Elements of voluminal hull SHB with 6.15 and 20 nodes

Summary:

We present in this document 3 new elements of voluminal hull intended to supplement modeling `SHB`, which comprises already the element `SHB8` having for mesh support a hexahedron with 8 nodes [R3.07.07]. These 3 elements are:

- the element `SHB6` who has as a mesh support a pentahedron with 6 nodes,
- the element `SHB15` who has as a mesh support a pentahedron with 15 nodes,
- the element `SHB20` who has as a mesh support a hexahedron with 20 nodes.

Just as it `SHB8`, these 3 elements have a called privileged direction thickness. Thus, they can be used to represent mean structures while correctly taking into account the phenomena through the thickness (inflection, elastoplasticity), grace a digital integration to 5 points of Gauss in this privileged direction.

Like `SHB8`, and in order to reduce time calculation, these elements under-are integrated but, contrary to him, they do not have modes of Hourglass (modes of deformation to worthless energy) and thus do not require a mechanism of stabilization. Nevertheless, to avoid blockings (in particular in transverse shearing), it `SHB6` is project following the method of the supposed deformations (assumed strain).
The quadratic elements neither are stabilized, nor projected.

In addition to their cost of relatively weak calculation and their good performances in elastoplasticity, these elements have another advantage. Since they are based on a three-dimensional formulation and that they have only degrees of freedom of translation, it is easy to couple them with voluminal elements 3D, which is very useful in systems where voluminal hulls and elements must cohabit.

Contents

1 Introduction

Element of a solid type - voluminal hull of hexahedral geometry at five points of Gauss was already established in ASTER. Good performances of this element, named SHB8, were put in obviousness by Abed-Meraim and Combescure [bib1], [bib2] like by Legay in [bib3]. This element represents a thick hull obtained starting from a purely three-dimensional formulation. It has eight nodes and five points of integration distributed according to the direction thickness. The three-dimensional law of behavior was also amended to approach the behavior of the hulls and to avoid certain lockings (shearing, membrane). To eliminate the modes with worthless energy due to under-integration, an effective technique of stabilization was used while following the approach of Belytschko and Bindeman [bib4]. In the same way, the operator discretized gradient was modified for the elimination of various blockings. Thus, the version obtained of this element has the following advantages:

- capacity to model mean three-dimensional structures with few elements of grid thanks to the tolerated important twinge (significant time-saver of calculations),
- simplified grid of complex geometries where solid hulls and elements must cohabit (reinforcements or supports for example) without having the classical problems of connections of made grids of various types of elements.

This hexahedral element was introduced into *Code_Aster* in version 7 (see [R3,07,07]). However, the hexahedral element SHB8 does not allow to net geometries of complex forms unspecified. The development of a similar element but of prismatic geometry was thus necessary. One describes in the beginning of this document this prismatic element (element SHB6).

The research tasks of Caironi and Abed-Meraim [bib5] proved that the element SHB6 did not present modes of hourglass, and after having established it, they as showed as this one presented a severe digital blocking, in particular in the requests in transverse shearing of the element. The element SHB6 established in Aster these digital blockings by using the method “assumed strain aims at eliminating”. The principle of this method consists in projecting the operator discretized gradient B on a suitable subspace in order to avoid the various problems involved in blocking. Several projections were tested before finding that which eliminates the maximum of lockings.

The element SHB6 fact the object of the §2.

The §3 presents an extension of this family of finite elements of standard solid-hull: two finite elements of prismatic and hexahedric geometry but of quadratic formulation named SHB15 and SHB20. They are respectively elements with 15 and 20 nodes. They under-are also integrated by 15 and 20 points of Gauss and have a direction privileged according to the thickness of the element. These elements not having blockings are not projected.

In addition, the element SHB8 initial had been coupled with the only laws of elastic and elastoplastic behaviour with isotropic work hardening of type Von-Put. The field of application of the element SHB8 as well as the other finite elements solid-hull SHB6, SHB15 and SHB20 was extended to the other laws of behavior of Code_Aster. The §4 presents the theoretical principle of this coupling.

Finally the §5 treats establishment of these elements in *Code_Aster*.

2 Element SHB6

2.1 Kinematics of the element

Element SHB6 is a pentahedron with 6 nodes. The five points of integration are selected along the direction ζ in the reference mark of the local coordinates of the element of reference: ξ, η, ζ (or $\hat{x}_{1,} \hat{x}_{2,} \hat{x}_3$ for certain expressions). The shape of the element of reference as well as the points of integration are represented on [Figure 1].

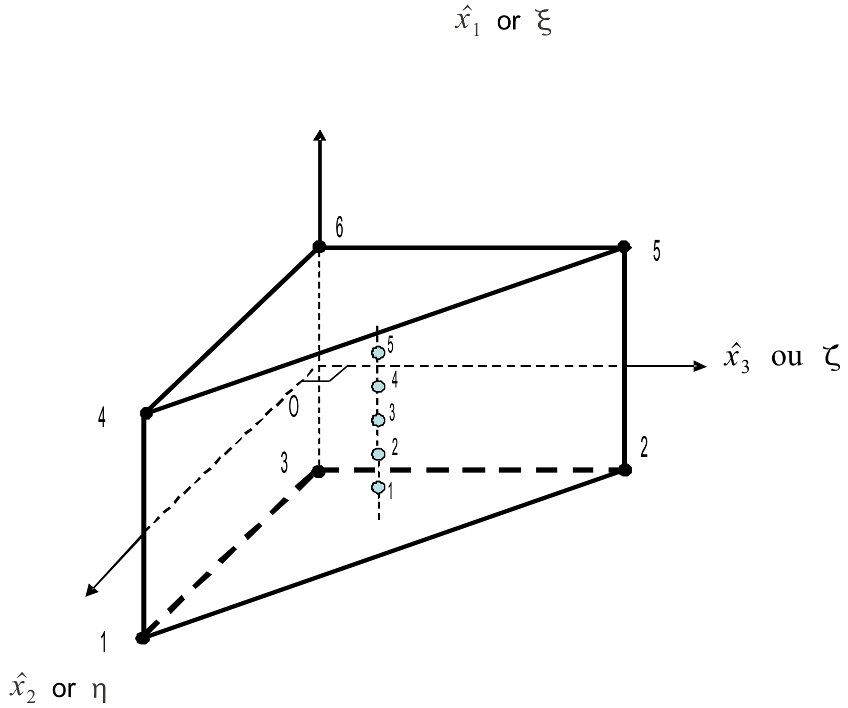


Figure 1: Geometry of the element of reference and points of integration

This element is isoparametric and has the same linear interpolation and same kinematics as the pentahedral elements with 6 standard nodes.

2.2 Discretization

2.2.1 Discretization of the field of displacement

Space coordinates x_i element are connected to the nodal coordinates x_{iI} by means of the isoparametric functions of forms N_I by the formulas:

$$\boldsymbol{x}_i = \boldsymbol{x}_{iI} \boldsymbol{N}_I(\hat{x}_{1,} \hat{x}_{2,} \hat{x}_3) = \sum_{i=1}^{6} \boldsymbol{x}_{iI} \boldsymbol{N}_I(\hat{x}_{1,} \hat{x}_{2,} \hat{x}_3)$$

In the continuation, and except contrary mention, one will adopt the convention of summation for the repeated indices. Indices in small letters i vary from one to three and represent the directions of the space coordinates. Those in capital letters I vary from one to six and correspond to the nodes of the element.

The same functions of forms are used to define the field of displacement of the element u_i according to nodal displacements u_{iI} :

$$\boldsymbol{u_i} = \boldsymbol{U_{iI}} \boldsymbol{N_I}(\hat{x}_1, \hat{x}_2, \hat{x}_3) = \sum_{i=1}^{6} \boldsymbol{U_{iI}} \boldsymbol{N_I}(\hat{x}_1, \hat{x}_2, \hat{x}_3)$$

To continue calculations, linear isoparametric functions of form are given $N_i(\xi, \eta, \zeta) = N_i(\hat{x}_1, \hat{x}_2, \hat{x}_3)$ associated with the prismatic element with six nodes:

$$N_1 = \frac{1}{2}\hat{x}_2\left(1-\hat{x}_1\right) \qquad N_4 = \frac{1}{2}\hat{x}_2\left(1+\hat{x}_1\right)$$

$$N_2 = \frac{1}{2}\hat{x}_3\left(1-\hat{x}_1\right) \qquad N_5 = \frac{1}{2}\hat{x}_3\left(1+\hat{x}_1\right)$$

$$N_3 = \frac{1}{2}\left(1-\hat{x}_2-\hat{x}_3\right)\left(1-\hat{x}_1\right) \qquad N_6 = \frac{1}{2}\left(1-\hat{x}_2-\hat{x}_3\right)\left(1+\hat{x}_1\right)$$

$$\hat{x}_1 = [-1,1]; \qquad \hat{x}_2 = [0,1]; \qquad \hat{x}_3 = [0,1-\hat{x}_2]$$

The origin of the reference mark is confused with the right corner of the triangle of the median plane of the element.

2.2.2 Operator discretized gradient

The gradient $u_{i,j}$ field of displacement is a function of displacements $\mathbf{U}_{iI}$ nodes I in the direction i

$$u_{i,j} = \mathbf{U}_{iI} N_{I,j}$$

The linear tensor of deformation is given by the symmetrical part of the gradient of displacement:

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

One now will build vectors allowing to express the matrix B connecting the deformations to displacements in a particular form.

In a way similar to Belytschko-Bindeman [bib6], the three vectors are introduced $\underline{\mathbf{b}}_i$, derived from the functions of form at the origin of the coordinates:

$$\boldsymbol{b_i^T} = \boldsymbol{N_{,i}}(0) = \left.\frac{\partial \boldsymbol{N}}{\partial x_i}\right]_{\hat{x}_1=\hat{x}_2=\hat{x}_3=0} \qquad i=1,2,3$$

These 3 vectors are constant and are given by the expression:

$$\boldsymbol{b}_i^T = \begin{pmatrix} j_{i1} & j_{i2} & j_{i3} \end{pmatrix} \cdot \begin{bmatrix} 0 & 0 & \frac{-1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & \frac{-1}{2} \\ 0 & \frac{1}{2} & \frac{-1}{2} & 0 & \frac{1}{2} & \frac{-1}{2} \end{bmatrix}$$

where coefficients j_{kl} are the coefficients of the matrix jacobienne evaluated in the beginning.

Also let us introduce the following vectors:

$$\begin{aligned} \mathbf{s}^T &= (1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1) \\ \mathbf{h}_1^T &= (-1 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0) \\ \mathbf{h}_2^T &= (0 \quad -1 \quad 0 \quad 0 \quad 1 \quad 0) \\ \mathbf{X}_i^T &= (x_{i1}, x_{i2}, x_{i3}, x_{i4}, x_{i5}, x_{i6}) \end{aligned}$$

Three vectors $\mathbf{X}_i^T$ the nodal coordinates of the six nodes represent. Two vectors $\mathbf{h}_\alpha^T$ the functions represent respectively h_1 and h_2 for each of the six nodes, which are defined by:

$$h_1 = \hat{x}_1 \hat{x}_2 \qquad h_2 = \hat{x}_1 \hat{x}_3$$

Let us introduce finally the two following vectors:

$$\boldsymbol{\gamma}_\alpha = \frac{1}{2}\left[\mathbf{h}_\alpha - \sum_{j=1}^{3} \left(\mathbf{h}_\alpha^T . \mathbf{X}_j\right) \mathbf{b}_j\right] \qquad \alpha = 1,2$$

One can check by algebraic considerations that the following conditions of orthogonality are satisfied:

$$\begin{aligned} &\mathbf{b}_i^T \cdot \mathbf{h}_\alpha = 0 \\ &\mathbf{b}_i^T \cdot \mathbf{s} = 0 \\ &\mathbf{h}_\alpha{}^T \cdot \mathbf{s} = 0 \\ &\mathbf{b}_i^T \cdot \mathbf{X}_j = \delta_{ij} \qquad i,j = 1,2,3 \quad \alpha,\beta = 1,2 \\ &\mathbf{h}_\alpha{}^T \cdot \mathbf{h}_\beta = 2\delta_{\alpha\beta} \qquad \textbf{(1)} \\ &\gamma_\alpha^T \cdot \mathbf{X}_j = 0 \\ &\gamma_\alpha{}^T \cdot \mathbf{h}_\beta = \delta_{\alpha\beta} \end{aligned}$$

where δ_{ij} is the symbol of Kronecker.

These vectors will make it possible to express the matrix B connecting the deformations to displacements in a particular form used thereafter.

The gradient of the field of displacement can be now written after calculations in the form (without any approximation [bib6]):

$$u_{i,j}=\left(\boldsymbol{b}_j^T+\boldsymbol{h}_{\alpha,j}\gamma_\alpha^T\right)\cdot\boldsymbol{U}_i$$

The symmetrical operator gradient (noted ∇_S) discretized connecting the tensor of deformation to the vector of nodal displacements

$$\nabla_s\boldsymbol{u}=\boldsymbol{B}\cdot\boldsymbol{u}$$

takes the matric shape then:

$$\mathbf{B}=\begin{bmatrix}\mathbf{b}_x^T+h_{\alpha,x}\gamma_\alpha^T & 0 & 0\\ 0 & \mathbf{b}_y^T+h_{\alpha,y}\gamma_\alpha^T & 0\\ 0 & 0 & \mathbf{b}_z^T+h_{\alpha,z}\gamma_\alpha^T\\ \mathbf{b}_y^T+h_{\alpha,y}\gamma_\alpha^T & \mathbf{b}_x^T+h_{\alpha,x}\gamma_\alpha^T & 0\\ \mathbf{b}_z^T+h_{\alpha,z}\gamma_\alpha^T & 0 & \mathbf{b}_x^T+h_{\alpha,x}\gamma_\alpha^T\\ 0 & \mathbf{b}_z^T+h_{\alpha,z}\gamma_\alpha^T & \mathbf{b}_y^T+h_{\alpha,y}\gamma_\alpha^T\end{bmatrix} \quad \mathbf{(2)}$$

2.3 Matrix of rigidity and stabilization

2.3.1 Matrix of rigidity

The matrix of rigidity of the element is given by:

$$\boldsymbol{K}_e=\int_{V_e}\boldsymbol{B}^T\cdot\boldsymbol{C}\cdot\boldsymbol{B}\,dV$$

Five points of integration considered $\mathbf{P_i}$ are on the same vertical line. Their coordinates and their weights of integration are the following:

	$\hat{\mathbf{x}}_2$	$\hat{\mathbf{x}}_3$	$\hat{\mathbf{x}}_1$	ω
$\mathbf{P}_1$	1/3	1/3	-0.906179845938664	0.236926885056189
$\mathbf{P}_2$	1/3	1/3	-0.538469310105683	0.478628670499366
$\mathbf{P}_3$	1/3	1/3	0	0.568888888888889
$\mathbf{P}_4$	1/3	1/3	0.538469310105683	0.478628670499366
$\mathbf{P}_5$	1/3	1/3	0.906179845938664	0.236926885056189

Thus, the expression of rigidity $\mathbf{K}_e$ is:

$$K_e=\sum_{j=1}^{5}\omega(P_j)J(P_j)B^T(P_j).C.B(P_j) \quad \mathbf{(3)}$$

where $J(P_j)$ is Jacobien, calculated at the point of Gauss j , transformation enters the element of reference and the current element. The elastic matrix of behavior $\mathbf{C}$ chosen has the following form:

$$\mathbf{C}=\begin{bmatrix} \bar{\lambda}+2\mu & \bar{\lambda} & 0 & 0 & 0 & 0 \\ \bar{\lambda} & \bar{\lambda}+2\mu & 0 & 0 & 0 & 0 \\ 0 & 0 & E & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix}$$

where E is the Young modulus, ν the Poisson's ratio, $\mu=\frac{E}{2(1+\nu)}$ the modulus of rigidity and $\bar{\lambda}=\frac{E\nu}{1-\nu^2}$ the coefficient of modified Lamé. This law is specific to elements SHB. It resembles that which one would have in the case of the assumption of the plane constraints, put except for the term (3.3).

Even if this choice involves an artificial anisotropic behavior, it makes it possible to satisfy all the tests without introducing blocking.

2.3.2 Analysis of the modes "hourglass" for element SHB6

The modes of "hourglass" are kinematics modes which are due to under-integration and are associated with a worthless energy whereas they induce a nonworthless deformation. This anomaly is explained by the difference, that induced under-integration, between the core of the continuous operator of rigidity discretized and that. Let us start initially by noticing that the operator discretized gradient under-integrated associated with the five points of integration defined above takes the shape of the equation **(2)** with $\alpha=1,2$.

Now let us analyze the core of the matrix of rigidity obtained by under-integration. According to (3), that returns under investigation from the row of the matrix B insofar as the matrix of behavior C is not singular. In other words, it is enough to search the modes of displacement $\boldsymbol{d}$ with worthless deformation, i.e. checking:

$$\nabla_s(\boldsymbol{u})=\boldsymbol{B}\cdot\boldsymbol{d}=\mathbf{0} \qquad (4)$$

We will seek from now on which are the modes of deformations which give a worthless deformation energy. The deformation energy is written $w(\boldsymbol{\epsilon})=\frac{1}{2}\int_V \boldsymbol{\epsilon}\cdot\boldsymbol{C}\cdot\epsilon\, dV$ and like $\boldsymbol{\epsilon}=\boldsymbol{B}\,.\,\boldsymbol{d}$. we thus have:

$$w(\boldsymbol{\epsilon})=\frac{1}{2}\int_V \boldsymbol{d}^T\boldsymbol{B}^T\cdot\boldsymbol{C}\cdot\boldsymbol{B}\,\boldsymbol{d}\,dV=\boldsymbol{d}^T\left[\frac{1}{2}\int_V \boldsymbol{B}^T\cdot\boldsymbol{C}\cdot\boldsymbol{B}\,dV\right]\boldsymbol{d}$$

and if we consider the following approximation: $\boldsymbol{B}$ is calculated at the points of integration of Gauss, we obtain:

$$w(\boldsymbol{\epsilon})=\frac{1}{2}\boldsymbol{d}^T\boldsymbol{K}_e\boldsymbol{d}$$

Thus to search the modes of deformations to worthless energy is to search the core of $\boldsymbol{K}_e$

$$K_e\cdot\boldsymbol{X}=\mathbf{0}\Leftrightarrow\boldsymbol{B}(\xi_{Gj})\cdot\boldsymbol{X}=\mathbf{0}$$

Thus to search the modes of hourglass is to search the vectors $\boldsymbol{X}$ such as:

$$\boldsymbol{B}(\xi_{Gj})\cdot\boldsymbol{X}=\mathbf{0}\quad\forall\xi_{Gj} \qquad (5)$$

It is natural to find in the core of rigidity $\boldsymbol{K}_e$ modes associated with the rigid movements of body. For a three-dimensional element such as the prism with 6 nodes, these movements rigidifying are composed of three translations and three rotations. Thus the core of the continuous operator of rigidity is of dimension six and is reduced to the only following modes:

$$\begin{pmatrix}\underline{\mathbf{S}} & \underline{\mathbf{0}} & \underline{\mathbf{0}}\\ \underline{\mathbf{0}} & \underline{\mathbf{S}} & \underline{\mathbf{0}}\\ \underline{\mathbf{0}} & \underline{\mathbf{0}} & \underline{\mathbf{S}}\end{pmatrix} \quad et \quad \begin{pmatrix}\underline{\mathbf{y}} & \underline{\mathbf{z}} & \underline{\mathbf{0}}\\ -\underline{\mathbf{x}} & \underline{\mathbf{0}} & \underline{\mathbf{z}}\\ \underline{\mathbf{0}} & -\underline{\mathbf{x}} & -\underline{\mathbf{y}}\end{pmatrix} \qquad (6)$$

One easily checks that each of the six vectors columns above satisfy the equation **(5)** and thus belongs to the core of $\boldsymbol{K}_e$. It is enough, to see it, to use the expression **(2)** of $\boldsymbol{B}$ and conditions of orthogonality **(1)**. The first three vectors columns correspond to the translations according to the axes Ox, Oy and Oz respectively. The three other vectors are relating to rotations around the axes Oz, Oy and Ox.

