# Voluminal element of hull SHB with 8 nodes

**Summary:**

We present in this document the theoretical formulation of element SHB8 and its digital establishment for implicit non-linear incremental analyses (great displacements, small rotations, small deformations).

It is about a three-dimensional cubic element with 8 nodes with a called privileged direction thickness. Thus, it can be used to represent mean structures while correctly taking into account the phenomena through the thickness (inflection, elastoplasticity), grace a digital integration to 5 points of Gauss in this privileged direction.

In order to reduce the computing time considerably and to draw aside various blockings likely to appear, this element under-is integrated. It requires consequently a mechanism of stabilization in order to control the modes of deformation to worthless energy (modes of Hourglass).

In addition to its cost of relatively weak calculation and its good performances in elastoplasticity, this element has another advantage. Since it is based on a three-dimensional formulation and that it has only degrees of freedom of translation, it is very easy to couple it with voluminal elements 3D, which is very useful in systems where voluminal hulls and elements must cohabit. Moreover, it makes it possible to easily model the plates and hulls with variable thickness.

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# 1 Introduction

Many recent work proposed to use a voluminal formulation for the mean structures. Two principal families of methods, which rest all on the introduction of a field of applied deformation (“assumed strain”), emerge. The methods of the first family consist in using a conventional digital integration with an adequate control of all the modes of blocking and locking (volume, transverse shearing, membrane). The methods of the second family consist under-integrating the elements to remove blockings and controlling the modes of *Hourglass* who rise from this under-integration (see [bib3] [bib4]). The two approaches were studied in details in the case of an elastic behavior. On the other hand, very little work treats elastoplastic case.

The element presented here rests on an under-integrated formulation especially developed for the elastoplastic behavior of the structures in inflection. The basic idea first of all consists in making sure that there are sufficient points of Gauss in the thickness to represent the phenomenon of inflection correctly, then to calculate rigidities of stabilization in an adaptive way according to the plastic state of the element. That represents an unquestionable improvement compared to the classical formulations for the forces of stabilization, because these last rest on an elastic stabilization which becomes too rigid when the effects of plasticity dominate the answer of the structure.

Element SHB8 is a continuous three-dimensional cube with eight nodes, in which a privileged direction, called thickness, was selected. It can thus be used to model the mean structures and to take into account the phenomena which develop in the thickness within the framework of the mechanics of the continuous mediums three-dimensional. Since this element under-is integrated, it displays modes of *Hourglass* who must be stabilized. We chose the method of stabilization introduced by Belytschko, Bindeman and Flanagan [bib3] [bib4]. This element (entitled SHB8PS then) and this method of stabilization were initially implemented in an explicit formulation by Abed-Meraim and Combescure [bib2]. The numeric work implementation of this element within an implicit non-linear framework was proposed by Legay and Combescure in [bib1].

This documentation describes the formulation of this element, its numeric work implementation for the prediction of elastic and elastoplastic structural instabilities, like its establishment in *Code_Aster*. For the non-linear problems, an implicit incremental formulation of Newton-Raphson type is used [R5.03.01]. The equilibrium equations are solved by the method of Lagrangian the update. The control of the increments of load and displacement is based on a method of piloting close to the algorithm to Riks [bib5].

## 2 Kinematics of the element

Element SHB8 is a hexahedron with 8 nodes. The five points of integration are selected along the direction $\zeta$ in the reference mark of the local coordinates. The shape of the element of reference as well as the points of integration are represented on [Figure 2-a].

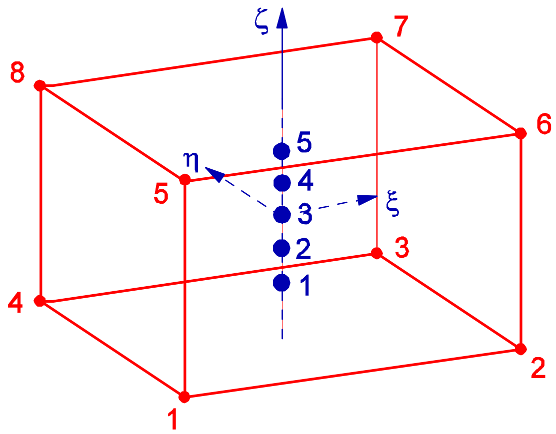


**Figure 2-a: Geometry of the element of reference and points of integration**

This element is isoparametric and has the same linear interpolation and same kinematics as the hexaèdraux elements with 8 standard nodes.

## 3 Variational formulation

The formulation used for the construction of element SHB8 differs from a classical formulation simply by the choice of an applied deformation $\dot{\bar{\varepsilon}}$, therefore of an operator discretized gradient, allowing to avoid the induced parasitic modes by under integration.

Thus, the variational principle is written:

$$\delta\pi\left(v,\dot{\bar{\varepsilon}}\right)=\int_V \delta\left(\dot{\bar{\varepsilon}}\right):\sigma\, dV-\delta\dot{u}\, f^{ext}=0$$

where $\pi$ represent the total virtual power, $\delta$ variation, $v$ the field speed, $\dot{u}$ nodal speeds, $\dot{\bar{\varepsilon}}$ the rate of applied deformation (assumed strain disastrous), $\sigma$ the constraint of Cauchy, $V$ updated volume and $f^{ext}$ external forces.

The discretized equations thus require the only interpolation speed $v$ and of the rate of applied deformation $\dot{\bar{\varepsilon}}$ in the element. We now will build element SHB8 starting from this equation. The complete developments and the demonstrations concerning this element are exposed in details in [bib2].

# 4 Discretization

## 4.1 Discretization of the field of displacement

Space coordinates $x_i$ element are connected to the nodal coordinates $x_{iI}$ by means of the isoparametric functions of forms $N_i$ by the formulas:

$$x_i = x_{iI} N_I(\xi, \eta, \zeta) = \sum_{I=1}^{8} N_I(\xi, \eta, \zeta) x_{iI}$$

In the continuation, and except contrary mention, one will adopt the convention of summation for the repeated indices. Indices in small letters $i$ vary from one to three and represent the directions of the space coordinates. Those in capital letters $I$ vary from one to eight and correspond to the nodes of the element.

