Treatment of offsetting for the elements of plate DKT, DST, DKQ, DSQ and Q4G

Summary:

The elements of plate [R3.07.03] are intended for the three-dimensional mean structural analyses. The average layer of these structures always does not coincide with the plan of diagram or plan of grid. One thus introduces the concept of offsetting of the average layer compared to the plan of diagram. It is usable for elements with taking into account of transverse shearing, or without this assumption.

Contents

1 Introduction

With an aim of being able to analyze the behavior of slim structures of type plates, or curved surfaces approximate by facets, whose average layer is excentré compared to the plan of load application, one introduces the concept of offsetting of the average layer compared to the surface of grid. The fields of displacement varying linearly in the thickness of the plate originate in the surface of grid, i.e. on the level of the surface of grid, the only degrees of freedom of translation are necessary to the description of displacement.

The introduction of kinematics into the expression of the work of deformation makes it possible to obtain rigidities of membrane, of inflection and transverse shearing of the excentré element from those of the element are equivalent nonexcentré and of the distance from offsetting. The whole of calculations (except specific postprocessing) is thus made in a reference mark of diagram attached to the plan of the grid. By defaults the results are thus obtained in the reference mark of the grid. For certain postprocessings, it is possible to have automatically these results in other reference marks insofar as the user indicates the position of the plan of postprocessing compared to the plan of the grid.

The distance from offsetting between the plan of the grid and the average layer of the plate is given in AFFE_CARA_ELEM on the same level as the thickness. A offsetting d positive means that the average surface of the plate is actually at a distance $d\,\boldsymbol{n}$ element of plate with a grid, direction $\boldsymbol{n}$ being given by the normal to the element (see [§4.1] reference material [R3.07.03] of the elements of plate for the construction of this normal).

The adopted notations are those of the note [R3.07.03] on the elements of plates DKT, DST, DKQ, DSQ and Q4G.

2 Formulation

2.1 Geometry

For the offset elements of plate, the surface of reference is given by the plan of diagram or plan of the grid (plan $x\,y$ for example). The average layer of the element is positioned compared to this surface of reference. The thickness $h(x, y)$ must be small compared to other dimensions (extensions, radii of curvature) of the structure to model. The figure [Figure 2.1-a] below illustrates our matter. Concerning the value of offsetting d, and because of the conditions of linearization of the inflection adopted in the theory, one will take d so that an element thickness $d+h$ remain in the theory of the plates.

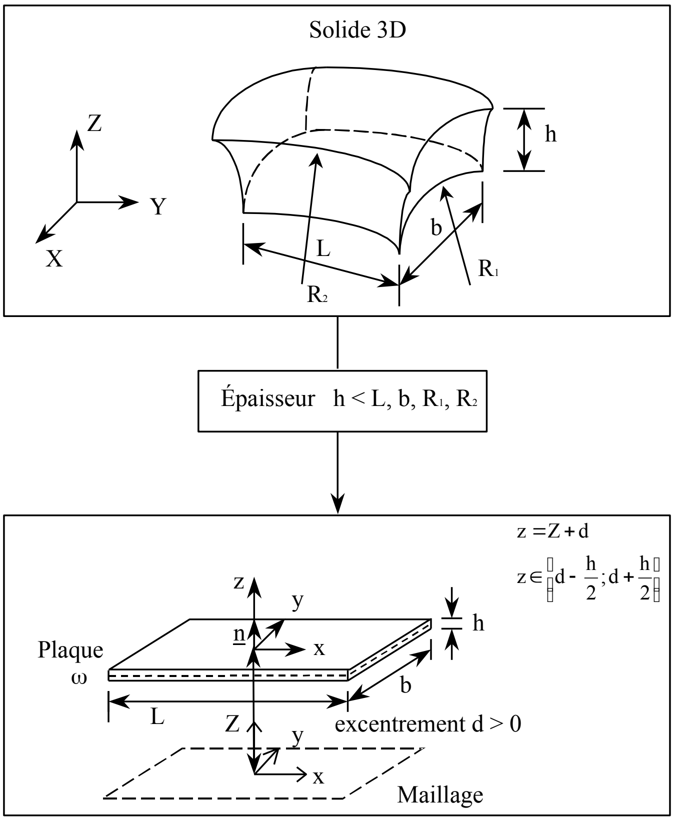


Figure 2.1-a

One attaches to the plan of diagram (the plan of the grid) a local orthonormal reference mark $0xyz$ associated with the plan of the grid different from the total reference mark $OXYZ$. The position of the points of the plate is given by the Cartesian coordinates (x, y) in the plan of diagram (plan of the grid) and rise z compared to this plan.

2.2 Kinematics

The cross-sections which are the sections perpendicular to the average layer of the plate remain right. The material points located on a normal at not deformed average surface remain on a line in the deformed configuration. It results from this approach that the fields of displacement vary linearly in the thickness of the plate. If one indicates by u, v, w displacements of a point of the plan of diagram $q(x, y, z)$ according to x, y and z, the kinematics of Hencky-Mindlin gives us:

$$\begin{pmatrix} u_x(x,y,z) \\ u_y(x,y,z) \\ u_z(x,y,z) \end{pmatrix} = \begin{pmatrix} u(x,y) \\ v(x,y) \\ w(x,y) \end{pmatrix} + z \begin{pmatrix} \theta_y(x,y) \\ -\theta_x(x,y) \\ 0 \end{pmatrix} = \begin{pmatrix} u(x,y) \\ v(x,y) \\ w(x,y) \end{pmatrix} + z \begin{pmatrix} \beta_x(x,y) \\ \beta_y(x,y) \\ 0 \end{pmatrix}$$

where: u, v, w are displacements of the plan of diagram;

θ_x and θ_y are respectively rotations of this plan compared to respectively the axis x and centers it y.

One prefers to introduce two rotations $\beta_x(x,y)=\theta_y(x,y), \beta_y(x,y)=-\theta_x(x,y)$. The three-dimensional deformations in any point, with kinematics introduced previously, are thus given by:

$$\left| \begin{array}{l} \varepsilon_{xx}=e_{xx}+z\kappa_{xx} \\ \varepsilon_{yy}=e_{yy}+z\kappa_{yy} \\ 2\,\varepsilon_{xy}=\gamma_{xy}=2\,e_{xy}+2\,z\kappa_{xy} \\ 2\,\varepsilon_{xz}=\gamma_x \\ 2\,\varepsilon_{yz}=\gamma_y \end{array} \right.$$

where: e_{xx}, e_{yy} and e_{xy} are the membrane deformations of average surface;

γ_x and γ_y deformations associated with transverse shearings;

κ_{xx}, κ_{yy}, κ_{xy} the deformations of inflection of average surface, which are written:

$$\left| \begin{array}{l} e_{xx}=\dfrac{\partial u}{\partial x} \\ e_{yy}=\dfrac{\partial v}{\partial y} \\ 2\,e_{xy}=\dfrac{\partial v}{\partial x}+\dfrac{\partial u}{\partial y} \\ \kappa_{xx}=\dfrac{\partial \beta_x}{\partial x} \\ \kappa_{yy}=\dfrac{\partial \beta_y}{\partial y} \\ 2\,\kappa_{xy}=\dfrac{\partial \beta_x}{\partial y}+\dfrac{\partial \beta_y}{\partial x} \\ \gamma_x=\beta_x+\dfrac{\partial w}{\partial x} \\ \gamma_y=\beta_y+\dfrac{\partial w}{\partial y} \end{array} \right.$$

Note:

- *In the theories of plate, the introduction of β_x and β_y allows to symmetrize the formulations of the deformations and the equilibrium equations [R3.07.03]. In the theories of hull, one uses rather θ_x and θ_y and associated couples M_x and M_y compared to x and y ,*
- *the degrees of freedom which one chose are displacements and rotations of the plan of diagram and not those of the average layer. Indeed if one considers the superposition of several plates offset to carry out a material sandwich it can correspond to the nodes of the grid one field of displacement and not the various fields of displacements of the layers composing material.*

2.3 Law of behavior

The behavior of the plates is a behavior 3D in "plane constraints". **The transverse constraint σ_{zz} is taken worthless** because negligible compared to the other components of the tensor of the constraints (assumption of the plane constraints). The most general law of behavior is written then as follows:

$$\begin{pmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\\ \sigma_{xz}\\ \sigma_{yz}\end{pmatrix}=C(\varepsilon,\alpha)\begin{pmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\\ \gamma_{x}\\ \gamma_{y}\end{pmatrix}=\mathbf{Ce}+zC\kappa+C\gamma \quad \text{with} \quad e=\begin{pmatrix}e_{xx}\\ e_{yy}\\ 2e_{xy}\\ 0\\ 0\end{pmatrix}, \quad \kappa=\begin{pmatrix}\kappa_{xx}\\ \kappa_{yy}\\ 2\kappa_{xy}\\ 0\\ 0\end{pmatrix} \quad \text{et} \quad \gamma=\begin{pmatrix}0\\ 0\\ 0\\ \gamma_{x}\\ \gamma_{y}\end{pmatrix}$$

where: $C(\varepsilon,\alpha)$ is the matrix of local tangent rigidity in plane constraints;
α represent the whole of the internal variables when the behavior is nonlinear.

For behaviors (for example of multi-layer) for which distortions are coupled with the deformations of membrane and inflection, $C(\varepsilon,\alpha)$ puts itself in the form:

$$C=\begin{pmatrix}H & H_{c\gamma}\\ H^T_{c\gamma} & H_{\gamma}\end{pmatrix}$$

where: $H(\varepsilon,\alpha)$ is a matrix 3×3 symmetrical;
$H_{\gamma}(\varepsilon,\alpha)$ a matrix 2×2 symmetrical;
$H_{c\gamma}(\varepsilon,\alpha)$ a matrix 3×2 of coupling between the effects of membrane or inflection and transverse shearing.

If it is uncoupled, one has $H_{c\gamma}(\varepsilon,\alpha)=0$. Determination of $H_{\gamma}(\varepsilon,\alpha)$ within the framework of the theory of Reissner ([§ 2.2.3.2] of [R3.07.03]) is given in appendix. It is shown that it is equivalent to that of the not offset plates.

3 Principle of virtual work

3.1 Work of deformation

The general expression of the work of deformation 3D for the element of excentré plate of the distance d compared to the datum-line is worth:

$$W_{\text{def}}=\int_S \int_{d-h/2}^{d+h/2} (\varepsilon_{xx}\sigma_{xx}+\varepsilon_{yy}\sigma_{yy}+\gamma_{xy}\sigma_{xy}+\gamma_x\sigma_{xz}+\gamma_y\sigma_{yz})\,dV$$

where S is average surface, $dV=dxdydz$ and where the position in the thickness of the plate varies between $d - h/2$ and $d+h/2$.

3.1.1 Expression of the resulting efforts

By adopting the kinematics of [R3.07.03], one identifies the work of the interior efforts:

$$W_{\text{def}}=\int_S (e_{xx}N_{xx}+e_{yy}N_{yy}+2e_{xy}N_{xy}+\kappa_{xx}M_{xx}+\kappa_{yy}M_{yy}+2\kappa_{xy}M_{xy}+\gamma_x T_x+\gamma_y T_y)\,dS$$

where:

$$N=\begin{pmatrix}N_{xx}\\N_{yy}\\N_{xy}\end{pmatrix}=\int_{d-h/2}^{d+h/2}\begin{pmatrix}\sigma_{xx}\\\sigma_{yy}\\\sigma_{xy}\end{pmatrix}dz$$

$$M=\begin{pmatrix}M_{xx}\\M_{yy}\\M_{xy}\end{pmatrix}=\int_{d-h/2}^{d+h/2}\begin{pmatrix}\sigma_{xx}\\\sigma_{yy}\\\sigma_{xy}\end{pmatrix}z\,dz$$

$$T=\begin{pmatrix}T_x\\T_y\end{pmatrix}=\int_{d-h/2}^{d+h/2}\begin{pmatrix}\sigma_{xz}\\\sigma_{yz}\end{pmatrix}dz$$

where: N_{xx} , N_{yy} , N_{xy} are the efforts resulting from membrane (in N/m);
M_{xx} , M_{yy} , M_{xy} are the efforts resulting from inflection or moments compared to the plan from diagram (in N);
T_x , T_y are the efforts resulting from shearing or efforts cutting-edges (in N/m).

3.1.2 Relation resulting efforts generalized deformations

The expression of the work of deformation is also written:

$$W_{\text{def}}=\int_S\int_{d-h/2}^{d+h/2}[\boldsymbol{\varepsilon}\mathbf{C}(\varepsilon,\alpha)\varepsilon]\,dV=\int_S\int_{d-h/2}^{d+h/2}[\mathbf{eCe}+z\,\mathbf{eC}\,\kappa+\mathbf{eC}\gamma+z\kappa\,\mathbf{Ce}+z^2\,\kappa C\kappa+z\kappa C\gamma+\gamma C(e+z\kappa+\gamma)]\,dSdz$$

where: $C(\varepsilon,\alpha)$ is the matrix of local tangent rigidity (symmetrical matrix).

This is still written:

$$W_{\text{def}}=\int_S \int_{-h/2}^{h/2} [\mathbf{eCe}+(\zeta+d)\mathbf{eC}\kappa+\mathbf{eC}\gamma+\kappa(\zeta+d)\mathbf{Ce}+(\zeta+d)^2\kappa C\kappa+\kappa(\zeta+d)C\gamma+\gamma C(e+(\zeta+d)\kappa+\gamma)]dSd\zeta$$

By using the expression obtained for W_{def} in the preceding paragraph, one finds the relation following between the resulting efforts and the généraliées deformations:

$$N=H_m e+(H_{\mathbf{mf}}+dH_m)\kappa+H_{m\gamma}\gamma$$
$$M=(H_{\mathbf{mf}}+dH_m)e+(H_f+2dH_{\mathbf{mf}}+d^2H_m)\kappa+(H_{f\gamma}+dH_{m\gamma})\gamma$$
$$T=H_{m\gamma}^T e+(H_{f\gamma}^T+dH_{m\gamma}^T)\kappa+H_{\mathbf{ct}}\gamma$$

with:

$$H_m=\int_{-h/2}^{+h/2} Hd\zeta \qquad H_{\mathbf{mf}}=\int_{-h/2}^{+h/2} H\zeta d\zeta \qquad H_f=\int_{-h/2}^{+h/2} H\zeta^2 d\zeta$$
$$H_{\mathbf{ct}}=\int_{-h/2}^{+h/2} H_\gamma d\zeta \qquad H_{m\gamma}=\int_{-h/2}^{+h/2} H_{c\gamma}d\zeta \qquad H_{f\gamma}=\int_{-h/2}^{+h/2} H_{c\gamma}\zeta d\zeta$$

and:

$$e=\begin{pmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{pmatrix}, \qquad \kappa=\begin{pmatrix} \kappa_{xx} \\ \kappa_{yy} \\ 2\kappa_{xy} \end{pmatrix}, \qquad \gamma=\begin{pmatrix} \gamma_x \\ \gamma_y \end{pmatrix}$$

Matrices H_m, H_f and H_{ct} are the matrices of rigidity out of membrane, inflection and transverse shearing, respectively, for the element of nonexcentré plate. The matrix H_{mf} is a matrix of rigidity of coupling between the membrane and the inflection for the element of nonexcentré plate. It is worthless if the element of plate is symmetrical compared to its average layer. The matrix $H_m\gamma$ is a matrix of rigidity of coupling between the membrane and the transverse distortion. The matrix $H_f\gamma$ is a matrix of rigidity of coupling between the inflection and the transverse distortion. These matrices are worthless for a null offsetting, except in the case of the multi-layer ones where they remain nonworthless.