We search from now on, in addition to the preceding rigid modes, of the modes which also cancel the operator discretized gradient given in **(2)**. Let us take a base of eighteen following vectors:

$$\begin{bmatrix}\underline{\mathbf{S}} & \underline{\mathbf{0}} & \underline{\mathbf{0}} & \underline{\mathbf{y}} & \underline{\mathbf{z}} & \underline{\mathbf{0}} & \underline{\mathbf{x}} & \underline{\mathbf{0}} & \underline{\mathbf{0}} & \underline{\mathbf{y}} & \underline{\mathbf{z}} & \underline{\mathbf{0}} & \underline{\mathbf{h}}_1 & \underline{\mathbf{0}} & \underline{\mathbf{0}} & \underline{\mathbf{h}}_2 & \underline{\mathbf{0}} & \underline{\mathbf{0}}\\ \underline{\mathbf{0}} & \underline{\mathbf{S}} & \underline{\mathbf{0}} & -\underline{\mathbf{x}} & \underline{\mathbf{0}} & \underline{\mathbf{z}} & \underline{\mathbf{0}} & \underline{\mathbf{y}} & \underline{\mathbf{0}} & \underline{\mathbf{x}} & \underline{\mathbf{0}} & \underline{\mathbf{z}} & \underline{\mathbf{0}} & \underline{\mathbf{h}}_1 & \underline{\mathbf{0}} & \underline{\mathbf{0}} & \underline{\mathbf{h}}_2 & \underline{\mathbf{0}}\\ \underline{\mathbf{0}} & \underline{\mathbf{0}} & \underline{\mathbf{S}} & \underline{\mathbf{0}} & -\underline{\mathbf{x}} & -\underline{\mathbf{y}} & \underline{\mathbf{0}} & \underline{\mathbf{0}} & \underline{\mathbf{z}} & \underline{\mathbf{0}} & \underline{\mathbf{x}} & \underline{\mathbf{y}} & \underline{\mathbf{0}} & \underline{\mathbf{0}} & \underline{\mathbf{h}}_1 & \underline{\mathbf{0}} & \underline{\mathbf{0}} & \underline{\mathbf{h}}_2\end{bmatrix}$$

One can show easily that the vectors above are linearly independent within the space of dimension eighteen. Elementary calculations using the conditions of orthogonality **(1)** show that the last twelve vectors columns do not check the equation **(5)**.

That wants to say that there are not other modes only the rigid modes which cancel the operator discretized gradient given in **(2)**. In other words, the element `SHB6` do not present hourglass mode.

2.3.3 Projection by "local Assumed strain method"

The first stage is to place itself in the local reference mark of the element defined by the reference mark $(\hat{x}_1,\hat{x}_2,\hat{x}_3)$ described in *Figure 1*. The deformations from now on will thus be calculated in this reference mark. The operator discretized gradient $\boldsymbol{B}$ will be project on under suitable space in order to avoiding the various problems of blocking. This method is variationnellement coherent with the principle of Hu-Washizu if the interpolation of the constraint is judiciously selected (Simo and Hughes [15]). However, it is very difficult to select in a general and systematic way the good field of applied deformation. The fields of applied deformation should present neither voluminal blocking nor blocking in shearing.

We present here an easy choice and acceptable. The operator $\boldsymbol{B}$ is first of all separate in two parts $\underline{\underline{\mathbf{B}}}_1$ and $\underline{\underline{\mathbf{B}}}_2$. The matrix $\underline{\underline{\mathbf{B}}}_1$ contains the gradients in the average plan of the hull and the perpendicular deformation, $\underline{\underline{\mathbf{B}}}_2$ contains the gradients associated with the shearing strains transverse.

$$\underline{\underline{\mathbf{B}}} = \begin{bmatrix} \underline{\mathbf{b}}_1^T + \mathrm{h}_{\alpha,x_1}\,\underline{\boldsymbol{\gamma}}_\alpha^T & \underline{\mathbf{0}} & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \underline{\mathbf{b}}_2^T + \mathrm{h}_{\alpha,x_2}\,\underline{\boldsymbol{\gamma}}_\alpha^T & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \underline{\mathbf{0}} & \underline{\mathbf{b}}_3^T + \mathrm{h}_{\alpha,x_3}\,\underline{\boldsymbol{\gamma}}_\alpha^T \\ \underline{\mathbf{b}}_2^T + \mathrm{h}_{\alpha,x_2}\,\underline{\boldsymbol{\gamma}}_\alpha^T & \underline{\mathbf{b}}_1^T + \mathrm{h}_{\alpha,x_1}\,\underline{\boldsymbol{\gamma}}_\alpha^T & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \underline{\mathbf{0}} & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \underline{\mathbf{0}} & \underline{\mathbf{0}} \end{bmatrix} + \begin{bmatrix} \underline{\mathbf{0}} & \underline{\mathbf{0}} & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \underline{\mathbf{0}} & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \underline{\mathbf{0}} & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \underline{\mathbf{0}} & \underline{\mathbf{0}} \\ \underline{\mathbf{b}}_3^T + \mathrm{h}_{\alpha,x_3}\,\underline{\boldsymbol{\gamma}}_\alpha^T & \underline{\mathbf{0}} & \underline{\mathbf{b}}_1^T + \mathrm{h}_{\alpha,x_1}\,\underline{\boldsymbol{\gamma}}_\alpha^T \\ \underline{\mathbf{0}} & \underline{\mathbf{b}}_3^T + \mathrm{h}_{\alpha,x_3}\,\underline{\boldsymbol{\gamma}}_\alpha^T & \underline{\mathbf{b}}_2^T + \mathrm{h}_{\alpha,x_2}\,\underline{\boldsymbol{\gamma}}_\alpha^T \end{bmatrix}$$

$$= \underline{\underline{\mathbf{B}}}_1 + \underline{\underline{\mathbf{B}}}_2$$

The blockings noted in the element come from transverse shearings. One will seek a diagram of integration which allows of under - to integrate this part of energy. With this intention one seeks to control each component entering the energy of transverse shearing. Being given the shape of the matrix $\underline{\underline{\mathbf{B}}}$ we thus have 12 nonworthless terms which intervene in the deformation. They will be controlled by the introduction of the parameter c in the matrices $\underline{\underline{\mathbf{B}}}_2$. The matrix $\underline{\underline{\mathbf{B}}}_2$ becomes then $\overline{\underline{\underline{\mathbf{B}}}}_2$:

$$\overline{\underline{\underline{\mathbf{B}}}}_2 = c \begin{bmatrix} \underline{\mathbf{0}} & \underline{\mathbf{0}} & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \underline{\mathbf{0}} & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \underline{\mathbf{0}} & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \underline{\mathbf{0}} & \underline{\mathbf{0}} \\ \underline{\mathbf{b}}_3^T + \mathrm{h}_{\alpha,x_3}\,\underline{\boldsymbol{\gamma}}_\alpha^T & \underline{\mathbf{0}} & \underline{\mathbf{b}}_1^T + \mathrm{h}_{\alpha,x_1}\,\underline{\boldsymbol{\gamma}}_\alpha^T \\ \underline{\mathbf{0}} & \underline{\mathbf{b}}_3^T + \mathrm{h}_{\alpha,x_3}\,\underline{\boldsymbol{\gamma}}_\alpha^T & \underline{\mathbf{b}}_2^T + \mathrm{h}_{\alpha,x_2}\,\underline{\boldsymbol{\gamma}}_\alpha^T \end{bmatrix}$$

The matrix of rigidity is written now:

$$\boldsymbol{K}_e = \int_V \overline{\boldsymbol{B}}^T \cdot \boldsymbol{C} \cdot \overline{\boldsymbol{B}}\, dV = \int_V \overline{\boldsymbol{B}}_1^T \cdot \boldsymbol{C} \cdot \overline{\boldsymbol{B}}_1\, dV + \int_V \overline{\boldsymbol{B}}_2^T \cdot \boldsymbol{C} \cdot \overline{\boldsymbol{B}}_1\, dV + \int_V \overline{\boldsymbol{B}}_1^T \cdot \boldsymbol{C} \cdot \overline{\boldsymbol{B}}_2\, dV + \int_V \overline{\boldsymbol{B}}_2^T \cdot \boldsymbol{C} \cdot \overline{\boldsymbol{B}}_2\, dV = \\ = \boldsymbol{K}_{e1} + \boldsymbol{K}_{e2} + \boldsymbol{K}_{e3} + \boldsymbol{K}_{e4}$$

Matrices $\boldsymbol{K}_{e1,} \boldsymbol{K}_{e2,} \boldsymbol{K}_{e3,} \boldsymbol{K}_{e4}$ are integrated with the five points of Gauss defined previously. Additive decomposition given higher, $\underline{\underline{\mathbf{B}}} = \underline{\underline{\mathbf{B}}}_1 + \underline{\underline{\mathbf{B}}}_2$, for the operator discretized gradient, makes that the cross terms $\mathbf{K}_{e2}$ and $\mathbf{K}_{e3}$ cancel themselves. Following many the test digital, it was selected to characterize the matrix $\overline{\underline{\underline{\mathbf{B}}}}_2$ by the coefficient: $c = 0{,}45$, which plays here the part of a factor of reduction of shearing.

This choice gives to the element a good behavior in the cases of reference. It is clear that this strategy, as that installation for the cubic elements voluminal hulls are adapted only to the quasi isotropic behavior of selected material.

2.4 Geometrical matrix of rigidity Ksigma

The matrix $\boldsymbol{K}_\sigma$ aims to solve the problems of buckling. We point out here that the modes of buckling are the clean vectors of the problem to the eigenvalues generalized according to:

$$\left(\boldsymbol{K}+\mu\,\boldsymbol{K}_\sigma\right)\cdot\boldsymbol{u}=\boldsymbol{0}\Leftrightarrow\boldsymbol{K}\cdot\boldsymbol{u}=\lambda\,\boldsymbol{K}_\sigma\cdot\boldsymbol{u}$$

with $\lambda=-\mu$, and μ is the multiplying coefficient of the loading.

By introducing the quadratic deformation $\underline{\mathbf{e}}^Q$ such as:

$$e_{ij}^Q(\delta\boldsymbol{u},\Delta\boldsymbol{u})=\sum_{k=1}^{3}\delta u_{k,i}.\Delta u_{k,j}$$

One can define this matrix of geometrical rigidity by:

$$\delta\boldsymbol{u}^T\cdot\boldsymbol{K}_\sigma\cdot\Delta\boldsymbol{u}=\int_{\Omega_0}\boldsymbol{\sigma}:\boldsymbol{e}^Q(\delta\boldsymbol{u},\Delta\boldsymbol{u})\,d\Omega=\int_{\Omega_0}\boldsymbol{\sigma}:\nabla\delta\boldsymbol{u}^T\nabla\Delta\boldsymbol{u}\;\;d\Omega$$

In order to express this matrix in discretized space, let us introduce the discretized operators quadratic gradient $\underline{\underline{\mathbf{B}}}^Q$ (in matric notation) such as:

$$\underline{\mathbf{e}}^Q(\delta\mathbf{u},\Delta\mathbf{u})=\begin{bmatrix}e_{11}^Q\\e_{22}^Q\\e_{33}^Q\\e_{12}^Q+e_{21}^Q\\e_{13}^Q+e_{31}^Q\\e_{23}^Q+e_{32}^Q\end{bmatrix}=\begin{bmatrix}\underline{\delta\mathbf{u}}^T\cdot\underline{\underline{\mathbf{B}}}_{11}^Q\cdot\underline{\Delta\mathbf{u}}\\\underline{\delta\mathbf{u}}^T\cdot\underline{\underline{\mathbf{B}}}_{22}^Q\cdot\underline{\Delta\mathbf{u}}\\\underline{\delta\mathbf{u}}^T\cdot\underline{\underline{\mathbf{B}}}_{33}^Q\cdot\underline{\Delta\mathbf{u}}\\\underline{\delta\mathbf{u}}^T\cdot\underline{\underline{\mathbf{B}}}_{12}^Q\cdot\underline{\Delta\mathbf{u}}\\\underline{\delta\mathbf{u}}^T\cdot\underline{\underline{\mathbf{B}}}_{13}^Q\cdot\underline{\Delta\mathbf{u}}\\\underline{\delta\mathbf{u}}^T\cdot\underline{\underline{\mathbf{B}}}_{23}^Q\cdot\underline{\Delta\mathbf{u}}\end{bmatrix}$$

Various terms $\underline{\underline{\mathbf{B}}}_{ij}^Q$ are given by the following equations:

$$\underline{\underline{\mathbf{B}}}_{11}^Q=\begin{bmatrix}\underline{\mathbf{B}}_1\underline{\mathbf{B}}_1^T&\underline{\underline{0}}&\underline{\underline{0}}\\\underline{\underline{0}}&\underline{\mathbf{B}}_1\underline{\mathbf{B}}_1^T&\underline{\underline{0}}\\\underline{\underline{0}}&\underline{\underline{0}}&\underline{\mathbf{B}}_1\underline{\mathbf{B}}_1^T\end{bmatrix}\;;\;\underline{\underline{\mathbf{B}}}_{22}^Q=\begin{bmatrix}\underline{\mathbf{B}}_2\underline{\mathbf{B}}_2^T&\underline{\underline{0}}&\underline{\underline{0}}\\\underline{\underline{0}}&\underline{\mathbf{B}}_2\underline{\mathbf{B}}_2^T&\underline{\underline{0}}\\\underline{\underline{0}}&\underline{\underline{0}}&\underline{\mathbf{B}}_2\underline{\mathbf{B}}_2^T\end{bmatrix}\;;\;\underline{\underline{\mathbf{B}}}_{33}^Q=\begin{bmatrix}\underline{\mathbf{B}}_3\underline{\mathbf{B}}_3^T&\underline{\underline{0}}&\underline{\underline{0}}\\\underline{\underline{0}}&\underline{\mathbf{B}}_3\underline{\mathbf{B}}_3^T&\underline{\underline{0}}\\\underline{\underline{0}}&\underline{\underline{0}}&\underline{\mathbf{B}}_3\underline{\mathbf{B}}_3^T\end{bmatrix}$$

$$\underline{\underline{\mathbf{B}}}_{12}^Q=\begin{bmatrix}\underline{\mathbf{B}}_1\underline{\mathbf{B}}_2^T+\underline{\mathbf{B}}_2\underline{\mathbf{B}}_1^T&\underline{\underline{0}}&\underline{\underline{0}}\\\underline{\underline{0}}&\underline{\mathbf{B}}_1\underline{\mathbf{B}}_2^T+\underline{\mathbf{B}}_2\underline{\mathbf{B}}_1^T&\underline{\underline{0}}\\\underline{\underline{0}}&\underline{\underline{0}}&\underline{\mathbf{B}}_1\underline{\mathbf{B}}_2^T+\underline{\mathbf{B}}_2\underline{\mathbf{B}}_1^T\end{bmatrix}$$

$$\underline{\underline{\mathbf{B}}}_{13}^Q=c^2\begin{bmatrix}\underline{\mathbf{B}}_1\underline{\mathbf{B}}_3^T+\underline{\mathbf{B}}_3\underline{\mathbf{B}}_1^T&\underline{\underline{0}}&\underline{\underline{0}}\\\underline{\underline{0}}&\underline{\mathbf{B}}_1\underline{\mathbf{B}}_3^T+\underline{\mathbf{B}}_3\underline{\mathbf{B}}_1^T&\underline{\underline{0}}\\\underline{\underline{0}}&\underline{\underline{0}}&\underline{\mathbf{B}}_1\underline{\mathbf{B}}_3^T+\underline{\mathbf{B}}_3\underline{\mathbf{B}}_1^T\end{bmatrix}$$

$$\underline{\underline{\mathbf{B}}}_{23}^Q=c^2\begin{bmatrix}\underline{\mathbf{B}}_2\underline{\mathbf{B}}_3^T+\underline{\mathbf{B}}_3\underline{\mathbf{B}}_2^T&\underline{\underline{0}}&\underline{\underline{0}}\\\underline{\underline{0}}&\underline{\mathbf{B}}_2\underline{\mathbf{B}}_3^T+\underline{\mathbf{B}}_3\underline{\mathbf{B}}_2^T&\underline{\underline{0}}\\\underline{\underline{0}}&\underline{\underline{0}}&\underline{\mathbf{B}}_2\underline{\mathbf{B}}_3^T+\underline{\mathbf{B}}_3\underline{\mathbf{B}}_2^T\end{bmatrix}$$

with the vectors $\underline{\mathbf{B}}_i$ (I = 1,2,3) defined as:

$$\underline{\mathbf{B}}_i = \left(\underline{\mathbf{b}}_i + h_{\alpha,i} \underline{\boldsymbol{\gamma}}_\alpha \right)$$

Note: We must multiply the matrices $\underline{\underline{\mathbf{B}}}_{13}^{\mathbf{Q}}$ and $\underline{\underline{\mathbf{B}}}_{23}^{\mathbf{Q}}$ by the coefficient $c^2 = 0{,}45^2 = 0{,}2025$ because element SHB6 is project by the technique " *Local Assumed strain method* " to see section 2.3.3 .