The same functions of forms are used to define the field of displacement of the element $u_i$ according to nodal displacements $u_{iI}$:

$$u_i = u_{iI} N_i(\xi, \eta, \zeta)$$

Trilinear isoparametric functions of form are chosen:

$$\begin{cases} N_I(\xi, \eta, \zeta) = \frac{1}{8}(1+\xi_I \xi)(1+\eta_I \eta)(1+\zeta_I \zeta) \\ \xi, \eta, \zeta \in [-1,1], \quad I=1,\dots,8 \end{cases}$$

These functions of form transform a unit cube in space $(\xi, \eta, \zeta)$ in an unspecified hexahedron in space $(x_1, x_2, x_3)$ .

## 4.2 Operator discretized gradient

The gradient $u_{i,j}$ field of displacement is a function of displacement $\mathbf{U}_{iI}$ node $I$ in the direction $i$ :

$$u_{i,j} = \mathbf{U}_{iI} N_{I,j}$$

The linear tensor of deformation is given by the symmetrical part of the gradient of displacement:

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

Let us introduce the three vectors $\mathbf{b}_i$ , derived from the functions of form at the points of Gauss $P_3$ :

$$\mathbf{b}_i^T(P_3) = \left. \frac{\partial N}{\partial x_i} \right|_{\xi=0,\ \eta=0,\ \zeta=0}$$

Also let us introduce the following vectors:

$$
\begin{array}{lcl}
\mathbf{s}^T &=& (\ 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1\ ) \\
\mathbf{h}_1^T &=& (\ 1 \quad 1 \quad -1 \quad -1 \quad -1 \quad -1 \quad 1 \quad 1\ ) \\
\mathbf{h}_2^T &=& (\ 1 \quad -1 \quad -1 \quad 1 \quad -1 \quad 1 \quad 1 \quad -1\ ) \\
\mathbf{h}_3^T &=& (\ 1 \quad -1 \quad 1 \quad -1 \quad 1 \quad -1 \quad 1 \quad -1\ ) \\
\mathbf{h}_4^T &=& (-1 \quad 1 \quad -1 \quad 1 \quad 1 \quad -1 \quad 1 \quad -1\ ) \\
\mathbf{X}_1^T &=& (-1 \quad 1 \quad 1 \quad -1 \quad -1 \quad 1 \quad 1 \quad -1\ ) \\
\mathbf{X}_2^T &=& (-1 \quad -1 \quad 1 \quad 1 \quad -1 \quad -1 \quad 1 \quad 1\ ) \\
\mathbf{X}_3^T &=& (-1 \quad -1 \quad -1 \quad -1 \quad 1 \quad 1 \quad 1 \quad 1\ )
\end{array}
$$

Three vectors $\mathbf{X}_i^T$ the nodal coordinates of the eight nodes represent. Four vectors $\mathbf{h}_\alpha^T$ the functions represent respectively $h_1$, $h_2$, $h_3$ and $h_4$ for each of the eight nodes, which are defined by:

$$h_1=\eta\zeta \qquad h_2=\zeta\xi \qquad h_3=\xi\eta \qquad h_4=\xi\eta\zeta$$

Let us introduce finally the four following vectors:

$$\boldsymbol{\gamma}_\alpha=\frac{1}{8}\left[\mathbf{h}_\alpha-\sum_{j=1}^{3}\left(\mathbf{h}_\alpha^T.\mathbf{X}_j\right)\mathbf{b}_j\right]$$

The gradient of the field of displacement can be now written in the form (without any approximation [bib3]):

$$u_{i,j}=\left(\mathbf{b}_j^T+\sum_{\alpha=1}^{4}\mathbf{h}_{\alpha,j}\boldsymbol{\gamma}_\alpha^T\right)\cdot\mathbf{U}_i=\left(\mathbf{b}_j^T+\mathbf{h}_{\alpha,j}\boldsymbol{\gamma}_\alpha^T\right)\cdot\mathbf{U}_i$$

Or, in the form of vector:

$$\nabla_S\mathbf{u}=\begin{bmatrix} u_{x,x} \\ u_{y,y} \\ u_{z,z} \\ u_{x,y}+u_{y,x} \\ u_{x,z}+u_{z,x} \\ u_{y,z}+u_{z,y} \end{bmatrix}$$

with $\mathbf{U}_i$ nodal displacement in the direction $i$. The symmetrical operator gradient (noted $\nabla_S$) discretized connecting the tensor of deformation to the vector of nodal displacements

$$\nabla_s\mathbf{u}=\mathbf{B}\cdot\mathbf{u}$$

takes the matric shape then:

$$\mathbf{B}=\begin{bmatrix} \mathbf{b}_x^T+h_{\alpha,x}\gamma_\alpha^T & 0 & 0 \\ 0 & \mathbf{b}_y^T+h_{\alpha,y}\gamma_\alpha^T & 0 \\ 0 & 0 & \mathbf{b}_z^T+h_{\alpha,z}\gamma_\alpha^T \\ \mathbf{b}_y^T+h_{\alpha,y}\gamma_\alpha^T & \mathbf{b}_x^T+h_{\alpha,x}\gamma_\alpha^T & 0 \\ \mathbf{b}_z^T+h_{\alpha,z}\gamma_\alpha^T & 0 & \mathbf{b}_x^T+h_{\alpha,x}\gamma_\alpha^T \\ 0 & \mathbf{b}_z^T+h_{\alpha,z}\gamma_\alpha^T & \mathbf{b}_y^T+h_{\alpha,y}\gamma_\alpha^T \end{bmatrix}$$

The detailed formulation was presented by Belytschko in [bib3].