For an isotropic homogeneous elastic behavior, these matrices have as an expression:

$$H_m=\frac{Eh}{1-\nu^2}\begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{pmatrix}, \qquad H_f=\frac{Eh^3}{12(1-\nu^2)}\begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{pmatrix}, \qquad H_{\mathbf{ct}}=\frac{kEh}{2(1+\nu)}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and $H_{mf}=H_m\gamma=H_f\gamma=0$ because there is a material symmetry compared to the plan $\zeta=0$.

For the determination of the coefficient of shearing k one returns to [§2.2.3] of [R3.07.03].

The system of relation between the resulting efforts and the generalized deformations can be also written:

$$
\begin{aligned}
N &= H_m e + H'_{\mathbf{mf}} \kappa + H_{m\gamma} \gamma \\
M &= H'_{\mathbf{mf}} e + H'_f \kappa + H'_{f\gamma} \gamma \\
& T \\
T &= H^T_{m\gamma} e + H^{¿}_{f\gamma} + H_{\mathbf{ct}} \gamma
\end{aligned}
$$

with:

$$
\begin{aligned}
H'_{\mathbf{mf}} &= H_{\mathbf{mf}} + dH_m \\
H'_f &= H_f + 2\,dH_{\mathbf{mf}} + d^2 H_m \\
H'_{fg} &= H_{fg} + dH_{mg}
\end{aligned}
$$

Thus, in the case of a plate having material symmetry compared to the plan $\zeta=0$, one has $H_{mf}=0$ but $H'_{\mathbf{mf}}=dH_m$. The offsetting of the plate involves a coupling between the terms of membrane and inflection.

Note:

Relations flexible H_m , H_f , H_{mf} with H and H_{ct} with H_γ are valid whatever the law of elastic behavior tangent, with unelastic deformations (thermoelasticity, plasticity,…).

For a plate made up of N orthotropic layers in elasticity, matrices H_m, H_f, H_{mf} and H_{ct} are written:

$$\mathbf{H_m} = \sum_{i=1}^{N} \mathbf{H_{mi}}, \qquad \mathbf{H_{mf}} = \sum_{i=1}^{N} \left(\mathbf{H_{mf\,i}} + \eta_i \mathbf{H_{mi}} \right), \quad \mathbf{H_f} = \sum_{i=1}^{N} \left(\mathbf{H_{fi}} + 2\,\eta_i \mathbf{H_{mf\,i}} + \eta_i^2 \mathbf{H_{mi}} \right), \ \mathbf{H_{ct}} = \sum_{i=1}^{N} \mathbf{H_{ct\,i}}$$

where: $\eta_i = \dfrac{1}{2}\left(z_{i+1} + z_i\right)$

H_{mi}, H_{fi}, H_{mfi}, $H_{\gamma i}$ represent the matrices of membrane, inflection, coupling membrane inflection and transverse shearing for the layer i. One notices the analogy between these expressions with the form established above:

$$
\begin{aligned}
H'_{\mathbf{mf}} &= H_{\mathbf{mf}} + dH_m \\
H'_f &= H_f + 2\,dH_{\mathbf{mf}} + d^2 H_m
\end{aligned}
$$

One from of deduced whereas offsetting for such a plate is obtained in substituent $\eta_i + d$ with η_i .

3.1.3 Energy interns elastic of plate

Taking into account the preceding remarks, energy interns elastic plate is more usually expressed for this kind of geometry in the following way:

$$\Phi_{\text{int}} = \frac{1}{2} \int_S \left[e\left(H_m e + H'_{\mathbf{mf}} \kappa + H_{m\gamma} \gamma \right) + \kappa \left(H'_{\mathbf{mf}} e + H'_f \kappa + H'_{f\gamma} \gamma \right) + \gamma \left(H^T_{m\gamma} e + H'^T_{f\gamma} \kappa + H_{\mathbf{ct}} \gamma \right) \right] dS \ .$$

3.1.4 Notice

One can choose to express the efforts resulting from inflection or moments compared to the average layer from the element and either compared to the datum-line. In this case one obtains:

$$N=\begin{pmatrix}N_{xx}\\N_{yy}\\N_{xy}\end{pmatrix}=\int_{d-h/2}^{d+h/2}\begin{pmatrix}\sigma_{xx}\\\sigma_{yy}\\\sigma_{xy}\end{pmatrix}dz\ ,\quad M'=\begin{pmatrix}M'_{xx}\\M'_{yy}\\M'_{xy}\end{pmatrix}=\int_{d-h/2}^{d+h/2}\begin{pmatrix}\sigma_{xx}\\\sigma_{yy}\\\sigma_{xy}\end{pmatrix}(z-d)\,dz\ ,\quad T=\begin{pmatrix}T_x\\T_y\end{pmatrix}=\int_{d-h/2}^{d+h/2}\begin{pmatrix}\sigma_{xz}\\\sigma_{yz}\end{pmatrix}dz$$

and the expression of the work of the interior efforts becomes:

$$W_{\text{def}}=\int_S\left(e_{xx}N_{xx}+e_{yy}N_{yy}+2\,e_{xy}N_{xy}+\kappa_{xx}(M'_{xx}+dN_{xx})+\kappa_{yy}(M'_{yy}+dN_{yy})\right)ds+$$
$$\int_S\left(2\,\kappa_{xy}(M'_{xy}+dN_{xy})+\gamma_x T_x+\gamma_y T_y\right)dS$$

One from of then deduced by using the expression 3D from the work of deformation that:

$$\begin{aligned}N&=H_m e+(H_{\mathbf{mf}}+dH_m)\kappa+H_{m\gamma}\gamma\\M'+dN&=(H_{\mathbf{mf}}+dH_m)e+(H_f+2\,dH_{\mathbf{mf}}+d^2H_m)\kappa+(H_{f\gamma}+dH_{m\gamma})\gamma\\T&=H_{m\gamma}^T e+(H_{f\gamma}^T+dH_{m\gamma}^T)\kappa+H_{\mathbf{ct}}\gamma\end{aligned}$$

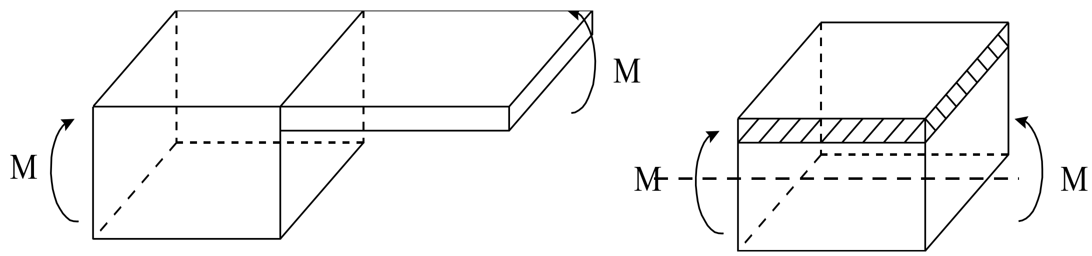
That is to say still:

$$\begin{aligned}N&=H_m e+(H_{\mathbf{mf}}+dH_m)\kappa+H_{m\gamma}\gamma\\M'&=H_{\mathbf{mf}}e+(H_f+dH_{\mathbf{mf}})\kappa+H_{f\gamma}\gamma\ .\\T&=H_{m\gamma}^T e+(H_{f\gamma}^T+dH_{m\gamma}^T)\kappa+H_{\mathbf{ct}}\gamma\end{aligned}$$

The expression of the internal energy of the plate remains unchanged of course as for it. In the case of elasticity, she is always written:

$$\Phi_{\text{int}}=\frac{1}{2}\int_S\left[e(H_m e+H'_{\mathbf{mf}}\kappa+H_{m\gamma}\gamma)+\kappa(H'_{\mathbf{mf}}e+H'_f\kappa+H'_{f\gamma}\gamma)+\gamma(H_{m\gamma}^T e+H'^T_{f\gamma}\kappa+H_{\mathbf{ct}}\gamma)\right]dS$$

The question of the choice of the plan interesting to use for the expression of the moments can vary from a situation to another.



In the case of the figure of right-hand side, the approach developed above is preferable because the expression of the loadings is defined compared to the average layer of each plate. In the case of the figure of left, if one wishes to replace the multi-layer hull by two offset hulls, the reference axis is the average layer of the multi-layer hull. One thus may find it beneficial with all to define compared to the plan of diagram. It is this approach which is adopted in the code. All the loadings applied are regarded as being defined by default in the reference mark of diagram or plan of the grid. If ever certain loadings are defined compared to other plans (average layer, higher layer or inferior) is to the user to make the adapted changes of reference mark, with the hand or by the means of the command file by specifying the plan of load application when that is possible (see [§5]), to bring back itself to a loading defined in the plan of the grid.

3.2 Work of the forces and external couples

The work of the forces and couples being exerted on the plate is expressed in the following way:

$$W_{\text{ext}}=\int_S \int_{d-h/2}^{d+h/2} F_v . U \, dV + \int_S F_s . U \, dS + \int_C \int_{d-h/2}^{d+h/2} F_c . U \, dz \, ds$$

where: F_v, F_s, F_c are the voluminal, surface efforts and of contour being exerted on the plate, respectively.
C is the part of the contour of the plate on which efforts of contour F_c are applied.

With the kinematics of [§2.2], one determines as follows:

$$\begin{aligned} W_{\text{ext}} &= \int_S (f_x u + f_y v + f_z w + c_x \theta_x + c_y \theta_y) dS + \int_C (\varphi_x u + \varphi_y v + \varphi_z w + \chi_x \theta_x + \chi_y \theta_y) ds \\ &= \int_S (f_x u + f_y v + f_z w + c_y \beta_x - c_x \beta_y) dS + \int_C (\varphi_x u + \varphi_y v + \varphi_z w + \chi_y \beta_x - \chi_x \beta_y) ds \end{aligned}$$

where are present on the plate:

- f_x, f_y, f_z surface forces acting according to x, y and z ;
- $f_i = \int_{-h/2}^{+h/2} F_v . e_i dz + F_s . e_i$ where e_x and e_y are the basic vectors of the tangent plan and e_z their normal vector;
- $\mathbf{c}_x, \mathbf{c}_y$: surface couples acting around the axes x and y ;
- $c_i = \int_{-h/2}^{+h/2} [(z+d) e_z \wedge F_v] . e_i dz + [(d \pm \frac{h}{2}) e_z \wedge F_s] . e_i$ where e_x, e_y, e_z are the basic vectors previously definite.

and where are present on the contour of the plate:

- ϕ_x, ϕ_y, ϕ_z linear forces acting according to x, y and z ;
- $\phi_i = \int_{-h/2}^{+h/2} F_c . e_i dz$ where e_x, e_y, e_z are the basic vectors previously definite;
- χ_x, χ_y linear couples around the axes x and y ;
- $\chi_i = \int_{-h/2}^{+h/2} [(z+d) e_z \wedge F_c] . e_i dz$ where e_x, e_y, e_z are the basic vectors previously definite.

Note:

Moments compared to z are worthless. The efforts and the couples are expressed in the reference mark of the grid. All the calculations are done by default in the reference mark of diagram. So efforts or couples are expressed in another reference mark (that of the average layer of the plate for example) the user will have to make conversions with the hand if it uses the options by default or to specify the plan of load application (see the paragraph [§ 5]).

3.3 Principle of virtual work and equilibrium equations

This paragraph is unchanged compared to the paragraph [§3.3] of [R3.07.03].

4 Digital discretization of the variational formulation resulting from the principle of virtual work

4.1 Introduction

The variational formulation for energy interns enables us to write:

$$\delta W_{\text{int}}=\int_S [\delta e(H_m e+H'_{\mathbf{mf}}\kappa+H_{m\gamma}\gamma)+\delta\kappa(H'_{\mathbf{mf}}e+H'_f\kappa+H'_{f\gamma}\gamma)+\delta\gamma(H^T_{m\gamma}e+H'^T_{f\gamma}\kappa+H_{\mathbf{ct}}\gamma)]dS$$

with:

$$e=\begin{pmatrix} u_{,x}\\ v_{,y}\\ u_{,y}+v_{,x}\end{pmatrix},\qquad \kappa=\begin{pmatrix}\beta_{x,x}\\ \beta_{y,y}\\ \beta_{x,y}+\beta_{y,x}\end{pmatrix},\qquad \gamma=\begin{pmatrix} w_{,x}+\beta_x\\ w_{,y}+\beta_y\end{pmatrix}$$

The five degrees of freedom are displacements in the plan of the grid u and v, except plan w and two rotations β_x and β_y.

