Element of beam to 7 degrees of freedom for the taking into account of warping

Summary:

This document presents the element POU_D_TG who is a finite element of right beam with taking into account of the warping of the sections. It allows the calculation of the beams mean transverse sections and opened profile, with constrained or free torsion.

With regard to the inflection, the normal efforts and cutting-edges, this element is based on the element POU_D_T, which is an element of right beam with transverse shearing (model of Timoshenko).

For the element POU_D_TG, the section is supposed to be constant (of an unspecified form) and the material is homogeneous and isotropic, of linear elastic behavior.
The element POU_D_TGM is to be used for the non-linear behaviors.

This reference material is based on the general reference material of the beams, in linear elasticity [R3.08.01]. It describes specificities of the element of right beam with warping POU_D_TG.

Contents

1 Field of application

The development of the elements of beam of Timoshenko with warping (modeling POU_D_TG) in *Code_Aster* was carried out initially with an aim of calculating the behavior of the pylons. It was mainly a question of calculating formed structures by beams with open mean profile (corner), for which warping is important. The non-linear behaviors are to be used with the element POU_D_TGM (beam multifibre). These nonlinear behaviors relate only to traction, the inflection. Shearing due to the shearing action, as well as warping and the bi--moment (effort related to warping) remain dependent by an elastic behavior, fault of being able to express a nonlinear behavior on these sizes. The description of torsion with warping is valid for the use of the elements POU_D_TG and POU_D_TGM with the linear operators (MECA_STATIQUE, DYNA_LINE_TRAN,…) or not linear (STAT_NON_LINE, DYNA_NON_LINE,…).

2 Notations

The notations used here correspond to those used in [R3.08.01] and [R3.08.03]. One gives here the correspondence between this notation and that of the documentation of use. DX , DY , DZ , DRX , DRY , DRZ et GRX are the names of the degrees of freedom associated with the components with displacement u, v, w, θ_x, θ_y, θ_z, $\theta_{x,x}$. They are expressed in total reference mark, except the degree of freedom associated with warping GRX, which is expressed in local reference mark.

Notation used	Significance	Notation of documentation of use
S	surface of the section	A
I_y, I_z	geometrical moments of inflection compared to the axes x and y .	IY , IZ
C	constant of torsion	JX
I_ω	constant of warping	JG
k_y, k_z	coefficients of shearing	$\frac{1}{AY}$ $\frac{1}{AZ}$
e_y, e_z	offsetting of the center of torsion/shearing compared to the centre of gravity of the cross-section	EY , EZ
N	normal effort with the section	N
V_y, V_z	efforts cutting-edges along the axes y and z	VY , VZ
M_x, M_y, M_z	moments around the axes x , y and z	MT , MFY , MFZ
M_ω	bi--moment	BX
u, v, w	translations on the axes x , y and z	DX DY DZ
θ_x, θ_y, θ_z	rotations around the axes x , y and z	DRX DRY DRZ
$\theta_{x,x}$	rotary derivative of torsion according to x	GRX
E	Young modulus	E
ν	Poisson's ratio	NU
$G=\frac{E}{2(1+\nu)}=\mu$	module of Coulomb (identical to the coefficient of Lamé)	G

3 Kinematics specific to torsion with warping

Kinematics used to represent the displacement of the sections of beam is identical to that of the right beams of Timoshenko [R3.08.01] with regard to the traction and compression, and the inflection - shearing. Only torsion here is detailed.

Two possibilities are to be considered for the modeling of behaviour in torsion of the noncircular sections [feeding-bottle 1], which always produces a warping of the cross-section.

- Torsion is free (torsion of Saint-Coming) : the warping of the cross-sections is nonnull (it can even be important for an open mean section), but it is independent of the position on the axis x beam, (constant according to x) and there is no axial stress which had with torsion.
- Torsion is constrained (Vlassov): warping is nonnull, and moreover of nonuniform axial stresses (from which the effort resulting bi--moment is called) exist in the beam.

The element `POU_D_TG` allows to treat these two configurations: torsion can be free or constrained. The user will have access to warping in both cases, on the other hand the bi--moment will be nonnull only in the case of constrained torsion. It should be noted that at the place of the connection of the beams, the transmission of warping depends on the type of connection. In general, torsion in an assembly of beams is constrained. Warping can then be blocked at the points of connection.

Note:
With elements without modeling of warping (`POU_D_T` , `POU_D_E`), one can treat the case of free torsion (displacements other than warping will be correct), but not the case of constrained torsion.

One can uncouple the effects of torsion and inflection in a local reference mark (relocated principal reference mark of inertia) having for origin the center from torsion. The center of torsion is the point which remains fixed when the section is subjected to the only torque. It is also called center of shearing because an effort applied in this point does not produce rotation around x .

Displacements in the plan of the section will thus be expressed in this reference mark. Axial displacements remain expressed in the principal reference mark of inertia related to the centre of gravity G , to keep a decoupling of displacements of inflection and traction and compression.

The displacement of an unspecified point of the cross-section is written then in general form (free or constrained torsion):

$$\begin{Bmatrix} u(x,y,z) \\ v(x,y,z) \\ w(x,y,z) \end{Bmatrix} = \begin{Bmatrix} u_G(x) \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} z\,\theta_y(x) \\ 0 \\ w(x) \end{Bmatrix} + \begin{Bmatrix} -y\,\theta_z(x) \\ v(x) \\ 0 \end{Bmatrix} + \begin{Bmatrix} \omega(y,z)\,\theta_{x,x}(x) \\ -(z-z_c)\,\theta_x(x) \\ (y-y_c)\,\theta_x(x) \end{Bmatrix}$$

Displacement = membrane + inflection y + inflection z + torsion with warping

The components are expressed in the principal reference mark of inertia (centered in G): x is directed along the axis of the beam, y and z are the two other main axes of inertia.