## 4.3 Matrix of rigidity

The matrix of rigidity of the element is given by:

$$\mathbf{K}_e=\int_{\Omega_e}\mathbf{B}^T\cdot\mathbf{C}\cdot\mathbf{B}\,d\,\Omega$$

The five points of integration considered are on the same vertical line. Their coordinates are $(\xi,\eta,\zeta)$ and their weights of integration are the roots of the polynomial of Gauss-Legendre:

| | $\xi$ | $\eta$ | $\zeta$ | $\omega$ |
|---|---|---|---|---|
| P (1) | 0 | 0 | $\zeta_1=0.91$ | $\omega_1=0.24$ |
| P (2) | 0 | 0 | $\zeta_2=0.54$ | $\omega_2=0.48$ |
| P (3) | 0 | 0 | 0 | 0.57 |
| P (4) | 0 | 0 | $-\zeta_2$ | $\omega_2$ |
| P (5) | 0 | 0 | $-\zeta_1$ | $\omega_1$ |

Thus, the expression of rigidity $\mathbf{K}_e$ is:

$$\mathbf{K}_e=\sum_{j=1}^{5}\omega(\zeta_j)J(\zeta_j)\mathbf{B}^T(\zeta_j)\cdot\mathbf{C}\cdot\mathbf{B}(\zeta_j)$$

where $J(\zeta_j)$ is Jacobien, calculated at the point of Gauss $j$, transformation enters the unit configuration of reference and an arbitrary hexahedron. The elastic matrix of behavior $\mathbf{C}$ chosen has the following form:

$$\mathbf{C}=\begin{bmatrix} \bar{\lambda}+2\mu & \bar{\lambda} & 0 & 0 & 0 & 0 \\ \bar{\lambda} & \bar{\lambda}+2\mu & 0 & 0 & 0 & 0 \\ 0 & 0 & E & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix}$$

where $E$ is the Young modulus, $\nu$ the Poisson's ratio, $\mu=\frac{E}{2(1+\nu)}$ the modulus of rigidity and $\bar{\lambda}=\frac{E\nu}{1-\nu^2}$ the coefficient of modified Lamé. This law is specific to element SHB8. It resembles that which one would have in the case of the assumption of the plane constraints, put except for the term (3.3). One can note that this choice involves an artificial anisotropic behavior.

This choice makes it possible to satisfy all the tests without introducing blocking.

## 4.4 Geometrical matrix of rigidity Kσ

By introducing the quadratic deformation $\mathbf{e}^Q$ :

$$\mathbf{e}^Q=\frac{1}{2}\left(\sum_{k=1}^{3} u_{k,i}u_{k,j}\right)$$

one can define this matrix of geometrical rigidity by:

$$\mathbf{u}^T\cdot\mathbf{K}_\sigma\cdot\mathbf{u}=\int_{\Omega_0}\sigma:\mathbf{e}^Q(\mathbf{u},\mathbf{u})\,d\Omega=\int_{\Omega_0}\sigma:(\nabla\mathbf{u}^T\cdot\nabla\mathbf{u})\,d\Omega$$

In order to express this matrix in discretized space, let us introduce the discretized operators quadratic gradient $\mathbf{B^Q}$ such as:

$$\mathbf{e}^Q(\mathbf{u}(\zeta_j),\mathbf{u}(\zeta_j))=\begin{bmatrix}\mathbf{e}^Q_{\mathbf{xx}} & \mathbf{e}^Q_{\mathbf{yy}} & \mathbf{e}^Q_{\mathbf{zz}}\\ \mathbf{e}^Q_{\mathbf{yy}} & \mathbf{e}^Q_{\mathbf{zz}} & \mathbf{e}^Q_{\mathbf{xy}}+\mathbf{e}^Q_{\mathbf{yx}}\\ \mathbf{e}^Q_{\mathbf{zz}} & \mathbf{e}^Q_{\mathbf{xy}}+\mathbf{e}^Q_{\mathbf{yx}} & \mathbf{e}^Q_{\mathbf{xz}}+\mathbf{e}^Q_{\mathbf{zx}}\\ \mathbf{e}^Q_{\mathbf{xy}}+\mathbf{e}^Q_{\mathbf{yx}} & \mathbf{e}^Q_{\mathbf{xz}}+\mathbf{e}^Q_{\mathbf{zx}} & \mathbf{e}^Q_{\mathbf{yz}}+\mathbf{e}^Q_{\mathbf{zy}}\\ \mathbf{e}^Q_{\mathbf{xz}}+\mathbf{e}^Q_{\mathbf{zx}} & \mathbf{e}^Q_{\mathbf{yz}}+\mathbf{e}^Q_{\mathbf{zy}} & \\ \mathbf{e}^Q_{\mathbf{yz}}+\mathbf{e}^Q_{\mathbf{zy}} & & \end{bmatrix}=\begin{bmatrix}\mathbf{U}^T\cdot\mathbf{B}^\mathbf{Q}_{\mathbf{xx}}(\zeta_j)\cdot\mathbf{U} & \mathbf{U}^T\cdot\mathbf{B}^\mathbf{Q}_{\mathbf{yy}}(\zeta_j)\cdot\mathbf{U} & \mathbf{U}^T\cdot\mathbf{B}^\mathbf{Q}_{\mathbf{zz}}(\zeta_j)\cdot\mathbf{U}\\ \mathbf{U}^T\cdot\mathbf{B}^\mathbf{Q}_{\mathbf{yy}}(\zeta_j)\cdot\mathbf{U} & \mathbf{U}^T\cdot\mathbf{B}^\mathbf{Q}_{\mathbf{zz}}(\zeta_j)\cdot\mathbf{U} & \mathbf{U}^T\cdot\mathbf{B}^\mathbf{Q}_{\mathbf{xy}}(\zeta_j)\cdot\mathbf{U}\\ \mathbf{U}^T\cdot\mathbf{B}^\mathbf{Q}_{\mathbf{zz}}(\zeta_j)\cdot\mathbf{U} & \mathbf{U}^T\cdot\mathbf{B}^\mathbf{Q}_{\mathbf{xy}}(\zeta_j)\cdot\mathbf{U} & \mathbf{U}^T\cdot\mathbf{B}^\mathbf{Q}_{\mathbf{xz}}(\zeta_j)\cdot\mathbf{U}\\ \mathbf{U}^T\cdot\mathbf{B}^\mathbf{Q}_{\mathbf{xy}}(\zeta_j)\cdot\mathbf{U} & \mathbf{U}^T\cdot\mathbf{B}^\mathbf{Q}_{\mathbf{xz}}(\zeta_j)\cdot\mathbf{U} & \mathbf{U}^T\cdot\mathbf{B}^\mathbf{Q}_{\mathbf{yz}}(\zeta_j)\cdot\mathbf{U}\\ \mathbf{U}^T\cdot\mathbf{B}^\mathbf{Q}_{\mathbf{xz}}(\zeta_j)\cdot\mathbf{U} & \mathbf{U}^T\cdot\mathbf{B}^\mathbf{Q}_{\mathbf{yz}}(\zeta_j)\cdot\mathbf{U} & \\ \mathbf{U}^T\cdot\mathbf{B}^\mathbf{Q}_{\mathbf{yz}}(\zeta_j)\cdot\mathbf{U} & & \end{bmatrix}$$