With these notations, the contribution to the geometrical matrix of rigidity, $\underline{\underline{\mathbf{k}}}_{\boldsymbol{\sigma}}$, at the point of Gauss ξ_j is given by:

$$\begin{aligned}\underline{\underline{\mathbf{k}}}_{\boldsymbol{\sigma}}\left(\xi_j\right) = {} & \sigma_{11}\left(\xi_j\right)\underline{\underline{\mathbf{B}}}_{11}^{Q}\left(\xi_j\right) + \sigma_{22}\left(\xi_j\right)\underline{\underline{\mathbf{B}}}_{22}^{Q}\left(\xi_j\right) + \sigma_{33}\left(\xi_j\right)\underline{\underline{\mathbf{B}}}_{33}^{Q}\left(\xi_j\right) \\ & + \sigma_{12}\left(\xi_j\right)\underline{\underline{\mathbf{B}}}_{12}^{Q}\left(\xi_j\right) + \sigma_{13}\left(\xi_j\right)\underline{\underline{\mathbf{B}}}_{13}^{Q}\left(\xi_j\right) + \sigma_{23}\left(\xi_j\right)\underline{\underline{\mathbf{B}}}_{23}^{Q}\left(\xi_j\right)\end{aligned}$$

By integration on the points of Gauss of the element, the geometrical matrix of rigidity is obtained by the formula:

$$\underline{\underline{K}}_\sigma = \sum_{j=1}^{5} \omega\left(\xi_j\right) J\left(\xi_j\right) \underline{\underline{k}}_\sigma\left(\xi_j\right)$$

2.5 Following forces and matrix of pressure K_p

The following compressive forces are present in the tangent matrix via the matrix $\underline{\underline{\mathbf{K}}}_P$, because the following external forces depend on displacement [R3.03.04]. The following compressive forces are written:

$$\int_{\partial\Omega} p\,\boldsymbol{n}^{\boldsymbol{T}} \cdot \boldsymbol{u}\, dS = \int_{\partial\Omega_0} p \det[\boldsymbol{F}(\boldsymbol{u})]\, \boldsymbol{n}_{\mathbf{0}}^{\boldsymbol{T}} \boldsymbol{F}(\boldsymbol{u})^{-\mathrm{T}} dS_0 = p\,\boldsymbol{F}_{\mathbf{0}} - p\,\boldsymbol{K}_{\boldsymbol{p}} \cdot \boldsymbol{u}$$
$$\boldsymbol{F}(\boldsymbol{u}) = \mathbf{1} + \nabla \boldsymbol{u}$$

by using the notations:

- $\underline{\mathbf{n}}_0{}^T = (n_1, n_2, n_3)$, normal on the surface external of the element in the configuration of reference;
- $\underline{\tilde{\mathbf{b}}}_i$, vector of size 3, derived from the functions of form to the 3 nodes of the face of the element charged in pressure;
- S_0 surface of the face charged in pressure. For the element `SHB6`, this surface S_0 is worth $\dfrac{1}{2}$.

The preceding formulation leads to a not-symmetrical matrix. It is known that one can nevertheless use a symmetrical formulation if the external forces due to the pressure derive from a potential. It is the case if the compressive forces do not work on the border of the modelled field. It is thus considered that the symmetrical part of the matrix is enough. The symmetrized matrix takes the following shape:

$$\boldsymbol{K}_p = S_0 \begin{bmatrix} 0 & \tilde{\boldsymbol{b}}_2^T n_1 - \tilde{\boldsymbol{b}}_1^T n_2 & \tilde{\boldsymbol{b}}_3^T n_1 - \tilde{\boldsymbol{b}}_1^T n_3 \\ 0 & \tilde{\boldsymbol{b}}_2^T n_1 - \tilde{\boldsymbol{b}}_1^T n_2 & \tilde{\boldsymbol{b}}_3^T n_1 - \tilde{\boldsymbol{b}}_1^T n_3 \\ 0 & \tilde{\boldsymbol{b}}_2^T n_1 - \tilde{\boldsymbol{b}}_1^T n_2 & \tilde{\boldsymbol{b}}_3^T n_1 - \tilde{\boldsymbol{b}}_1^T n_3 \\ \tilde{\boldsymbol{b}}_1^T n_2 - \tilde{\boldsymbol{b}}_2^T n_1 & 0 & \tilde{\boldsymbol{b}}_3^T n_2 - \tilde{\boldsymbol{b}}_2^T n_3 \\ \tilde{\boldsymbol{b}}_1^T n_2 - \tilde{\boldsymbol{b}}_2^T n_1 & 0 & \tilde{\boldsymbol{b}}_3^T n_2 - \tilde{\boldsymbol{b}}_2^T n_3 \\ \tilde{\boldsymbol{b}}_1^T n_2 - \tilde{\boldsymbol{b}}_2^T n_1 & 0 & \tilde{\boldsymbol{b}}_3^T n_2 - \tilde{\boldsymbol{b}}_2^T n_3 \\ \tilde{\boldsymbol{b}}_1^T n_3 - \tilde{\boldsymbol{b}}_3^T n_1 & \tilde{\boldsymbol{b}}_2^T n_3 - \tilde{\boldsymbol{b}}_3^T n_2 & 0 \\ \tilde{\boldsymbol{b}}_1^T n_3 - \tilde{\boldsymbol{b}}_3^T n_1 & \tilde{\boldsymbol{b}}_2^T n_3 - \tilde{\boldsymbol{b}}_3^T n_2 & 0 \\ \tilde{\boldsymbol{b}}_1^T n_3 - \tilde{\boldsymbol{b}}_3^T n_1 & \tilde{\boldsymbol{b}}_2^T n_3 - \tilde{\boldsymbol{b}}_3^T n_2 & 0 \end{bmatrix}$$

It is a matrix (9, 9), that it is necessary to multiply by displacements of the 3 nodes of the face to which one applies a pressure.

3 Elements SHB15 and SHB20

In this paragraph, one presents modelings of the finite elements quadratic voluminal hulls `SHB15` and `SHB20`.

The element `SHB15` is a purely three-dimensional prism with fifteen nodes with three degrees of freedom in displacement to each node, and it also has a called privileged direction "thickness" which is normal with the average plan of the prism. Reduced digital integration is used (3 points of Gauss in the plan). Integration through the thickness is based on 5 points of Gauss.

The element `SHB20` is a purely three-dimensional hexahedron with twenty nodes with three degrees of freedom in displacement to each node, and it has also a called privileged direction "thickness" which is normal with the average plan of the hexahedron. Reduced digital integration is used (4 points of Gauss in the plan). Integration through the thickness is based on 5 points of Gauss.

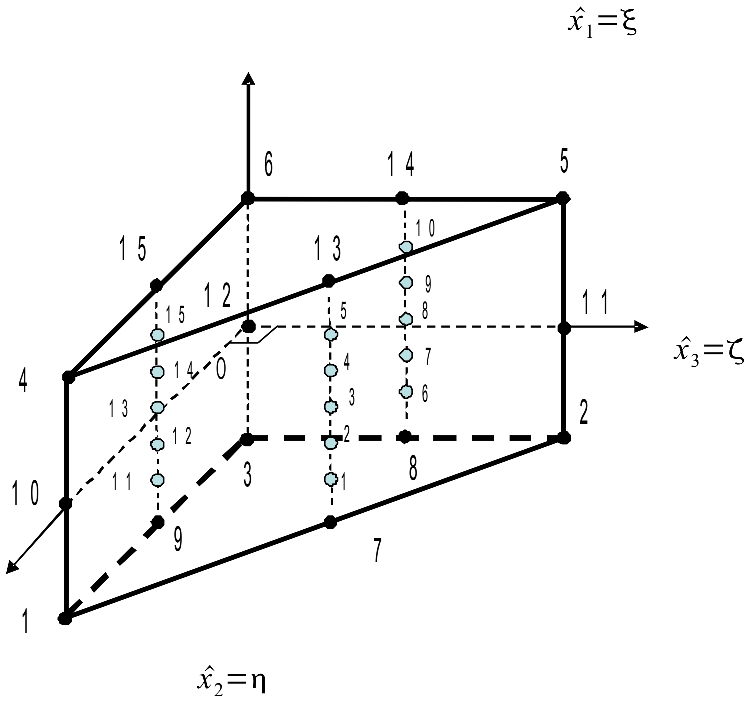
Contrary to the linear elements these finite elements have neither stabilization nor projection.

3.1 Kinematics and interpolation of elements SHB15 and SHB20

3.1.1 Element SHB15

The element `SHB15` is formulated in the local axes of the average plan. F igure 3.1.1-a represent the geometry of an element of reference `SHB15` and its points of integration.

The reference mark of the local coordinates of the element of reference is defined by: ξ, η, ζ ou $\hat{x}_{1,} \hat{x}_{2,} \hat{x}_3$



F igure 3.1.1-a . Geometry of the element of reference SHB15 and its points of integration

Coordonnées des noeuds:

$1(-1,1,0)$; $2(-1,0,1)$; $3(-1,0,0)$; $4(1,1,0)$; $5(1,0,1)$; $6(1,0,0)$;

$7\left(-1,\frac{1}{2},\frac{1}{2}\right)$; $8\left(-1,0,\frac{1}{2}\right)$; $9\left(-1,\frac{1}{2},0\right)$;

$10(0,1,0)$; $11(0,0,1)$; $12(0,0,0)$; $13\left(1,\frac{1}{2},\frac{1}{2}\right)$; $14\left(1,0,\frac{1}{2}\right)$; $15\left(1,\frac{1}{2},0\right)$.

Element SHB15 is an isoparametric quadratic element. Space coordinates $\mathbf{x}_i$ are connected to the nodal coordinates $\mathbf{x}_{iI}$ by means of the functions of form $\mathbf{N}_I$ by the formulas:

$$\boldsymbol{x}_i=\boldsymbol{x}_{iI}\boldsymbol{N}_I(\hat{x}_1,\hat{x}_2,\hat{x}_3)=\sum_{i=1}^{15}\boldsymbol{x}_{iI}\boldsymbol{N}_I(\hat{x}_1,\hat{x}_2,\hat{x}_3)$$

The same functions of form are used to define the field of displacement of the element $\mathbf{u}_i$ in terms of nodal displacements $\boldsymbol{U}_{iI}$:

:

$$\boldsymbol{u}_i=\boldsymbol{u}_{iI}\boldsymbol{N}_I(\hat{x}_1,\hat{x}_2,\hat{x}_3)=\sum_{i=1}^{15}\boldsymbol{U}_{iI}\boldsymbol{N}_I(\hat{x}_1,\hat{x}_2,\hat{x}_3) \quad \textbf{(6)}$$

3.1.2 Element SHB20

Element SHB20 is formulated in the local axes of the average plan. Figure 3.1.2-a represent the geometry of an element of reference SHB20 and its points of integration.
The reference mark of the local coordinates of the element of reference is defined by: ξ,η,ζ ou $\hat{x}_1,\hat{x}_2,\hat{x}_3$

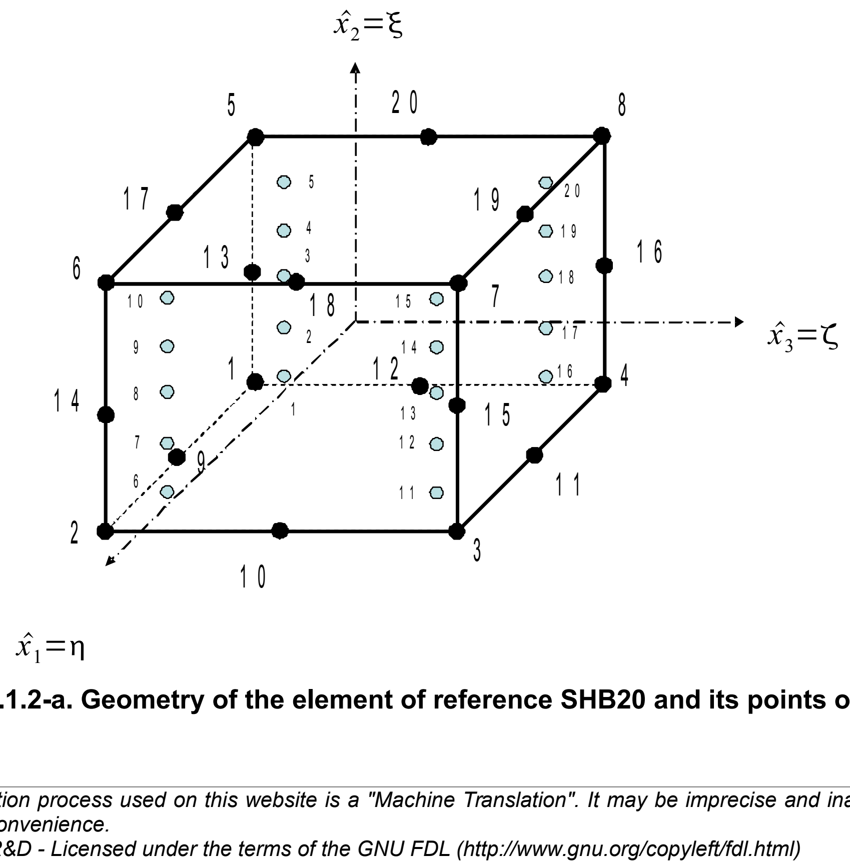


Figure 3.1.2-a. Geometry of the element of reference SHB20 and its points of integration

Coordonnées des noeuds:

$1(-1,-1,-1)$ $2(1,-1,-1)$ $3(1,1,-1)$ $4(-1,1,-1)$
$5(-1,-1,1)$ $6(1,-1,1)$ $7(1,1,1)$ $8(-1,1,1)$
$9(0,-1,-1)$ $10(1,0,-1)$ $11(0,1,-1)$ $12(-1,0,-1)$
$13(-1,-1,0)$ $14(1,-1,0)$ $15(1,1,0)$ $16(-1,1,0)$
$17(0,-1,1)$ $18(1,0,1)$ $19(0,1,1)$ $20(-1,0,1)$

Element SHB20 is also an isoparametric quadratic element. Space coordinates $\mathbf{x}_i$ are connected to the nodal coordinates $\mathbf{x}_{iI}$ by means of the functions of form $\mathbf{N}_I$ by the formulas:

$$\boldsymbol{x}_i=\boldsymbol{x}_{iI}\,\boldsymbol{N}_I(\hat{x}_{1,}\hat{x}_{2,}\hat{x}_3)=\sum_{i=1}^{20}\boldsymbol{x}_{iI}\,\boldsymbol{N}_I(\hat{x}_{1,}\hat{x}_{2,}\hat{x}_3)$$

The same functions of form are used to define the field of displacement of the element $\mathbf{u}_i$ in terms of nodal displacements $\boldsymbol{U}_{iI}$:

$$\boldsymbol{u}_i=\boldsymbol{u}_{iI}\,\boldsymbol{N}_I(\hat{x}_{1,}\hat{x}_{2,}\hat{x}_3)=\sum_{i=1}^{20}\boldsymbol{U}_{iI}\,\boldsymbol{N}_I(\hat{x}_{1,}\hat{x}_{2,}\hat{x}_3) \quad (6)$$

3.2 Operator discretized gradient

3.2.1 Element SHB15

The interpolation of the field of displacement of the element (**6**) will allow us to define the rate of deformation and to write the relations connecting the deformations to nodal displacements. One starts initially by writing the gradient $\mathbf{u}_{i,j}$ field of displacement:

$$\boldsymbol{u}_{i,j}=\boldsymbol{U}_{iI}\,\boldsymbol{N}_{I,j} \quad (7)$$

The tensor of deformation ε_{ij} is given then by the symmetrical part of the gradient of displacement:

$$\varepsilon_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i}) \quad (8)$$

To continue calculations, quadratic isoparametric functions of form are given $N_i(\hat{x}_{1,}\hat{x}_{2,}\hat{x}_3)$, associated with the prismatic element with fifteen nodes:

$$\begin{aligned}
N_1&=\frac{1}{2}\hat{x}_2\left(1-\hat{x}_1\right)(2\hat{x}_2-2-\hat{x}_1) & N_4&=\frac{1}{2}\hat{x}_2\left(1+\hat{x}_1\right)(2\hat{x}_2-2+\hat{x}_1)\\
N_2&=\frac{1}{2}\hat{x}_3\left(1-\hat{x}_1\right)(2\hat{x}_3-2-\hat{x}_1) & N_5&=\frac{1}{2}\hat{x}_3\left(1+\hat{x}_1\right)(2\hat{x}_3-2+\hat{x}_1) \quad (9)\\
N_3&=-\frac{1}{2}\left(1-\hat{x}_2-\hat{x}_3\right)\left(1-\hat{x}_1\right)\hat{x}_1 & N_6&=\frac{1}{2}\left(1-\hat{x}_2-\hat{x}_3\right)\left(1+\hat{x}_1\right)\hat{x}_1
\end{aligned}$$

$$N_7 = 2(1-\hat{x}_1)\hat{x}_2\hat{x}_3 \qquad N_8 = 2(1-\hat{x}_1)\left(1-\hat{x}_2-\hat{x}_3\right)\hat{x}_3 \qquad N_9 = 2(1-\hat{x}_1)\hat{x}_2\left(1-\hat{x}_2-\hat{x}_3\right)$$

$$N_{10} = (1-\hat{x}_1)(1+\hat{x}_1)\hat{x}_2 \qquad N_{11} = (1-\hat{x}_1)(1+\hat{x}_1)\hat{x}_3 \qquad N_{12} = (1-\hat{x}_1)(1+\hat{x}_1)\left(1-\hat{x}_2-\hat{x}_3\right)$$

$$N_{13} = 2(1+\hat{x}_1)\hat{x}_2\hat{x}_3 \qquad N_{14} = 2(1+\hat{x}_1)\left(1-\hat{x}_2-\hat{x}_3\right)\hat{x}_3 \qquad N_{15} = 2(1+\hat{x}_1)\hat{x}_2\left(1-\hat{x}_2-\hat{x}_3\right)$$

$$\hat{x}_1 = [-1,1]; \qquad \hat{x}_2 = [0,1]; \qquad \hat{x}_3 = [0,1-\hat{x}_2]$$

While combining the preceding equations one manages to develop the field of displacement as being the sum of a constant term, linear terms in x_i , and of terms utilizing functions h_α
To simplify the writings, one will note $\xi=\hat{x}_1, \quad \eta=\hat{x}_2, \quad \zeta=\hat{x}_3$

$$\begin{cases} u_i = a_{0i} + a_{1i}x_1 + a_{2i}x_2 + a_{3i}x_3 + c_{1i}\,h_1 + c_{2i}\,h_2 + c_{3i}\,h_3 + \\ \qquad c_{4i}\,h_4 + c_{5i}\,h_5 + c_{6i}\,h_6 + c_{7i}\,h_7 + c_{8i}\,h_8 + c_{9i}\,h_9 + c_{10i}\,h_{10} + c_{11i}\,h_{11} \\ i=1,2,3 \\ h_1 = \xi\zeta, h_2 = \eta\zeta, h_3 = \xi\eta, h_4 = \xi\eta\zeta, h_5 = \xi^2, h_6 = \eta^2, \\ h_7 = \zeta^2, h_8 = \xi^2\zeta, h_9 = \eta^2\zeta, h_{10} = \xi\zeta^2, h_{11} = \eta\zeta^2 \end{cases} \quad (10)$$