The term $\omega(y,z)\theta_{x,x}(x)$ represent axial displacement due to the warping of the cross-section. $\omega(y,z)$ is the function of warping (expressed in m^2 , but which does not have obvious physical interpretation).

The deformations of an unspecified point of the section are then:

$$\begin{Bmatrix} \varepsilon_{xx}(x,y,z) \\ 2\varepsilon_{xy}(x,y,z) \\ 2\varepsilon_{xz}(x,y,z) \end{Bmatrix} = \begin{Bmatrix} u_{G,x}(x) \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} z\,\theta_{y,x}(x) \\ 0 \\ \gamma_{xz}(x) \end{Bmatrix} + \begin{Bmatrix} -y\,\theta_{z,x}(x) \\ \gamma_{xy}(x) \\ 0 \end{Bmatrix} + \begin{Bmatrix} \omega(y,z)\,\theta_{x,xx}(x) \\ \left(\omega_{,y}-(z-z_c)\right)\theta_{x,x}(x) \\ \left(\omega_{,z}+(y-y_c)\right)\theta_{x,x}(x) \end{Bmatrix}$$

$$\gamma_{xy}(x) = v_{,x} - \theta_z$$

$$\gamma_{xz}(x) = w_{,x} + \theta_y$$

$$\text{Déformation} = \text{membrane} + \text{flexion}/y + \text{flexion}/z + \text{torsion avec gauchissement}$$

The term $\omega(y,z)\theta_{x,xx}(x)$ is null in the case of free torsion: one has indeed $\theta_{x,xx}(x)=0$, since warping is independent of x . It is considerable in the case of constrained torsion.

The law of elastic behavior isotropic is written (by making the assumption of the plane constraints in the directions y and z):

$$\begin{Bmatrix} \sigma_{xx}(x,y,z) \\ \sigma_{xy}(x,y,z) \\ \sigma_{xz}(x,y,z) \end{Bmatrix} = \begin{Bmatrix} E.\varepsilon_{xx}(x,y,z) \\ G.2\varepsilon_{xy}(x,y,z) \\ G.2\varepsilon_{xz}(x,y,z) \end{Bmatrix}$$

The efforts generalized in the section are expressed according to the constraints for a homogeneous section by [feeding-bottle 1]:

$$N(x) = \int_S \sigma_{xx}(x,y,z)\,ds$$ Normal effort

$$V_y(x) = \int_S \sigma_{xy}(x,y,z)\,ds$$ Shearing action according to y

$$V_z(x) = \int_S \sigma_{xz}(x,y,z)\,ds$$ Shearing action according to z

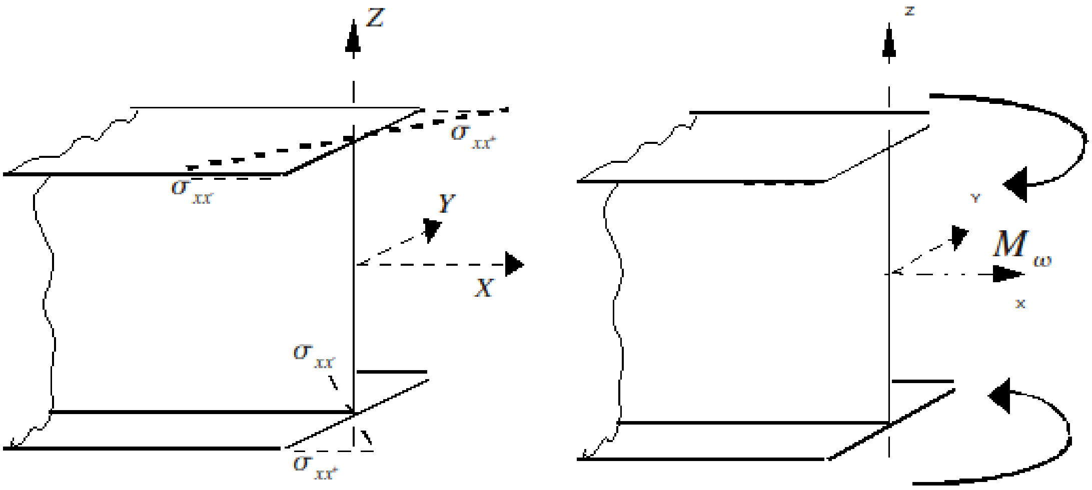
$$M_y(x) = \int_S z.\sigma_{xx}(x,y,z)\,ds$$ Bending moment around y

$$M_z(x) = \int_S -y.\sigma_{xx}(x,y,z)\,ds$$ Bending moment around z

$$M_x(x) = \int_S \left((y-y_c).\sigma_{xz}(x,y,z) - (z-z_c).\sigma_{xy}(x,y,z)\right)ds$$ Torque

$$M_\omega(x) = \int_S \omega.\sigma_{xx}(x,y,z)\,ds$$ Bi--moment (associate with warping)

$M_\omega(x)$ represent the generalized effort associated with warping. It is expressed in $N.m^2$. One can give of it an illustration as in [feeding-bottle 1] for a beam with section in I (the bi--moment acts here according to z only):



For an isotropic and homogeneous elastic behavior in the section, the generalized efforts are thus expressed directly according to displacements by the following relations:

$$N(x)=E.S.u_{,x}$$
$$V_y(x)=Gk_y S\left(v_{,x}-\theta_z\right)$$
$$V_z(x)=Gk_z S\left(w_{,x}+\theta_y\right)$$
$$M_y(x)=E.I_y\theta_{y,x}$$
$$M_z(x)=E.I_z\theta_{z,x}$$
$$M_x(x)=G.J.\theta_{x,x}$$
$$M_\omega(x)=E.I_\omega.\theta_{x,xx}$$

where k_y, k_z are the coefficients of shearing. Warping does not intervene on the level of the efforts cutting-edges, because those are expressed in the reference mark related to the center of shearing. Indeed, the function of warping ω is such as:

$$\int_S \omega\left(y,z\right)ds=0$$
$$\int_S y.\omega\left(y,z\right)ds=0$$
$$\int_S z.\omega\left(y,z\right)ds=0$$

And the constant of warping is expressed according to ω by: $\int_S \omega^2\left(y,z\right)ds=I_\omega$

4 Element of right beam with warping: matrices of rigidity and mass

Elementary matrices of rigidity and mass for the element POU_D_TG are identical to those of the element of right beam of Timoshenko (POU_D_T) with regard to the terms of traction and compression and inflection - shearing [R3.08.01]. The approach is identical, one points out simply the result.