Elements `DKT` and `DST` are triangular isoparametric elements. Elements `DKQ`, `DSQ` and `Q4gamma` are quadrilateral isoparametric elements. They are represented below:

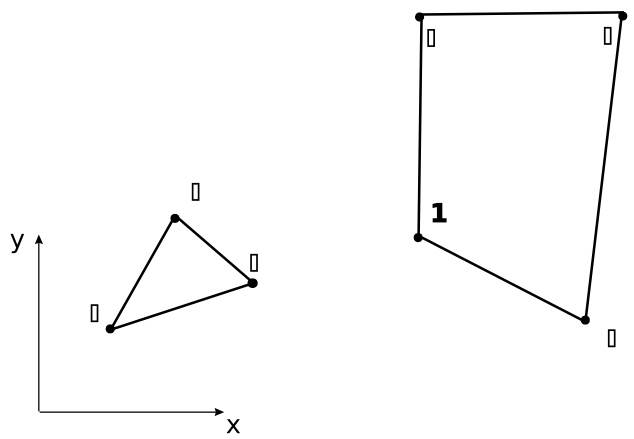


Figure 4.1-a: Real elements

The elements of reference are presented below:

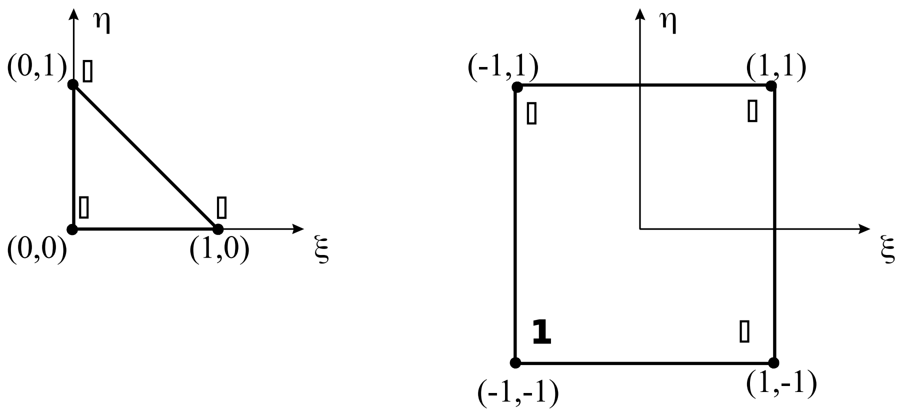


Figure 4.1-b: Elements of reference triangle and quadrangle

One defines the reduced reference mark of the element as the reference mark $(\xi.\eta)$ element of reference. The local reference mark of the element, in the plan of diagram (x, y) is defined by the user, by the keyword ANGLE_REP. Direction $X1$ this local reference mark is the projection of a direction of reference $\underline{d}$ as regards the element. This direction of reference $\underline{d}$ is chosen by the user who defines it by two nautical angles in the total reference mark. The normal N with the plan of the element ($12\wedge 13$ for a numbered triangle 123 and $12\wedge 14$ for a numbered quadrangle 1234) fix the second direction. The vector product of the two vectors previously definite $Y1=N\wedge X1$ allows to define the local trihedron in which will be expressed the generalized efforts representing the state of stresses. The user will have to take care that the selected reference axis is not found parallel with the normal of certain elements of plate. By default, direction of reference $\underline{d}$ is the axis X total reference mark of definition of the grid.

Note:

For the elements of plate QUAD4 , the use of a noncoplanar element can lead to irregularities ([bib1]). In this case, the user is alerted.

4.2 Discretization of the field of displacement

The matrix jacobienne $J(\xi,\eta)$ is:

$$J=\begin{pmatrix} x_{,\xi} & y_{,\xi} \\ x_{,\eta} & y_{,\eta} \end{pmatrix}=\begin{pmatrix} \sum_{i=1}^{N} N_{i,\xi}x_i & \sum_{i=1}^{N} N_{i,\xi}y_i \\ \sum_{i=1}^{N} N_{i,\eta}x_i & \sum_{i=1}^{N} N_{i,\eta}y_i \end{pmatrix}=\begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}$$

Moreover:

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}=j\begin{pmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{pmatrix} \quad \text{avec} \quad j=\begin{pmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{pmatrix}=J^{-1}=\frac{1}{J}\begin{pmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{pmatrix} \quad \text{où } J=\det J=J_{11}J_{22}\text{-}J_{12}J_{21}$$

The field of displacement is discretized by:

$$\begin{pmatrix} u \\ v \end{pmatrix} = \sum_{k=1}^{N} N_k(\xi,\eta) \begin{pmatrix} u^k \\ v^k \end{pmatrix}$$

$$\begin{pmatrix} w \\ \beta_x \\ \beta_y \end{pmatrix} = \sum_{k=1}^{N} N_k(x,h) \begin{pmatrix} w_k \\ \beta_{xk} \\ \beta_{yk} \end{pmatrix} + [\sum_{k=N+1}^{2N} \begin{pmatrix} 0 \\ P_{xk}(x,h) \\ P_{yk}(x,h) \end{pmatrix} \alpha_k]$$

In this last expression, the term between hooks is present for the elements of the type `DKT`, `DST`, `DKQ` or `DSQ`, but not for the elements `Q4γ`.

4.3 Taking into account of the transverse distortion

It is pointed out that the essential difference between the elements `DKT`, `DKQ` on the one hand and `DST`, `DSQ`, `Q4γ` in addition comes owing to the fact that for the first the transverse distortion is worthless is still $\gamma = 0$. The difference enters `Q4γ` and elements `DST` and `DSQ` comes from a choice different of interpolation for the representation of transverse shearing. The introduction of offsetting leads to a particular treatment of transverse shearing.

One replaces in the expression of the internal energy established with [§4.1] γ by $\bar{\gamma}$ where them $\bar{\gamma}$ are deformations of substitution checking $\bar{\gamma} = \gamma$ in a weak way (integral on the sides of the element), and such as:

$$N = H_m e + H'_{\mathbf{mf}} \kappa + H_{m\gamma} \bar{\gamma}$$
$$M = H'_{\mathbf{mf}} e + H'_f \kappa + H'_{f\gamma} \bar{\gamma}$$
$$T$$
$$T = H^T_{m\gamma} e + H^{¿}_{f\gamma} + H_{\mathbf{ct}} \bar{\gamma}$$

One checks thus that on the sides ij of the element, one a: $\int_i^j (\bar{\gamma}_s - \gamma_s) ds = 0$ with $\gamma_s = w_{,s} + \beta_s$.

4.3.1 For the elements Q4γ

The field linearly is discretized $\bar{\gamma}$ constant by side so that:

$$\bar{\gamma}^{\xi} = \begin{pmatrix} \bar{\gamma}_\xi \\ \bar{\gamma}_\eta \end{pmatrix} = \begin{pmatrix} \frac{1-\eta}{2} \gamma_\xi^{12} + \frac{1+\eta}{2} \gamma_\xi^{34} \\ \frac{1-\xi}{2} \gamma_\eta^{23} + \frac{1-\xi}{2} \gamma_\eta^{41} \end{pmatrix}$$

By using the relations then:

$$\left| \begin{array}{l} \int_{-1}^{+1} (\bar{g}_x - (w_{,x} + \beta_x)) dx = 0 \, ; \\ \int_{-1}^{+1} (\bar{g}_h - (w_{,h} + \beta_h)) dh = 0 \end{array} \right.$$

it is established that:

$$\left|\begin{array}{l}\gamma_{\xi}^{ij}=\frac{1}{2}(w_j-w_i+\beta_{\xi i}+\beta_{\xi j})\\ \gamma_{\eta}^{kp}=\frac{1}{2}(w_p-w_k+\beta_{\eta p}+\beta_{\eta k})\end{array}\right.$$ for $(ij)=(12{,}34)$ and $(kp)=(23{,}41)$.

By deferring the two results above in the expression of $\bar{\gamma}^{\xi}$, it is established that:

$$\bar{g}^{x}=\begin{pmatrix}\bar{g}_x\\ \bar{g}_h\end{pmatrix}=B'_x u_x$$

where: $u_x=\begin{pmatrix}w_1\\ \beta_{x1}\\ \beta_{h1}\\ M\\ w_N\\ \beta_{xN}\\ \beta_{xN}\end{pmatrix}$ and $B'_x=(B'_{x1}, L, B'_{xN})$ with $B'_{xk}=\begin{pmatrix}N_{k,x} & x_k N_{k,x} & 0\\ N_{k,h} & 0 & h_k N_{k,h}\end{pmatrix}$

Moreover, like:

$$\begin{pmatrix}\beta_{xi}\\ \beta_{hi}\end{pmatrix}=\begin{pmatrix}J_{11} & J_{12}\\ J_{21} & J_{22}\end{pmatrix}\begin{pmatrix}\beta_{xi}\\ \beta_{yi}\end{pmatrix}$$

one from of deduced that

$$\bar{g}^{x}=B_x u_f$$

where: $u_f=\begin{pmatrix}w_1\\ \beta_{x1}\\ \beta_{y1}\\ M\\ w_N\\ \beta_{xN}\\ \beta_{yN}\end{pmatrix}$ and $B_x=(B_{x1}, L, B_{xN})$

with: $B_{xk}=\begin{pmatrix}N_{k,x} & x_k N_{k,x} J_{11} & x_k N_{k,x} J_{12}\\ N_{k,h} & h_k N_{k,h} J_{21} & h_k N_{k,h} J_{22}\end{pmatrix}$

Finally:

$$\bar{g}=\begin{pmatrix}\bar{g}_x\\ \bar{g}_y\end{pmatrix}=\begin{pmatrix}j_{11} & j_{12}\\ j_{21} & j_{22}\end{pmatrix}\bar{g}^{x}=B_c u_f \quad \text{with} \quad B_{c[2'3N]}=\mathbf{j}\mathbf{B}_{\boldsymbol{x}}$$

Note:

> *This treatment is equivalent to that of the elements $Q4\ \gamma$ not offset of [§ 4.3.2.1] of [R3.07.03].*

4.3.2 For the elements of the type DKT,DST,DKQ,DSQ

With regard to the transverse distortions, one knows that:

$$T_x = M_{xx,x} + M_{xy,y} \text{ et } T_y = M_{yy,y} + M_{xy,x} \text{ with } M = H'_{\mathbf{mf}} e + H'_f \kappa + H'_{f\gamma} \bar{\gamma}$$

One from of deduced that:

$$T = \bar{H}^c_m u_{,xx} + \bar{H}^c_f \beta_{,xx}$$

Calculation of: $\bar{H}^c_m \bar{H}^c_f$

where: $\beta^T_{,xx} = \begin{pmatrix} \beta_{x,xx} & \beta_{x,yy} & \beta_{x,xy} & \beta_{y,xx} & \beta_{y,yy} & \beta_{y,xy} \end{pmatrix}$

$u^T_{,xx} = \begin{pmatrix} u_{,xx} & u_{,yy} & u_{,xy} & v_{,xx} & v_{,yy} & v_{,xy} \end{pmatrix}$

with: $$\bar{H}^c_m = \begin{pmatrix} H^{mf}_{11} & H^{mf}_{33} & 2H^{mf}_{13} & H^{mf}_{13} & H^{mf}_{23} & H^{mf}_{12} + H^{mf}_{33} \\ H^{mf}_{13} & H^{mf}_{23} & H^{mf}_{12} + H^{mf}_{33} & H^{mf}_{33} & H^{mf}_{22} & 2H^{mf}_{23} \end{pmatrix}$$

$$\bar{H}^c_f = \begin{pmatrix} H^f_{11} & H^f_{33} & 2H^f_{13} & H^f_{13} & H^f_{23} & H^f_{12} + H^f_{33} \\ H^f_{13} & H^f_{23} & H^f_{12} + H^f_{33} & H^f_{33} & H^f_{22} & 2H^f_{23} \end{pmatrix}$$

where them H^{mf}_{ij} are the terms (i, j) of $H'_{\mathbf{mf}}$ and where them $\mathbf{H}^{\mathbf{f}}_{\mathbf{ij}}$ are the terms (i, j) of H'_f .

Like:

$$\beta_{x,xx} = \sum_{k=1}^{N} N_{k,xx}(\xi,\eta)\beta_{xk} + \sum_{k=N+1}^{2N} P_{xk,xx}(\xi,\eta)\alpha_k = \sum_{k=1}^{N} N_{k,xx}(\xi,\eta)\beta_{xk} + \sum_{k=N+1}^{2N} (j_{11}^2 P_{xk,\xi\xi} + 2 j_{11} j_{12} P_{xk,\xi\eta} + j_{12}^2 P_{xk,\eta\eta})\alpha_k$$

$$\beta_{x,yy} = \sum_{k=1}^{N} N_{k,yy}(\xi,\eta)\beta_{xk} + \sum_{k=N+1}^{2N} P_{xk,yy}(\xi,\eta)\alpha_k = \sum_{k=1}^{N} N_{k,yy}(\xi,\eta)\beta_{xk} + \sum_{k=N+1}^{2N} (j_{21}^2 P_{xk,\xi\xi} + 2 j_{21} j_{22} P_{xk,\xi\eta} + j_{22}^2 P_{xk,\eta\eta})\alpha_k$$

$$\beta_{x,xy} = \sum_{k=1}^{N} N_{k,xy}(\xi,\eta)\beta_{xk} + \sum_{k=N+1}^{2N} P_{xk,xy}(\xi,\eta)\alpha_k = \sum_{k=1}^{N} N_{k,xy}(\xi,\eta)\beta_{xk} + \sum_{k=N+1}^{2N} (j_{11} j_{21} P_{xk,\xi\xi} + [j_{11} j_{22} + j_{12} j_{21}] P_{xk,\xi\eta} + j_{11} j_{21} P_{xk,\eta\eta})\alpha_k$$

$$\beta_{y,xx} = \sum_{k=1}^{N} N_{k,xx}(\xi,\eta)\beta_{yk} + \sum_{k=N+1}^{2N} P_{yk,xx}(\xi,\eta)\alpha_k = \sum_{k=1}^{N} N_{k,xx}(\xi,\eta)\beta_{yk} + \sum_{k=N+1}^{2N} (j_{11}^2 P_{yk,\xi\xi} + 2 j_{11} j_{12} P_{yk,\xi\eta} + j_{12}^2 P_{yk,\eta\eta})\alpha_k$$

$$\beta_{y,yy} = \sum_{k=1}^{N} N_{k,yy}(\xi,\eta)\beta_{yk} + \sum_{k=N+1}^{2N} P_{yk,yy}(\xi,\eta)\alpha_k = \sum_{k=1}^{N} N_{k,yy}(\xi,\eta)\beta_{yk} + \sum_{k=N+1}^{2N} (j_{21}^2 P_{yk,\xi\xi} + 2 j_{21} j_{22} P_{yk,\xi\eta} + j_{22}^2 P_{yk,\eta\eta})\alpha_k$$

$$\beta_{y,xy} = \sum_{k=1}^{N} N_{k,xy}(\xi,\eta)\beta_{yk} + \sum_{k=N+1}^{2N} P_{yk,xy}(\xi,\eta)\alpha_k = \underbrace{\sum_{k=1}^{N} N_{k,xy}(\xi,\eta)\beta_{yk}}_{\beta^{1_{xx}}} + \sum_{k=N+1}^{2N} (j_{11} j_{21} P_{yk,\xi\xi} + [j_{11} j_{22} + j_{12} j_{21}] P_{yk,\xi\eta} + j_{11} j_{21} P_{yk,\eta\eta})\alpha_k$$

with:

$$\beta^{1}{}_{,xx}=\sum_{k=1}^{N}\begin{pmatrix}0 & j_{11}^2 N_{k,\xi\xi}+2j_{11}j_{12}N_{k,\xi\eta}+j_{12}^2 N_{k,\eta\eta} & 0\\ 0 & j_{21}^2 N_{k,\xi\xi}+2j_{21}j_{22}N_{k,\xi\eta}+j_{22}^2 N_{k,\eta\eta} & 0\\ 0 & j_{11}j_{21}N_{k,\xi\xi}+[j_{11}j_{22}+j_{12}j_{21}]N_{k,\xi\eta}+j_{12}j_{22}N_{k,\eta\eta} & 0\\ 0 & 0 & j_{11}^2 N_{k,\xi\xi}+2j_{11}j_{12}N_{k,\xi\eta}+j_{12}^2 N_{k,\eta\eta}\\ 0 & 0 & j_{21}^2 N_{k,\xi\xi}+2j_{21}j_{22}N_{k,\xi\eta}+j_{22}^2 N_{k,\eta\eta}\\ 0 & 0 & j_{11}j_{21}N_{k,\xi\xi}+[j_{11}j_{22}+j_{12}j_{21}]N_{k,\xi\eta}+j_{12}j_{22}N_{k,\eta\eta}\end{pmatrix}\begin{pmatrix}w_k\\ \beta_{xk}\\ \beta_{yk}\end{pmatrix}$$

the first contribution to $\beta_{,xx}$ in the expression above and:

$$u_{,xx}=\sum_{k=1}^{n}\begin{pmatrix} j_{11}^2 N_{k,\xi\xi}+2j_{11}j_{12}N_{k,\xi\eta}+j_{12}^2 N_{k,\eta\eta} & 0 \\ j_{21}^2 N_{k,\xi\xi}+2j_{21}j_{22}N_{k,\xi\eta}+j_{22}^2 N_{k,\eta\eta} & 0 \\ j_{11}j_{21}N_{k,\xi\xi}+[j_{11}j_{22}+j_{12}j_{21}]N_{k,\xi\eta}+j_{12}j_{22}N_{k,\eta\eta} & 0 \\ 0 & j_{11}^2 N_{k,\xi\xi}+2j_{11}j_{12}N_{k,\xi\eta}+j_{12}^2 N_{k,\eta\eta} \\ 0 & j_{21}^2 N_{k,\xi\xi}+2j_{21}j_{22}N_{k,\xi\eta}+j_{22}^2 N_{k,\eta\eta} \\ 0 & j_{11}j_{21}N_{k,\xi\xi}+[j_{11}j_{22}+j_{12}j_{21}]N_{k,\xi\eta}+j_{12}j_{22}N_{k,\eta\eta} \end{pmatrix}\begin{pmatrix} u_k \\ v_k \end{pmatrix}$$

that is to say still in matric form that:

$$\mathbf{T}=\bar{\mathbf{H}}_m^c\begin{pmatrix} u_{,xx} \\ u_{,yy} \\ u_{,xy} \\ v_{,xx} \\ v_{,yy} \\ v_{,xy} \end{pmatrix}+\bar{\mathbf{H}}_f^c\begin{pmatrix} \beta_{x,xx} \\ \beta_{x,yy} \\ \beta_{x,xy} \\ \beta_{y,xx} \\ \beta_{y,yy} \\ \beta_{y,xy} \end{pmatrix}$$

$$\mathbf{T}=\bar{\mathbf{H}}_m^c\sum_{k=1}^{N}\mathbf{P_{cm}^k}\begin{pmatrix} u_k \\ v_k \end{pmatrix}+\bar{\mathbf{H}}_f^c\sum_{k=1}^{N}\mathbf{P_{c\beta}^k}\begin{pmatrix} w_k \\ \beta_{xk} \\ \beta_{yk} \end{pmatrix}+\bar{\mathbf{H}}_f^c\sum_{k=N+1}^{2N}\alpha_k\begin{pmatrix} C_k(j_{11}^2 P_{k,\xi\xi}+2j_{11}j_{12}P_{k,\xi\eta}+j_{12}^2 P_{k,\eta\eta}) \\ C_k(j_{21}^2 P_{k,\xi\xi}+2j_{21}j_{22}P_{k,\xi\eta}+j_{22}^2 P_{k,\eta\eta}) \\ C_k(j_{11}j_{21}P_{k,\xi\xi}+[j_{11}j_{22}+j_{12}j_{21}]P_{k,\xi\eta}+j_{11}j_{21}P_{k,\eta\eta}) \\ S_k(j_{11}^2 P_{k,\xi\xi}+2j_{11}j_{12}P_{k,\xi\eta}+j_{12}^2 P_{k,\eta\eta}) \\ S_k(j_{21}^2 P_{k,\xi\xi}+2j_{21}j_{22}P_{k,\xi\eta}+j_{22}^2 P_{k,\eta\eta}) \\ S_k(j_{11}j_{21}P_{k,\xi\xi}+[j_{11}j_{22}+j_{12}j_{21}]P_{k,\xi\eta}+j_{11}j_{21}P_{k,\eta\eta}) \end{pmatrix}$$

$$\mathbf{T}=\bar{\mathbf{H}}_m^c\sum_{k=1}^{N}\mathbf{P_{cm}^k}\begin{pmatrix} u_k \\ v_k \end{pmatrix}+\bar{\mathbf{H}}_f^c\sum_{k=1}^{N}\mathbf{P_{c\beta}^k}\begin{pmatrix} w_k \\ \beta_{xk} \\ \beta_{yk} \end{pmatrix}+\sum_{k=N+1}^{2N}\bar{\mathbf{H}}_f^c T_2\begin{pmatrix} C_k P_{k,\xi\xi} \\ C_k P_{k,\eta\eta} \\ C_k P_{k,\xi\eta} \\ S_k P_{k,\xi\xi} \\ S_k P_{k,\eta\eta} \\ S_k P_{k,\xi\eta} \end{pmatrix}\alpha_k$$

$$\mathbf{T}=\bar{\mathbf{H}}_m^c\sum_{k=1}^{N}\mathbf{P_{cm}^k}\begin{pmatrix} u_k \\ v_k \end{pmatrix}+\bar{\mathbf{H}}_f^c\sum_{k=1}^{N}\mathbf{P_{c\beta}^k}\begin{pmatrix} w_k \\ \beta_{xk} \\ \beta_{yk} \end{pmatrix}+\bar{\mathbf{H}}_f^c\mathbf{T_2}\sum_{k=N+1}^{2N}\mathbf{T_{ck}}\,\alpha_k$$

$$\mathbf{T}=\bar{\mathbf{H}}_m^c\mathbf{P_{cm}}\mathbf{U_m}+\bar{\mathbf{H}}_f^c\mathbf{P_{c\beta}}\mathbf{U_{f\beta}}+\bar{\mathbf{H}}_f^c\mathbf{T_2}\mathbf{T_\alpha}\alpha=\mathbf{B_{cm}}\mathbf{U_m}+\mathbf{B_{c\beta}}\mathbf{U_{f\beta}}+\mathbf{B_{c\alpha}}\alpha$$

Where:

$$U_m=\begin{pmatrix} u_1 \\ v_1 \\ M \\ u_N \\ v_N \end{pmatrix}$$

$$\mathbf{T}_\alpha = (\mathbf{T}_{c(N+1)} \cdots \mathbf{T}_{c2N})$$

$$T_2=\begin{pmatrix} t_2 & 0 \\ 0 & t_2 \end{pmatrix} \quad \text{with} \quad t_2=\begin{pmatrix} j_{11}^2 & j_{12}^2 & 2\,j_{11}\,j_{12} \\ j_{21}^2 & j_{22}^2 & 2\,j_{21}\,j_{22} \\ j_{11}\,j_{21} & j_{12}\,j_{22} & j_{11}\,j_{22}+j_{12}\,j_{21} \end{pmatrix}$$

$$u_{f\beta}=\begin{pmatrix} w_1 \\ \beta_{x1} \\ \beta_{y1} \\ \vdots \\ w_N \\ \beta_{xN} \\ \beta_{yN} \end{pmatrix}$$

One can also write:

$$T=\bar{H}_m^c u_{,xx}+\bar{H}_f^c \beta_{,xx}=B_{cm} U_m+B_{c\beta} U_{f\beta}+B_{c\alpha}\,\alpha$$

By using the relation $\int_i^j (\bar{\gamma}_s-\gamma_s)\,ds=0$ with $\gamma_s = w_{,s} + \beta_s$ for each side ij element, one can obtain them α_K since this relation is still written:

$$w_j-w_i+\frac{L_k}{2}(C_k\,\beta_{xi}+S_k\,\beta_{yi}+C_k\,\beta_{xj}+S_k\,\beta_{yj})+\frac{2}{3}L_k\,\alpha_k=L_k\,\bar{\gamma}_{sk}$$

where:

$$\begin{aligned}\bar{\gamma}_{sk}&=\begin{pmatrix} C_k & S_k \end{pmatrix}\bar{\gamma}=\begin{pmatrix} C_k & S_k \end{pmatrix} H_{ct}^{-1}\left[T-H_{m\gamma}^T e-H'^{T}_{f\gamma}\kappa\right]\\ &=\begin{pmatrix} C_k & S_k \end{pmatrix} H_{ct}^{-1}\left[\left(B_{cm}-H_{m\gamma}{}^T B_m\right)U_m+\left(B_{c\beta}-H'_{f\gamma}{}^T B_{f\beta}\right)U_{f\beta}+\left(B_{c\alpha}-H'^{T}_{f\gamma}B_{f\alpha}\right)\alpha\right]\end{aligned}$$

The relation above is still written in matric form:

$$\mathbf{A}_\alpha \alpha = (\mathbf{A}_w + \mathbf{A}_\beta)\mathbf{U}_{f\beta} + \mathbf{A}_m \mathbf{U}_m$$

with:

$$\mathbf{A}_\alpha=\frac{2}{3}\begin{pmatrix} L_{N+1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & L_{2N} \end{pmatrix}-\begin{pmatrix} L_{N+1}C_{N+1} & L_{N+1}S_{N+1} \\ \vdots & \vdots \\ L_{2N}C_{2N} & L_{2N}S_{2N} \end{pmatrix}\mathbf{H}_{ct}^{-1}\left(\mathbf{B}_{c\alpha}-\mathbf{H}'_{f\gamma}{}^T\mathbf{B}_{f\alpha}\right)$$

$$A_w=-\frac{1}{2}\begin{pmatrix} -2 & L_{N+1}C_{N+1} & L_{N+1}S_{N+1} & 2 & L_{N+1}C_{N+1} & L_{N+1}S_{N+1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & L_{k+1}C_{k+1} & L_{k+1}S_{k+1} & 2 & L_{k+1}C_{k+1} & L_{k+1}S_{k+1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & L_{2N-1}C_{2N-1} & L_{2N-1}S_{2N-1} & 2 & L_{2N-1}C_{2N-1} & L_{2N-1}S_{2N-1} \\ 2 & L_{2N}C_{2N} & L_{2N}S_{2N} & 0 & 0 & \cdots & \cdots & 0 & 0 & -2 & L_{2N}C_{2N} & L_{2N}S_{2N} \end{pmatrix}$$

$$A_\beta=\begin{pmatrix} L_{N+1}C_{N+1} & L_{N+1}S_{N+1} \\ \vdots & \vdots \\ L_{2N}C_{2N} & L_{2N}S_{2N} \end{pmatrix}\mathbf{H}_{ct}^{-1}\left(\mathbf{B}_{c\beta}-\mathbf{H}'^{T}_{f\gamma}\mathbf{B}_{f\beta}\right)$$

$$A_m=\begin{pmatrix} L_{N+1}C_{N+1} & L_{N+1}S_{N+1} \\ \vdots & \vdots \\ L_{2N}C_{2N} & L_{2N}S_{2N} \end{pmatrix}H_{ct}^{-1}\left(B_{cm}-H_{m\gamma}^{T}B_m\right)$$

As follows:

$$\boldsymbol{\alpha}=\mathbf{P}_\beta\mathbf{U}_{f\beta}+\mathbf{P}_\mathbf{m}\mathbf{U}_\mathbf{m}$$

with:

$$\mathbf{P}_\beta=\mathbf{A}_\alpha^{-1}(\mathbf{A}_w+\mathbf{A}_\beta)$$

$$\mathbf{P}_m=\mathbf{A}_\alpha^{-1}\mathbf{A}_m$$

what implies:

$$T=\left(B_{cm}+B_{c\alpha}P_m\right)U_m+\left(B_{c\beta}+B_{c\alpha}P_\beta\right)U_{f\beta}$$

Note:

For the elements of the type `DKT` and `DST` , one has $\mathbf{B}_{\mathbf{cm}}=\mathbf{B}_{c\beta}=0$. It results from it from the simplified expressions of the preceding equations.

4.4 Elementary matrix of rigidity

4.4.1 Elementary matrix of rigidity for the elements Q4γ

One takes again the forms of the matrices of rigidity given to [§4.4.1] of the reference material [R3.07.03] and one replaces $H_{\mathbf{mf}}$ by $H'_{\mathbf{mf}}$, H_f by H'_f and $H_{f\gamma}$ by $H'_{f\gamma}$. It will be noted that in [R3.07.03] the results were presented without term of coupling transverse membrane shearing or transverse inflection shearing. They here are added.

4.4.2 Elementary matrix of rigidity for the elements DKT, DKQ

One takes again the forms of the matrices of rigidity given to [§4.4.1] of the reference material [R3.07.03] and one replaces $H_{\mathbf{mf}}$ by $H'_{\mathbf{mf}}$, H_f by H'_f . Since the relation $\bar{\gamma}=0$ is satisfied the couplings transverse membrane shearing or transverse inflection shearing are non-existent.

Note:

Offsetting introducing a coupling membrane-inflection, it can appear an incompatibility between spaces of approximations. That is due to the fact that the "contributions of pure membrane" are constant by element and they result from the linear approximation from translations while the "contributions of membrane related to offsetting" are linear by element and they result from the approximation of rotations.

To avoid this incompatibility, one uses a method of approximation mixed to calculate the term of coupling. With this intention one makes a distinction between the term of coupling contributing to the membrane κ^{tran} and the term of coupling contributing to the inflection κ^{rot}. In the simplest case of excentré homogeneous plate, one a:

$$
\begin{aligned}
&N = H_m e + d H_m \kappa^{tran}\\
&M = d H_m e + (H_f + d^2 H_m)\kappa^{rot}\\
&T = H_{\mathbf{ct}} \gamma
\end{aligned}
$$

To avoid the incompatibility, one discretizes κ^{tran} in the same space of approximation as e while κ^{rot} is discretized in a classical way.