By evaluating the equation **(6)** at the nodes of the element, one arrives at the three systems of fifteen equations following:

$$\begin{cases} \underline{\mathbf{d}}_i = a_{0i}\underline{\mathbf{S}} + a_{1i}\underline{\mathbf{x}}_1 + a_{2i}\underline{\mathbf{x}}_2 + a_{3i}\underline{\mathbf{x}}_3 + c_{1i}\underline{\mathbf{h}}_1 + c_{2i}\underline{\mathbf{h}}_2 + c_{3i}\underline{\mathbf{h}}_3 + \\ \qquad c_{4i}\underline{\mathbf{h}}_4 + c_{5i}\underline{\mathbf{h}}_5 + c_{6i}\underline{\mathbf{h}}_6 + c_{7i}\underline{\mathbf{h}}_7 + c_{8i}\underline{\mathbf{h}}_8 + c_{9i}\underline{\mathbf{h}}_9 + c_{10i}\underline{\mathbf{h}}_{10} + c_{11i}\underline{\mathbf{h}}_{11} \\ i=1,2,3 \end{cases} \quad (11)$$

Thus vectors $\underline{\mathbf{d}}_i$ and $\underline{\mathbf{x}}_i$ represent, respectively, displacements and the nodal coordinates and are given by:

$$\begin{cases} \underline{\mathbf{d}}_i^T = \left(u_{i1}, u_{i2}, u_{i3}, u_{i4}, u_{i5}, u_{i6}, u_{i>7}, u_{i8}, u_{i9}, u_{i10}, u_{i11}, u_{i12}, u_{i13}, u_{i14}, u_{i15}\right) \\ \underline{\mathbf{x}}_i^T = \left(x_{i1}, x_{i2}, x_{i3}, x_{i4}, x_{i5}, x_{i6}, x_{i7}, x_{i8}, x_{i9}, x_{i10}, x_{i11}, x_{i12}, x_{i13}, x_{i14}, x_{i15}\right) \end{cases} \quad (12)$$

Vectors **S** and $\mathbf{h}_\alpha$ $\left(\alpha = 1,2,3,\ldots,11\right)$ are given as for them by:

$$
\begin{cases}
\underline{\mathbf{S}}^T = (1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1) \\
\underline{\mathbf{h}}_1^T = \left(0\ -\frac{1}{2}\ -1\ -\frac{1}{2}\ 0\ 0\ 0\ 0\ 0\ 0\ \frac{1}{2}\ 1\ \frac{1}{2}\ 0\ 0\right) \\
\underline{\mathbf{h}}_2^T = \left(0\ 0\ 0\ -\frac{1}{2}\ -1\ -\frac{1}{2}\ 0\ 0\ 0\ 0\ 0\ 0\ \frac{1}{2}\ 1\ \frac{1}{2}\right) \\
\underline{\mathbf{h}}_3^T = \left(0\ 0\ 0\ \frac{1}{4}\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ \frac{1}{4}\ 0\ 0\right) \\
\underline{\mathbf{h}}_4^T = \left(0\ 0\ 0\ -\frac{1}{4}\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ \frac{1}{4}\ 0\ 0\right) \\
\underline{\mathbf{h}}_5^T = \left(0\ \frac{1}{4}\ 1\ \frac{1}{4}\ 0\ 0\ 0\ 1\ 0\ 0\ \frac{1}{4}\ 1\ \frac{1}{4}\ 0\ 0\right) \\
\underline{\mathbf{h}}_6^T = \left(0\ 0\ 0\ \frac{1}{4}\ 1\ \frac{1}{4}\ 0\ 0\ 1\ 0\ 0\ 0\ \frac{1}{4}\ 1\ \frac{1}{4}\right) \\
\underline{\mathbf{h}}_7^T = (1\ 1\ 1\ 1\ 1\ 1\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 1\ 1) \\
\underline{\mathbf{h}}_8^T = \left(0\ -\frac{1}{4}\ -1\ -\frac{1}{4}\ 0\ 0\ 0\ 0\ 0\ 0\ \frac{1}{4}\ 1\ \frac{1}{4}\ 0\ 0\right) \\
\underline{\mathbf{h}}_9^T = \left(0\ 0\ 0\ -\frac{1}{4}\ -1\ -\frac{1}{4}\ 0\ 0\ 0\ 0\ 0\ 0\ \frac{1}{4}\ 1\ \frac{1}{4}\right) \\
\underline{\mathbf{h}}_{10}^T = \left(0\ \frac{1}{2}\ 1\ \frac{1}{2}\ 0\ 0\ 0\ 0\ 0\ 0\ \frac{1}{2}\ 1\ \frac{1}{2}\ 0\ 0\right) \\
\underline{\mathbf{h}}_{11}^T = \left(0\ 0\ 0\ \frac{1}{2}\ 1\ \frac{1}{2}\ 0\ 0\ 0\ 0\ 0\ 0\ \frac{1}{2}\ 1\ \frac{1}{2}\right)
\end{cases}
\qquad (13)
$$

To arrive at an advantageous writing of the operator discretized gradient $\mathbf{B}$, one will introduce the three vectors $\mathbf{b_i}$ defined by:

$$\mathbf{b_i^T} = \mathbf{N^T}_{,\mathbf{i}}(0) = {}^{\mathbf{T}}\frac{\partial \mathbf{N}}{\partial \mathbf{x_i}}(0) \quad \text{I} = 1,2,3 \qquad (14)$$

If we place ourselves in $(\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2, \hat{\mathbf{x}}_3) = (0,0,0)$ then we obtain:

$$\mathbf{b_i^T} = \mathbf{N^T}_{,\mathbf{i}}(0) = \mathbf{cste}$$

where $\mathbf{N^T}$ represent: $(N_1 \quad N_2 \quad N_3 \quad . \quad . \quad N_{15})$.

$$\mathbf{b_i^T} = \mathbf{N^T}_{,\mathbf{i}}(0) = \left(\frac{\partial N_1}{\partial x_i}(0), \ldots\ldots, \frac{\partial N_{15}}{\partial x_i}(0)\right)$$

$$\frac{\partial \mathrm{N}_I}{\partial x_j} = \left(\frac{\partial \mathrm{N}_I}{\partial \xi} . \frac{\partial \xi}{\partial x_j} + \frac{\partial \mathrm{N}_I}{\partial \eta} . \frac{\partial \eta}{\partial x_j} + \frac{\partial \mathrm{N}_I}{\partial \zeta} . \frac{\partial \zeta}{\partial x_j} \right)_{|\xi=\eta=\zeta=0} = \left(\frac{\partial \mathrm{N}_I}{\partial \xi} . j_{1j} + \frac{\partial \mathrm{N}_I}{\partial \eta} . j_{2j} + \frac{\partial \mathrm{N}_I}{\partial \zeta} . j_{3j} \right)_{\xi=\eta=\zeta=0}$$

$$avec \quad I = 1,2,...,15 \quad et \quad j = 1,2,3$$

$$\mathbf{F}^{-1}_{|\xi=\eta=\zeta=0} = \begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} & \frac{\partial \xi}{\partial z} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} & \frac{\partial \eta}{\partial z} \\ \frac{\partial \zeta}{\partial x} & \frac{\partial \zeta}{\partial y} & \frac{\partial \zeta}{\partial z} \end{pmatrix}_{|\xi=\eta=\zeta=0} = \begin{pmatrix} j_{11} & j_{12} & j_{13} \\ j_{21} & j_{22} & j_{23} \\ j_{31} & j_{32} & j_{33} \end{pmatrix}$$

We have:

$$\mathrm{N}_{1,\xi} = \frac{(1-\zeta)}{2}(4\xi + 4\eta + \zeta\text{-}2) \qquad \mathrm{N}_{1,\eta} = \frac{(1-\zeta)}{2}(4\xi + 4\eta + \zeta\text{-}2) \qquad \mathrm{N}_{1,\zeta} = \frac{(1-\xi-\eta)}{2}(2\xi + 2\eta + 2\zeta\text{-}1)$$

$$\mathrm{N}_{2,\xi} = 2(1-\zeta)(1-2\xi-\eta) \qquad \mathrm{N}_{2,\eta} = -2(1-\zeta)\xi \qquad \mathrm{N}_{2,\zeta} = -2\xi(1-\xi-\eta)$$

$$\mathrm{N}_{3,\xi} = \frac{(1-\zeta)}{2}(-2+4\xi-\zeta) \qquad \mathrm{N}_{3,\eta} = 0 \qquad \mathrm{N}_{3,\zeta} = \frac{\xi}{2}(1-2\xi+2\zeta)$$

$$\mathrm{N}_{4,\xi} = 2(1-\zeta)\eta \qquad \mathrm{N}_{4,\eta} = 2(1-\zeta)\xi \qquad \mathrm{N}_{4,\zeta} = -2\xi\eta$$

$$\mathrm{N}_{5,\xi} = 0 \qquad \mathrm{N}_{5,\eta} = \frac{(1-\zeta)}{2}(-2+4\eta-\zeta) \qquad \mathrm{N}_{5,\zeta} = \frac{\eta}{2}(1-2\eta+2\zeta)$$

$$\mathrm{N}_{6,\xi} = -2(1-\zeta)\eta \qquad \mathrm{N}_{6,\eta} = 2(1-\zeta)(1-\xi-2\eta) \qquad \mathrm{N}_{6,\zeta} = -2\eta(1-\xi-\eta)$$

$$\mathrm{N}_{7,\xi} = -1+\zeta^2 \qquad \mathrm{N}_{7,\eta} = -1+\zeta^2 \qquad \mathrm{N}_{7,\zeta} = -2\zeta(1-\xi-\eta)$$

$$\mathrm{N}_{8,\xi} = 1-\zeta^2 \qquad \mathrm{N}_{8,\eta} = 0 \qquad \mathrm{N}_{8,\zeta} = -2\zeta\xi$$

$$\mathrm{N}_{9,\xi} = 0 \qquad \mathrm{N}_{9,\eta} = 1-\zeta^2 \qquad \mathrm{N}_{9,\zeta} = -2\zeta\eta$$

$$\mathrm{N}_{10,\xi} = \frac{(1+\zeta)}{2}(4\xi + 4\eta - \zeta\text{-}2) \qquad \mathrm{N}_{10,\eta} = \frac{(1+\zeta)}{2}(4\xi + 4\eta - \zeta\text{-}2) \qquad \mathrm{N}_{10,\zeta} = \frac{(1-\xi-\eta)}{2}(-2\xi - 2\eta + 2\zeta + 1)$$

$$\mathrm{N}_{11,\xi} = 2(1+\zeta)(1-2\xi-\eta) \qquad \mathrm{N}_{11,\eta} = -2(1+\zeta)\xi \qquad \mathrm{N}_{11,\zeta} = 2\xi(1-\xi-\eta)$$

$$\mathrm{N}_{12,\xi} = \frac{(1+\zeta)}{2}(-2+4\xi+\zeta) \qquad \mathrm{N}_{12,\eta} = 0 \qquad \mathrm{N}_{12,\zeta} = \frac{\xi}{2}(-1+2\xi+2\zeta)$$

$$\mathrm{N}_{13,\xi} = 2(1+\zeta)\eta \qquad \mathrm{N}_{13,\eta} = 2(1+\zeta)\xi \qquad \mathrm{N}_{13,\zeta} = 2\xi\eta$$

$$\mathrm{N}_{14,\xi} = 0 \qquad \mathrm{N}_{14,\eta} = \frac{(1+\zeta)}{2}(-2+4\eta+\zeta) \qquad \mathrm{N}_{14,\zeta} = \frac{\eta}{2}(-1+2\eta+2\zeta)$$

$$\mathrm{N}_{15,\xi} = -2(1+\zeta)\eta \qquad \mathrm{N}_{15,\eta} = 2(1+\zeta)(1-2\eta-\xi) \qquad \mathrm{N}_{15,\zeta} = 2\eta(1-\xi-\eta)$$

Therefore, in $(\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2, \hat{\mathbf{x}}_3) = (0,0,0)$, we have:

$$\underline{\mathbf{b}}^T{}_i = \begin{pmatrix} j_{i1} & j_{i2} & j_{i3} \end{pmatrix} . \begin{bmatrix} -1 & 2 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 2 \\ -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Moreover, one can check by algebraic considerations that the following conditions of orthogonality are satisfied:

$$\begin{cases} \underline{\mathbf{b}}_i^T . \underline{\mathbf{h}}_\alpha = 0; \quad \underline{\mathbf{b}}_i^T . \underline{\mathbf{S}} = 0; \quad \underline{\mathbf{b}}_i^T . \underline{\mathbf{x}}_j = \delta_{ij} \\ \underline{\mathbf{h}}_1^T . \underline{\mathbf{S}} = 0; \quad \underline{\mathbf{h}}_2^T . \underline{\mathbf{S}} = 0; \quad \underline{\mathbf{h}}_3^T . \underline{\mathbf{S}} = \frac{1}{2}; \quad \underline{\mathbf{h}}_4^T . \underline{\mathbf{S}} = 0; \quad \underline{\mathbf{h}}_5^T . \underline{\mathbf{S}} = 4; \quad \underline{\mathbf{h}}_6^T . \underline{\mathbf{S}} = 4; \\ \underline{\mathbf{h}}_7^T . \underline{\mathbf{S}} = 12; \quad \underline{\mathbf{h}}_8^T . \underline{\mathbf{S}} = 0 \quad \underline{\mathbf{h}}_9^T . \underline{\mathbf{S}} = 0 \quad \underline{\mathbf{h}}_{10}^T . \underline{\mathbf{S}} = 4 \quad \underline{\mathbf{h}}_{11}^T . \underline{\mathbf{S}} = 4 \\ \underline{\mathbf{h}}_m^T . \underline{\mathbf{h}}_n = \begin{bmatrix} 3 & -\frac{1}{2} & 0 & \frac{1}{4} & 0 & 0 & 0 & \frac{5}{2} & \frac{1}{4} & 0 & 0 \\ -\frac{1}{2} & 3 & 0 & \frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} & \frac{5}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{8} & 0 & \frac{1}{8} & \frac{1}{8} & \frac{1}{2} & 0 & 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{8} & 0 & 0 & 0 & \frac{1}{8} & \frac{1}{8} & 0 & 0 \\ 0 & 0 & \frac{1}{8} & 0 & \frac{13}{4} & \frac{1}{8} & 3 & 0 & 0 & \frac{5}{2} & \frac{1}{4} \\ 0 & 0 & \frac{1}{8} & 0 & \frac{1}{8} & \frac{13}{4} & 3 & 0 & 0 & \frac{1}{4} & \frac{5}{2} \\ 0 & 0 & \frac{1}{2} & 0 & 3 & 3 & 12 & 0 & 0 & 4 & 4 \\ \frac{5}{2} & \frac{1}{4} & 0 & \frac{1}{8} & 0 & 0 & 0 & \frac{9}{4} & \frac{1}{8} & 0 & 0 \\ \frac{1}{4} & \frac{5}{2} & 0 & \frac{1}{8} & 0 & 0 & 0 & \frac{1}{8} & \frac{9}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 & \frac{5}{2} & \frac{1}{4} & 4 & 0 & 0 & 3 & \frac{1}{2} \\ 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & \frac{5}{2} & 4 & 0 & 0 & \frac{1}{2} & 3 \end{bmatrix} \\ \qquad \alpha = 1,2,...,11 \qquad\qquad i,j = 1,2,3 \\ \delta \quad \textit{désigne le symbol de Kronecker}; m,n = 1,2,...,11 \end{cases} \tag{15}$$

This stage, one can determine the constant unknown factors which intervene in the writing (10) of the field of displacement by multiplying scalairement the equation (11) by $\underline{\mathbf{b}}_j^T$, $\underline{\mathbf{S}}^T$ and $\underline{\mathbf{h}}_\alpha^T$ respectively, and by using the relations of orthogonality (15).