This implies that, in the case of free torsion, one preserves the properties of exactitude of the solution at the nodes for the degrees of freedom of inflection and traction and compression.

On the other hand, we will see that with regard to obstructed torsion, one carries out an approximation which does not make it possible to find this property in the case general.

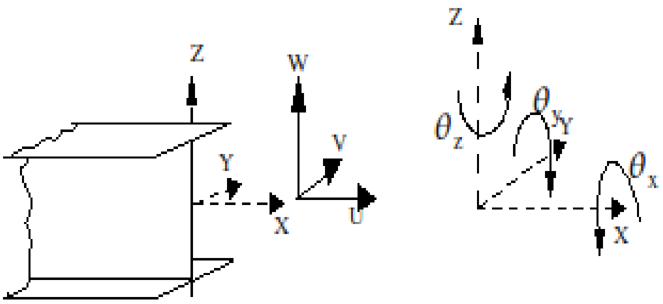
The matrices of rigidity are always calculated with the option 'RIGI_MECA', and matrices of mass with the option 'MASS_MECA'. But the option 'MASS_MECA_DIAG' (matrix of diagonalized mass) was not realized for this element (this option is especially useful for the fast problem of dynamics, which is not the preferential field of application of this element).

The degrees of freedom of the element are those of the beams of Timoshenko, plus a degree of freedom per node making it possible to calculate the terms relating to warping:

In each of the two nodes of the element, the degrees of freedom are:

u, v, w	translations on the axes x, y, z	$DX\ DY\ DZ$
$\theta_x, \theta_y, \theta_z$	rotations around the axes x, y, z	$DRX\ DRY\ DRZ$
$\theta_{x,x}$	rotary derivative of torsion according to x	GRX

The local coordinates are expressed in the principal reference mark of inertia. The element POU_D_TG thus comprise 14 degrees of freedom. The element of reference is defined by: $0<x<L$



4.1 Traction and compression the degree of freedom are u or DX

The matrix of rigidity of the element is: $K = \dfrac{ES}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

The matrix of mass (coherent) is written: $M = \dfrac{\rho\ SL}{6}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

4.2 Inflection in the plan Gxz, degrees of freedom: Ω, Θthere or DZ, Dry

The matrix of rigidity is written for the movement of inflection in the principal plan of inertia (Gxz) :

$$\mathbf{K}=\frac{12\ EI_y}{L^3\left(1+\varphi_y\right)}\begin{pmatrix} 1 & -\frac{L}{2} & -1 & -\frac{L}{2} \\ & \frac{\left(4+\varphi_y\right)L^2}{12} & \frac{L}{2} & \frac{\left(2-\varphi_y\right)L^2}{12} \\ \text{Sym} & & 1 & \frac{L}{2} \\ & & & \frac{\left(4+\varphi_y\right)L^2}{12} \end{pmatrix}$$

Transverse shearing is taken into account by the term: $\varphi_y=\dfrac{12\,EI_y}{k_z SGL^2}$

For the matrix of mass, $w\left(x,\ t\right)$ and $\theta_y\left(x,\ t\right)$ are discretized on the basis of function tests introduced for the calculation of the matrix of rigidity, that is to say:

$$w(x,\ t)=\xi_1(x)w_1(t)+\xi_2(x)\theta_{y_1}(t)+\xi_3(x)w_2(t)+\xi_4(x)\theta_{y_2}(t)$$
$$\theta_y(x,\ t)=\xi_5(x)w_1(t)+\xi_6(x)\theta_{y_1}(t)+\xi_7(x)w_2(t)+\xi_8(x)\theta_{y_2}(t)$$

The function of interpolation used for the translations (ξ_1 with ξ_4) are polynomials of Hermit of degree 3, that which are used for rotations (ξ_5 with ξ_8) are of degree 2: for $0<x<L$, they are defined by [R3.08.01]:

$$\begin{aligned}
&\xi_1(x)=\frac{1}{1+\varphi_y}\left[2\left(\frac{x}{L}\right)^3-3\left(\frac{x}{L}\right)^2-\varphi_y\frac{x}{L}+\left(1+\phi_y\right)\right] && \xi_5(x)=\frac{6}{L\left(1+\varphi_y\right)}\frac{x}{L}\left[1-\frac{x}{L}\right]\\
&\xi_2(x)=\frac{L}{1+\varphi_y}\left[-\left(\frac{x}{L}\right)^3+\frac{4+\varphi_y}{2}\left(\frac{x}{L}\right)^2-\frac{2+\varphi_y}{2}\left(\frac{x}{L}\right)\right] && \xi_6(x)=\frac{1}{1+\varphi_y}\left[3\left(\frac{x}{L}\right)^2-\left(4+\varphi_y\right)\left(\frac{x}{L}\right)+\left(1+\varphi_y\right)\right]\\
&\xi_3(x)=\frac{1}{1+\varphi_y}\left[-2\left(\frac{x}{L}\right)^3+3\left(\frac{x}{L}\right)^2+\varphi_y\left(\frac{x}{L}\right)\right] && \xi_7(x)=\frac{-6}{L\left(1+\varphi_y\right)}\frac{x}{L}\left[1-\frac{x}{L}\right]\\
&\xi_4(x)=\frac{L}{1+\varphi_y}\left[-\left(\frac{x}{L}\right)^3+\frac{2-\varphi_y}{2}\left(\frac{x}{L}\right)^2+\frac{\varphi_y}{2}\left(\frac{x}{L}\right)\right] && \xi_8(x)=\frac{1}{1+\varphi_y}\left[3\left(\frac{x}{L}\right)^2+\left(-2+\varphi_y\right)\left(\frac{x}{L}\right)\right]
\end{aligned}\qquad [1]$$