4.4.3 Elementary matrix of rigidity for the elements DST, DSQ

One a:

$$
\begin{aligned}
\delta W^e_{\mathrm{int}} = &\int_e \boldsymbol{\delta e}\left(\mathbf{H_m e} + \mathbf{H}'_{mf}\boldsymbol{\kappa} + \mathbf{H}_{m\gamma}\boldsymbol{\gamma} - \mathbf{H}_{m\gamma}\mathbf{H}_{ct}^{-1}\mathbf{T}\right) + \boldsymbol{\delta\kappa}\left(\mathbf{H}'_{mf}\mathbf{e} + \mathbf{H}'_{f}\boldsymbol{\kappa} + \mathbf{H}'_{f\gamma}\boldsymbol{\gamma} - \mathbf{H}'_{f\gamma}\mathbf{H}_{ct}^{-1}\mathbf{T}\right) + \boldsymbol{\delta}\,\mathbf{T}\mathbf{H}_{ct}^{-1}\mathbf{T}\,dS =\\
&\int_e \boldsymbol{\delta e}\left(\left[\mathbf{H_m} - \mathbf{H}_{m\gamma}\mathbf{H}_{ct}^{-1}\mathbf{H}^{\mathbf{T}}_{m\gamma}\right]\mathbf{e} + \left[\mathbf{H}'_{mf} - \mathbf{H}_{m\gamma}\mathbf{H}_{ct}^{-1}\mathbf{H}'^{\mathbf{T}}_{f\gamma}\right]\boldsymbol{\kappa}\right) + \boldsymbol{\delta\kappa}\left(\left[\mathbf{H}'_{mf} - \mathbf{H}'_{f\gamma}\mathbf{H}_{ct}^{-1}\mathbf{H}^{\mathbf{T}}_{m\gamma}\right]\mathbf{e}\,dS\right.\\
&+\int_e \left[\mathbf{H}'_{f} - \mathbf{H}'_{f\gamma}\mathbf{H}_{ct-1}\mathbf{H}'^{\mathbf{T}}_{f\gamma}\right]\boldsymbol{\kappa}\Big) + \boldsymbol{\delta}\,\mathbf{T}\mathbf{H}_{ct-1}\mathbf{T}\,dS
\end{aligned}
$$

That is to say still:

$$
\delta W^e_{\mathrm{int}} = \int_e \delta e\left(H''_m e + H''_{mf}\kappa\right) + \delta\kappa\left(H^{\text{¿}\,T}_{mf} + H''_f \kappa\right) + \delta\,\mathbf{T}\mathbf{H}_{ct-1} T\,dS
$$

where:

$$
\begin{aligned}
H''_m &= H_m - H_{m\gamma} H_{ct}^{-1} H^T_{m\gamma}\\
H''_{\mathbf{mf}} &= H'_{\mathbf{mf}} - H_{m\gamma} H_{ct}^{-1} H^{\text{¿}\,T}_{f\gamma}\\
\text{¿}\,H''_f &= H'_f - H'_{f\gamma} H_{ct}^{-1} H^{\text{¿}\,T}_{f\gamma}\text{¿¿¿}
\end{aligned}
$$

From where:

$$
\begin{aligned}
\delta W^e_{\mathrm{int}} = \int_e \Big(&\delta\,\mathbf{U}^T_m \mathbf{B}^T_m \mathbf{H}''_m \mathbf{B}_m \mathbf{U}_m + \delta\,\mathbf{U}^T_m \mathbf{B}^T_m \mathbf{H}''_{mf} \mathbf{B}_f \mathbf{U}_f + \delta\,\mathbf{U}^T_f \mathbf{B}^T_f \mathbf{H}''^{\,T}_{mf} \mathbf{B}_m \mathbf{U}_m + \delta\,\mathbf{U}^T_f \mathbf{B}^T_f \mathbf{H}''_f \mathbf{B}_f \mathbf{U}_f\\
&+\delta\alpha^T \mathbf{B}^T_{c\alpha}\mathbf{H}_{ct}^{-1}\mathbf{B}_{c\alpha}\alpha + \delta\alpha^T \mathbf{B}^T_{c\alpha}\mathbf{H}_{ct}^{-1}\mathbf{B}_{cm}\mathbf{U}_m + \delta\,\mathbf{U}^T_m \mathbf{B}^T_{cm}\mathbf{H}_{ct}^{-1}\mathbf{B}_{c\alpha}\alpha + \delta\,\mathbf{U}^T_m \mathbf{B}^T_{cm}\mathbf{H}_{ct}^{-1}\mathbf{B}_{cm}\mathbf{U}_m\\
&+\delta\alpha^T \mathbf{B}^T_{c\alpha}\mathbf{H}_{ct}^{-1}\mathbf{B}_{c\beta}\mathbf{U}_{f\beta} + \delta\,\mathbf{U}^T_{f\beta}\mathbf{B}^T_{c\beta}\mathbf{H}_{ct}^{-1}\mathbf{B}_{c\alpha}\alpha + \delta\,\mathbf{U}^T_{f\beta}\mathbf{B}^T_{c\beta}\mathbf{H}_{ct}^{-1}\mathbf{B}_{c\beta}\mathbf{U}_{f\beta}\\
&+\delta\,\mathbf{U}^T_m \mathbf{B}^T_{cm}\mathbf{H}_{ct}^{-1}\mathbf{B}_{c\beta}\mathbf{U}_{f\beta} + \delta\,\mathbf{U}^T_{f\beta}\mathbf{B}^T_{c\beta}\mathbf{H}_{ct}^{-1}\mathbf{B}_{cm}\mathbf{U}_m\Big)\,dS
\end{aligned}
$$

$$
\begin{aligned}
&= \delta\,\mathbf{U}^T_m\left(\int_e \mathbf{B}^T_m \mathbf{H}''_m \mathbf{B}_m\,\mathrm{dS}\right)\mathbf{U}_m + \delta\,\mathbf{U}^T_f\left(\int_e \mathbf{B}^T_f \mathbf{H}''_f \mathbf{B}_f\,\mathrm{dS}\right)\mathbf{U}_f\\
&+\delta\,\mathbf{U}^T_m\left(\int_e \mathbf{B}^T_m \mathbf{H}''_{\mathrm{mf}} \mathbf{B}_f\,\mathrm{dS}\right)\mathbf{U}_f + \delta\,\mathbf{U}^T_f\left(\int_e \mathbf{B}^T_f \mathbf{H}''^{T}_{\mathrm{mf}} \mathbf{B}_m\,\mathrm{dS}\right)\mathbf{U}_m\\
&+\delta\,\alpha^T\left(\int_e \mathbf{B}^T_{c\alpha}\mathbf{H}_{\mathrm{ct}}^{-1}\mathbf{B}_{c\alpha}\,\mathrm{dS}\right)\alpha + \delta\,\alpha^T\left(\int_e \mathbf{B}^T_{c\alpha}\mathbf{H}_{\mathrm{ct}}^{-1}\mathbf{B}_{cm}\,\mathrm{dS}\right)\mathbf{U_m}\\
&+\delta\,\mathbf{U}_{\text{¿}}{}_m{}^T\left(\int_e \mathbf{B}^T_{cm}\mathbf{H}_{ct}^{-1}\mathbf{B}_{c\alpha}\,dS\right)\alpha + \delta\,\mathbf{U}^T_m\left(\int_e \mathbf{B}^T_{\mathrm{cm}}\mathbf{H}_{\mathrm{ct}}^{-1}\mathbf{B}_{\mathrm{cm}}\,\mathrm{dS}\right)\mathbf{U}_m
\end{aligned}
$$

$$+\delta\alpha^T\left(\int_e \mathbf{B}_{c\alpha}^T \mathbf{H}_{ct}^{-1} \mathbf{B}_{c\beta} dS\right)\mathbf{U}_{f\beta}+\delta\mathbf{U}_{f\beta}^T\left(\int_e \mathbf{B}_{c\beta}^T \mathbf{H}_{ct}^{-1} \mathbf{B}_{c\alpha} dS\right)\alpha$$
$$+\delta\mathbf{U}_{f\beta}^T\left(\int_e \mathbf{B}_{c\beta}^T \mathbf{H}_{ct}^{-1} \mathbf{B}_{c\beta} dS\right)\mathbf{U}_{f\beta}$$
$$+\delta\mathbf{U}_m^T\left(\int_e \mathbf{B}_{cm}^T \mathbf{H}_{ct}^{-1} \mathbf{B}_{c\beta} dS\right)\mathbf{U}_{f\beta}+\delta\mathbf{U}_{f\beta}^T\left(\int_e \mathbf{B}_{c\beta}^T \mathbf{H}_{ct}^{-1} \mathbf{B}_{cm} dS\right)\mathbf{U}_m$$
$$=\delta\mathbf{U}_m^T\mathbf{K}_m\mathbf{U}_m+\delta\mathbf{U}_f^T\mathbf{K}_f\mathbf{U}_f+\delta\mathbf{U}_m^T\mathbf{K}_{mf}\mathbf{U}_f+\delta\mathbf{U}_f^T\mathbf{K}_{fm}\mathbf{U}_m+\delta\alpha^T\mathbf{K}_{\alpha\alpha}\alpha+\delta\mathbf{U}_m^T\mathbf{K}_{m\alpha}\alpha+\delta\alpha^T\mathbf{K}_{m\alpha}^T\mathbf{U}_m$$
$$+\delta\mathbf{U}_{f\beta}^T\mathbf{K}_{\beta\alpha}\alpha+\delta\alpha^T\mathbf{K}_{\beta\alpha}^T\mathbf{U}_{f\beta}+\delta\mathbf{U}_m^T\mathbf{K}_{m\beta}\mathbf{U}_{f\beta}+\delta\mathbf{U}_{f\beta}^T\mathbf{K}_{\beta m}\mathbf{U}_m+\delta\mathbf{U}_{f\beta}^T\mathbf{K}_{\beta\beta}\mathbf{U}_{f\beta}$$

with:

$$K_m=\int_s [B_m^T H_m^{''} B_m+B_{cm}^T H_{ct}^{-1} B_{cm}]\, dS$$

It is also known that $U_f=(U_{f\beta},\alpha)$ from where it results that:

$$K_f=\begin{pmatrix} K_{f11} & K_{f12} \\ K_{f12}^T & K_{22}\end{pmatrix} \quad \text{with:} \quad \left| \begin{array}{l} K_{f11}=\int_s B_{f\beta}^T H_f^{''} B_{f\beta}\, dS \\ K_{f12}=\int_s B_{f\beta}^T H_f^{''} B_{f\alpha}\, dS \\ K_{f22}=\int_s B_{f\alpha}^T H_f^{''} B_{f\alpha}\, dS \end{array}\right.$$

$$K_{mf}=\begin{pmatrix} K_{mf11} & K_{mf12}\end{pmatrix} \quad \text{with:} \quad \left| \begin{array}{l} K_{mf11}=\int_s B_m^T H_{mf}^{''} B_{f\beta}\, dS \\ K_{mf12}=\int_s B_m^T H_{mf}^{''} B_{f\alpha}\, dS \end{array}\right.$$

$$K_{fm}=K_{mf}^T$$

Using the fact that $\alpha=P_\beta U_{f\beta}+P_m U_m$ one from of deduced that:

$$\delta W_{\text{int}}=\delta U_m^T K_m^{'} U_m+\delta U_{f\beta}^T K_f^{'} U_{f\beta}+\delta U_m^T K_{mf}^{'} U_{f\beta}+\delta U_{f\beta}^T K_{fm}^{'} U_m$$

where:

$$K_m^{'}=K_m+P_m^T(K_{f22}+K_{\alpha\alpha})P_m+(K_{mf12}+K_{m\alpha})P_m+P_m^T(K_{mf12}^T+K_{m\alpha}^T)$$
$$K_f^{'}=K_{f11}+K_{\beta\beta}+P_\beta^T(K_{f22}+K_{\alpha\alpha})P_\beta+(K_{f12}+K_{\beta\alpha})P_\beta+P_\beta^T(K_{f12}^T+K_{\beta\alpha}^T)$$
$$K_{mf}^{'}=K_{mf11}+K_{m\beta}+(K_{mf12}+K_{m\alpha})P_\beta+P_m^T(K_{f12}^T+K_{\beta\alpha}^T)+P_m^T(K_{f22}+K_{\alpha\alpha})P_\beta$$
$$K_{fm}^{'}=K_{mf}^{'T}$$

This is still written:

$$\delta W_{\text{int}}^e=(\delta U_m,\delta U_{f\beta})\,K\begin{pmatrix}U_m\\ U_{f\beta}\end{pmatrix}$$

where: $\mathbf{K}_{[5N\times5N]}=\begin{pmatrix}\mathbf{K}_{m[2N\times2N]}^{'} & \mathbf{K}_{mf[2N\times3N]}^{'} \\ \mathbf{K}_{mf[3N\times2N]}^{'T} & \mathbf{K}_{f[3N\times3N]}^{'}\end{pmatrix}$ is the elementary matrix of rigidity for an element of excentré plate DST.

4.5 Elementary matrix of mass

The terms of the matrix of mass are obtained after discretization of the following variational formulation:

$$\delta W_{mass}^{ac} = \int_{d-h/2}^{d+h/2} \int_S \rho\, \ddot{u}\, \delta u\, dz dS$$
$$= \int_S \rho_m (\ddot{u}\,\delta u + \ddot{v}\,\delta v + \ddot{w}\,\delta w) + (\rho_{mf} + d\rho_m)(\ddot{u}\,\delta\beta_x + \ddot{v}\,\delta\beta_y + \ddot{\beta}_x \delta u + \ddot{\beta}_y \delta v)\, dS +$$
$$\int_S (\rho_f + 2\, d\rho_{mf} + d^2 \rho_m)(\ddot{\beta}_x \delta\beta_x + \ddot{\beta}_y \delta\beta_y)\, dS$$

¿

with $\rho_m = \int_{-h/2}^{+h/2} \rho\, dz$, $\rho_{mf} = \int_{-h/2}^{+h/2} \rho\, z dz$, and $\rho_f = \int_{-h/2}^{+h/2} \rho z^2 dz$.

Note:

If the plate is homogeneous or symmetrical compared to its average layer then $\rho_{mf} = 0$.

4.5.1 Matrix of elementary mass classical

4.5.1.1 Element Q4gamma

The discretization of displacement for this isoparametric element is:

$$\mathbf{u} = \sum_{k=1}^{N} \mathbf{N_k} \begin{pmatrix} u_k \\ v_k \\ w_k \\ \beta_{xk} \\ \beta_{yk} \end{pmatrix} k = 1, \ldots, N$$

The matrix of mass, in the base where the degrees of freedom are gathered according to the directions of translation and rotation, has then as an expression:

$$M = \begin{pmatrix} M_m & 0 & 0 & M_{mf} & 0 \\ 0 & M_m & 0 & 0 & M_{mf} \\ 0 & 0 & M_m & 0 & 0 \\ M_{mf}^T & 0 & 0 & M_f & 0 \\ 0 & M_{mf}^T & 0 & 0 & M_f \end{pmatrix}$$

with: $M_m = \int_S \rho_m N^T N\, dS$

$$M_{mf} = \int_S (\rho_{mf} + d\rho_m) N^T N\, dS$$

$$M_f = \int_S (\rho_f + 2\mathrm{d}\rho_{mf} + d^2 \rho_m) N^T N\, dS$$

where: $N = (N_1 \ln_k)$.