One obtains after calculations: $a_{ji} = \underline{\mathbf{b}}_j^T . \underline{\mathbf{d}}_i \qquad c_{\alpha i} = \underline{\boldsymbol{\gamma}}_\alpha^T . \underline{\mathbf{d}}_i$

with

$$
\begin{aligned}
\underline{\boldsymbol{\gamma}}_\alpha^T = {} & n_{\alpha 1}\left(\underline{\mathbf{h}}_1^T - \left(\underline{\mathbf{h}}_1^T . \underline{\mathbf{x}}_j\right)\underline{\mathbf{b}}_j^T\right) + n_{\alpha 2}\left(\underline{\mathbf{h}}_2^T - \left(\underline{\mathbf{h}}_2^T . \underline{\mathbf{x}}_j\right)\underline{\mathbf{b}}_j^T\right) + \\
& + n_{\alpha 3}\left[\left(\underline{\mathbf{h}}_3^T - \frac{1}{30}\underline{\mathbf{S}}^T\right) - \left(\left(\underline{\mathbf{h}}_3^T - \frac{1}{30}\underline{\mathbf{S}}^T\right) . \underline{\mathbf{x}}_j\right)\underline{\mathbf{b}}_j^T\right] + n_{\alpha 4}\left(\underline{\mathbf{h}}_4^T - \left(\underline{\mathbf{h}}_4^T . \underline{\mathbf{x}}_j\right)\underline{\mathbf{b}}_j^T\right) \\
& + n_{\alpha 5} \cdot \left[\left(\underline{\mathbf{h}}_5^T - \frac{4}{15}\underline{\mathbf{S}}^T\right) - \left(\left(\underline{\mathbf{h}}_5^T - \frac{4}{15}\underline{\mathbf{S}}^T\right) . \underline{\mathbf{x}}_j\right)\underline{\mathbf{b}}_j^T\right] + n_{\alpha 6}\left(\left(\underline{\mathbf{h}}_6^T - \frac{4}{15}\underline{\mathbf{S}}^T\right) - \left(\left(\underline{\mathbf{h}}_6^T - \frac{4}{15}\underline{\mathbf{S}}^T\right) . \underline{\mathbf{x}}_j\right)\underline{\mathbf{b}}_j^T\right) \\
& + n_{\alpha 7}\left(\left(\underline{\mathbf{h}}_7^T - \frac{4}{5}\underline{\mathbf{S}}^T\right) - \left(\left(\underline{\mathbf{h}}_7^T - \frac{4}{5}\underline{\mathbf{S}}^T\right) . \underline{\mathbf{x}}_j\right)\underline{\mathbf{b}}_j^T\right) + n_{\alpha 8}\left(\underline{\mathbf{h}}_8^T - \left(\underline{\mathbf{h}}_8^T . \underline{\mathbf{x}}_j\right)\underline{\mathbf{b}}_j^T\right) + n_{\alpha 9}\left(\underline{\mathbf{h}}_9^T - \left(\underline{\mathbf{h}}_9^T . \underline{\mathbf{x}}_j\right)\underline{\mathbf{b}}_j^T\right) \\
& + n_{\alpha 10}\left(\left(\underline{\mathbf{h}}_{10}^T - \frac{4}{15}\underline{\mathbf{S}}^T\right) - \left(\left(\underline{\mathbf{h}}_{10}^T - \frac{4}{15}\underline{\mathbf{S}}^T\right) . \underline{\mathbf{x}}_j\right)\underline{\mathbf{b}}_j^T\right) + n_{\alpha 11}\left(\left(\underline{\mathbf{h}}_{11}^T - \frac{4}{15}\underline{\mathbf{S}}^T\right) - \left(\left(\underline{\mathbf{h}}_{11}^T - \frac{4}{15}\underline{\mathbf{S}}^T\right) . \underline{\mathbf{x}}_j\right)\underline{\mathbf{b}}_j^T\right)
\end{aligned}
$$

$$
\left[\mathbf{n}_{\alpha\beta}\right] = \begin{bmatrix}
\frac{17}{2} & 0 & 0 & -8 & 0 & 0 & 0 & -9 & 0 & 0 & 0 \\
0 & \frac{17}{2} & 0 & -8 & 0 & 0 & 0 & 0 & -9 & 0 & 0 \\
0 & 0 & \frac{256}{17} & 0 & \frac{36}{17} & \frac{36}{17} & 2 & 0 & 0 & -\frac{58}{17} & -\frac{58}{17} \\
-8 & -8 & 0 & 24 & 0 & 0 & 0 & 8 & 8 & 0 & 0 \\
0 & 0 & \frac{36}{17} & 0 & \frac{316}{187} & \frac{146}{187} & 1 & 0 & 0 & -\frac{324}{187} & -\frac{171}{187} \\
0 & 0 & \frac{36}{17} & 0 & \frac{146}{187} & \frac{316}{187} & 1 & 0 & 0 & -\frac{171}{187} & -\frac{324}{187} \\
0 & 0 & 2 & 0 & 1 & 1 & \frac{3}{2} & 0 & 0 & -\frac{3}{2} & -\frac{3}{2} \\
-9 & 0 & 0 & 8 & 0 & 0 & 0 & 10 & 0 & 0 & 0 \\
0 & -9 & 0 & 8 & 0 & 0 & 0 & 0 & 10 & 0 & 0 \\
0 & 0 & -\frac{58}{17} & 0 & -\frac{324}{187} & -\frac{171}{187} & -\frac{3}{2} & 0 & 0 & \frac{505}{187} & \frac{585}{374} \\
0 & 0 & -\frac{58}{17} & 0 & -\frac{171}{187} & -\frac{324}{187} & -\frac{3}{2} & 0 & 0 & \frac{585}{374} & \frac{505}{187}
\end{bmatrix} \qquad \alpha, \beta = 1, 2, \ldots, 11
$$

The field of displacement is put finally in the following form:

$$
\begin{aligned}
u_i &= a_{0i} + a_{1i}x_1 + a_{2i}x_2 + a_{3i}x_3 + c_{1i}\,h_1 + c_{2i}\,h_2 + c_{3i}\,h_3 + c_{4i}\,h_4 + c_{5i}\,h_5 + c_{6i}\,h_6 \\
&\quad + c_{7i}\,h_7 + c_{8i}\,h_8 + c_{9i}\,h_9 + c_{10i}\,h_{10} + c_{11i}\,h_{11} \\
u_i &= a_{0i} + \underline{\mathbf{b}}_1^T.\underline{\mathbf{d}}_i x_1 + \underline{\mathbf{b}}_2^T.\underline{\mathbf{d}}_i x_2 + \underline{\mathbf{b}}_3^T.\underline{\mathbf{d}}_i x_3 + \underline{\gamma}_1^T.\underline{\mathbf{d}}_i\,h_1 + \underline{\gamma}_2^T.\underline{\mathbf{d}}_i\,h_2 + \\
&\quad + \underline{\gamma}_3^T.\underline{\mathbf{d}}_i\,h_3 + \underline{\gamma}_4^T.\underline{\mathbf{d}}_i\,h_4 + \underline{\gamma}_5^T.\underline{\mathbf{d}}_i\,h_5 + \underline{\gamma}_6^T.\underline{\mathbf{d}}_i\,h_6 + \underline{\gamma}_7^T.\underline{\mathbf{d}}_i\,h_7 \\
&\quad + \underline{\gamma}_8^T.\underline{\mathbf{d}}_i\,h_8 + \underline{\gamma}_9^T.\underline{\mathbf{d}}_i\,h_9 + \underline{\gamma}_{10}^T.\underline{\mathbf{d}}_i\,h_{10} + \underline{\gamma}_{11}^T.\underline{\mathbf{d}}_i\,h_{11} \\
u_i &= a_{0i} + (\underline{\mathbf{b}}_1^T x_1 + \underline{\mathbf{b}}_2^T x_2 + \underline{\mathbf{b}}_3^T x_3 + \underline{\gamma}_1^T\,h_1 + \underline{\gamma}_2^T\,h_2 + \underline{\gamma}_3^T\,h_3 + \underline{\gamma}_4^T\,h_4 + \underline{\gamma}_5^T\,h_5 \\
&\quad + \underline{\gamma}_6^T\,h_6 + \underline{\gamma}_7^T\,h_7 + \underline{\gamma}_8^T\,h_8 + \underline{\gamma}_9^T\,h_9 + \underline{\gamma}_{10}^T\,h_{10} + \underline{\gamma}_{11}^T\,h_{11}).\underline{\mathbf{d}}_i
\end{aligned}
$$

(16)

By deriving the formula above compared to x_j, the gradient of displacement is obtained:

$$u_{i,j} = \left(\boldsymbol{b}_j^t + \sum_{\alpha=1}^{11} h_{\alpha,j}\,\gamma_\alpha^T\right)\cdot \boldsymbol{d}_i \qquad (17)$$

3.2.2 Element SHB20

The interpolation of the field of displacement of the element will enable us to define the rate of deformation and to write the relations connecting the deformations to nodal displacements. One starts initially by writing the gradient $\mathbf{u}_{i,j}$ field of displacement:

$$\mathbf{u}_{i,j} = \mathbf{u}_{iI}\mathbf{N}_{I,j} \qquad (18)$$

The tensor of deformation $\boldsymbol{\varepsilon}_{\mathbf{ij}}$ is given then by the symmetrical part of the gradient of displacement:

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \qquad (19)$$

To continue calculations, quadratic isoparametric functions of form are given $\boldsymbol{N_i}(\hat{x}_1, \hat{x}_2, \hat{x}_3)$ associated with the hexahedral element with twenty nodes:

$$\mathbf{N}_1=\frac{1}{8}\left(1-\hat{\mathbf{x}}_1\right)\left(1-\hat{\mathbf{x}}_2\right)\left(1-\hat{\mathbf{x}}_3\right)\left(-2-\hat{\mathbf{x}}_1-\hat{\mathbf{x}}_2-\hat{\mathbf{x}}_3\right)$$
$$\mathbf{N}_2=\frac{1}{8}\left(1+\hat{\mathbf{x}}_1\right)\left(1-\hat{\mathbf{x}}_2\right)\left(1-\hat{\mathbf{x}}_3\right)\left(-2+\hat{\mathbf{x}}_1-\hat{\mathbf{x}}_2-\hat{\mathbf{x}}_3\right)$$
$$\mathbf{N}_3=\frac{1}{8}\left(1+\hat{\mathbf{x}}_1\right)\left(1+\hat{\mathbf{x}}_2\right)\left(1-\hat{\mathbf{x}}_3\right)\left(-2+\hat{\mathbf{x}}_1+\hat{\mathbf{x}}_2-\hat{\mathbf{x}}_3\right)$$
$$\mathbf{N}_4=\frac{1}{8}\left(1-\hat{\mathbf{x}}_1\right)\left(1+\hat{\mathbf{x}}_2\right)\left(1-\hat{\mathbf{x}}_3\right)\left(-2-\hat{\mathbf{x}}_1+\hat{\mathbf{x}}_2-\hat{\mathbf{x}}_3\right)$$
$$\mathbf{N}_5=\frac{1}{8}\left(1-\hat{\mathbf{x}}_1\right)\left(1-\hat{\mathbf{x}}_2\right)\left(1+\hat{\mathbf{x}}_3\right)\left(-2-\hat{\mathbf{x}}_1-\hat{\mathbf{x}}_2+\hat{\mathbf{x}}_3\right)$$
$$\mathbf{N}_6=\frac{1}{8}\left(1+\hat{\mathbf{x}}_1\right)\left(1-\hat{\mathbf{x}}_2\right)\left(1+\hat{\mathbf{x}}_3\right)\left(-2+\hat{\mathbf{x}}_1-\hat{\mathbf{x}}_2+\hat{\mathbf{x}}_3\right) \quad (20)$$
$$\mathbf{N}_7=\frac{1}{8}\left(1+\hat{\mathbf{x}}_1\right)\left(1+\hat{\mathbf{x}}_2\right)\left(1+\hat{\mathbf{x}}_3\right)\left(-2+\hat{\mathbf{x}}_1+\hat{\mathbf{x}}_2+\hat{\mathbf{x}}_3\right)$$
$$\mathbf{N}_8=\frac{1}{8}\left(1-\hat{\mathbf{x}}_1\right)\left(1+\hat{\mathbf{x}}_2\right)\left(1+\hat{\mathbf{x}}_3\right)\left(-2-\hat{\mathbf{x}}_1+\hat{\mathbf{x}}_2+\hat{\mathbf{x}}_3\right)$$
$$\mathbf{N}_9=\frac{1}{4}\left(1-\hat{\mathbf{x}}_1^{\,2}\right)\left(1-\hat{\mathbf{x}}_2\right)\left(1-\hat{\mathbf{x}}_3\right)$$
$$\mathbf{N}_{10}=\frac{1}{4}\left(1-\hat{\mathbf{x}}_2^{\,2}\right)\left(1+\hat{\mathbf{x}}_1\right)\left(1-\hat{\mathbf{x}}_3\right)$$
$$\mathbf{N}_{11}=\frac{1}{4}\left(1-\hat{\mathbf{x}}_1^{\,2}\right)\left(1+\hat{\mathbf{x}}_2\right)\left(1-\hat{\mathbf{x}}_3\right) \qquad \mathbf{N}_{12}=\frac{1}{4}\left(1-\hat{\mathbf{x}}_2^{\,2}\right)\left(1-\hat{\mathbf{x}}_1\right)\left(1-\hat{\mathbf{x}}_3\right)$$
$$\mathbf{N}_{13}=\frac{1}{4}\left(1-\hat{\mathbf{x}}_3^{\,2}\right)\left(1-\hat{\mathbf{x}}_1\right)\left(1-\hat{\mathbf{x}}_2\right) \qquad \mathbf{N}_{14}=\frac{1}{4}\left(1-\hat{\mathbf{x}}_3^{\,2}\right)\left(1+\hat{\mathbf{x}}_1\right)\left(1-\hat{\mathbf{x}}_2\right)$$
$$\mathbf{N}_{15}=\frac{1}{4}\left(1-\hat{\mathbf{x}}_3^{\,2}\right)\left(1+\hat{\mathbf{x}}_1\right)\left(1+\hat{\mathbf{x}}_2\right) \qquad \mathbf{N}_{16}=\frac{1}{4}\left(1-\hat{\mathbf{x}}_3^{\,2}\right)\left(1-\hat{\mathbf{x}}_1\right)\left(1+\hat{\mathbf{x}}_2\right)$$
$$\mathbf{N}_{17}=\frac{1}{4}\left(1-\hat{\mathbf{x}}_1^{\,2}\right)\left(1-\hat{\mathbf{x}}_2\right)\left(1+\hat{\mathbf{x}}_3\right) \qquad \mathbf{N}_{18}=\frac{1}{4}\left(1-\hat{\mathbf{x}}_2^{\,2}\right)\left(1+\hat{\mathbf{x}}_1\right)\left(1+\hat{\mathbf{x}}_3\right)$$
$$\mathbf{N}_{19}=\frac{1}{4}\left(1-\hat{\mathbf{x}}_1^{\,2}\right)\left(1+\hat{\mathbf{x}}_2\right)\left(1+\hat{\mathbf{x}}_3\right) \qquad \mathbf{N}_{20}=\frac{1}{4}\left(1-\hat{\mathbf{x}}_2^{\,2}\right)\left(1-\hat{\mathbf{x}}_1\right)\left(1+\hat{\mathbf{x}}_3\right)$$

$$\hat{\mathbf{x}}_1=[-1,1]; \qquad \hat{\mathbf{x}}_2=[-1,1]; \qquad \hat{\mathbf{x}}_3=[-1,1]$$

While combining the preceding equations one manages to develop the field of displacement as being the sum of a constant term, linear terms in x_i , and of terms utilizing functions h_α □

$$\begin{cases} \mathrm{u}_i = a_{0i} + a_{1i}x_1 + a_{2i}x_2 + a_{3i}x_3 + c_{1i}\,\mathrm{h}_1 + c_{2i}\,\mathrm{h}_2 + c_{3i}\,\mathrm{h}_3 + c_{4i}\,\mathrm{h}_4 + c_{5i}\,\mathrm{h}_5 + c_{6i}\,\mathrm{h}_6 + c_{7i}\,\mathrm{h}_7 \\ \quad + c_{8i}\,\mathrm{h}_8 + c_{9i}\,\mathrm{h}_9 + c_{10i}\,\mathrm{h}_{10} + c_{11i}\,\mathrm{h}_{11} + c_{12i}\,\mathrm{h}_{12} + c_{13i}\,\mathrm{h}_{13} + c_{14i}\,\mathrm{h}_{14} + c_{15i}\,\mathrm{h}_{15} + c_{16i}\,\mathrm{h}_{16} \\ i = 1,2,3 \\ \mathrm{h}_1 = \xi\zeta, \mathrm{h}_2 = \eta\zeta, \mathrm{h}_3 = \xi\eta, \mathrm{h}_4 = \xi^2, \mathrm{h}_5 = \eta^2, \mathrm{h}_6 = \zeta^2, \mathrm{h}_7 = \xi\eta\zeta, \mathrm{h}_8 = \xi^2\eta, \mathrm{h}_9 = \xi^2\zeta, \\ \mathrm{h}_{10} = \eta^2\xi, \mathrm{h}_{11} = \eta^2\zeta, \mathrm{h}_{12} = \zeta^2\xi, \mathrm{h}_{13} = \zeta^2\eta, \mathrm{h}_{14} = \xi^2\eta\zeta, \mathrm{h}_{15} = \xi\eta^2\zeta, \mathrm{h}_{16} = \xi\eta\zeta^2 \end{cases} \quad (21)$$

To simplify the writings, one will note $\xi = \hat{x}_1$, $\eta = \hat{x}_2$, $\zeta = \hat{x}_3$

By evaluating the equation **16** at the nodes of the element, one arrives at the three systems of twenty equations following:

$$\begin{cases} \underline{\mathbf{d}}_i = a_{0i}\underline{\mathbf{S}} + a_{1i}\underline{\mathbf{x}}_1 + a_{2i}\underline{\mathbf{x}}_2 + a_{3i}\underline{\mathbf{x}}_3 + c_{1i}\underline{\mathbf{h}}_1 + c_{2i}\underline{\mathbf{h}}_2 + c_{3i}\underline{\mathbf{h}}_3 + \ldots + c_{16i}\underline{\mathbf{h}}_{16} \\ i = 1,2,3 \end{cases} \quad (22)$$

Thus vectors $\underline{\mathbf{d}}_i$ and $\underline{\mathbf{x}}_i$ represent, respectively, displacements and the nodal coordinates and are given by:

$$\begin{cases} \underline{\mathbf{d}}_i^T = (u_{i1}, u_{i2}, u_{i3}, \ldots, u_{i20}) \\ \underline{\mathbf{x}}_i^T = (x_{i1}, x_{i2}, x_{i3}, \ldots, x_{i20}) \end{cases} \quad (23)$$

Vectors $\underline{\mathbf{S}}$ and $\underline{\mathbf{h}}_\alpha$ $(\alpha = 1,2,3,\ldots,16)$ are given as for them by:

$$\begin{cases} \underline{\mathbf{S}}^T = (1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1) \\ \underline{\mathbf{h}}_1^T = (1\ -1\ -1\ 1\ -1\ 1\ 1\ -1\ 0\ -1\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ -1) \\ \underline{\mathbf{h}}_2^T = (1\ 1\ -1\ -1\ -1\ -1\ 1\ 1\ 1\ 0\ -1\ 0\ 0\ 0\ 0\ 0\ -1\ 0\ 1\ 0) \\ \underline{\mathbf{h}}_3^T = (1\ -1\ 1\ -1\ 1\ -1\ 1\ -1\ 0\ 0\ 0\ 0\ 1\ -1\ 1\ -1\ 0\ 0\ 0\ 0) \\ \underline{\mathbf{h}}_4^T = (1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 0\ 1\ 0\ 1\ 1\ 1\ 1\ 1\ 0\ 1\ 0\ 1) \\ \underline{\mathbf{h}}_5^T = (1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 0\ 1\ 0\ 1\ 1\ 1\ 1\ 1\ 0\ 1\ 0) \\ \underline{\mathbf{h}}_6^T = (1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 1) \\ \underline{\mathbf{h}}_7^T = (-1\ 1\ -1\ 1\ 1\ -1\ 1\ -1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) \\ \underline{\mathbf{h}}_8^T = (-1\ -1\ 1\ 1\ -1\ -1\ 1\ 1\ 0\ 0\ 0\ 0\ -1\ -1\ 1\ 1\ 0\ 0\ 0\ 0) \\ \underline{\mathbf{h}}_9^T = (-1\ -1\ -1\ -1\ 1\ 1\ 1\ 1\ 0\ -1\ 0\ -1\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 1) \\ \underline{\mathbf{h}}_{10}^T = (-1\ 1\ 1\ -1\ -1\ 1\ 1\ -1\ 0\ 0\ 0\ 0\ -1\ 1\ 1\ -1\ 0\ 0\ 0\ 0) \\ \underline{\mathbf{h}}_{11}^T = (-1\ -1\ -1\ -1\ 1\ 1\ 1\ 1\ -1\ 0\ -1\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 1\ 0) \\ \underline{\mathbf{h}}_{12}^T = (-1\ 1\ 1\ -1\ -1\ 1\ 1\ -1\ 0\ 1\ 0\ -1\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ -1) \\ \underline{\mathbf{h}}_{13}^T = (-1\ -1\ 1\ 1\ -1\ -1\ 1\ 1\ -1\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ -1\ 0\ 1\ 0) \\ \underline{\mathbf{h}}_{14}^T = (1\ 1\ -1\ -1\ -1\ -1\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) \\ \underline{\mathbf{h}}_{15}^T = (1\ -1\ -1\ 1\ -1\ 1\ 1\ -1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) \\ \underline{\mathbf{h}}_{16}^T = (1\ -1\ 1\ -1\ 1\ -1\ 1\ -1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) \end{cases} \quad (24)$$