The form of the matrix of mass is:

$$\mathbf{M}=\frac{\rho S}{\left(1+\phi_y\right)^2}\begin{pmatrix} \frac{13L}{35}+\frac{7L\phi_y}{10}+\frac{L\phi_y^2}{3} & -\frac{11L^2}{210}-\frac{11L^2\phi_y}{120}-\frac{L^2\phi_y^2}{24} & \frac{9L}{70}+\frac{3L\phi_y}{10}+\frac{L\phi_y^2}{6} & \frac{13L^2}{420}+\frac{3L^2\phi_y}{40}+\frac{L^2\phi_y^2}{24} \\ & \frac{L^3}{105}+\frac{L^3\phi_y}{60}+\frac{L^3\phi_y^2}{120} & -\frac{13L^2}{420}-\frac{3L^2\phi_y}{40}-\frac{L^2\phi_y^2}{24} & -\frac{L^3}{140}-\frac{L^3\phi_y}{60}-\frac{L^3\phi_y^2}{120} \\ & & \frac{13L}{35}+\frac{7L\phi_y}{10}+\frac{L\phi_y^2}{3} & -\frac{11L^2}{210}+\frac{11L^2\phi_y}{120}+\frac{L^2\phi_y^2}{24} \\ sym & & & \frac{L^3}{105}+\frac{L^3\phi_y}{60}+\frac{L^3\phi_y^2}{120} \end{pmatrix}$$

$$+\frac{\rho I_y}{\left(1+\phi_y\right)^2}\begin{pmatrix} \frac{6}{5L} & -\frac{1}{10}+\frac{\phi_y}{2} & -\frac{6}{5L} & -\frac{1}{10}+\frac{\phi_y}{2} \\ & \frac{2L}{15}+\frac{L\phi_y}{6}+\frac{L\phi_y^2}{3} & \frac{1}{10}-\frac{\phi_y}{2} & -\frac{L}{30}-\frac{L\phi_y}{6}+\frac{L\phi_y^2}{6} \\ & & \frac{6}{5L} & \frac{1}{10}-\frac{\phi_y}{2} \\ sym & & & \frac{2L}{15}+\frac{L\phi_y}{6}+\frac{L\phi_y^2}{3} \end{pmatrix}$$

4.3 Inflection in the plan Gxy, degrees of freedom: Γ, ΘZ or Dthere, Drz

In the same way, for the movement of inflection around the axis (Gz), in the principal plan of inertia (Gxy), the matrix of rigidity is written:

$$\mathbf{K}=\frac{12\ EI_z}{L^3\left(1+\varphi_z\right)}\begin{pmatrix} 1 & \frac{L}{2} & -1 & \frac{L}{2} \\ & \frac{\left(4+\varphi_z\right)L^2}{12} & -\frac{L}{2} & \frac{\left(2-\varphi_z\right)L^2}{12} \\ & & 1 & -\frac{L}{2} \\ \text{sym} & & & \frac{\left(4+\varphi_z\right)L^2}{12} \end{pmatrix}$$

Transverse shearing is taken into account by the term:

$$\varphi_z=\frac{12EI_z}{k_y SGL^2}$$

To calculate the matrix of mass, $v(x,\ t)$ and $\theta_z(x,\ t)$ are discretized by:

$$v(x,\ t)=\xi_1(x)\ v_1(t)-\xi_2(x)\theta_{z_1}(t)+\ \xi_3(x)\ v_2(t)-\xi_4(x)\theta_{z_2}(t)$$
$$\theta_z(x,\ t)=-\xi_5(x)\ v_1(t)+\xi_6(x)\theta_{z_1}(t)-\xi_7(x)\ v_2(t)+\xi_8(x)\theta_{z_2}(t)$$

We obtain the matrix of following mass then:

$$\mathbf{M}=\frac{\rho S}{\left(1+\varphi_z\right)^2}\begin{pmatrix} \frac{13L}{35}+\frac{7L\varphi_z}{10}+\frac{L\varphi_z^2}{3} & \frac{11L^2}{210}+\frac{11L^2\varphi_z}{120}+\frac{L^2\varphi_z^2}{24} & \frac{9L}{70}+\frac{3L\varphi_z}{10}+\frac{L\varphi_z^2}{6} & -\frac{13L^2}{420}-\frac{3L^2\varphi_z}{40}-\frac{L^2\varphi_z^2}{24} \\ & \frac{L^3}{105}+\frac{L^3\varphi_z}{60}+\frac{L^3\varphi_z^2}{120} & \frac{13L^2}{420}+\frac{3L^2\varphi_z}{40}+\frac{L^2\varphi_z^2}{24} & -\frac{L^3}{140}-\frac{L^3\varphi_z}{60}-\frac{L^3\varphi_z^2}{120} \\ & & \frac{13L}{35}+\frac{7L\varphi_z}{10}+\frac{L\varphi_z^2}{3} & \frac{11L^2}{210}-\frac{11L^2\varphi_z}{120}-\frac{L^2\varphi_z^2}{24} \\ sym & & & \frac{L^3}{105}+\frac{L^3\varphi_z}{60}+\frac{L^3\varphi_z^2}{120} \end{pmatrix}$$

$$+\frac{\rho I_z}{\left(1+\varphi_z\right)^2}\begin{pmatrix} \frac{6}{5L} & \frac{1}{10}-\frac{\varphi_z}{2} & -\frac{6}{5L} & \frac{1}{10}-\frac{\varphi_z}{2} \\ & \frac{2L}{15}+\frac{L\varphi_z}{6}+\frac{L\varphi_z^2}{3} & -\frac{1}{10}+\frac{\varphi_z}{2} & -\frac{L}{30}-\frac{L\varphi_z}{6}+\frac{L\varphi_z^2}{6} \\ & & \frac{6}{5L} & -\frac{1}{10}+\frac{\varphi_z}{2} \\ sym & & & \frac{2L}{15}+\frac{L\varphi_z}{6}+\frac{L\varphi_z^2}{3} \end{pmatrix}$$