Various terms $\mathbf{B^Q_{ij}}$ are given by the following equations:

$$\mathbf{B}^\mathbf{Q}_{\mathbf{xx}}(\zeta_j)=\begin{bmatrix}\mathbf{B_x}\cdot\mathbf{B_x^T} & 0 & 0\\ 0 & \mathbf{B_x}\cdot\mathbf{B_x^T} & 0\\ 0 & 0 & \mathbf{B_x}\cdot\mathbf{B_x^T}\end{bmatrix}$$

$$\mathbf{B}^\mathbf{Q}_{\mathbf{yy}}(\zeta_j)=\begin{bmatrix}\mathbf{B_y}\cdot\mathbf{B_y^T} & 0 & 0\\ 0 & \mathbf{B_y}\cdot\mathbf{B_y^T} & 0\\ 0 & 0 & \mathbf{B_y}\cdot\mathbf{B_y^T}\end{bmatrix}$$

$$\mathbf{B}^\mathbf{Q}_{\mathbf{zz}}(\zeta_j)=\begin{bmatrix}\mathbf{B_z}\cdot\mathbf{B_z^T} & 0 & 0\\ 0 & \mathbf{B_z}\cdot\mathbf{B_z^T} & 0\\ 0 & 0 & \mathbf{B_z}\cdot\mathbf{B_z^T}\end{bmatrix}$$

$$\mathbf{B}^\mathbf{Q}_{\mathbf{xy}}(\zeta_j)=\begin{bmatrix}\mathbf{B_x}\cdot\mathbf{B_y^T}+\mathbf{B_y}\cdot\mathbf{B_x^T} & 0 & 0\\ 0 & \mathbf{B_x}\cdot\mathbf{B_y^T}+\mathbf{B_y}\cdot\mathbf{B_x^T} & 0\\ 0 & 0 & \mathbf{B_x}\cdot\mathbf{B_y^T}+\mathbf{B_y}\cdot\mathbf{B_x^T}\end{bmatrix}$$

$$\mathbf{B}^{\mathbf{Q}}_{\mathbf{xz}}(\zeta_j)=\begin{bmatrix}\mathbf{B_x}\cdot\mathbf{B_z^T}+\mathbf{B_z}\cdot\mathbf{B_x^T} & 0 & 0\\ 0 & \mathbf{B_x}\cdot\mathbf{B_z^T}+\mathbf{B_z}\cdot\mathbf{B_x^T} & 0\\ 0 & 0 & \mathbf{B_x}\cdot\mathbf{B_z^T}+\mathbf{B_z}\cdot\mathbf{B_x^T}\end{bmatrix}$$

$$\mathbf{B}^{\mathbf{Q}}_{\mathbf{yz}}(\zeta_j)=\begin{bmatrix}\mathbf{B_y}\cdot\mathbf{B_z^T}+\mathbf{B_z}\cdot\mathbf{B_y^T} & 0 & 0\\ 0 & \mathbf{B_y}\cdot\mathbf{B_z^T}+\mathbf{B_z}\cdot\mathbf{B_y^T} & 0\\ 0 & 0 & \mathbf{B_y}\cdot\mathbf{B_z^T}+\mathbf{B_z}\cdot\mathbf{B_y^T}\end{bmatrix}$$

where $\mathbf{B}_i=\mathbf{b}_i^T+h_{\alpha,i}\gamma_\alpha^T$ avec $i=x,y,z$

With these notations, the geometrical matrix of rigidity $\mathbf{k}_\sigma$ at the point of Gauss $\zeta_j$ is given by:

$$\mathbf{k}_\sigma(\zeta_j)=\sigma_{xx}(\zeta_j).\mathbf{B}^{\mathbf{Q}}_{\mathbf{xx}}(\zeta_j)+\sigma_{yy}(\zeta_j).\mathbf{B}^{\mathbf{Q}}_{\mathbf{yy}}(\zeta_j)+\sigma_{zz}(\zeta_j).\mathbf{B}^{\mathbf{Q}}_{\mathbf{zz}}(\zeta_j)$$
$$+\sigma_{xy}(\zeta_j).\mathbf{B}^{\mathbf{Q}}_{\mathbf{xy}}(\zeta_j)+\sigma_{xz}(\zeta_j).\mathbf{B}^{\mathbf{Q}}_{\mathbf{xz}}(\zeta_j)+\sigma_{yz}(\zeta_j).\mathbf{B}^{\mathbf{Q}}_{\mathbf{yz}}(\zeta_j)$$

and geometrical rigidity of the element stamps it is given by:

$$\mathbf{K}_\sigma=\sum_{j=1}^{5}\omega(\zeta_j)J(\zeta_j)\mathbf{k}_\sigma(\zeta_j)$$

## 4.5 Matrix of pressure $K_p$

The following compressive forces are present in the tangent matrix via the matrix $\mathbf{K_p}$, because the following external forces depend on displacement. The following compressive forces are written:

$$\int_{\partial\Omega} p\,\mathbf{n}\cdot\mathbf{u}\,dS=\int_{\partial\Omega\,0} p\,\mathbf{det}[F(\mathbf{u})]\mathbf{n}_0^T\cdot F(\mathbf{u})^{-1T}\,dS_0=p\,\mathbf{F}_0-p\,\mathbf{K_p}\cdot\mathbf{U}$$
$$F(\mathbf{u})=1+\nabla\mathbf{u}$$

by using the notations:

- $\mathbf{n}_0{}^T=(n_x,n_y,n_z)$, normal on the surface external of the element in the configuration of reference
- $\mathbf{b}_{\mathrm{i}}$%, vector of dimension 4, drift of the functions of form to the 4 nodes of the face of the element charged in pressure
- $S_0$ surface of the face charged in pressure