To arrive at an advantageous writing of the operator discretized gradient $\mathbf{B}$, one will introduce the three vectors $\mathbf{b_i}$ defined by:

$$\mathbf{b}_\mathbf{i}^\mathbf{T} = \mathbf{N}^\mathbf{T}{}_{,\mathbf{i}}(0) = {}^\mathbf{T}\frac{\partial \mathbf{N}}{\partial \mathbf{x}_\mathbf{i}}(0) \quad \text{I = 1,2,3} \qquad (25)$$

If we place ourselves in $(\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2, \hat{\mathbf{x}}_3) = (0,0,0)$ then we obtain:

$$\mathbf{b}_\mathbf{i}^\mathbf{T} = \mathbf{N}^\mathbf{T}{}_{,\mathbf{i}}(0) = \mathbf{cste}$$

where $\mathbf{N}^\mathbf{T}$ represent: $(\mathrm{N}_1 \ \ \mathrm{N}_2 \ \ \mathrm{N}_3 \ldots \mathrm{N}_{20})$

$$\mathbf{b}_\mathbf{i}^\mathbf{T} = \mathbf{N}^\mathbf{T}{}_{,\mathbf{i}}(0) = \left(\frac{\partial \mathrm{N}_I}{\partial \mathrm{x}_\mathrm{i}}(0), \ldots, \frac{\partial \mathrm{N}_{20}}{\partial \mathrm{x}_\mathrm{i}}(0)\right)$$

$$\frac{\partial \mathrm{N}_I}{\partial x_\mathrm{j}} = \left(\frac{\partial \mathrm{N}_I}{\partial \xi} \cdot \frac{\partial \xi}{\partial x_\mathrm{j}} + \frac{\partial \mathrm{N}_I}{\partial \eta} \cdot \frac{\partial \eta}{\partial x_\mathrm{j}} + \frac{\partial \mathrm{N}_I}{\partial \zeta} \cdot \frac{\partial \zeta}{\partial x_\mathrm{j}}\right)_{|\xi=\eta=\zeta=0} = \left(\frac{\partial \mathrm{N}_I}{\partial \xi} \cdot j_{1j} + \frac{\partial \mathrm{N}_I}{\partial \eta} \cdot j_{2j} + \frac{\partial \mathrm{N}_I}{\partial \zeta} \cdot j_{3j}\right)_{\xi=\eta=\zeta=0}$$

$$avec \quad I = 1, 2, \ldots, 20 \qquad et \qquad j = 1, 2, 3$$

$$\mathbf{F}^{-1}{}_{|\xi=\eta=\zeta=0} = \begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} & \frac{\partial \xi}{\partial z} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} & \frac{\partial \eta}{\partial z} \\ \frac{\partial \zeta}{\partial x} & \frac{\partial \zeta}{\partial y} & \frac{\partial \zeta}{\partial z} \end{pmatrix}_{|\xi=\eta=\zeta=0} = \begin{pmatrix} j_{11} & j_{12} & j_{13} \\ j_{21} & j_{22} & j_{23} \\ j_{31} & j_{32} & j_{33} \end{pmatrix}$$

After calculations one finds:

$$\underline{\mathbf{b}}_i^T = \left(j_{i1} \ \ j_{i2} \ \ j_{i3}\right) \cdot \begin{bmatrix} \frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & \frac{1}{8} & 0 & \frac{1}{4} & 0 & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & 0 & -\frac{1}{4} \\ \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{4} & 0 & \frac{1}{4} & 0 & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & 0 \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

Moreover, one can check by algebraic considerations that the following conditions of orthogonality are satisfied:

$$
\underline{\mathbf{h}}_m^T.\underline{\mathbf{h}}_n = \begin{bmatrix}
\mathbf{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{8} & 0 \\
0 & \mathbf{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{8} & 0 & 0 \\
0 & 0 & \mathbf{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{8} \\
0 & 0 & 0 & \mathbf{16} & \mathbf{12} & \mathbf{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \mathbf{12} & \mathbf{16} & \mathbf{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \mathbf{12} & \mathbf{12} & \mathbf{16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \mathbf{8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{12} & 0 & 0 & 0 & 0 & \mathbf{8} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{12} & 0 & \mathbf{8} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{12} & 0 & \mathbf{8} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{8} & 0 & \mathbf{12} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{8} & 0 & \mathbf{12} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{8} & 0 & 0 & 0 & 0 & \mathbf{12} & 0 & 0 & 0 \\
0 & \mathbf{8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{8} & 0 & 0 \\
\mathbf{8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{8} & 0 \\
0 & 0 & \mathbf{8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{8}
\end{bmatrix}
$$

$$m, n = 1, 2, ..., 16$$

$$
\begin{aligned}
&\underline{\mathbf{b}}_i^T\,\underline{\mathbf{h}}_\alpha = 0; \quad \underline{\mathbf{b}}_i^T.\underline{\mathbf{S}} = 0; \; \underline{\mathbf{b}}_i^T.\underline{\mathbf{x}}_j = \delta_{ij} \\
&\underline{\mathbf{h}}_1^T.\underline{\mathbf{S}} = 0; \; \underline{\mathbf{h}}_2^T.\underline{\mathbf{S}} = 0; \; \underline{\mathbf{h}}_3^T.\underline{\mathbf{S}} = 0; \; \underline{\mathbf{h}}_4^T.\underline{\mathbf{S}} = 16; \; \underline{\mathbf{h}}_5^T.\underline{\mathbf{S}} = 16; \; \underline{\mathbf{h}}_6^T.\underline{\mathbf{S}} = 16; \; \underline{\mathbf{h}}_7^T.\underline{\mathbf{S}} = 0; \; \underline{\mathbf{h}}_8^T.\underline{\mathbf{S}} = 0 \\
&\underline{\mathbf{h}}_9^T.\underline{\mathbf{S}} = 0 \;\; \underline{\mathbf{h}}_{10}^T.\underline{\mathbf{S}} = 0; \; \underline{\mathbf{h}}_{11}^T.\underline{\mathbf{S}} = 0 \;\; \underline{\mathbf{h}}_{12}^T.\underline{\mathbf{S}} = 0 \;\; \underline{\mathbf{h}}_{13}^T.\underline{\mathbf{S}} = 0 \;\; \underline{\mathbf{h}}_{14}^T.\underline{\mathbf{S}} = 0 \;\; \underline{\mathbf{h}}_{15}^T.\underline{\mathbf{S}} = 0 \;\; \underline{\mathbf{h}}_{16}^T.\underline{\mathbf{S}} = 0
\end{aligned} \quad \textbf{(26)}
$$

This stage, one can determine the constant unknown factors which intervene in the writing **(21)** field of displacement by multiplying scalairement the equation (22) by $\underline{\mathbf{b}}_j^T$, $\underline{\mathbf{S}}^T$ and $\underline{\mathbf{h}}_\alpha^T$ respectively, and by using the relations of orthogonality **(26)**. One obtains:

$$a_{ji} = \underline{\mathbf{b}}_j^T.\underline{\mathbf{d}}_i \qquad c_{\alpha i} = \underline{\boldsymbol{\gamma}}_\alpha^T.\underline{\mathbf{d}}_i$$

with:

$$
\begin{aligned}
\underline{\boldsymbol{\gamma}}_\alpha^T = {} & n_{\alpha 1}\left(\underline{\mathbf{h}}_1^T - \left(\underline{\mathbf{h}}_1^T.\underline{\mathbf{x}}_j\right)\underline{\mathbf{b}}_j^T\right) + n_{\alpha 2}\left(\underline{\mathbf{h}}_2^T - \left(\underline{\mathbf{h}}_2^T.\underline{\mathbf{x}}_j\right)\underline{\mathbf{b}}_j^T\right) + n_{\alpha 3}\left(\underline{\mathbf{h}}_3^T - \left(\underline{\mathbf{h}}_3^T.\underline{\mathbf{x}}_j\right)\underline{\mathbf{b}}_j^T\right) \\
& + n_{\alpha 4}\left[\left(\underline{\mathbf{h}}_4^T - \frac{4}{5}\underline{\mathbf{S}}^T\right) - \left(\left(\underline{\mathbf{h}}_4^T - \frac{4}{5}\underline{\mathbf{S}}^T\right).\underline{\mathbf{x}}_j\right)\underline{\mathbf{b}}_j^T\right] + n_{\alpha 5}\left[\left(\underline{\mathbf{h}}_5^T - \frac{4}{5}\underline{\mathbf{S}}^T\right) - \left(\left(\underline{\mathbf{h}}_5^T - \frac{4}{5}\underline{\mathbf{S}}^T\right).\underline{\mathbf{x}}_j\right)\underline{\mathbf{b}}_j^T\right] + \\
& + n_{\alpha 6}\left[\left(\underline{\mathbf{h}}_6^T - \frac{4}{5}\underline{\mathbf{S}}^T\right) - \left(\left(\underline{\mathbf{h}}_6^T - \frac{4}{5}\underline{\mathbf{S}}^T\right).\underline{\mathbf{x}}_j\right)\underline{\mathbf{b}}_j^T\right] + n_{\alpha 7}\left(\underline{\mathbf{h}}_7^T - \left(\underline{\mathbf{h}}_7^T.\underline{\mathbf{x}}_j\right)\underline{\mathbf{b}}_j^T\right) \\
& + n_{\alpha 8}\left(\underline{\mathbf{h}}_8^T - \left(\underline{\mathbf{h}}_8^T.\underline{\mathbf{x}}_j\right)\underline{\mathbf{b}}_j^T\right) + n_{\alpha 9}\left(\underline{\mathbf{h}}_9^T - \left(\underline{\mathbf{h}}_9^T.\underline{\mathbf{x}}_j\right)\underline{\mathbf{b}}_j^T\right) + n_{\alpha 10}\left(\underline{\mathbf{h}}_{10}^T - \left(\underline{\mathbf{h}}_{10}^T.\underline{\mathbf{x}}_j\right)\underline{\mathbf{b}}_j^T\right) \\
& + n_{\alpha 11}\left(\underline{\mathbf{h}}_{11}^T - \left(\underline{\mathbf{h}}_{11}^T.\underline{\mathbf{x}}_j\right)\underline{\mathbf{b}}_j^T\right) + n_{\alpha 12}\left(\underline{\mathbf{h}}_{12}^T - \left(\underline{\mathbf{h}}_{12}^T.\underline{\mathbf{x}}_j\right)\underline{\mathbf{b}}_j^T\right) + n_{\alpha 13}\left(\underline{\mathbf{h}}_{13}^T - \left(\underline{\mathbf{h}}_{13}^T.\underline{\mathbf{x}}_j\right)\underline{\mathbf{b}}_j^T\right) \\
& + n_{\alpha 14}\left(\underline{\mathbf{h}}_{14}^T - \left(\underline{\mathbf{h}}_{14}^T.\underline{\mathbf{x}}_j\right)\underline{\mathbf{b}}_j^T\right) + n_{\alpha 15}\left(\underline{\mathbf{h}}_{15}^T - \left(\underline{\mathbf{h}}_{15}^T.\underline{\mathbf{x}}_j\right)\underline{\mathbf{b}}_j^T\right) + n_{\alpha 16}\left(\underline{\mathbf{h}}_{16}^T - \left(\underline{\mathbf{h}}_{16}^T.\underline{\mathbf{x}}_j\right)\underline{\mathbf{b}}_j^T\right)
\end{aligned}
$$

$$\left[\mathbf{n}_{\alpha\beta}\right] = \begin{bmatrix}
\frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{4} & 0 \\
0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{4} & 0 & 0 \\
0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{4} \\
0 & 0 & 0 & \frac{3}{8} & \frac{1}{8} & \frac{1}{8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{8} & \frac{3}{8} & \frac{1}{8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{8} & \frac{1}{8} & \frac{3}{8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{20} & 0 & 0 & 0 & 0 & -\frac{1}{10} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{20} & 0 & -\frac{1}{10} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{20} & 0 & -\frac{1}{10} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{10} & 0 & \frac{3}{20} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{10} & 0 & \frac{3}{20} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{10} & 0 & 0 & 0 & 0 & \frac{3}{20} & 0 & 0 & 0 \\
0 & -\frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{8} & 0 & 0 \\
-\frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{8} & 0 \\
0 & 0 & -\frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{8}
\end{bmatrix} \qquad \alpha,\beta = 1,2,\ldots,16$$

For 2 elements SHB15 and SHB20, the operator discretized gradient connecting the tensor of deformation to the vector of nodal displacements is given by:

$$\underline{\underline{\nabla}}_S\left(\underline{\mathbf{u}}\right) = \underline{\underline{\mathbf{B}}}.\underline{\mathbf{d}} \tag{27}$$

where:

$$\underline{\underline{\nabla}}_S\left(\underline{\mathbf{u}}\right) = \begin{bmatrix} u_{x,x} \\ u_{y,y} \\ u_{z,z} \\ u_{x,y}+u_{y,x} \\ u_{x,z}+u_{z,x} \\ u_{y,z}+u_{z,y} \end{bmatrix} \quad , \quad \underline{\mathbf{d}} = \begin{bmatrix} \underline{\mathbf{d}}_1 \\ \underline{\mathbf{d}}_2 \\ \underline{\mathbf{d}}_3 \end{bmatrix} \tag{28}$$

and takes the practical matric shape then:

$$\underline{\underline{\mathbf{B}}}=\begin{bmatrix} \underline{\mathbf{b}}_1^T+\mathrm{h}_{\alpha,x_1}\underline{\gamma}_\alpha^T & \underline{0} & \underline{0} \\ \underline{0} & \underline{\mathbf{b}}_2^T+\mathrm{h}_{\alpha,x_2}\underline{\gamma}_\alpha^T & \underline{0} \\ \underline{0} & \underline{0} & \underline{\mathbf{b}}_3^T+\mathrm{h}_{\alpha,x_3}\underline{\gamma}_\alpha^T \\ \underline{\mathbf{b}}_2^T+\mathrm{h}_{\alpha,x_2}\underline{\gamma}_\alpha^T & \underline{\mathbf{b}}_1^T+\mathrm{h}_{\alpha,x_1}\underline{\gamma}_\alpha^T & \underline{0} \\ \underline{\mathbf{b}}_3^T+\mathrm{h}_{\alpha,x_3}\underline{\gamma}_\alpha^T & \underline{0} & \underline{\mathbf{b}}_1^T+\mathrm{h}_{\alpha,x_1}\underline{\gamma}_\alpha^T \\ \underline{0} & \underline{\mathbf{b}}_3^T+\mathrm{h}_{\alpha,x_3}\underline{\gamma}_\alpha^T & \underline{\mathbf{b}}_2^T+\mathrm{h}_{\alpha,x_2}\underline{\gamma}_\alpha^T \end{bmatrix} \quad \textbf{(29)}$$

This writing of the operator discretized gradient using the formulas of Hallquist [4] is very convenient because the vectors $\underline{\gamma}_\alpha$, which intervenes in the expression of $\underline{\underline{\mathbf{B}}}$, check the following conditions of orthogonality:

$$\underline{\gamma}_\alpha^T.\underline{\mathbf{x}}_j=0 \quad , \quad \underline{\gamma}_\alpha^T.\underline{\mathbf{h}}_\beta=\delta_{\alpha\beta} \quad (30)$$

This makes it possible to separately handle each mode of the deformation to simply obtain the shape of the field of applied deformation. Let us note that an element based on the formulation (29) is convergent when it is evaluated exactly. However, the evaluation of this operator $\underline{\underline{\mathbf{B}}}$, given in (29), of each point of integration makes this element expensive in computing times for the practical applications, and the simplified shape of this element is essential.

3.3 Variational formulation used for elements SHB15 and SHB20

The extension of the weak form of the variational principle of Hu-Washizu to the case of the mechanics of the nonlinear solids is due to Fish and Belytschko [6]. For a simple element, one a:

$$\delta\pi(\boldsymbol{u},\dot{\boldsymbol{\epsilon}},\bar{\boldsymbol{\sigma}})=\int_{V_e}\delta\dot{\boldsymbol{\epsilon}}\cdot\bar{\boldsymbol{\sigma}}\,dV+\delta\int_{V_e}\bar{\boldsymbol{\sigma}}\cdot\left(\nabla_s\boldsymbol{u}-\dot{\boldsymbol{\epsilon}}\right)dV-\delta\,\boldsymbol{d}^T\cdot\boldsymbol{f}^{ext}=0 \quad \textbf{(31)}$$

where δ represent a variation, $\underline{\mathbf{u}}$ the field of displacement, $\dot{\boldsymbol{\epsilon}}$ is the rate of applied deformation, $\underline{\underline{\bar{\boldsymbol{\sigma}}}}$ the applied constraint, $\underline{\underline{\boldsymbol{\sigma}}}$ the constraint evaluated by the constitutive law, $\underline{\mathbf{d}}$ nodal displacements, $\underline{\mathbf{f}}^{ext}$ external nodal forces, and $\nabla_s\boldsymbol{u}$ the symmetrical part of the gradient of the field of displacement.

