ N^{(2)}_1 n^{\varphi}_y & -N^{(2)}_1 n^{\varphi}_x & 0 \end{bmatrix}\right] LI=1,NB1$$

$$|\frac{h}{2}\begin{bmatrix} 0 & \xi_3 N^{(2)}_{NB2,\xi_1} n^{\varphi}_z & -\xi_3 N^{(2)}_{NB2,\xi_1} n^{\varphi}_y \\ 0 & \xi_3 N^{(2)}_{NB2,\xi_2} n^{\varphi}_z & -\xi_3 N^{(2)}_{NB2,\xi_2} n^{\varphi}_y \\ 0 & N^{(2)}_{NB2} n^{\varphi}_z & -N^{(2)}_{NB2} n^{\varphi}_y \\ -\xi_3 N^{(2)}_{NB2,\xi_1} n^{\varphi}_z & 0 & \xi_3 N^{(2)}_{NB2,\xi_1} n^{\varphi}_x \\ -\xi_3 N^{(2)}_{NB2,\xi_2} n^{\varphi}_z & 0 & \xi_3 N^{(2)}_{NB2,\xi_2} n^{\varphi}_x \\ -N^{(2)}_{NB2} n^{\varphi}_z & 0 & N^{(2)}_{NB2} n^{\varphi}_x \\ \xi_3 N^{(2)}_{NB2,\xi_1} n^{\varphi}_y & -\xi_3 N^{(2)}_{NB2,\xi_1} n^{\varphi}_x & 0 \\ \xi_3 N^{(2)}_{NB2,\xi_2} n^{\varphi}_y & -\xi_3 N^{(2)}_{NB2,\xi_2} n^{\varphi}_x & 0 \\ N^{(2)}_{NB2} n^{\varphi}_y & -N^{(2)}_{NB2} n^{\varphi}_x & 0 \end{bmatrix}$$

$$\left[\frac{\partial N}{\partial \xi}\right]_2=¿[¿][¿][\,]$$

- calculation of the second operator of the deformations

$$\tilde{B}_2 = H\left(Q + A\left(\frac{\partial u}{\partial x}\right)\right)\tilde{J}^{-1}\left[\frac{\partial N}{\partial \xi}\right]_2$$

- calculation and digital integration $r^e = \int_{\Omega_\zeta} \tilde{\boldsymbol{B}}_2^T \tilde{\boldsymbol{S}} \det \boldsymbol{J}\, d\xi_1 d\xi_2 d\xi_3$

- calculation of the matrix of behavior $\boldsymbol{D}$

- calculation and digital integration $\boldsymbol{K}_m^e = \int_{\Omega_\zeta} \tilde{\boldsymbol{B}}_2^T \boldsymbol{D}\, \tilde{\boldsymbol{B}}_2 \det \boldsymbol{J}\, d\xi_1 d\xi_2 d\xi_3$

- construction of the symmetrical tensor 3×3 local constraints $[\tilde{\boldsymbol{S}}]$

- calculation of the symmetrical tensor 3×3 total constraints $[\boldsymbol{S}] = \boldsymbol{P}^T [\tilde{\boldsymbol{S}}]\boldsymbol{P}$

- calculation of $\tilde{\boldsymbol{B}}_3^T = \boldsymbol{H}[\boldsymbol{S}]\tilde{\boldsymbol{J}}^{-1}\left[\frac{\partial N}{\partial \xi}\right]_3$

calculation and digital integration $\boldsymbol{Z}_I = \int_{\Omega_z} \tilde{B}_3^T \tilde{S} \det \boldsymbol{J}\, d\xi_1 d\xi_2 d\xi_3$

- calculation of the generalized tensor of the constraints **total** $\overset{9\times9}{\bar{S}} = \begin{bmatrix} \overset{3\times3}{[\,S\,]} & 0 & 0 \\ 0 & [S] & 0 \\ 0 & 0 & [S] \end{bmatrix}$

- calculation and digital integration of classical rigidity

$$\boldsymbol{K}_g^{e\,\text{classique}} = \int_{\Omega_z} \left[\tilde{\boldsymbol{J}}^{-1}\left[\frac{\partial N}{\partial \xi}\right]_2\right]^T \bar{\boldsymbol{S}} \left[\tilde{\boldsymbol{J}}^{-1}\left[\frac{\partial N}{\partial \xi}\right]_2\right] \det \boldsymbol{J}\, d\xi_1 d\xi_2 d\xi_3$$

Fine INTSN buckles on the points of integration

Beginning Buckle JN on $NB2$ **nodes**

- calculation of $[z_I \times]$ knowing that $Z_I = \begin{pmatrix} . \\ . \\ . \\ z_I \\ . \\ . \\ . \\ I = 1, NB\,2 \end{pmatrix}$

- calculation of $[n_I \times]$

- calculation of the nonsymmetrical matrix $\overset{3\times3}{K^{e}_{\text{gnon classique}}}(I,I)=[z_I\times][n_I\times]$

IF JN $\leq$ NB1

- addition of $\overset{3\times3}{K^{e}_{\text{gnon classique}}}(I,I)$ with distinction of the extra-node

ELSE JN

- assignment of $\overset{3\times3}{K^{e}_{\text{gnon classique}}}(I,I)$ with distinction of the extra-node

END IF JN

Storage of all the nonsymmetrical matrix K^{e^T}