For the continuation, one poses $\rho'_{mf} = \rho_{mf} + d\rho_m$ and $\rho'_f = \rho_f + 2\mathrm{d}\rho_{mf} + d^2 \rho_m$.

4.5.1.2 Elements of the type DKT, DST

Like:

$$\begin{pmatrix} w \\ \beta_x \\ \beta_y \end{pmatrix} = \sum_{k=1}^{N} N_k(\xi,\eta) \begin{pmatrix} w_k \\ \beta_{xk} \\ \beta_{yk} \end{pmatrix} + \sum_{k=N+1}^{2N} \begin{pmatrix} 0 \\ P_{xk}(\xi,\eta) \\ P_{yk}(\xi,\eta) \end{pmatrix} \alpha_k$$

where: $\alpha = P_m U_m + P_\beta U_{f\beta}$

one from of deduced that:

$$\begin{pmatrix} w \\ \beta_x \\ \beta_y \end{pmatrix} = \sum_{k=1}^{N} \begin{pmatrix} 0 & 0 & N_k(\xi,\eta) & 0 & 0 \\ N_{kxu}(\xi,\eta) & N_{kxv}(\xi,\eta) & N_{kxw}(\xi,\eta) & N_{kxx}(\xi,\eta) & N_{kxy}(\xi,\eta) \\ N_{kyu}(\xi,\eta) & N_{kyv}(\xi,\eta) & N_{kyw}(\xi,\eta) & N_{kyx}(\xi,\eta) & N_{kyy}(\xi,\eta) \end{pmatrix} \begin{pmatrix} u_k \\ v_k \\ w_k \\ \beta_{xk} \\ \beta_{yk} \end{pmatrix}.$$

The matrix of mass has then as an expression:

$$M = \begin{pmatrix} M'_m & M'_{mf} \\ M'_{fm} & M'_f \end{pmatrix}$$

The membrane part M'_m elementary matrix of mass is composed of the blocks kp (k ième line and p ième column) following:

$$\rho_m \begin{pmatrix} N_k N_p & 0 \\ 0 & N_k N_p \end{pmatrix} + \rho'_{mf} \begin{pmatrix} N_k N_{pxu} + N_{kxu} N_p & N_k N_{pxv} + N_{kyu} N_p \\ N_k N_{pyu} + N_{kxv} N_p & N_k N_{pyv} + N_{kyv} N_p \end{pmatrix}$$
$$+ \rho'_f \begin{pmatrix} N_{kxu} N_{pxu} + N_{kyu} N_{pyu} & N_{kxu} N_{pxv} + N_{kyu} N_{pyv} \\ N_{pxu} N_{kxv} + N_{pyu} N_{kyv} & N_{kxv} N_{pxv} + N_{kyv} N_{pyv} \end{pmatrix}$$

The inflection part M'_f is composed of the blocks kp (k ième line and p ième column) following:

$$\rho_m \begin{pmatrix} N_k N_p & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \rho'_f \begin{pmatrix} N_{kxw} N_{pxw} + N_{kyw} N_{pyw} & N_{kxw} N_{pxx} + N_{kyw} N_{pyx} & N_{kxw} N_{pxy} + N_{kyw} N_{pyy} \\ N_{kxx} N_{pxw} + N_{kyx} N_{pyw} & N_{kxx} N_{pxx} + N_{kyx} N_{pyx} & N_{kxx} N_{pxy} + N_{kyx} N_{pyy} \\ N_{kxy} N_{pxw} + N_{kyy} N_{pyw} & N_{kxy} N_{pxx} + N_{kyy} N_{pyx} & N_{kxy} N_{pxy} + N_{kyy} N_{pyy} \end{pmatrix}$$

The coupling part between the membrane and the inflection M'_{mf} is composed of the blocks kp (k ième line and p ième column) following:

$$\rho'_{mf}\begin{pmatrix} N_k N_{pxw} & N_k N_{pxx} & N_k N_{pxy} \\ N_k N_{pyw} & N_k N_{pyx} & N_k N_{pyy}\end{pmatrix}+$$

$$\rho'_f\begin{pmatrix} N_{kxu}N_{pxw}+N_{kyu}N_{pyw} & N_{kxu}N_{pxx}+N_{kyu}N_{pyx} & N_{kxu}N_{pxy}+N_{kyu}N_{pyy} \\ N_{kxv}N_{pxw}+N_{kyv}N_{pyw} & N_{kxv}N_{pxx}+N_{kyv}N_{pyx} & N_{kxv}N_{pxy}+N_{kyv}N_{pyy}\end{pmatrix}$$

The coupling part between the inflection and the membrane M'_{fm} is composed of the blocks kp (k ième line and p ième column) following:

$$\rho'_{mf}\begin{pmatrix} N_{kxw}N_p & N_{kyw}N_p \\ N_{kxx}N_p & N_{kyx}N_p \\ N_{kxy}N_p & N_{kyy}N_p\end{pmatrix}+\rho'_f\begin{pmatrix} N_{kxw}N_{pxu}+N_{kyw}N_{pyu} & N_{kxw}N_{pxv}+N_{kyw}N_{pyv} \\ N_{kxx}N_{pxu}+N_{kyx}N_{pyu} & N_{kxx}N_{pxv}+N_{kyx}N_{pyv} \\ N_{kxy}N_{pxu}+N_{kyy}N_{pyu} & N_{kxy}N_{pxv}+N_{kyy}N_{pyv}\end{pmatrix}$$

4.5.2 Elementary matrix of improved mass

As the arrow of a flexbeam only can be represented by a linear approximation with difficulty, one can enrich the functions by form for the terms of inflection. This approach is used in *Code_Aster* for the elements of the type DKT, DST and Q4G where the functions of form used in the calculation of the matrix of mass of inflection are of order 3. The interpolation for w is written as follows:

$$\mathrm{w} = \sum_{k=1}^{N} \mathrm{N}_{3(k-1)+1}(\xi,\eta)\mathrm{w}_k + \mathrm{N}_{3(k-1)+2}(\xi,\eta)\mathrm{w}_{,\xi k} + \mathrm{N}_{3(k-1)+3}(\xi,\eta)\mathrm{w}_{,\eta k}$$

4.5.2.1 Elements of the type DKT

It is known that in the approximation of one Coils-Kirchhoff has $\beta_x=-w_{,x}$ and $\beta_y=-w_{,y}$ in any point of the element.
Because of discretization stated above one a:

$$w=\sum_{k=1}^{N} N_{3(k-1)+1}(\xi,\eta)w_k+(J_{11}N_{3(k-1)+2}(\xi,\eta)+J_{21}N_{3(k-1)+3}(\xi,\eta))w_{,xk}+ \\ (J_{12}N_{3(k-1)+2}(\xi,\eta)+J_{22}N_{3(k-1)+3}(\xi,\eta))w_{,yk}$$

since:

$$\begin{pmatrix} w_{,\xi k} \\ w_{,\eta k}\end{pmatrix}=\begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22}\end{pmatrix}\begin{pmatrix} w_{,xk} \\ w_{,yk}\end{pmatrix}$$

This is still written:

$$w \quad ¿\sum_{k=1}^{N} N'_{3(k-1)+1}(\xi,\eta)w_k+N'_{3(k-1)+2}(\xi,\eta)\beta_{xk}+N'_{3(k-1)+3}(\xi,\eta)\beta_{yk}$$

$$=\sum_{k=1}^{N} N_{kww}(\xi,\eta)w_k+N_{kwx}(\xi,\eta)\beta_{xk}+N_{kwy}(\xi,\eta)\beta_{yk}$$

where:

$$\begin{array}{l} N'_{3(k-1)+1}(\xi,\eta)=N_{3(k-1)+1}(\xi,\eta) \\ N'_{3(k-1)+2}(\xi,\eta)=-J_{11}N_{3(k-1)+2}(\xi,\eta)-J_{21}N_{3(k-1)+3}(\xi,\eta) \\ N'_{3(k-1)+3}(\xi,\eta)=-J_{12}N_{3(k-1)+2}(\xi,\eta)-J_{22}N_{3(k-1)+3}(\xi,\eta) \end{array}.$$

As follows:

$$\begin{pmatrix} w \\ \beta_x \\ \beta_y \end{pmatrix}=\sum_{k=1}^{N}\begin{pmatrix} 0 & 0 & N_{kww}(\xi,\eta) & N_{kwx}(\xi,\eta) & N_{kwy}(\xi,\eta) \\ N_{kxu}(\xi,\eta) & N_{kxv}(\xi,\eta) & N_{kxw}(\xi,\eta) & N_{kxx}(\xi,\eta) & N_{kxy}(\xi,\eta) \\ N_{kyu}(\xi,\eta) & N_{kyv}(\xi,\eta) & N_{kyw}(\xi,\eta) & N_{kyx}(\xi,\eta) & N_{kyy}(\xi,\eta) \end{pmatrix}\begin{pmatrix} u_k \\ v_k \\ w_k \\ \beta_{xk} \\ \beta_{yk} \end{pmatrix}$$

By not taking account of the effects of inertia, the matrix of mass has the following form thus:

$$M=\begin{pmatrix} M'_m & M'_{mf} \\ M'_{fm} & M'_f \end{pmatrix}$$

The membrane part M'_m elementary matrix of mass is composed of the blocks kp (k ième line and p ième column) following:

$$\rho_m\begin{pmatrix} N_k N_p & 0 \\ 0 & N_k N_p \end{pmatrix}+\rho'_{mf}\begin{pmatrix} N_k N_{pxu}+N_{kxu}N_p & N_k N_{pxv}+N_{kyu}N_p \\ N_k N_{pyu}+N_{kxv}N_p & N_k N_{pyv}+N_{kyv}N_p \end{pmatrix}$$
$$+\rho'_f\begin{pmatrix} N_{kxu}N_{pxu}+N_{kyu}N_{pyu} & N_{kxu}N_{pxv}+N_{kyu}N_{pyv} \\ N_{pxu}N_{kxv}+N_{pyu}N_{kyv} & N_{kxv}N_{pxv}+N_{kyv}N_{pyv} \end{pmatrix}$$

The membrane-inflection part M'_{mf} is composed of the blocks kp (k ième line and p ième column) following:

$$\rho'_{mf}\begin{pmatrix} N_k N_{pxw} & N_k N_{pxx} & N_k N_{pxy} \\ N_k N_{pyw} & N_k N_{pyx} & N_k N_{pyy} \end{pmatrix}$$
$$+\rho'_f\begin{pmatrix} N_{kxu}N_{pxw}+N_{kyu}N_{pyw} & N_{kxu}N_{pxx}+N_{kyu}N_{pyx} & N_{kxu}N_{pxy}+N_{kyu}N_{pyy} \\ N_{kxv}N_{pxw}+N_{kyv}N_{pyw} & N_{kxv}N_{pxx}+N_{kyv}N_{pyx} & N_{kxv}N_{pxy}+N_{kyv}N_{pyy} \end{pmatrix}$$

The inflection-membrane part M'_{fm} is composed of the blocks kp (k ième line and p ième column) following:

$$\rho'_{mf}\begin{pmatrix} N_{kxw}N_p & N_{kyw}N_p \\ N_{kxx}N_p & N_{kyx}N_p \\ N_{kxy}N_p & N_{kyy}N_p \end{pmatrix}+\rho'_f\begin{pmatrix} N_{kxw}N_{pxu}+N_{kyw}N_{pyu} & N_{kxw}N_{pxv}+N_{kyw}N_{pyv} \\ N_{kxx}N_{pxu}+N_{kyx}N_{pyu} & N_{kxx}N_{pxv}+N_{kyx}N_{pyv} \\ N_{kxy}N_{pxu}+N_{kyy}N_{pyu} & N_{kxy}N_{pxv}+N_{kyy}N_{pyv} \end{pmatrix}$$

The term M'_f of inflection is composed of the blocks kp (k ième line and p ième column) following:

$$\rho_m \begin{pmatrix} N_{kww} N_{pww} & N_{kww} N_{pwx} & N_{kww} N_{pwy} \\ N_{kwx} N_{pww} & N_{kwx} N_{pwx} & N_{kwx} N_{pwy} \\ N_{kwy} N_{pww} & N_{kwy} N_{pwx} & N_{kwy} N_{pwy} \end{pmatrix} +$$

$$\rho'_f \begin{pmatrix} N_{kxw} N_{pxw} + N_{kyw} N_{pyw} & N_{kxw} N_{pxx} + N_{kyw} N_{pyx} & N_{kxw} N_{pxy} + N_{kyw} N_{pyy} \\ N_{kxx} N_{pxw} + N_{kyx} N_{pyw} & N_{kxx} N_{pxx} + N_{kyx} N_{pyx} & N_{kxx} N_{pxy} + N_{kyx} N_{pyy} \\ N_{kxy} N_{pxw} + N_{kyy} N_{pyw} & N_{kxy} N_{pxx} + N_{kyy} N_{pyx} & N_{kxy} N_{pxy} + N_{kyy} N_{pyy} \end{pmatrix}$$

4.5.2.2 Elements of the type DST

It is known that for these elements one has $\beta_x = \gamma_x - w_{,x}$ and $\beta_y = \gamma_y - w_{,y}$ where the distortion γ is constant on the element.