The preceding formulation leads to a not-symmetrical matrix. It is known that one can nevertheless use a symmetrical formulation if the external forces due to the pressure derive from a potential. It is the case if the compressive forces do not work on the border of the modelled field. It is thus considered that the symmetrical part of the matrix is enough. The symmetrized matrix takes the following shape:

$$
\mathbf{K_p}=S_0\left|\begin{array}{ccc}
0 & \tilde{\mathbf{b}}_{\mathbf{y}}^T n_x-\tilde{\mathbf{b}}_{\mathbf{x}}^T n_y & \tilde{\mathbf{b}}_{\mathbf{z}}^T n_x-\tilde{\mathbf{b}}_{\mathbf{x}}^T n_z\\
0 & \tilde{\mathbf{b}}_{\mathbf{y}}^T n_x-\tilde{\mathbf{b}}_{\mathbf{x}}^T n_y & \tilde{\mathbf{b}}_{\mathbf{z}}^T n_x-\tilde{\mathbf{b}}_{\mathbf{x}}^T n_z\\
0 & \tilde{\mathbf{b}}_{\mathbf{y}}^T n_x-\tilde{\mathbf{b}}_{\mathbf{x}}^T n_y & \tilde{\mathbf{b}}_{\mathbf{z}}^T n_x-\tilde{\mathbf{b}}_{\mathbf{x}}^T n_z\\
0 & \tilde{\mathbf{b}}_{\mathbf{y}}^T n_x-\tilde{\mathbf{b}}_{\mathbf{x}}^T n_y & \tilde{\mathbf{b}}_{\mathbf{z}}^T n_x-\tilde{\mathbf{b}}_{\mathbf{x}}^T n_z\\
\tilde{\mathbf{b}}_{\mathbf{x}}^T n_y-\tilde{\mathbf{b}}_{\mathbf{y}}^T n_x & 0 & \tilde{\mathbf{b}}_{\mathbf{z}}^T n_y-\tilde{\mathbf{b}}_{\mathbf{y}}^T n_z\\
\tilde{\mathbf{b}}_{\mathbf{x}}^T n_y-\tilde{\mathbf{b}}_{\mathbf{y}}^T n_x & 0 & \tilde{\mathbf{b}}_{\mathbf{z}}^T n_y-\tilde{\mathbf{b}}_{\mathbf{y}}^T n_z\\
\tilde{\mathbf{b}}_{\mathbf{x}}^T n_y-\tilde{\mathbf{b}}_{\mathbf{y}}^T n_x & 0 & \tilde{\mathbf{b}}_{\mathbf{z}}^T n_y-\tilde{\mathbf{b}}_{\mathbf{y}}^T n_z\\
\tilde{\mathbf{b}}_{\mathbf{x}}^T n_y-\tilde{\mathbf{b}}_{\mathbf{y}}^T n_x & 0 & \tilde{\mathbf{b}}_{\mathbf{z}}^T n_y-\tilde{\mathbf{b}}_{\mathbf{y}}^T n_z\\
\tilde{\mathbf{b}}_{\mathbf{x}}^T n_z-\tilde{\mathbf{b}}_{\mathbf{z}}^T n_x & \tilde{\mathbf{b}}_{\mathbf{y}}^T n_z-\tilde{\mathbf{b}}_{\mathbf{z}}^T n_y & 0\\
\tilde{\mathbf{b}}_{\mathbf{x}}^T n_z-\tilde{\mathbf{b}}_{\mathbf{z}}^T n_x & \tilde{\mathbf{b}}_{\mathbf{y}}^T n_z-\tilde{\mathbf{b}}_{\mathbf{z}}^T n_y & 0\\
\tilde{\mathbf{b}}_{\mathbf{x}}^T n_z-\tilde{\mathbf{b}}_{\mathbf{z}}^T n_x & \tilde{\mathbf{b}}_{\mathbf{y}}^T n_z-\tilde{\mathbf{b}}_{\mathbf{z}}^T n_y & 0\\
\tilde{\mathbf{b}}_{\mathbf{x}}^T n_z-\tilde{\mathbf{b}}_{\mathbf{z}}^T n_x & \tilde{\mathbf{b}}_{\mathbf{y}}^T n_z-\tilde{\mathbf{b}}_{\mathbf{z}}^T n_y & 0
\end{array}\right|
$$

It is a matrix $(12 \text{x} 12)$, that it is necessary to multiply by displacements of the 4 nodes of the face to which one applies a pressure.

The formulation is similar to that used in 3D, described in [R3.03.04].

# 5 Stabilization of the element

## 5.1 Motivations

The under-integration of element SHB8 (5 points of Gauss only) aims at reducing the computing time considerably (gradient displacement, law of behavior,…). It also makes it possible to draw aside the various blockings met in the numeric work implementation of the finite elements.

However, this under-integration does not have only advantages: it unfortunately introduces parasitic modes associated with a worthless energy (mode with *Hourglass* or of sand glass). In statics, that can lead to a singularity of the matrix of total stiffness for certain boundary conditions. In transitory dynamics, on the other hand, that led to modes of sand glass which will deform the grid in an unrealistic way and which end up exploding the solution. This deficiency of the matrix of stiffness, due to under-integration, must thus be compensated by adding to elementary rigidity a matrix of stabilization. The core of the new rigidity, thus obtained, must be reduced to the only modes corresponding to the rigid movements of solids.