The formulation " *Assumed strain* " (projection of the operator discretized gradient $\underline{\underline{\mathbf{B}}}$ on under suitable space in order to avoiding the various problems of blocking) is based on a simplified form of the variational principle of Hu-Washizu as it was described by Simo and Hughes [7]. In this simplified form, the applied constraint is selected orthogonal with the difference between the symmetrical part of the gradient of displacement and the rate of applied deformation. Thus, the second term in the equation **(31**) be eliminated and one obtains:

$$\delta\pi(\boldsymbol{u},\dot{\boldsymbol{\epsilon}},\bar{\boldsymbol{\sigma}})=\int_{V_e}\delta\dot{\boldsymbol{\epsilon}}\cdot\boldsymbol{\sigma}\,dV-\delta\,\boldsymbol{d}^T\cdot\boldsymbol{f}^{ext}=0 \quad (32)$$

In this form, the variational principle is independent of the interpolation of the constraint, since the applied constraint does not intervene any more and thus need does not have to be defined. The discretized equations thus require the only interpolation of displacement $\underline{\mathbf{u}}$ and of the rate of applied deformation $\dot{\boldsymbol{\epsilon}}$ in the element. With the preceding vectorial notations one a:

$$\boldsymbol{u}(x,t)=\sum_{i=1}^{15}\boldsymbol{d}_I(t)N_I(x) \text{ (element SHB15)}$$

Or $$\boldsymbol{u}(x,t)=\sum_{i=1}^{20}\boldsymbol{d}_{\boldsymbol{I}}(t)N_I(x)$$ (element SHB20) (33)

This led to:

$$\nabla_s\boldsymbol{u}(x,t)=\boldsymbol{B}(x)\boldsymbol{d}(t) \quad (34)$$

The applied deformation is defined as for it by:

$$\dot{\boldsymbol{\epsilon}}(x,t)=\bar{\boldsymbol{B}}(x)\boldsymbol{d}(t) \quad (35)$$

Replacing the expression (35) in the variational principle (32), one obtains:

$$\delta\,\boldsymbol{d}^{\boldsymbol{T}}\left(\int_{V_e}\bar{\boldsymbol{B}}\cdot\boldsymbol{\sigma}\,dV-\boldsymbol{F}^{\boldsymbol{ext}}\right)=0 \quad (36)$$

Like $\delta\underline{d}$ can be arbitrarily selected, the preceding equation leads to:

$$\underline{\mathbf{f}}^{\text{int}}=\underline{\mathbf{f}}^{ext} \quad (37)$$

with:

$$\boldsymbol{f}^{\textbf{int}}=\int_{V_e}\bar{\boldsymbol{B}}(x)\cdot\boldsymbol{\sigma}(\dot{\boldsymbol{\epsilon}})\,dV \quad (38)$$

In the equation above, it is well specified that the constraint $\underline{\underline{\boldsymbol{\sigma}}}$ is calculated by the law constitutive starting from the rate of applied deformation. $\dot{\boldsymbol{\epsilon}}$ For the nonlinear problems, $\underline{\underline{\boldsymbol{\sigma}}}$ can also be an integral function of the rate of applied deformation and other internal variables:

$$\boldsymbol{\sigma}=F(\dot{\boldsymbol{\epsilon}},\boldsymbol{\alpha},...) \quad (39)$$

where $\underline{\underline{\boldsymbol{\alpha}}}$ represent the internal variables. The formulation thus obtained is valid for problems including the two types of nonlinearities: geometrical and material. In the case of linear problems, one a:

$$\underline{\underline{\boldsymbol{\sigma}}}=\underline{\underline{\mathbf{C}}}.\underline{\underline{\bar{\boldsymbol{\varepsilon}}}}=\underline{\underline{\mathbf{C}}}.\underline{\underline{\bar{\mathbf{B}}}}.\underline{\mathbf{d}} \quad (40)$$

The elastic matrix of behavior $\underline{\underline{\mathbf{C}}}$, in the case of an isotropic material, is selected like following:

$$\underline{\underline{\mathbf{C}}}=\begin{bmatrix}\frac{E}{1-\nu^2} & \frac{E\nu}{1-\nu^2} & 0 & 0 & 0 & 0\\ \frac{E\nu}{1-\nu^2} & \frac{E}{1-\nu^2} & 0 & 0 & 0 & 0\\ 0 & 0 & E & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{E}{2(1+\nu)} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{E}{2(1+\nu)} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{E}{2(1+\nu)}\end{bmatrix}$$

In this matrix, E is the modulus Young and v is the Poisson's ratio. This law is specific to elements SHB. It resembles that which one would have in the case of the assumption of the plane constraints, put except for the term (3.3). We can note that this choice involves an artificial anisotropic behavior. This choice makes it possible to satisfy all the tests without introducing blocking.

The internal forces of the element are written then simply in terms of the elementary matrix of rigidity:

$$\boldsymbol{f}^{\text{int}}=\boldsymbol{K}_e\cdot\boldsymbol{d} \quad (41)$$

where:

$$\boldsymbol{K}_e=\int_{V_e}\bar{\boldsymbol{B}}^T\cdot\boldsymbol{C}\cdot\bar{\boldsymbol{B}}\,dV \quad (42)$$

In a standard approach in displacement, the rate of applied deformation is identified with the symmetrical part of the gradient speed, which amounts replacing $\bar{\boldsymbol{B}}$ by $\boldsymbol{B}$ in the preceding expressions. One thus obtains simply:

$$\boldsymbol{K}_e=\int_{V_e}\boldsymbol{B}^T\cdot\boldsymbol{C}\cdot\boldsymbol{B}\,dV \quad (43)$$

The prismatic element with 15 nodes, named "SHB15", has 15 points of integration. Their coordinates (ξ,η,ζ) and their weights of integration are the roots of the polynomial of Gauss-Legendre given in the following table:

Not Gauss	ξ	η	ζ	$w(\xi,\eta,\zeta)$
P (1)	1/2	1/2	$\xi_{G1}=-0.906179845938664$	0.236926885056189/6
P (2)	1/2	1/2	$\xi_{G2}=-0.538469310105683$	0.478628670499366/6
P (3)	1/2	1/2	$\xi_{G3}=0$	0.568888888888889/6
P (4)	1/2	1/2	$\xi_{G4}=0.538469310105683$	0.478628670499366/6
P (5)	1/2	1/2	$\xi_{G5}=0.906179845938664$	0.236926885056189/6
P (6)	0	1/2	$\xi_{G6}=-0.906179845938664$	0.236926885056189/6
P (7)	0	1/2	$\xi_{G7}=-0.538469310105683$	0.478628670499366/6
P (8)	0	1/2	$\xi_{G8}=0$	0.568888888888889/6
P (9)	0	1/2	$\xi_{G9}=0.538469310105683$	0.478628670499366/6
P (10)	0	1/2	$\xi_{G10}=0.906179845938664$	0.236926885056189/6
P (11)	1/2	0	$\xi_{G11}=-0.906179845938664$	0.236926885056189/6
P (12)	1/2	0	$\xi_{G12}=-0.538469310105683$	0.478628670499366/6
P (13)	1/2	0	$\xi_{G13}=0$	0.568888888888889/6
P (14)	1/2	0	$\xi_{G14}=0.538469310105683$	0.478628670499366/6
P (15)	1/2	0	$\xi_{G15}=0.906179845938664$	0.236926885056189/6

Thus, the expression of rigidity is: $\boldsymbol{K}_e=\sum_{j=1}^{15} w(P(j))J(P(j))\bar{\boldsymbol{B}}^T(P(j))\cdot\boldsymbol{C}\cdot\bar{\boldsymbol{B}}(P(j))$

The coordinates of the points of Gauss and their weights for element SHB20 are given in the table below:

Not Gauss	ξ	η	ζ	$w(\xi,\eta,\zeta)$
P (1)	$-1/\sqrt{3}$	$-1/\sqrt{3}$	$\zeta_{G1}=-0.906179845938664$	0.236926885056189
P (2)	$-1/\sqrt{3}$	$-1/\sqrt{3}$	$\zeta_{G2}=-0.538469310105683$	0.478628670499366
P (3)	$-1/\sqrt{3}$	$-1/\sqrt{3}$	$\zeta_{G3}=0$	0.568888888888889
P (4)	$-1/\sqrt{3}$	$-1/\sqrt{3}$	$\zeta_{G4}=0.538469310105683$	0.478628670499366
P (5)	$-1/\sqrt{3}$	$-1/\sqrt{3}$	$\zeta_{G5}=0.906179845938664$	0.236926885056189
P (6)	$1/\sqrt{3}$	$-1/\sqrt{3}$	$\zeta_{G6}=-0.906179845938664$	0.236926885056189
P (7)	$1/\sqrt{3}$	$-1/\sqrt{3}$	$\zeta_{G7}=-0.538469310105683$	0.478628670499366
P (8)	$1/\sqrt{3}$	$-1/\sqrt{3}$	$\zeta_{G8}=0$	0.568888888888889
P (9)	$1/\sqrt{3}$	$-1/\sqrt{3}$	$\zeta_{G9}=0.538469310105683$	0.478628670499366
P (10)	$1/\sqrt{3}$	$-1/\sqrt{3}$	$\zeta_{G10}=0.906179845938664$	0.236926885056189
P (11)	$1/\sqrt{3}$	$1/\sqrt{3}$	$\zeta_{G11}=-0.906179845938664$	0.236926885056189
P (12)	$1/\sqrt{3}$	$1/\sqrt{3}$	$\zeta_{G12}=-0.538469310105683$	0.478628670499366
P (13)	$1/\sqrt{3}$	$1/\sqrt{3}$	$\zeta_{G13}=0$	0.568888888888889
P (14)	$1/\sqrt{3}$	$1/\sqrt{3}$	$\zeta_{G14}=0.538469310105683$	0.478628670499366
P (15)	$1/\sqrt{3}$	$1/\sqrt{3}$	$\zeta_{G15}=0.906179845938664$	0.236926885056189
P (16)	$-1/\sqrt{3}$	$1/\sqrt{3}$	$\zeta_{G16}=-0.906179845938664$	0.236926885056189
P (17)	$-1/\sqrt{3}$	$1/\sqrt{3}$	$\zeta_{G17}=-0.538469310105683$	0.478628670499366
P (18)	$-1/\sqrt{3}$	$1/\sqrt{3}$	$\zeta_{G18}=0$	0.568888888888889
P (19)	$-1/\sqrt{3}$	$1/\sqrt{3}$	$\zeta_{G19}=0.538469310105683$	0.478628670499366
P (20)	$-1/\sqrt{3}$	$1/\sqrt{3}$	$\zeta_{G20}=0.906179845938664$	0.236926885056189

Thus, the expression of rigidity is: $\boldsymbol{K_e}=\sum_{j=1}^{20} w(P(j))J(P(j))\,\bar{\boldsymbol{B}}^T(P(j))\cdot\boldsymbol{C}\cdot\bar{\boldsymbol{B}}(P(j))$

3.4 Geometrical matrix of rigidity Ksigma

The matrix $\boldsymbol{K}_\sigma$ aims to solve the problems of buckling. We point out here that the modes of buckling are the clean vectors of the problem to the eigenvalues generalized according to:

$$\left(\boldsymbol{K}+\mu\,\boldsymbol{K}_\sigma\right)\cdot\boldsymbol{u}=\boldsymbol{0}\Leftrightarrow\boldsymbol{K}\cdot\boldsymbol{u}=\lambda\,\boldsymbol{K}_\sigma\cdot\boldsymbol{u}$$

with $\lambda=-\mu$, and μ is the multiplying coefficient of the loading.

By introducing the quadratic deformation $\underline{\mathbf{e}}^Q$ such as:

$$e_{ij}^Q(\delta\boldsymbol{u},\Delta\boldsymbol{u})=\sum_{k=1}^{3}\delta u_{k,i}.\Delta u_{k,j}$$

One can define this matrix of geometrical rigidity by:

$$\delta \boldsymbol{u}^T \cdot \boldsymbol{K}_\sigma \cdot \Delta \boldsymbol{u} = \int_{\Omega_0} \boldsymbol{\sigma} : \boldsymbol{e}^Q(\delta \boldsymbol{u}, \Delta \boldsymbol{u})\, d\Omega = \int_{\Omega_0} \boldsymbol{\sigma} : \nabla \delta \boldsymbol{u}^T \nabla \Delta \boldsymbol{u} \;\; d\Omega$$

In order to express this matrix in discretized space, let us introduce the discretized operators quadratic gradient $\underline{\underline{\mathbf{B}}}^Q$ (in matric notation) such as:

$$\underline{\mathbf{e}}^Q(\delta\mathbf{u}, \Delta\mathbf{u}) = \begin{bmatrix} e^Q_{11} \\ e^Q_{22} \\ e^Q_{33} \\ e^Q_{12} + e^Q_{21} \\ e^Q_{13} + e^Q_{31} \\ e^Q_{23} + e^Q_{32} \end{bmatrix} = \begin{bmatrix} \underline{\delta\mathbf{u}}^T \times \underline{\underline{\mathbf{B}}}^Q_{11} \times \underline{\Delta\mathbf{u}} \\ \underline{\delta\mathbf{u}}^T \times \underline{\underline{\mathbf{B}}}^Q_{22} \times \underline{\Delta\mathbf{u}} \\ \underline{\delta\mathbf{u}}^T \times \underline{\underline{\mathbf{B}}}^Q_{33} \times \underline{\Delta\mathbf{u}} \\ \underline{\delta\mathbf{u}}^T \times \underline{\underline{\mathbf{B}}}^Q_{12} \times \underline{\Delta\mathbf{u}} \\ \underline{\delta\mathbf{u}}^T \times \underline{\underline{\mathbf{B}}}^Q_{13} \times \underline{\Delta\mathbf{u}} \\ \underline{\delta\mathbf{u}}^T \times \underline{\underline{\mathbf{B}}}^Q_{23} \times \underline{\Delta\mathbf{u}} \end{bmatrix}$$

Various terms $\underline{\underline{\mathbf{B}}}^Q_{ij}$ are given by the following equations:

$$\underline{\underline{\mathbf{B}}}^Q_{11} = \begin{bmatrix} \underline{\mathbf{B}}_1\underline{\mathbf{B}}_1^\mathbf{T} & \underline{\underline{\mathbf{0}}} & \underline{\underline{\mathbf{0}}} \\ \underline{\underline{\mathbf{0}}} & \underline{\mathbf{B}}_1\underline{\mathbf{B}}_1^\mathbf{T} & \underline{\underline{\mathbf{0}}} \\ \underline{\underline{\mathbf{0}}} & \underline{\underline{\mathbf{0}}} & \underline{\mathbf{B}}_1\underline{\mathbf{B}}_1^\mathbf{T} \end{bmatrix} \;;\; \underline{\underline{\mathbf{B}}}^Q_{22} = \begin{bmatrix} \underline{\mathbf{B}}_2\underline{\mathbf{B}}_2^\mathbf{T} & \underline{\underline{\mathbf{0}}} & \underline{\underline{\mathbf{0}}} \\ \underline{\underline{\mathbf{0}}} & \underline{\mathbf{B}}_2\underline{\mathbf{B}}_2^\mathbf{T} & \underline{\underline{\mathbf{0}}} \\ \underline{\underline{\mathbf{0}}} & \underline{\underline{\mathbf{0}}} & \underline{\mathbf{B}}_2\underline{\mathbf{B}}_2^\mathbf{T} \end{bmatrix} \;;\; \underline{\underline{\mathbf{B}}}^Q_{33} = \begin{bmatrix} \underline{\mathbf{B}}_3\underline{\mathbf{B}}_3^\mathbf{T} & \underline{\underline{\mathbf{0}}} & \underline{\underline{\mathbf{0}}} \\ \underline{\underline{\mathbf{0}}} & \underline{\mathbf{B}}_3\underline{\mathbf{B}}_3^\mathbf{T} & \underline{\underline{\mathbf{0}}} \\ \underline{\underline{\mathbf{0}}} & \underline{\underline{\mathbf{0}}} & \underline{\mathbf{B}}_3\underline{\mathbf{B}}_3^\mathbf{T} \end{bmatrix}$$

$$\underline{\underline{\mathbf{B}}}^Q_{12} = \begin{bmatrix} \underline{\mathbf{B}}_1\underline{\mathbf{B}}_2^\mathbf{T} + \underline{\mathbf{B}}_2\underline{\mathbf{B}}_1^\mathbf{T} & \underline{\underline{\mathbf{0}}} & \underline{\underline{\mathbf{0}}} \\ \underline{\underline{\mathbf{0}}} & \underline{\mathbf{B}}_1\underline{\mathbf{B}}_2^\mathbf{T} + \underline{\mathbf{B}}_2\underline{\mathbf{B}}_1^\mathbf{T} & \underline{\underline{\mathbf{0}}} \\ \underline{\underline{\mathbf{0}}} & \underline{\underline{\mathbf{0}}} & \underline{\mathbf{B}}_1\underline{\mathbf{B}}_2^\mathbf{T} + \underline{\mathbf{B}}_2\underline{\mathbf{B}}_1^\mathbf{T} \end{bmatrix}$$

$$\underline{\underline{\mathbf{B}}}^Q_{13} = c^2 \begin{bmatrix} \underline{\mathbf{B}}_1\underline{\mathbf{B}}_3^\mathbf{T} + \underline{\mathbf{B}}_3\underline{\mathbf{B}}_1^\mathbf{T} & \underline{\underline{\mathbf{0}}} & \underline{\underline{\mathbf{0}}} \\ \underline{\underline{\mathbf{0}}} & \underline{\mathbf{B}}_1\underline{\mathbf{B}}_3^\mathbf{T} + \underline{\mathbf{B}}_3\underline{\mathbf{B}}_1^\mathbf{T} & \underline{\underline{\mathbf{0}}} \\ \underline{\underline{\mathbf{0}}} & \underline{\underline{\mathbf{0}}} & \underline{\mathbf{B}}_1\underline{\mathbf{B}}_3^\mathbf{T} + \underline{\mathbf{B}}_3\underline{\mathbf{B}}_1^\mathbf{T} \end{bmatrix}$$

$$\underline{\underline{\mathbf{B}}}^Q_{23} = c^2 \begin{bmatrix} \underline{\mathbf{B}}_2\underline{\mathbf{B}}_3^\mathbf{T} + \underline{\mathbf{B}}_3\underline{\mathbf{B}}_2^\mathbf{T} & \underline{\underline{\mathbf{0}}} & \underline{\underline{\mathbf{0}}} \\ \underline{\underline{\mathbf{0}}} & \underline{\mathbf{B}}_2\underline{\mathbf{B}}_3^\mathbf{T} + \underline{\mathbf{B}}_3\underline{\mathbf{B}}_2^\mathbf{T} & \underline{\underline{\mathbf{0}}} \\ \underline{\underline{\mathbf{0}}} & \underline{\underline{\mathbf{0}}} & \underline{\mathbf{B}}_2\underline{\mathbf{B}}_3^\mathbf{T} + \underline{\mathbf{B}}_3\underline{\mathbf{B}}_2^\mathbf{T} \end{bmatrix}$$

with the vectors $\underline{\mathbf{B}}_i$ (I = 1,2,3) defined as:

$$\underline{\mathbf{B}}_i = \left(\underline{\mathbf{b}}_i \boldsymbol{\gamma} + h_{\alpha,i\_\alpha} \right)$$

With these notations, the contribution to the geometrical matrix of rigidity, $\underline{\underline{\mathbf{k}}}_{\boldsymbol{\sigma}}$, at the point of Gauss ξ_j is given by:

$$\underline{\underline{\mathbf{k}}}_{\boldsymbol{\sigma}}(\xi_j)=\sigma_{11}(\xi_j)\underline{\underline{\mathbf{B}}}_{11}^{Q}(\xi_j)+\sigma_{22}(\xi_j)\underline{\underline{\mathbf{B}}}_{22}^{Q}(\xi_j)+\sigma_{33}(\xi_j)\underline{\underline{\mathbf{B}}}_{33}^{Q}(\xi_j)$$
$$+\sigma_{12}(\xi_j)\underline{\underline{\mathbf{B}}}_{12}^{Q}(\xi_j)+\sigma_{13}(\xi_j)\underline{\underline{\mathbf{B}}}_{13}^{Q}(\xi_j)+\sigma_{23}(\xi_j)\underline{\underline{\mathbf{B}}}_{23}^{Q}(\xi_j)$$

By integration on the points of Gauss of the element, the geometrical matrix of rigidity is obtained by the formula:

$$\boldsymbol{K}_{\boldsymbol{\sigma}}=\sum_{j=1}^{5} w(\xi_j)J(\xi_j)\boldsymbol{k}_{\boldsymbol{\sigma}}(\xi_j)$$ for element SHB15 and element SHB20

3.5 Following forces and matrix of pressure K_p

The following compressive forces are present in the tangent matrix via the matrix K_p , because the following external forces depend on displacement. The following compressive forces are written:

$$\int_{\partial\Omega} p\,\boldsymbol{n}^{\boldsymbol{T}}\cdot\boldsymbol{u}\,dS=\int_{\partial\Omega_0} p\det[\boldsymbol{F}(\boldsymbol{u})]\boldsymbol{n}_{\boldsymbol{0}}^{\boldsymbol{T}}\boldsymbol{F}(\boldsymbol{u})^{\text{-T}}dS_0=p\boldsymbol{F}_{\boldsymbol{0}}-p\boldsymbol{K}_{\boldsymbol{p}}\cdot\boldsymbol{u}$$
$$\boldsymbol{F}(\boldsymbol{u})=\boldsymbol{1}+\nabla\boldsymbol{u}$$

by using the notations:

- $\underline{\mathbf{n}}_0{}^T=(n_1,n_2,n_3)$, normal on the surface external of the element in the configuration of reference;
- $\tilde{\boldsymbol{b}}_{\boldsymbol{i}}$, vector of size 6 (for SHB15) or 8 (for SHB20), derived from the functions of form to the 6 (for SHB15) or 8 (for SHB20) nodes of the face of the element charged in pressure;
- S_0 surface of the face charged in pressure.