Like:

$$w = \sum_{k=1}^{N} N_{3(k-1)+1}(\xi,\eta) w_k + (J_{11} N_{3(k-1)+2}(\xi,\eta) + J_{21} N_{3(k-1)+3}(\xi,\eta)) w_{,xk} + (J_{12} N_{3(k-1)+2}(\xi,\eta) + J_{22} N_{3(k-1)+3}(\xi,\eta)) w_{,yk}$$

one can also write:

$$w \; ¿ \sum_{k=1}^{N} N'_{3(k-1)+1}(\xi,\eta) w_k + N'_{3(k-1)+2}(\xi,\eta) \beta_{xk} + N'_{3(k-1)+3}(\xi,\eta) \beta_{yk} + (J_{11} \overline{\gamma}_x + J_{12} \overline{\gamma}_y) \Sigma N_{3(k-1)+2}(\xi,\eta) + (J_{21} \overline{\gamma}_x + J_{22} \overline{\gamma}_y) \Sigma N_{3(k-1)+3}(\xi,\eta)$$

where:

$$\left| \begin{array}{l} N'_{3(k-1)+1}(\xi,\eta) = N_{3(k-1)+1}(\xi,\eta) \\ N'_{3(k-1)+2}(\xi,\eta) = -J_{11} N_{3(k-1)+2}(\xi,\eta) - J_{21} N_{3(k-1)+3}(\xi,\eta) \\ N'_{3(k-1)+3}(\xi,\eta) = -J_{12} N_{3(k-1)+2}(\xi,\eta) - J_{22} N_{3(k-1)+3}(\xi,\eta) \end{array} \right.$$

$$\left| \begin{array}{l} \Sigma N_{3(k-1)+1}(\xi,\eta) = \sum_{k=1}^{N} N_{3(k-1)+1}(\xi,\eta) \\ \Sigma N_{3(k-1)+2}(\xi,\eta) = \sum_{k=1}^{N} N_{3(k-1)+2}(\xi,\eta) \\ \Sigma N_{3(k-1)+3}(\xi,\eta) = \sum_{k=1}^{N} N_{3(k-1)+3}(\xi,\eta) \end{array} \right.$$

$$\begin{pmatrix} \overline{\gamma}_x \\ \overline{\gamma}_y \end{pmatrix} = H_{ct^{-1}} [(B_{cm} + B_{c\alpha} P_m) \begin{pmatrix} u_1 \\ v_1 \\ \vdots \\ u_N \\ v_N \end{pmatrix} + (B_{c\beta} + B_{c\alpha} P_\beta) \begin{pmatrix} w_1 \\ \beta_{x1} \\ \beta_{y1} \\ \vdots \\ w_N \\ \beta_{xN} \\ \beta_{yN} \end{pmatrix}] = T_{\gamma u} \begin{pmatrix} u_1 \\ v_1 \\ \vdots \\ u_N \\ v_N \end{pmatrix} + T_{\gamma w} \begin{pmatrix} w_1 \\ \beta_{x1} \\ \beta_{y1} \\ \vdots \\ w_N \\ \beta_{xN} \\ \beta_{yN} \end{pmatrix}$$

One obtains the interpolation then for w :

$$w=\sum_{k=1}^{N} N''_{5(k-1)+1}(\xi,\eta)u_k+N''_{5(k-1)+2}(\xi,\eta)v_k+$$
$$\sum_{k=1}^{N} N''_{5(k-1)+3}(\xi,\eta)w_k+N''_{5(k-1)+4}(\xi,\eta)\beta_{xk}+N''_{5(k-1)+5}(\xi,\eta)\beta_{yk}$$

where:

$$N''_{5(k-1)+1}(\xi,\eta) ¿ (J_{11}T_{\gamma u}(1,2(k-1)+1)+J_{12}T_{\gamma u}(2,2(k-1)+1))\Sigma N_{3(j-1)+2}(\xi,\eta) + (J_{21}T_{\gamma u}(1,2(k-1)+1)+J_{22}T_{\gamma u}(2,2(k-1)+1))\Sigma N_{3(j-1)+3}(\xi,\eta)$$

$$N''_{5(k-1)+2}(\xi,\eta) ¿ (J_{11}T_{\gamma u}(1,2(k-1)+2)+J_{12}T_{\gamma u}(2,2(k-1)+2))\Sigma N_{3(j-1)+2}(\xi,\eta) + (J_{21}T_{\gamma u}(1,2(k-1)+2)+J_{22}T_{\gamma u}(2,2(k-1)+2))\Sigma N_{3(j-1)+3}(\xi,\eta)$$

$$N''_{5(k-1)+3}(\xi,\eta) ¿ N'_{3(k-1)+1}(\xi,\eta) + (J_{11}T_{\gamma w}(1,3(k-1)+1)+J_{12}T_{\gamma w}(2,3(k-1)+1))\Sigma N_{3(j-1)+2}(\xi,\eta) + (J_{21}T_{\gamma w}(1,3(k-1)+1)+J_{22}T_{\gamma w}(2,3(k-1)+1))\Sigma N_{3(j-1)+3}(\xi,\eta)$$

$$N''_{5(k-1)+4}(\xi,\eta) ¿ N'_{3(k-1)+2}(\xi,\eta) + (J_{11}T_{\gamma w}(1,3(k-1)+2)+J_{12}T_{\gamma w}(2,3(k-1)+2))\Sigma N_{3(j-1)+2}(\xi,\eta) + (J_{21}T_{\gamma w}(1,3(k-1)+2)+J_{22}T_{\gamma w}(2,3(k-1)+2))\Sigma N_{3(j-1)+3}(\xi,\eta)$$

$$N''_{5(k-1)+5}(\xi,\eta) ¿ N'_{3(k-1)+3}(\xi,\eta) + (J_{11}T_{\gamma w}(1,3(k-1)+3)+J_{12}T_{\gamma w}(2,3(k-1)+3))\Sigma N_{3(j-1)+2}(\xi,\eta) + (J_{21}T_{\gamma w}(1,3(k-1)+3)+J_{22}T_{\gamma w}(2,3(k-1)+3))\Sigma N_{3(j-1)+3}(\xi,\eta)$$

This can be still written in the following way:

$$\begin{pmatrix} w \\ \beta_x \\ \beta_y \end{pmatrix}=\sum_{k=1}^{N}\begin{pmatrix} N_{kwu}(\xi,\eta) & N_{kwv}(\xi,\eta) & N_{kww}(\xi,\eta) & N_{kwx}(\xi,\eta) & N_{kwy}(\xi,\eta) \\ N_{kxu}(\xi,\eta) & N_{kxv}(\xi,\eta) & N_{kxw}(\xi,\eta) & N_{kxx}(\xi,\eta) & N_{kxy}(\xi,\eta) \\ N_{kyu}(\xi,\eta) & N_{kyv}(\xi,\eta) & N_{kyw}(\xi,\eta) & N_{kyx}(\xi,\eta) & N_{kyy}(\xi,\eta) \end{pmatrix}\begin{pmatrix} u_k \\ v_k \\ w_k \\ \beta_{xk} \\ \beta_{yk} \end{pmatrix}$$

The matrix of mass has the following form thus:

$$M=\begin{pmatrix} M'_m & M'_{mf} \\ M'_{fm} & M'_f \end{pmatrix}$$

The membrane part M'_m elementary matrix of mass is composed of the blocks kp (k ième line and p ième column) following:

$$\rho_m\begin{pmatrix} N_k N_p + N_{kwu} N_{pwu} & N_{kwu} N_{pwv} \\ N_{kwv} N_{pwu} & N_k N_p + N_{kwv} N_{pwv} \end{pmatrix} + \rho'_{mf}\begin{pmatrix} N_k N_{pxu} + N_{kxu} N_p & N_k N_{pxv} + N_{kyu} N_p \\ N_k N_{pyu} + N_{kxv} N_p & N_k N_{pyv} + N_{kyv} N_p \end{pmatrix}$$
$$+\rho'_f\begin{pmatrix} N_{kxu} N_{pxu} + N_{kyu} N_{pyu} & N_{kxu} N_{pxv} + N_{kyu} N_{pyv} \\ N_{pxu} N_{kxv} + N_{pyu} N_{kyv} & N_{kxv} N_{pxv} + N_{kyv} N_{pyv} \end{pmatrix}$$

The membrane-inflection part M'_{mf} is composed of the blocks kp (k ième line and p ième column) following:

$$\rho_m\begin{pmatrix} N_{kwu} N_{pww} & N_{kwu} N_{pwx} & N_{kwu} N_{pwy} \\ N_{kwv} N_{pww} & N_{kwv} N_{pwx} & N_{kwv} N_{pwy} \end{pmatrix} + \rho'_{mf}\begin{pmatrix} N_k N_{pxw} & N_k N_{pxx} & N_k N_{pxy} \\ N_k N_{pyw} & N_k N_{pyx} & N_k N_{pyy} \end{pmatrix}$$
$$+\rho'_f\begin{pmatrix} N_{kxu} N_{pxw} + N_{kyu} N_{pyw} & N_{kxu} N_{pxx} + N_{kyu} N_{pyx} & N_{kxu} N_{pxy} + N_{kyu} N_{pyy} \\ N_{kxv} N_{pxw} + N_{kyv} N_{pyw} & N_{kxv} N_{pxx} + N_{kyv} N_{pyx} & N_{kxv} N_{pxy} + N_{kyv} N_{pyy} \end{pmatrix}$$

The inflection-membrane part M'_{fm} is composed of the blocks kp (k ième line and p ième column) following:

$$\rho_m\begin{pmatrix} N_{kww} N_{pwu} & N_{kww} N_{pwv} \\ N_{kwx} N_{pwu} & N_{kwx} N_{pwv} \\ N_{kwy} N_{pwu} & N_{kwy} N_{pwv} \end{pmatrix} + \rho'_{mf}\begin{pmatrix} N_{kxw} N_p & N_{kyw} N_p \\ N_{kxx} N_p & N_{kyx} N_p \\ N_{kxy} N_p & N_{kyy} N_p \end{pmatrix}$$
$$+\rho'_f\begin{pmatrix} N_{kxw} N_{pxu} + N_{kyw} N_{pyu} & N_{kxw} N_{pxv} + N_{kyw} N_{pyv} \\ N_{kxx} N_{pxu} + N_{kyx} N_{pyu} & N_{kxx} N_{pxv} + N_{kyx} N_{pyv} \\ N_{kxy} N_{pxu} + N_{kyy} N_{pyu} & N_{kxy} N_{pxv} + N_{kyy} N_{pyv} \end{pmatrix}$$

The term M'_f of inflection is composed of the blocks kp (k ième line and p ième column) following:

$$\rho_m\begin{pmatrix} N_{kww} N_{pww} & N_{kww} N_{pwx} & N_{kww} N_{pwy} \\ N_{kwx} N_{pww} & N_{kwx} N_{pwx} & N_{kwx} N_{pwy} \\ N_{kwy} N_{pww} & N_{kwy} N_{pwx} & N_{kwy} N_{pwy} \end{pmatrix} +$$
$$\rho'_f\begin{pmatrix} N_{kxw} N_{pxw} + N_{kyw} N_{pyw} & N_{kxw} N_{pxx} + N_{kyw} N_{pyx} & N_{kxw} N_{pxy} + N_{kyw} N_{pyy} \\ N_{kxx} N_{pxw} + N_{kyx} N_{pyw} & N_{kxx} N_{pxx} + N_{kyx} N_{pyx} & N_{kxx} N_{pxy} + N_{kyx} N_{pyy} \\ N_{kxy} N_{pxw} + N_{kyy} N_{pyw} & N_{kxy} N_{pxx} + N_{kyy} N_{pyx} & N_{kxy} N_{pxy} + N_{kyy} N_{pyy} \end{pmatrix}$$

4.5.2.3 Elements of the type Q4Γ

One proceeds in the same way that for the elements of the type DST but with:

$$\begin{pmatrix}\bar{\gamma}_x\\ \bar{\gamma}_y\end{pmatrix}=B_c\begin{pmatrix}w_1\\ \beta_{x1}\\ \beta_{y1}\\ \vdots\\ w_N\\ \beta_{xN}\\ \beta_{yN}\end{pmatrix}$$

where: B_c is the matrix established with [§4.3.1].

One from of deduced that:

$$\begin{pmatrix}w\\ \beta_x\\ \beta_y\end{pmatrix}=\sum_{k=1}^{N}\begin{pmatrix}0 & 0 & N_{kww}(\xi,\eta) & N_{kwx}(\xi,\eta) & N_{kwy}(\xi,\eta)\\ 0 & 0 & 0 & N_k(\xi,\eta) & 0\\ 0 & 0 & 0 & 0 & N_k(\xi,\eta)\end{pmatrix}\begin{pmatrix}u_k\\ v_k\\ w_k\\ \beta_{xk}\\ \beta_{yk}\end{pmatrix}$$

The matrix of mass has the following form thus:

$$M=\begin{pmatrix}M'_m & 0\\ 0 & M'_f\end{pmatrix}$$

The membrane part M'_m elementary matrix of mass is composed of the blocks kp (k ième line and p ième column) following:

$$\rho_m\begin{pmatrix}N_kN_p & 0\\ 0 & N_kN_p\end{pmatrix}$$

The term M'_f of inflection is composed of the blocks kp (k ième line and p ième column) following:

$$\rho_m\begin{pmatrix}N_{kww}N_{pww} & N_{kww}N_{pwx} & N_{kww}N_{pwy}\\ N_{kwx}N_{pww} & N_{kwx}N_{pwx} & N_{kwx}N_{pwy}\\ N_{kwy}N_{pww} & N_{kwy}N_{pwx} & N_{kwy}N_{pwy}\end{pmatrix}+\rho'_f\begin{pmatrix}0 & 0 & 0\\ 0 & N_kN_p & 0\\ 0 & 0 & N_kN_p\end{pmatrix}$$

4.5.2.4 Notice

One neglects in the form of the elementary matrix of mass without offsetting the terms of inertia of rotation $\int_S \rho_f(\ddot{\beta}_x\delta\beta_x+\ddot{\beta}_y\delta\beta_y)\,dS$ because the latter are negligible compared to the different one. Indeed a multiplicative factor of $h^2/12$ the dregs with the other terms and they become negligible for a thickness report over characteristic length lower than $1/10$. When offsetting is introduced, these terms of the form $\int_S (\rho_f + 2\mathrm{d}\,\rho_{mf} + d^2\,\rho_m)(\ddot{\beta}_x\,\delta\,\beta_x + \ddot{\beta}_y\,\delta\,\beta_y)\,dS$ are not negligible any more and are introduced into the form of the matrix of mass.

5 Postprocessing and implementation

Offsetting is introduced by the optional keyword `OFFSETTING` on the level of `AFFE_CARA_ELEM` same manner as the thickness according to the methods defined in introduction. When this keyword is not present offsetting is worth zero by default.

5.1 Couple and load application

All the calculations are done in the reference mark of diagram (plan of the grid). If one defines forces or couples compared to another reference mark, the user will have to make for `FORCE_ARETE` and `FORCE_NODALE` transformations necessary to be reduced to the reference mark grid. For `FORCE_COQUE` the user will be able to specify the plan of load application and conversion towards the reference mark of calculation will be automatic.

One introduces thus into `AFFE_CHAR_MECA` concept of plan of load application by the keyword `PLAN` under `FORCE_COQUE`. This plan of application is different from the datum-line or plan from diagram on which the grid is pressed. For this keyword one will define the four following possibilities of application of the forces: `'INF' 'MOY' 'SUP' 'E-MAIL'`. `'INF' 'MOY'` and `'SUP'` mean that one applies the efforts in lower skin, average and higher of plate respectively. `'E-MAIL'` mean that one applies the efforts to the level of the datum-line or plan of the grid. By defaults the efforts will be applied as regards grid of the plate. The efforts of the type are concerned `FORCE_COQUE TE0032`.

In local reference mark with the element, when the forces and the couples are brackets on `'MOY'` the simple relation of passage is used:

$$c'_x = c_x - d f_y$$
$$c'_y = c_y + d f_x$$

to bring back the efforts and the couples in the reference mark of the grid where the calculations are done.