## 5.2 Modes of "Hourglass"

Since the points of integration are on the same vertical line (privileged direction), the derivative of the functions $h_3$ and $h_4$ cancel themselves in these points. The operator discretized gradient is thus reduced to:

$$\mathbf{B}=\begin{bmatrix} \mathbf{b}_x^T+\sum_{\alpha=1}^{2} h_{\alpha,x}\gamma_\alpha^T & 0 & 0 \\ 0 & \mathbf{b}_y^T+\sum_{\alpha=1}^{2} h_{\alpha,y}\gamma_\alpha^T & 0 \\ 0 & 0 & \mathbf{b}_z^T+\sum_{\alpha=1}^{2} h_{\alpha,z}\gamma_\alpha^T \\ \mathbf{b}_y^T+\sum_{\alpha=1}^{2} h_{\alpha,y}\gamma_\alpha^T & \mathbf{b}_x^T+\sum_{\alpha=1}^{2} h_{\alpha,x}\gamma_\alpha^T & 0 \\ \mathbf{b}_z^T+\sum_{\alpha=1}^{2} h_{\alpha,z}\gamma_\alpha^T & 0 & \mathbf{b}_x^T+\sum_{\alpha=1}^{2} h_{\alpha,x}\gamma_\alpha^T \\ 0 & \mathbf{b}_z^T+\sum_{\alpha=1}^{2} h_{\alpha,z}\gamma_\alpha^T & \mathbf{b}_y^T+\sum_{\alpha=1}^{2} h_{\alpha,y}\gamma_\alpha^T \end{bmatrix}$$

Modes of *Hourglass* are modes of displacement to worthless energy, i.e they check $\mathbf{Bu}=0$. Six modes, others that rigid modes of solids, which check this equation are:

$$\begin{bmatrix} \mathbf{h}_3 & 0 & 0 \\ 0 & \mathbf{h}_3 & 0 \\ 0 & 0 & \mathbf{h}_3 \end{bmatrix}\begin{bmatrix} \mathbf{h}_4 & 0 & 0 \\ 0 & \mathbf{h}_4 & 0 \\ 0 & 0 & \mathbf{h}_4 \end{bmatrix}$$

## 5.3 Stabilization of the type "Assumed Strain Method"

In this approach, inspired of work of Belytschko, Bindeman and Flanagan [bib3] [bib4], the derivative $\mathbf{b_i}$ functions of form are not calculated at the points of Gauss but are not realised on the element:

$$\hat{\mathbf{b}}_i^T=\frac{1}{V}\int_{\Omega_e}\mathbf{N}_{,i}(\xi,\eta,\zeta)d\Omega \quad , \quad i=1,2,3$$

Thus, the new operator discretized gradient can be written:

$$\hat{\mathbf{B}}=\mathbf{B}+\hat{\mathbf{B}}_{\mathbf{stab}}$$

The expression of $\hat{\mathbf{B}}_{\mathbf{stab}}$ is given by:

$$\hat{\mathbf{B}}_{\mathbf{stab}}=\begin{bmatrix} \sum_{\alpha=3}^{4} h_{\alpha,x}\hat{\gamma}_{\alpha}^{T} & 0 & 0 \\ 0 & \sum_{\alpha=3}^{4} h_{\alpha,y}\hat{\gamma}_{\alpha}^{T} & 0 \\ 0 & 0 & h_{3,z}\hat{\gamma}_{3}^{T} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & h_{4,y}\hat{\gamma}_{4}^{T} \end{bmatrix}$$

and that of the vectors $\hat{\gamma}_{\alpha}$ by:

$$\hat{\gamma}_{\alpha}=\frac{1}{8}\left[\mathbf{h}_{\alpha}-\sum_{j=1}^{3}\left(\mathbf{h}_{\alpha}^{T}\cdot\mathbf{X}_{j}\right)\hat{\mathbf{b}}_{j}\right]$$

The new matrix of rigidity becomes:

$$\mathbf{K}=\int_{\Omega_e}\mathbf{B}^{T}\cdot\mathbf{C}\cdot\mathbf{B}\,d\,\Omega+\int_{\Omega_e}\mathbf{B}^{T}\cdot\mathbf{C}\cdot\hat{\mathbf{B}}_{\mathbf{stab}}\,d\,\Omega+\int_{\Omega_e}\hat{\mathbf{B}}_{\mathbf{stab}}{}^{T}\cdot\mathbf{C}\cdot\mathbf{B}\,d\,\Omega+\underbrace{\int_{\Omega_e}\hat{\mathbf{B}}_{\mathbf{stab}}{}^{T}\cdot\mathbf{C}\cdot\hat{\mathbf{B}}_{\mathbf{stab}}}_{\mathbf{K}^{\mathbf{stab}}}\,d\,\Omega$$

The last term of the preceding equation ($\mathbf{K}^{\mathbf{stab}}$) is enough to stabilize the element. One can thus reduce the matrix of rigidity stabilized to:

$$\mathbf{K}=\mathbf{K}_{e}+\mathbf{K}^{\mathbf{stab}}$$

$$\mathbf{K}^{\mathbf{stab}}=\int_{\Omega_e}\hat{\mathbf{B}}_{\mathbf{stab}}{}^{T}\cdot\mathbf{C}\cdot\hat{\mathbf{B}}_{\mathbf{stab}}\,d\,\Omega$$

The many cases which were studied showed that it is enough to calculate the diagonal terms of the matrix of stabilization $\mathbf{K}^{\mathbf{stab}}{}_{ii}$, $i=1,2,3$, which is given by:

$$\mathbf{K}_{11}^{\mathbf{stab}}=H_{11}(\bar{\lambda}+2\mu)[\hat{\gamma}_{3}\hat{\gamma}_{3}^{T}+\frac{1}{3}\hat{\gamma}_{4}\hat{\gamma}_{4}^{T}]$$

$$\mathbf{K}_{22}^{\mathbf{stab}}=H_{22}(\bar{\lambda}+2\mu)[\hat{\gamma}_{3}\hat{\gamma}_{3}^{T}+\frac{1}{3}\hat{\gamma}_{4}\hat{\gamma}_{4}^{T}]$$

$$\mathbf{K}_{33}^{\mathbf{stab}}=\mu\frac{H_{33}}{3}E\hat{\gamma}_{4}\hat{\gamma}_{4}^{T}$$

Coefficients $H_{ii}$ themselves are given by the following equation, in which there is no summation on the repeated indices:

$$H_{ii}=\frac{1}{3}\frac{\left(\mathbf{X}_{j}^{T}\cdot\mathbf{X}_{j}\right)\left(\mathbf{X}_{k}^{T}\cdot\mathbf{X}_{k}\right)}{\left(\mathbf{X}_{i}^{T}\cdot\mathbf{X}_{i}\right)}$$

# 6 Strategy for non-linear calculations

## 6.1 Geometrical non-linearities

One treats here the case of great displacements, but with weak rotations (see further) and small deformations. One adopts for that an up to date put Lagrangian formulation.