4.4 Torsion and warping, degrees of freedom: Θx, Θx, x or Dx-ray, Gx-ray

With regard to torsion, the formulation is obviously different from that of the beams without warping of the reference [R3.08.01]. The virtual work of the interior efforts is written for torsion [feeding-bottle 1]:

$$W_{\text{int}}=\int_o^L \theta^*_{x,x}.G.J.\theta_{x,x}+\theta^*_{x,xx}.E.I_\omega.\theta_{x,xx}\,dx$$

The functions of interpolation of the rotation of torsion must be of class $C2$, since they must make it possible to interpolate the derivative second of rotation.

By using the equilibrium equations, one shows in [feeding-bottle 1] that the analytical solution utilizes function of interpolation hyperbolic in x . This then makes it possible to get exact results with the nodes. It is not the choice made for *Code_Aster* : one chose, by preoccupation with a simplicity for digital integration like avoiding the digital problems of evaluation of the function hyperbolic, of the polynomials of degree 3 of type Hermit, of the same kind as those used for the inflection [éq 1]. One writes them here on the element of reference [- 1.1] according to [feeding-bottle 1] (instead of $0<x<L$ previously):

$$\begin{matrix}\xi=\frac{2x}{L}-1 \\ -1\le\xi\le 1\end{matrix}\;,\quad \begin{aligned} N_1(\xi)&=\frac{1}{4}(1-\xi)^2(2+\xi) \\ N_2(\xi)&=\frac{L}{8}(1-\xi)(1-\xi^2) \\ N_3(\xi)&=\frac{1}{4}(1+\xi)^2(2-\xi) \\ N_4(\xi)&=\frac{L}{8}(1+\xi)(-1+\xi^2)\end{aligned}$$

The interpolation for the rotation of torsion and its derivative is:

$$\theta_x(\xi)=N_1(\xi)\,\theta^1_x+N_2(\xi)\,\theta^1_{x,x}+N_3(\xi)\,\theta^2_x+N_4(\xi)\,\theta^2_{x,x}$$
$$\theta_{x,x}(\xi)=N_{1,x}(\xi)\,\theta^1_x+N_{2,x}(\xi)\,\theta^1_{x,x}+N_{3,x}(\xi)\,\theta^2_x+N_{4,x}(\xi)\,\theta^2_{x,x}$$

The reference [feeding-bottle 1] note which this approximation corresponds to a borderline case of the hyperbolic interpolation, obtained for $\sqrt{\frac{GJ}{EI_{\omega}}}\rightarrow 0$. However, this parameter not being without dimension, it is difficult to define a priori the values for which the approximation is acceptable. The digital tests carried out show that one converges quickly towards the solution when the size of the elements decreases.
The matrix of rigidity corresponding to this approximation is written then:

$$K=K_T+K_{\omega}=\frac{GJ}{30L}\begin{pmatrix}36 & 3L & -36 & 6L\\ & 4L^2 & -3L & -L^2\\ & & 36 & -3L\\ \text{sym} & & & 4L^2\end{pmatrix}+\frac{EI_{\omega}}{L^3}\begin{pmatrix}12 & 6L & -12 & 6L\\ & 4L^2 & -6L & 2L^2\\ & & 12 & -6L\\ \text{sym} & & & 4L^2\end{pmatrix}$$

The matrix of mass can be obtained [feeding-bottle in several ways 1]:

- the most complete method would consist in calculating the terms of inertia with the functions of interpolation above, by taking account of the additional term:
- $W_{iner}=-\int_o^L \theta^*_{x,x}.\rho.I_{\omega}.\ddot{\theta}_{x,x}\,dx$
- in *Code_Aster*, the simplest method was selected: the matrix of mass is identical to that of the element `POU_D_T`. One preserves the already definite terms for the traction and compression and the inflection - shearing and one uses a linear approximation for torsion. The coefficients of the matrix of mass associated with warping are worthless with this approach.

4.5 Offsetting of the axis of torsion compared to the neutral axis

In the center of torsion C, the effects of inflection and torsion are uncoupled, one can thus use the results established in the preceding chapter.

Coordinates of the point C are to be provided to `AFFE_CARA_ELEM` : the components of the vector are given $\boldsymbol{GC}$ (G being the centre of gravity of the cross-section) in the principal reference mark of inertia:

$$\mathbf{GC}=\begin{pmatrix}0\\ e_y\\ e_z\end{pmatrix}$$

One can numerically determine them starting from the grid plan of the section using the operator `MACR_CARA_POUTRE` [R3.08.03].

Once the point C determined, one finds as in [R3.08.01] the components of displacement in the centre of gravity G by considering the rigid relation of body:

$$u(G)=u(C)+\mathbf{GC}\wedge\Omega \text{ with } \Omega=\begin{pmatrix}\theta_x\\ 0\\ 0\end{pmatrix} \text{ vector rotation, } \begin{cases}u_G=u_C\\ v_G=v_C+e_z\theta_x\\ w_G=w_C-e_y\theta_x\end{cases} .$$