In local reference mark with the element, when the forces and the couples are applied to `'SUP'` the simple relation of passage is used:

$$c'_x = c_x - (d + h/2)\, f_y$$
$$c'_y = c_y + (d + h/2)\, f_x$$

In local reference mark with the element, when the forces and the couples are applied to `'INF'` the simple relation of passage is used:

$$c'_x = c_x - (d - h/2)\, f_y$$
$$c'_y = c_y + (d - h/2)\, f_x$$

If the efforts are given in the total reference mark of the element, relations of passage of the type are used: $c' = c + (d + \varepsilon\, h/2)\, n \wedge f$ where c is defined compared to the reference mark `'INF' 'MOY' 'SUP'` with ε equal to -1.0 and 1, respectively. When there is no offsetting, the preceding formula is reduced to $c' = c + \varepsilon\, h/2 \mathrm{n} \wedge f$.

Note:

For the loadings of the type `FORCE_ARETE` or `FORCE_NODALE` the efforts and couples can be expressed only compared to the reference mark of the grid. If the user knows them only compared to the average layer of the plate, it will have to carry out the change of reference *mark itself to have the expression of the efforts and the couples compared to the surface of grid. The relation to be used is* $c' = c + dn \wedge f$ *where* d *is the distance between the plan of calculation and the loading plan directed by the normal with the hull. It is obvious that the user has interest so that the loading plan is the plan of the grid, but it is not always possible to make coincide these two plans as one can see it on the left part of the Figure 2.1-a.*

5.2 Application of the boundary conditions in displacement

For the boundary conditions of type displacement the user will have to pay attention to the fact that they can apply only to the reference mark of grid. The relations of passage compared to conditions given on the average layer are the following ones:

$$\theta_{ref} = \theta_{moy}$$
$$u_{ref} = u_{moy} - \theta_{moy} \wedge dn$$

5.3 Postprocessings

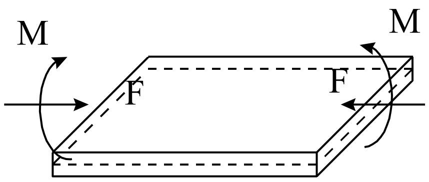
For postprocessings, the by default results of generalized the efforts type are given in the reference mark corresponding to the plan of diagram. To have them in the other reference marks, it will be necessary that the user indicates the plan of postprocessing and the changes of reference mark will be automatic.

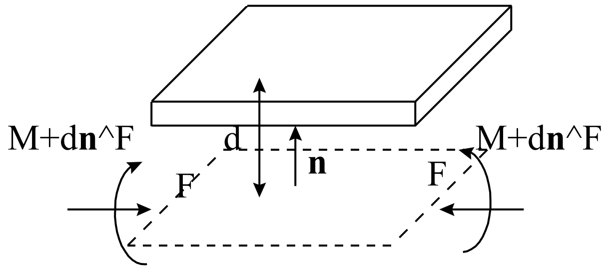
For the postprocessing of the generalized efforts, one will be able to calculate them in the average layer of the hull via the order `POST_CHAMP/COQUE_EXCENT`.

6 Static and modal validation

6.1 Initial validation

The first part of the validation consists in testing a simple plate subjected to forces and couples and whose plan of grid does not coincide with the plan of the average layer on which the efforts are applied. For the plate subjected to forces and couples, the results with and without offsetting must take account of the change of reference mark for the couples as indicated below.





Displacements are in the following way dependent for a point located at a height Z compared to the average layer:

$$u = u_{moy} + \theta_{moy} \wedge zn = u_{ref} + \theta_{ref} \wedge (z+d) n$$

what is still written:

$$\theta_{moy} = \theta_{ref}$$
$$u_{moy} = u_{ref} + \theta_{ref} \wedge dn$$

what enables us to establish the relations of passage between displacements compared to the average layer and those compared to the datum-line.

For the generalized efforts, in the two preceding cases, there are the same results on the layers means, inferior and superior of plate.

6.2 CAS-test SSLS111: offsetting for simple plates

It is about a calculation in inflection of double-layered made up by two different isotropic materials. The coupling membrane-inflection is studied. The calculation of reference is that of double-layered defined by `DEFI_COMPOSITE` composed of two different isotropic materials (not symmetry according to z). Other modeling is made up of two plates offset compared to average fibre of the plate used with `DEFI_COMPOSITE` . The results, identical of one modeling to the other, are given in term of displacements and generalized efforts. Moreover one carries out on the geometry of this test a modal analysis for two modelings: the found Eigen frequencies are identical.

6.3 CAS-test SSLS112: offsetting for composite plates

It is about a calculation in inflection of a quadricouche having a material not-symmetry compared to its average plan. The calculation of reference uses a definite quadricouches by `DEFI_COMPOSITE`. Other modeling uses two double-layered definite by `DEFI_COMPOSITE` but offset compared to average fibre of the quadricouche. The results, identical of one modeling to the other, are given in term of displacements.

7 Conclusion

The finite elements of plate which we describe here are used in the slim mean structural analyses whose thickness report over characteristic length is lower than $1/10$. The average layer of these structures does not coincide with the plan of the grid (plan of diagram). Offsetting thus corresponds to the distance from the average layer compared to the layer of diagram. A offsetting d positive means that the average surface of the plate is at a distance $d\,\mathbf{n}$ element of plate with a grid, direction $\mathbf{n}$ being given by the normal to the element.
The values of displacements and generalized efforts obtained are given by default in the reference mark of the grid. For the generalized efforts, one can however define a reference mark of postprocessing - reference mark associated with the average layer - different from the reference mark of diagram. Same manner, the efforts applied are regarded as being given by default in the reference mark of diagram. In the case of `FORCE_COQUE`, one can however specify a reference mark of load application and couples - reference mark associated with the average layer - different from the reference mark of diagram.
Equivalent elements are not available in thermics; the thermomechanical chainings are thus not available for the offset elements of plates.

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9 Description of the versions of the document

Version Aster	Author (S) Organization (S)	Description of the modifications

6.3	P.MASSIN EDF-R&D/AMA	Initial text

Annexe 1 Factors of transverse correction of shearing for orthotropic or laminated plates offset

The matrix $H_{\mathbf{ct}}$ is defined so that the surface density of transverse energy of shearing obtained in the case of the three-dimensional distribution of the constraints resulting from the resolution of balance is equal to that of the model of plate based on the assumptions of Reissner, for a behavior in pure bending. One must thus find $H_{\mathbf{ct}}$ such as:

$$\frac{1}{2}\int_{-h/2}^{+h/2} \tau\, H_g^{-1}\tau = \frac{1}{2}\mathbf{T}\mathbf{H}_{\mathbf{ct}}^{-1}T = \frac{1}{2}\gamma\, H_{\mathbf{ct}}\,\gamma \text{ with } \tau = \begin{pmatrix}\sigma_{xz}\\ \sigma_{yz}\end{pmatrix} \text{ and } T = \int_{-h/2}^{+h/2}\tau\, dz = H_{\mathbf{ct}}\,\gamma\ .$$

To obtain $H_{\mathbf{ct}}$ one uses the distribution of τ according to z obtained starting from the resolution of the equilibrium equations 3D without external couples:

$$\sigma_{xz} = -\int_{-h/2}^{z}(\sigma_{xx,x}+\sigma_{xy,y})d\zeta\,;\, s_{yz} = -\int_{-h/2}^{z}(\sigma_{xy,x}+\sigma_{yy,y})d\zeta \text{ with } \sigma_{xz}=\sigma_{yz}=0 \text{ for } z=\pm h/2\,.$$

If there is no coupling membrane inflection (symmetry compared to $z=0$), constraints in the plan of the element $\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$ in the case of have as an expression a behavior of pure inflection:

$$\sigma = zA(z)M \text{ with } A(z) = H(z)H_f^{-1}\,.$$

If $H(z)$ and H_f do not depend on x and y one can determine $H_{\mathbf{ct}}$. Indeed:

$$\tau(z) = D_1(z)T + D_2(z)\lambda \text{ where } T = \begin{pmatrix}T_x\\ T_y\end{pmatrix} = \begin{pmatrix}M_{xx,x}+M_{xy,y}\\ M_{xy,x}+M_{yy,y}\end{pmatrix} \text{ and } \lambda = \begin{pmatrix}M_{xx,x}-M_{xy,y}\\ M_{xy,x}-M_{yy,y}\\ M_{yy,x}\\ M_{xx,y}\end{pmatrix}$$

like:

$$\mathbf{D}_1 = -\int_{-h/2}^{z}\frac{z}{2}\begin{pmatrix}A_{11}+A_{33} & A_{13}+A_{32}\\ A_{31}+A_{23} & A_{22}+A_{33}\end{pmatrix}dz\ ,$$

$$\mathbf{D}_2 = -\int_{-h/2}^{z}\frac{z}{2}\begin{pmatrix}A_{11}-A_{33} & A_{13}-A_{32} & 2\mathrm{A}_{12} & 2\mathrm{A}_{31}\\ A_{31}-A_{23} & A_{33}-A_{22} & 2\mathrm{A}_{32} & 2\mathrm{A}_{21}\end{pmatrix}dz\ .$$

It results from it that $\frac{1}{2}\int_{-h/2}^{+h/2} t\,\mathbf{H}_g^{-1}t = \frac{1}{2}\begin{pmatrix}\mathbf{T}\\ \lambda\end{pmatrix}\begin{pmatrix}\mathbf{C_{11}} & \mathbf{C_{12}}\\ \mathbf{C_{12}^{T}} & \mathbf{C_{22}}\end{pmatrix}\begin{pmatrix}\mathbf{T}\\ \lambda\end{pmatrix}$ with:

$$\mathbf{C_{11}} = \int_{-h/2}^{+h/2}\mathbf{D}_1^T\mathbf{H}_g^{-1}\mathbf{D}_1\,dz\,;$$

$$\mathbf{C_{12}} = \int_{-h/2}^{+h/2}\mathbf{D}_1^T\mathbf{H}_g^{-1}\mathbf{D}_2\,dz\,;$$

$$\mathbf{C_{22}} = \int_{-h/2}^{+h/2}\mathbf{D}_2^T\mathbf{H}_g^{-1}\mathbf{D}_2\,dz$$

As in addition $\frac{1}{2}\int_{-h/2}^{+h/2} t\,H_g^{-1}\,t=\frac{1}{2}\mathbf{T}\mathbf{H}_{\mathbf{ct}}^{-1}T$ one proposes to take $H_{\mathbf{ct}}=C_{\mathbf{11}}^{-1}$ to satisfy the two equations as well as possible whatever T and λ.

While comparing $H_{\mathbf{ct}}$ thus calculated with $\bar{H}_{\mathbf{ct}}=\int_{-h/2}^{+h/2} H_g\,dz$ one reveals the coefficients of correction of following transverse shearing: $k_1=H_{ct}^{11}/\bar{H}_{ct}^{11};k_{12}=H_{ct}^{12}/\bar{H}_{ct}^{12};k_2=H_{ct}^{22}/\bar{H}_{ct}^{22}$.

For a homogeneous, isotropic or anisotropic plate, one finds as follows: $\mathbf{H}_{\mathbf{ct}}=kh\,\mathbf{H}_g$ with $k=5/6$.

Note:

This method is valid only when the composite plate is symmetrical compared to z=0.

- For a multi-layer material, one establishes that:

$$\mathbf{C}_{\mathbf{11}}=\sum_{i=1}^{N}\frac{h_i}{4}\left(\sum_{p=1}^{i-1}h_p h_p\mathbf{A}_p^T-\frac{1}{2}z_i^2\mathbf{A}_i^T\right)\mathbf{H}_g^{-1}\left(\sum_{p=1}^{i-1}h_p h_p\mathbf{A}_p-\frac{1}{2}z_i^2\mathbf{A}_i\right)+$$
$$\frac{1}{24}(z_{i+1}^3-z_i^3)\left[\mathbf{A}_i^T\mathbf{H}_g^{-1}\left(\sum_{p=1}^{i-1}h_p h_p\mathbf{A}_p-\frac{1}{2}z_i^2\mathbf{A}_i\right)+\left(\sum_{p=1}^{i-1}h_p h_p\mathbf{A}_p^T-\frac{1}{2}z_i^2\mathbf{A}_i^T\right)\mathbf{H}_g^{-1}\mathbf{A}_i\right]$$
$$+\frac{1}{80}(z_{i+1}^5-z_i^5)\mathbf{A}_i^T\mathbf{H}_g^{-1}\mathbf{A}_i$$

  where: $h_i=z_{i+1}-z_i, h_i=\frac{1}{2}\left(z_{i+1}+z_i\right)$ and $\mathbf{A}_i$ represent the matrix $\begin{pmatrix}A_{11}+A_{33} & A_{13}+A_{32}\\ A_{31}+A_{23} & A_{22}+A_{33}\end{pmatrix}$ for layer I.

- Validity of the choice $\mathbf{H}_{\mathbf{ct}}=\mathbf{C}_{\mathbf{11}}^{-1}$ can be examined a posteriori when one has an estimate of the solution (fields of displacements and plane constraints, in particular). One can then estimate the difference between the two estimates on energy. A approach of calculation in two stages for the multi-layer plates and hulls (with $\mathbf{H}_{\mathbf{ct}}$ diagonal and two coefficients k_1 and k_2) was developed besides by Noor and Burton [bib10] [bib11].

- In the case of an isotropic or anisotropic homogeneous plate the equality between two energies is satisfied in a strict sense since $D_2=0$. The choice makes above is then valid and no examination a posteriori is necessary.

Annexe 2 Calculation of shear stress

The term here is clarified $d1iel(z)$ who allows to calculate the constraint and the shearing strain in the presence of offsetting.

Indeed, according to R3.07.03 documentation, relations stress shear – efforts cutting-edges are:

$$\sigma_{xz}=T_x*d1iel(z) \quad \sigma_{yz}=T_y*d1iel(z)$$

In the classical cases without offsetting, one a: $d1iel(z)=3/2\mathrm{h}*(1-4z^2/h^2)$.
In cases plus generals (in the presence of offsetting for example) $d1iel(z)$ must be modified. To approximate the shear stress correctly, one makes the choice to apply a general quadratic form for $d1iel(z)=a*z^2+b*z+c$ such that the following conditions are observed:

- $\int_{-h/2}^{h/2} d1iel(z+d)dz=1$ relation effort slice-constraints
- $d1iel(z+d=-h/2)=0\,;\,d1iel(z+d=h/2)=0$ condition of free edges

Maybe with a offsetting of d, coefficients of $d1iel(z)$ are:

$$a=-6/quotient \quad , \quad b=(6*(zmin+zmax))/quotient \quad , \quad b=(6*zmin*zmax)/quotient$$

$$quotient=(zmax^3-3*zmax^2*zmin+3*zmax*zmin^2-zmin^3),\ zmin=-h/2+d\ ,\ zmax=h/2+d$$