Into nonlinear we seek to write balance between internal forces and force external at the end of the increment of load (located by index 2):

$$F_2^{\text{int}} = F_2^{extr}$$

The expression of the internal forces is written:

$$F_2^{\text{int}} = \int_{\Omega_2} \mathbf{B}_2^T \sigma_2 \, dV$$

In the preceding equation the operator $\mathbf{B}_2$ is the operator allowing to pass from the displacement to the linear deformation calculated on the geometry at the end of the step, the constraint $\sigma_2$ is the constraint of Cauchy at the end of the step and integration is made on volume $\Omega_2$ deformed at the end of the step.

The element is programmed in small rotations. Indeed the increment of deformation is calculated by using only the linear deformation:

$$\Delta\mathbf{E} = \frac{1}{2}\left(\underline{\underline{\nabla_1(\Delta\underline{\mathbf{u}})}} + \underline{\underline{\nabla_1^T(\Delta\underline{\mathbf{u}})}}\right)$$

The operator gradient is calculated on the geometry of beginning of step. This writing of the deformation is limited to small rotations (<5 degrees).

One can without difficulty of extending the formulation to great rotations by including in the deformation the terms of second order:

$$\Delta \underline{\underline{E}} = \frac{1}{2}\left( \underline{\underline{\nabla_1(\Delta \underline{\mathbf{u}})}} + \underline{\underline{\nabla_1^T(\Delta \underline{\mathbf{u}})}} + \underline{\underline{\nabla_1^T(\Delta \underline{\mathbf{u}})}} \cdot \underline{\underline{\nabla_1(\Delta \underline{\mathbf{u}})}} \right)$$

In elasticity, the law of behavior is written:

$$\Delta\underline{\underline{\pi}} = \underline{\underline{\mathbf{C}}}'\Delta\underline{\underline{E}}$$

where $\mathbf{C}$ is the matrix of Hooke. Let us notice that for the SHB8 this matrix is a transverse orthotropic matrix which is written in the axes of the lamina:

$$[\mathrm{C}'] = \begin{bmatrix} \lambda+2\mu & \mu & 0 & 0 & 0 & 0 \\ \mu & \lambda+2\mu & 0 & 0 & 0 & 0 \\ 0 & 0 & E & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix}$$

The formula allowing to calculate the constraint of Cauchy $\underline{\underline{\sigma}}_2$ starting from the constraint of Piola Kirchhoff II $\underline{\underline{\pi}}_2$ is:

$$\begin{cases} \underline{\underline{\pi}}_2 = \underline{\underline{\sigma}}_1 + \Delta\underline{\underline{\pi}} \\ \underline{\underline{\sigma}}_2 = \dfrac{1}{\det\left(\underline{\underline{F}}\right)} \underline{\underline{F}}^T \underline{\underline{\pi}}_2 \underline{\underline{F}} \\ \underline{\underline{F}} = \underline{\underline{I}} + \underline{\underline{\nabla_1 \Delta \mathbf{u}}} \end{cases}$$

combination of the four last equations with the expression of the internal forces gives the formulation of the element in great deformations into Lagrangian updated.

Let us notice that this up to date put Lagrangian formulation is completely equivalent to the total Lagrangian formulation for which the internal forces are written:

$$F_2^{\text{int}} = \int_{\Omega_0} \left( \mathbf{B} + \mathbf{B}^{NL}(\mathbf{u}) \right)_0^T \pi_2 \, dV$$

In this case all integrations are made on the initial geometry $\Omega_0$ the constraint $\pi_2$ used is the constraint of Piola Kirchhoff II. This last method is probably preferable when the grid becomes deformed significantly and thus makes it possible to deal with the problems in great deformations but requires the development of the operator $\mathbf{B}^{NL}(\mathbf{u})$ .

The increment of deformation in Lagrangian total is expressed on the initial geometry of the structure.

$$\Delta \underline{\underline{E}} = \frac{1}{2} \left( \underline{\underline{\nabla_0 (\Delta \mathbf{u})}} + \underline{\underline{\nabla_0^T (\Delta \mathbf{u})}} + \underline{\underline{\nabla_0^T (\Delta \mathbf{u})}} \cdot \underline{\underline{\nabla_0 (\Delta \mathbf{u})}} \right)$$

The combination of the two preceding equations gives the formulation of the element in great deformations in linear behavior material.

## 6.2 Small displacements

In the case of small displacements one confuses geometry in the beginning and end of step, constraint of Cauchy and Piola Kirchhoff II, moreover one uses the linear expression of the deformations.

## 6.3 Forces of stabilization

The forces of stabilization make it possible to avoid the modes of sand glass and are added in the calculation of the residues to balance the contribution of the matrix of stiffness of stabilization to the first member. Forces of stabilization $\mathbf{F}^{\mathbf{stab}}$ , to add to the internal forces $F_2^{\text{int}}$ , are written:

$$\mathbf{F}^{\mathbf{stab}} = \mathbf{K}^{\mathbf{stab}} \mathbf{U}$$

For reasons of effectiveness, one chooses not to assemble again $\mathbf{K}^{\mathbf{stab}}$ to calculate $\mathbf{F}^{\mathbf{stab}}$ at the end of the step, but rather to build $\mathbf{F}^{\mathbf{stab}}$ from $\hat{\mathbf{B}}_{\mathbf{stab}}$ that one calculated previously. One must for that place oneself in the reference frame corotationnel of medium of step suggested in [bib3]. For this reason, one does not obtain an exact expression of $\mathbf{F}^{\mathbf{stab}}$, and some additional iterations are generally necessary to converge. These some iterations are however unimportant compared to the cost of calculation saved while not assembling $\mathbf{K}^{\mathbf{stab}}$.

## 6.4 Plasticity

A first version of the element treated only the elastoplastic behavior of Von Mises, with isotropic work hardening. In each of the 5 points of integration, the formulas and the usual programming of plasticity 3D was used, with the linear matrix of behavior $C'$ orthotropic. This resulted in modifying the usual algorithm of three-dimensional elastoplastic flow by replacing the usual matrix of Hooke $C$ by the matrix of orthotropic behavior transverse $C'$. The nonlinear problem was solved by a method of Newton.