The change of variables is in the same way written that for POU_D_T, with 2 additional degrees of freedom:

$$\begin{Bmatrix} u_{x_{c_1}} \\ u_{y_{c_1}} \\ u_{z_{c_1}} \\ \theta_{x_{c_1}} \\ \theta_{y_{c_1}} \\ \theta_{z_{c_1}} \\ \theta_{x,x_{c_1}} \\ u_{x_{c_2}} \\ u_{y_{c_2}} \\ u_{z_{c_2}} \\ \theta_{x_{c_2}} \\ \theta_{y_{c_2}} \\ \theta_{z_{c_2}} \\ \theta_{x,x_{c_2}} \end{Bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & & & \\ 0 & 1 & 0 & -e_z & 0 & 0 & 0 & & & & & & & \\ 0 & 0 & 1 & e_y & 0 & 0 & 0 & & & & 0 & & & \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & & & & & & & \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & & & & & & & \\ & & & & & & & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & & & 0 & 1 & 0 & -e_z & 0 & 0 & 0 \\ & & & & & & & 0 & 0 & 1 & e_y & 0 & 0 & 0 \\ & & & 0 & & & & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ & & & & & & & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ & & & & & & & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ & & & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{\mathbf{P}} \begin{Bmatrix} u_{x_1} \\ u_{y_1} \\ u_{z_1} \\ \theta_{x_1} \\ \theta_{y_1} \\ \theta_{z_1} \\ \theta_{x,x_1} \\ u_{x_2} \\ u_{y_2} \\ u_{z_2} \\ \theta_{x_2} \\ \theta_{y_2} \\ \theta_{z_2} \\ \theta_{x,x_2} \end{Bmatrix}$$

From the elementary matrices of mass and rigidity calculated previously in the reference mark (C,x,y,z) where the movements of inflection and torsion are uncoupled, one obtains these matrices in the reference mark related to the neutral axis (G,x,y,z) by the following transformations:

$$\mathbf{K}=\mathbf{P}^T\,\mathbf{K_c}\,\mathbf{P}$$
$$\mathbf{M}=\mathbf{P}^T\,\mathbf{M_c}\,\mathbf{P}.$$

5 Geometrical rigidity - prestressed Structure

This matrix is calculated by the option `RIGI_GEOM`. It is used to deal with problems of buckling or vibrations of prestressed structures. In the case of a prestressed structure, therefore subjected to initial efforts (known and independent of time), one cannot neglect in the equilibrium equation the terms introduced by the change of geometry of the unconstrained state in a prestressed state. This change of geometry modifies the equilibrium equation only by the addition of a function term of displacements and prestressing whose associated matrix is called geometrical matrix of rigidity and who expresses himself by:

$$W_G = \int_{V_o} \frac{\partial u_k^{3D}}{\partial x_i} \sigma_{ij}^o \frac{\partial v_k^{3D}}{\partial x_j} dV$$

where σ_{ij}^o indicate the tensor of prestressing. This term appears naturally if one introduces the tensor of the deformations of GREEN-LAGRANGE into the virtual work of the deformation:

$$E_{xx} = \varepsilon_{xx} + \eta_{xx} = \frac{\partial u_x^{3D}}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u_x^{3D}}{\partial x}\right)^2 + \left(\frac{\partial u_y^{3D}}{\partial x}\right)^2 + \left(\frac{\partial u_z^{3D}}{\partial x}\right)^2\right]$$

$$2E_{xy} = 2\varepsilon_{xy} + 2\eta_{xy} = \frac{\partial u_x^{3D}}{\partial y} + \frac{\partial u_y^{3D}}{\partial x} + \left[\frac{\partial u_x^{3D}}{\partial x}\frac{\partial u_x^{3D}}{\partial y} + \frac{\partial u_y^{3D}}{\partial x}\frac{\partial u_y^{3D}}{\partial y} + \frac{\partial u_z^{3D}}{\partial x}\frac{\partial u_z^{3D}}{\partial y}\right]$$

$$2E_{xz} = 2\varepsilon_{xz} + 2\eta_{xz} = \frac{\partial u_x^{3D}}{\partial z} + \frac{\partial u_z^{3D}}{\partial x} + \left[\frac{\partial u_x^{3D}}{\partial x}\frac{\partial u_x^{3D}}{\partial z} + \frac{\partial u_y^{3D}}{\partial x}\frac{\partial u_y^{3D}}{\partial z} + \frac{\partial u_z^{3D}}{\partial x}\frac{\partial u_z^{3D}}{\partial z}\right]$$

In the expression of these deformations, the terms quadratic $\left(\frac{\partial u_x^{3D}}{\partial x}\right)^2$, $\frac{\partial u_x^{3D}}{\partial x}\frac{\partial u_x^{3D}}{\partial y}$ et $\frac{\partial u_x^{3D}}{\partial x}\frac{\partial u_x^{3D}}{\partial z}$ are neglected here, according to the assumption usually carried out by most authors [feeding-bottle 3]. For a model of beam, the tensor of initial constraints is reduced in the local axes of the beam to the components σ_{xx}, σ_{xy} and σ_{xz}. One uses the kinematics introduced with [§2]:

$$\begin{cases} u_x^{3D}(x,y,z) &=& u_G(x) + z\,\theta_y(x) - y\,\theta_z(x) + \omega(y,z)\theta_{x,x}(x) \\ u_y^{3D}(x,y,z) &=& v_C(x) - (z - z_c)\,\theta_x(x) \\ u_z^{3D}(x,y,z) &=& w_C(x) + (y - y_c)\,\theta_x(x) \end{cases}$$

and the expression of the efforts generalized according to the constraints:

$$N^0 = \int_S \sigma_{xx}^o\, ds \qquad V_y^0 = \int_S \sigma_{xy}^o\, ds \qquad V_z^0 = \int_S \sigma_{xz}^o\, ds \qquad M_y^0 = \int_S z\,\sigma_{xx}^0\, ds \qquad M_z^0 = \int_S -y\,\sigma_{xx}^0\, ds$$

It is supposed, moreover, that N^0, V_y^0, V_z^0 are constant in the discretized element (what is inaccurate for example for a vertical beam subjected to its actual weight). The moments are supposed to vary linearly:

$$M_y^0 = \left(M_{y2}^0 - M_{y1}^0\right)\frac{x}{L} + M_{y1}^0 \qquad \frac{\partial M_y^0}{\partial x} - V_z^0 = 0$$

$$M_z^0 = \left(M_{z2}^0 - M_{z1}^0\right)\frac{x}{L} + M_{z1}^0 \qquad \frac{\partial M_z^0}{\partial x} + V_y^0 = 0$$