The preceding formulation leads to a not-symmetrical matrix. It is known that one can nevertheless use a symmetrical formulation if the external forces due to the pressure derive from a potential. It is the case if the compressive forces do not work on the border of the modelled field. It is thus considered that the symmetrical part of the matrix is enough. The symmetrized matrix takes the following shape:

$$\underline{\underline{\mathbf{K}}}_{\mathbf{P}}=S_0\begin{bmatrix}\mathbf{0} & \tilde{\mathbf{b}}_2^T n_1-\tilde{\mathbf{b}}_1^T n_2 & \tilde{\mathbf{b}}_3^T n_1-\tilde{\mathbf{b}}_1^T n_3\\ \mathbf{0} & \tilde{\mathbf{b}}_2^T n_1-\tilde{\mathbf{b}}_1^T n_2 & \tilde{\mathbf{b}}_3^T n_1-\tilde{\mathbf{b}}_1^T n_3\\ \mathbf{0} & \tilde{\mathbf{b}}_2^T n_1-\tilde{\mathbf{b}}_1^T n_2 & \tilde{\mathbf{b}}_3^T n_1-\tilde{\mathbf{b}}_1^T n_3\\ \tilde{\mathbf{b}}_1^T n_2-\tilde{\mathbf{b}}_2^T n_1 & \mathbf{0} & \tilde{\mathbf{b}}_3^T n_2-\tilde{\mathbf{b}}_2^T n_3\\ \tilde{\mathbf{b}}_1^T n_2-\tilde{\mathbf{b}}_2^T n_1 & \mathbf{0} & \tilde{\mathbf{b}}_3^T n_2-\tilde{\mathbf{b}}_2^T n_3\\ \tilde{\mathbf{b}}_1^T n_2-\tilde{\mathbf{b}}_2^T n_1 & \mathbf{0} & \tilde{\mathbf{b}}_3^T n_2-\tilde{\mathbf{b}}_2^T n_3\\ \tilde{\mathbf{b}}_3^T n_1-\tilde{\mathbf{b}}_1^T n_3 & \tilde{\mathbf{b}}_2^T n_3-\tilde{\mathbf{b}}_3^T n_2 & \mathbf{0}\\ \tilde{\mathbf{b}}_3^T n_1-\tilde{\mathbf{b}}_1^T n_3 & \tilde{\mathbf{b}}_2^T n_3-\tilde{\mathbf{b}}_3^T n_2 & \mathbf{0}\\ \tilde{\mathbf{b}}_3^T n_1-\tilde{\mathbf{b}}_1^T n_3 & \tilde{\mathbf{b}}_2^T n_3-\tilde{\mathbf{b}}_3^T n_2 & \mathbf{0}\end{bmatrix}$$

It is a matrix 18×18 or 24×24 , that it is necessary to multiply by displacements of the 6 (for SHB15) or 8 (for SHB20) nodes of the face to which one applies a pressure.

4 Strategy for non-linear calculations

4.1 Geometrical non-linearities

One treats here the case of great displacements, but weak rotations and small deformations. One adopts for that an up to date put Lagrangian formulation.

Into nonlinear, we seek to write balance between internal forces and force external at the end of the increment of load (located by index 2):

$$\mathbf{F}_2^{\text{int}} = \mathbf{F}_2^{\textbf{ext}}$$

The expression of the internal forces is written:

$$\underline{\mathbf{F}}_2^{\text{int}} = \int_{\Omega_2} \underline{\underline{\mathbf{B}}}_2^T \cdot \underline{\boldsymbol{\sigma}}_2 \, dV$$

In the preceding equation the operator $\underline{\underline{\mathbf{B}}}_2$ is the operator allowing to pass from the displacement to the linear deformation calculated on the geometry at the end of the step, the constraint $\underline{\boldsymbol{\sigma}}_2$ is the constraint of Cauchy at the end of the step and integration is made on volume Ω_2 deformed at the end of the step.

Important remarks:

- For the element SHB6, the matrix $\underline{\underline{\mathbf{B}}}_2$ is also modified by " *Local Assumed strain method ".* It takes the following shape:

$$\underline{\underline{\mathbf{B}}}_2 = \begin{bmatrix} \underline{\mathbf{b}}_1^T + \mathrm{h}_{\alpha,x_1} \underline{\boldsymbol{\gamma}}_\alpha^T & \underline{\mathbf{0}} & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \underline{\mathbf{b}}_2^T + \mathrm{h}_{\alpha,x_2} \underline{\boldsymbol{\gamma}}_\alpha^T & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \underline{\mathbf{0}} & \underline{\mathbf{b}}_3^T + \mathrm{h}_{\alpha,x_3} \underline{\boldsymbol{\gamma}}_\alpha^T \\ \underline{\mathbf{b}}_2^T + \mathrm{h}_{\alpha,x_2} \underline{\boldsymbol{\gamma}}_\alpha^T & \underline{\mathbf{b}}_1^T + \mathrm{h}_{\alpha,x_1} \underline{\boldsymbol{\gamma}}_\alpha^T & \underline{\mathbf{0}} \\ c*\left(\underline{\mathbf{b}}_3^T + \mathrm{h}_{\alpha,x_3} \underline{\boldsymbol{\gamma}}_\alpha^T\right) & \underline{\mathbf{0}} & c*\left(\underline{\mathbf{b}}_1^T + \mathrm{h}_{\alpha,x_1} \underline{\boldsymbol{\gamma}}_\alpha^T\right) \\ \underline{\mathbf{0}} & c*\left(\underline{\mathbf{b}}_3^T + \mathrm{h}_{\alpha,x_3} \underline{\boldsymbol{\gamma}}_\alpha^T\right) & c*\left(\underline{\mathbf{b}}_2^T + \mathrm{h}_{\alpha,x_2} \underline{\boldsymbol{\gamma}}_\alpha^T\right) \end{bmatrix} \quad \text{où} \quad c = 0{,}45$$

- For elements SHB15 or SHB20, the matrix $\underline{\underline{\mathbf{B}}}_2$ need for modification does not have. It thus takes, the following form:

$$\underline{\underline{\mathbf{B}}}_2 = \begin{bmatrix} \underline{\mathbf{b}}_1^T + \mathrm{h}_{\alpha,x_1} \underline{\boldsymbol{\gamma}}_\alpha^T & \underline{\mathbf{0}} & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \underline{\mathbf{b}}_2^T + \mathrm{h}_{\alpha,x_2} \underline{\boldsymbol{\gamma}}_\alpha^T & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \underline{\mathbf{0}} & \underline{\mathbf{b}}_3^T + \mathrm{h}_{\alpha,x_3} \underline{\boldsymbol{\gamma}}_\alpha^T \\ \underline{\mathbf{b}}_2^T + \mathrm{h}_{\alpha,x_2} \underline{\boldsymbol{\gamma}}_\alpha^T & \underline{\mathbf{b}}_1^T + \mathrm{h}_{\alpha,x_1} \underline{\boldsymbol{\gamma}}_\alpha^T & \underline{\mathbf{0}} \\ \left(\underline{\mathbf{b}}_3^T + \mathrm{h}_{\alpha,x_3} \underline{\boldsymbol{\gamma}}_\alpha^T\right) & \underline{\mathbf{0}} & \left(\underline{\mathbf{b}}_1^T + \mathrm{h}_{\alpha,x_1} \underline{\boldsymbol{\gamma}}_\alpha^T\right) \\ \underline{\mathbf{0}} & \left(\underline{\mathbf{b}}_3^T + \mathrm{h}_{\alpha,x_3} \underline{\boldsymbol{\gamma}}_\alpha^T\right) & \left(\underline{\mathbf{b}}_2^T + \mathrm{h}_{\alpha,x_2} \underline{\boldsymbol{\gamma}}_\alpha^T\right) \end{bmatrix}$$

The element available to date in *Aster* is programmed in small rotations. Indeed, the increment of deformation is calculated by using only the linear deformation:

$$\Delta \boldsymbol{E}=\frac{1}{2}\left(\nabla_1(\Delta \boldsymbol{u})+\nabla_1^T(\Delta \boldsymbol{u})\right)$$

The operator gradient is calculated on the geometry of beginning of step. This writing of the deformation is limited to small rotations (lower than 5 degrees).

One could without difficulty of extending the formulation to great rotations by including in the deformation the terms of second order (tensor of Green-Lagrange):

$$\Delta \boldsymbol{E}=\frac{1}{2}\left(\nabla_1(\Delta \boldsymbol{u})+\nabla_1^T(\Delta \boldsymbol{u})+\nabla_1^T(\Delta \boldsymbol{u})\cdot\nabla_1(\Delta \boldsymbol{u})\right)$$

The associated tensor of constraint is the second tensor of Piola Kirchhoff II [R5.03.22]. But this is not available in version 12 of Code_Aster.

In elasticity, the law of behavior is written:

$$\Delta \underline{\underline{C}}=\underline{\underline{C'}}\Delta \underline{\underline{E}}$$

where C' is the matrix of Hooke. Let us notice that for the elements SHB, this matrix is a transverse orthotropic matrix which is written in the axes of the lamina:

$$\underline{\underline{\mathbf{C}}}'=\begin{bmatrix}\lambda+2\mu & \mu & 0 & 0 & 0 & 0\\ \mu & \lambda+2\mu & 0 & 0 & 0 & 0\\ 0 & 0 & E & 0 & 0 & 0\\ 0 & 0 & 0 & \mu & 0 & 0\\ 0 & 0 & 0 & 0 & \mu & 0\\ 0 & 0 & 0 & 0 & 0 & \mu\end{bmatrix}$$

4.2 Non-linearities materials

Into non-linear materials, we propose a method of particular construction of the tangent matrix $\underline{\underline{C}}^T$. It consists in supposing initially that the element is in a state of plane constraint in the local reference mark of each point of integration of Gauss and the deformations except plan are elastic. That involves then immediately that the total deflections except plan are equal to the elastic strain. Let us call $\underline{\underline{C}}^{CPT}$ the tangent matrix in plane constraints. The tangent matrix of behavior for the selected behavior and is written:

$$\underline{\underline{C}}^T = \begin{bmatrix} C^{CPT}_{xxxx} & C^{CPT}_{xxyy} & 0 & C^{CPT}_{xxxy} & 0 & 0 \\ C^{CPT}_{xyyx} & C^{CPT}_{yyyy} & 0 & C^{CPT}_{yyxy} & 0 & 0 \\ 0 & 0 & E & 0 & 0 & 0 \\ C^{CPT}_{xyxx} & C^{CPT}_{xyyy} & 0 & C^{CPT}_{xyxy} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix}$$

Then the constraints except plan are calculated in an elastic way. This method thus makes it possible to connect the elements `SHB` with all the laws of behavior available in *Code Aster*.

5 Establishment of elements SHB in *Code_Aster*

5.1 Description

These elements are pressed on the voluminal meshs 3D PENTA6, HEXA8, PENTA15 and HEXA20.

5.2 Use

These elements are used in the following way:

5.2.1 Grid

To check the good orientation of the faces of the elements indicated (compatibility with the privileged direction) while using ORIE_SHB of the operator MODI_MAILLAGE.

5.2.2 Modeling

The name of modeling SHB was preserved. It is of course an abuse language, this modeling now gathering the 4 finite elements SHB6, SHB8, SHB15, SHB20.

5.2.3 Material

For a homogeneous isotropic elastic behavior in the thickness one uses the keyword ELAS in DEFI_MATERIAU where the coefficients are defined E, Young modulus and NAKED, Poisson's ratio.

To define a plastic behavior the keyword is used TRACTION in DEFI_MATERIAU where one defines the name of a traction diagram. Only this kind of definition is available for the moment.

It should be noted that thermal dilation is not taken into account in version 12 of Code_Aster for elements SHB.

5.2.4 Boundary conditions and loading

One imposes the boundary conditions on the degrees of freedom of volume 3D (AFFE_CHAR_MECA / DDL_IMPO), and efforts in the total reference mark (FORCE_NODALE).

One defines the efforts of pressure distributed on the faces of the element (under the keyword PRES_REP). One will have taken first care to define meshs of skin QUAD4 and to suitably direct the outgoing normals with these meshs of skin using the order
MODI_MAILLAGE keyword ORIE_PEAU_3D

5.2.5 Calculation in linear elasticity

Order MECA_STATIQUE
The options of postprocessing available are SIEF_ELNO and SIEQ_ELNO.

5.2.6 Calculation in linear buckling

The option RIGI_MECA_GE being activated in the catalogue of the element, it is possible to carry out a classical calculation of buckling after assembly of the matrices of elastic and geometrical rigidity.

5.2.7 Calculation in geometrical nonlinear "elasticity"

The behavior is chosen ELAS under the keyword BEHAVIOR of STAT_NON_LINE, in small deformations ('SMALL') or in great displacements and small rotations ('GROT_GDEP') under the keyword DEFORMATION. In this last case, only the geometry is brought up to date at the beginning of step of time, the behavior remains calculated in small deformations.

The strategy used being based on the use of a matrix of tangent rigidity during iterations (reactualization at the beginning of step only), one will take care to use another option only that which is activated by default, namely REAC_ITER = 0 under NEWTON.

Digital integration in the thickness is carried out with 5 points of Gauss, just like in nonlinear material.

5.2.8 Calculation nonlinear plastic

Only the criterion of Von Mises is available to date (RELATION = 'VMIS_ISOT_TRAC' under BEHAVIOR). One defines the way of calculating of the deformations as in the case of nonlinear elasticity (DEFORMATION = 'GROT_GDEP' or 'SMALL').

The strategy used being based on the use of a matrix of tangent rigidity during iterations (reactualization at the beginning of step only), one will take care to use another option only that which is activated by default, namely REAC_ITER = 0 under NEWTON.

5.3 Establishment

Options RIGI_MECA, RIGI_MECA_GE, FORC_NODA, FULL_MECA, RIGI_MECA_TANG, RAPH_MECA, SIEF_ELGA, SIEF_ELNO were activated in the catalogue gener_shb3d_3.catastrophes.

No development was necessary for the compressive forces distributed and the following compressive forces. Indeed, these loadings are pressed on meshs of skin identical to those of the voluminal elements 3D.

5.4 Validation

The tests validating these elements are:

Tests into linear:

- SSLS101 C, D, K, L: circular plate simply posed subjected to a uniform pressure [V3.03.101]
  Modeling C: SHB8, Modeling D: SHB20, Modeling K: SHB6, Modeling L: SHB15.
- SSLS105 C: hemisphere doubly pinch [V3.03.105] classical test to check the convergence of element (SHB8)
- SSLS108 C with H: beam bored in inflection, test allowing to check the absence of blocking [V3.03.108]
  Modelings C, D: SHB8, Modeling G: SHB20, Modelings E, F: SHB6, Modeling H: SHB15.
- SSLS123 a: sphere under external pressure [V3.03.123] to validate the loadings of pressure and the orthotropic behavior particular to this element
  Modeling A: SHB8, Modelings C, D: SHB6.
- SSLS124 A with G: thin section in inflection with various twinges, to delimit the field of use of the element [V3.03.124].
  Modelings A, B: SHB8, Modeling C: SHB6, Modelings D, E: SHB20, Modelings F, G: SHB15.
- SSLS125 A, b: buckling (modes of Euler) of a free cylinder under external pressure [V3.03.125] this test makes it possible to validate the geometrical nature of rigidity
  Modeling A: SHB8, Modeling b: SHB20.

Tests into nonlinear:

- SSNS101 C with G: breakdown of a cylindrical roof [V6.03.101]. This test makes it possible to validate geometrical nonlinear calculation and elastoplasticity
Modelings C, D: SHB8, Modeling E: SHB20, Modeling F: SHB6, Modeling G: SHB15.

- SSNS102 A, b: buckling of a hull with stiffeners in great displacements and following pressure [V6.03.102].
Modeling A: SHB8, Modeling b: SHB20.

6 Bibliography

[1] F. Abed-Meraim and A. Combescure. SHB8PS, has new adaptive, assumed strain continuum mechanics Shell element for impact analysis. *Computers and Structures* , 80:791-803, 2002.

[2] F. Abed-Meraim and A. Combescure. Stabilization of the under-integrated finite elements. Report interns n° 247, LMT of Cachan, 2001.

[3] A. Legay. An effective method of calculating for parametric study of non-linear buckling of the three-dimensional structures: application to reliability. *Doctorate of the LMT of Cachan*, 2002.

[4] T. Belytschko, J.S. - J. Ong, W.K. Liu, and J.M. Kennedy. Hourglass control in linear and nonlinear problems. *Comput. Methods Appl. Mech. Eng.,* 43:251 - 276, 1984.

[5] NR. Caironi and F. Abed-Meraim. Implementation of element SHB6 to integration reduced in the INCA code and analyzes instabilities of the hourglass type. *Report of project of end of studies at laboratory LPMM, ENSAM Metz*, June 2003.

Description of the versions of the document:

Version Aster	Author (S), Organization (S)	Description of the modifications
9.5	Trinh Vuong Dieu (thesis) X Desroches EDF R & D AMA	Initial version