For more general information allows, and to give access to the whole of the laws of behavior, another strategy from now on is used, similar to that used for `COQUE_3D` [R3.07.04]. It is a question of uncoupling the behavior according to average surface, of the transverse behavior, according to the normal with the element. The method consists in supposing that the element is in a state of plane constraint in the local reference mark of each point of integration of Gauss and that the deformations except plan are elastic. That involves then immediately that the total deflections except plan are equal to the elastic strain. After integration of the behavior in plane constraints, the constraints except plan are calculated in an elastic way. Let us call $\underline{\underline{C}}^{CPT}$ the tangent matrix in plane constraints. The tangent matrix of behavior for the selected behavior is written:

$$\underline{\underline{C}}^T = \begin{bmatrix} C^{CPT}_{xxxx} & C^{CPT}_{xxyy} & 0 & C^{CPT}_{xxxy} & 0 & 0 \\ C^{CPT}_{xyyx} & C^{CPT}_{yyyy} & 0 & C^{CPT}_{yyxy} & 0 & 0 \\ 0 & 0 & E & 0 & 0 & 0 \\ C^{CPT}_{xyxx} & C^{CPT}_{xyyy} & 0 & C^{CPT}_{xyxy} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix}$$

This method thus makes it possible to connect elements SHB to all the laws of behavior available in plane constraints in the Code Aster (in an analytical way or via the method due to Borst).

# 7 Establishment of element SHB8 in *Code_Aster*

## 7.1 Description

This element is pressed on the voluminal meshs 3D HEXA8.

## 7.2 Use

This element is used in the following way:

### 7.2.1 Grid

It is necessary to check the good orientation of the faces of the elements indicated (compatibility with the privileged direction) while using ORIE_SHB of the operator MODI_MAILLAGE.

### 7.2.2 Modeling

The name of modeling SHB is used to affect finite element SHB8 with the meshs HEXA8 indicated.

### 7.2.3 Material

The element supposes that them coefficients E, Young modulus and NAKED, Poisson's ratio are given (as well into linear as into nonlinear) for the calculation of the matrix of elastic or tangent rigidity (transverse terms). It is thus necessary to inform the keyword ELAS in DEFI_MATERIAU.

All the nonlinear behaviors (compatible with a modeling in plane constraints) are usable.

It should be noted that thermal dilation is not taken into account in version 12 of Code_Aster for elements SHB.

### 7.2.4 Boundary conditions and loading

The usual voluminal loadings are usable: forces of volume, gravity.
The efforts of pressure (and other efforts surface) are applied to elements of faces, as in 3D (under the keyword PRES_REP). One will have taken first care to define meshs of skin QUAD4 and to suitably direct the outgoing normals with these meshs of skin using the order MODI_MAILLAGE keyword ORIE_PEAU_3D.

No development was necessary for the compressive forces distributed and the following compressive forces. Indeed, these loadings are pressed on meshs of skin identical to those of the voluminal elements 3D.

### 7.2.5 Calculation in linear elasticity

The options of postprocessing available are SIEF_ELNO and SIEQ_ELNO.

### 7.2.6 Calculation in linear buckling

The option RIGI_MECA_GE being activated in the catalogue of the element, it is possible to carry out a classical calculation of buckling after assembly of the matrices of elastic and geometrical rigidity.

### 7.2.7 Calculation nonlinear geometrical

One can carry out nonlinear calculations in small deformations (DEFORMATION='SMALL') or of great displacements and small rotations (DEFORMATION= ' GROT_GDEP') in STAT_NON_LINE or DYNA_NON_LINE.

The strategy used being based on the use of a matrix of tangent rigidity during iterations (reactualization at the beginning of step only), one will take care to use another option only that which is activated by default, namely REAC_ITER = 0 under NEWTON.

### 7.2.8 Calculation nonlinear material

All the nonlinear behaviors of continuous mediums are usable (keyword RELATION = under BEHAVIOR). If the behavior is not integrated in an analytical way in plane constraints, the method of Borst [R5.03.03] is automatically used.

The strategy used being based on the use of a matrix of tangent rigidity during iterations (reactualization at the beginning of step only), one will take care to use another option only that which is activated by default, namely REAC_ITER = 0 under NEWTON.

## 7.3 Characteristics of the establishment

The forces of stabilization of the element require the storage of a vector of size 12 for each point of Gauss. We chose to store these terms like additional components of the stress field.

## 7.4 Validation

The tests validating this element are the following:

1) SDLS109: Eigen frequencies of a thick cylindrical ring
2) SSLS101: Circular plate posed under pressure
3) SSLS105: hemisphere doubly pinch [V3.03.105] classical test to check the convergence of the element,
4) SSLS108: beam bored in inflection, test allowing to check the absence of blocking [V3.03.108],
5) SSLS123: sphere under external pressure [V3.03.123] to validate the loadings of pressure and the orthotropic behavior particular to this element,
6) SSLS124: thin section in inflection with various twinges, to delimit the field of use of the element [V3.03.124]. The results are correct (less than 1% with the analytical solution) for reports of twinge (thickness/width) going from 1 to 5 $10^{-3}$,
7) SSLS125: buckling (modes of Euler) of a free cylinder under external pressure [V3.03.125] this test makes it possible to validate the geometrical nature of rigidity,
8) SSLS129: Modeling D a doubly sinusoidal hull
9) SSNS101: breakdown of a cylindrical roof [V6.03.101]. This test makes it possible to validate geometrical nonlinear calculation and elastoplasticity,
10) SSNS102: buckling of a hull with stiffeners in great displacements and following pressure [V6.03.102].

# 8 Bibliography

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# 9 Description of the versions

| **Index document** | **Version Aster** | **Author (S) Organization (S)** | **Description of the modifications** |
|---|---|---|---|
| With | 7.2 | S.BAGUET, A.COMBESCURE INSA Lyon<br>J.M. PROIX EDF R & D AMA | Initial version |
| B | 9.4 | Trinh Vuong Dieu, X Desroches EDF R & D AMA | Modification of Bstat §5.3, and the integration of the nonlinear behaviors §6.4 |