These assumptions make it possible to express W_G for a right beam with warping in the following way:

$$
\begin{aligned}
W_G=&\int_0^L N^0\left(v_{,x}\delta v_{,x}+w_{,x}\delta w_{,x}\right)+N^0\left(\frac{I_y+I_z}{A}+y_c^2+z_c^2\right)\theta_{x,x}\delta\theta_{x,x}\\
&+z_c N^0\left(v_{,x}\delta\theta_{x,x}+\theta_{x,x}\delta v_{,x}\right)-y_c N^0\left(w_{,x}\delta\theta_{x,x}+\theta_{x,x}\delta w_{,x}\right)\\
&-M_y^0\left(w_{,x}\delta\theta_{,x}+\theta_{,x}\delta w_{,x}\right)-M_z^0\left(v_{,x}\delta\theta_{,x}+\theta_{,x}\delta v_{,x}\right)\\
&-\left(y_c V_y^0-z_c V_z^0\right)\left(\theta_{x,x}\delta\theta_x+\theta_x\delta\theta_{x,x}\right)+V_y^0\left(w_{,x}\delta\theta_x+\theta_x\delta w_{,x}\right)\\
&-V_z^0\left(v_{,x}\delta\theta_x+\theta_x\delta v_{,x}\right)+\left(\left(2\mathrm{y}_c-\frac{I_{yr^2}}{I_z}\right)M_z^0+\left(-2\mathrm{z}_c+\frac{I_{zr^2}}{I_y}\right)M_y^0\right)\theta_{x,x}\delta\theta_{x,x}\\
&+\left(-\frac{I_{yr2}}{I_z}\frac{dM_z^0}{dx}+\frac{I_{zr2}}{I_y}\frac{dM_y^0}{dx}\right)\left(\theta_{x,x}\delta\theta_x+\theta_x\delta\theta_{x,x}\right)
\end{aligned}
$$

with the terms:

$$I_{yr^2}=\int_S y\left(y^2+z^2\right)ds$$

$$I_{zr^2}=\int_S z\left(y^2+z^2\right)ds$$

who represent it not - symmetry of the section. If the section thus has two axes of symmetry (C is confused with G), these terms are worthless.

Attention, these terms (which are named `IYR2` and `IZR2` in the order `AFFE_CARA_ELEM`) are not currently calculated by `MACR_CARA_POUTRE`. The user must thus inform them starting from values tubulate for each type of section (corner, right-angled,…).

Moreover, to be able to deal with the problems of discharge of thin beams, requested primarily by normal effort and bending moments, it is necessary to add the assumption of rotations moderated in torsion [feeding-bottle 2], [feeding-bottle 3].

This results in the following modification of the field of displacements (only for the calculation of geometrical rigidity):

$$u_x^{3\mathrm{D}}(x,y,z)=u_G(x)+z\left(\theta_y(x)+\theta_x(x)\ \theta_z(x)\right)-y\left(\theta_z(x)-\theta_x(x)\ \theta_y(x)\right)+\omega(y,z)\,\theta_{x,x}(x)$$

The origin of this expression cannot be here detailed. It is the object of the thesis of TOWN OF GOYET [feeding-bottle 2] on the buckling of the beams with open mean sections. The assumption of rotations of torsion moderate (and not infinitesimal) makes it possible to correctly model the discharge of a thin beam of section in torsion (coupling torsion - inflection).
The assumption of moderate rotations results in adding with W_G^0 the term W_G^1 :

$$W_G^1=\frac{1}{2}\int_o^L -M_z^o\delta\left(\theta_x\ \theta_{y,x}+\theta_y\ \theta_{x,x}\right)+M_y^o\delta\left(\theta_x\ \theta_{z,x}+\theta_z\ \theta_{x,x}\right)+V_y^o\ \delta\left(\theta_x\theta_y\right)+V_z^o\delta\left(\theta_x\theta_z\right)$$

Finally, one obtains the geometrical matrix of rigidity while discretizing $W_G=W_G^0+W_G^1$ using the same functions of interpolation as the matrix of rigidity of [§4.4]. For having calculated these matrices, it is necessary to carry out a change of reference mark as with [§4.5]. One then obtains a geometrical matrix of rigidity of the form:

$$\mathbf{K_G}=\begin{pmatrix}\mathbf{A_1} & \mathbf{A_2}\\ \mathbf{A_2} & \mathbf{A_3}\end{pmatrix}$$

The blocks of the matrix are clarified hereafter. One uses to simplify the expressions:

$$N_{ey}^{0}=1.2\frac{N^{o}e_{y}}{L}\qquad N_{ez}^{0}=1.2\frac{N^{o}e_{z}}{L}$$

$$\bar{M}_{y}^{0}=\frac{M_{y1}^{0}+M_{y2}^{0}}{L}\qquad \bar{M}_{z}^{0}=\frac{M_{z1}^{0}+M_{z2}^{0}}{L}$$

$$\Delta M_{y}^{0}=\frac{M_{y1}^{0}-M_{y2}^{0}}{2}\qquad \Delta M_{z}^{0}=\frac{M_{z1}^{0}-M_{z2}^{0}}{2}$$

$$\tilde{k}=N^{o}\left(\frac{I_{y}+I_{z}}{S}+e_{y}^{2}+e_{z}^{2}\right)$$

$$\tilde{I}_{y}=-\frac{I_{yr2}}{I_{z}}+2e_{y}$$

$$\tilde{I}_{z}=\frac{I_{zr2}}{I_{y}}-2e_{z}$$

A1	2 v_1	3 w_1	4 θ_{x1}	5 θ_{y1}	6 q_{z1}	7 $q_{x,x1}$
2 v_1	$1.2\dfrac{N^o}{L}$		$N^0_{ez}+\dfrac{\bar{M}^0_y}{2}+1.2\dfrac{\Delta M^0_y}{L}$		$\dfrac{N^o}{10}$	$\dfrac{e_z N^0+L\bar{M}^0_y-M^0_{y2}}{10}$
3 w_1		$1.2\dfrac{N^o}{L}$	$-N^0_{ey}+\dfrac{\bar{M}^0_z}{2}+1.2\dfrac{\Delta M^0_z}{L}$	$-\dfrac{N^o}{10}$		$\dfrac{-e_y N^0+L\bar{M}^0_z-M^0_{z2}}{10}$
4 θ_{x1}			$\dfrac{1.2}{L}\left(\tilde{k}-\Delta M^0_z\tilde{I}_y-\Delta M^0_y\tilde{I}_z\right)$ $-\left(e_y+\dfrac{I_{yr2}}{2I_z}\right)\bar{M}^0_z+\left(e_z+\dfrac{I_{zr2}}{2I_y}\right)\bar{M}^0_y$	$\dfrac{e_y N^0+L\bar{M}^0_z+M^0_{z2}}{10}$ $-\dfrac{M^0_{z1}}{2}$	$\dfrac{e_z N^0-L\bar{M}^0_y-M^0_{y2}}{10}$ $+\dfrac{M^0_{y1}}{2}$	$\dfrac{\tilde{k}+M^0_{z2}\tilde{I}_y+M^0_{y2}\tilde{I}_z}{10}$
5 θ_{y1}				$\frac{2LN^0}{15}$		$\dfrac{2e_y LN^0}{15}-\dfrac{L\left(3M^0_{z1}-M^0_{z2}\right)}{30}$
6 θ_{z1}			sym		$\frac{2LN^0}{15}$	$\dfrac{2e_z LN^0}{15}-\dfrac{L\left(3M^0_{y1}-M^0_{y2}\right)}{30}$
7 $\theta_{x,x1}$						$\dfrac{4\tilde{k}L-L\tilde{I}_y\left(3M^0_{z1}-M^0_{z2}\right)}{30}$ $-\dfrac{L\tilde{I}_z\left(3M^0_{y1}-M^0_{y2}\right)}{30}$

A2	9 v_2	10: w_2	11: θ_{x2}	12: θ_{y2}	13: θ_{z2}	14: $\theta_{x,x2}$
2 v_1	$1.2\dfrac{N^o}{L}$		$-N^0_{ez}+\dfrac{\bar{M}^0_y}{2}-1.2\dfrac{\Delta M^0_y}{L}$		$\dfrac{N^o}{10}$	$\dfrac{e_z N^0-L\bar{M}^0_y+M^0_{y1}}{10}$
3 w_1		$-1.2\dfrac{N^o}{L}$	$N^0_{ey}+\dfrac{\bar{M}^0_z}{2}-1.2\dfrac{\Delta M^0_z}{L}$	$-\dfrac{N^o}{10}$		$\dfrac{-e_y N^0-L\bar{M}^0_z+M^0_{z1}}{10}$
4 θ_{x1}		$N^0_{ey}-\dfrac{\bar{M}^0_z}{2}-1.2\dfrac{\Delta M^0_z}{L}$	**- A1 (4.4)**	$\dfrac{e_y N^0-L\bar{M}^0_z-M^0_{z1}}{10}$	$\dfrac{e_z N^0+L\bar{M}^0_y+M^0_{y1}}{10}$	$\dfrac{\tilde{k}-M^0_{y1}\tilde{I}_z-M^0_{z1}\tilde{I}_y}{10}$
5 θ_{y1}		$\dfrac{N^o}{10}$	$\dfrac{-e_y N^0-L\bar{M}^0_z-M^0_{z2}}{10}$	$-\dfrac{LN^o}{30}$		$\dfrac{-e_y LN^0+L\Delta M^0_z}{30}+\dfrac{L^2\bar{M}^0_z}{60}$
6 θ_{z1}	$-\dfrac{N^o}{10}$		$\dfrac{-e_z N^0+L\bar{M}^0_y+M^0_{y2}}{10}$		$-\dfrac{LN^o}{30}$	$\dfrac{-e_z LN^0-L\Delta M^0_y}{30}-\dfrac{L^2\bar{M}^0_y}{60}$
7 $\theta_{x,x}$	$\dfrac{-e_z N^0-L\bar{M}^0_y+M^0_{y2}}{10}$	$\dfrac{e_y N^0-L\bar{M}^0_z+M^0_{z2}}{10}$	$\dfrac{-\tilde{k}-M^0_{y2}\tilde{I}_z-M^0_{z2}\tilde{I}_y}{10}$	$\dfrac{-e_y LN^0+L\Delta M^0_z}{30}-\dfrac{L^2\bar{M}^0_z}{60}$	$\dfrac{-e_z LN^0-L\Delta M^0_y}{30}+\dfrac{L^2\bar{M}^0_y}{60}$	$\dfrac{-\tilde{k}L+L\Delta M^0_y\tilde{I}_z}{30}+\dfrac{L\Delta M^0_z\tilde{I}_y}{30}$

A3	9: v_2	10: w_2	11: θ_{x2}	12: θ_{y2}	13: θ_{z2}	14: $\theta_{x,x2}$
2 v_2	$1.2\dfrac{N^o}{L}$		$N^0_{ez}-\dfrac{\bar{M}^0_y}{2}+1.2\dfrac{\Delta M^0_y}{L}$		$-\dfrac{N^o}{10}$	$\dfrac{-e_z N^0+L\bar{M}^0_y-M^0_{y1}}{10}$
3 w_2		$1.2\dfrac{N^o}{L}$	$-N^0_{ey}-\dfrac{\bar{M}^0_z}{2}+1.2\dfrac{\Delta M^0_z}{L}$	$\dfrac{N^o}{10}$		$\dfrac{e_y N^0+L\bar{M}^0_z-M^0_{z1}}{10}$
4 θ_{x2}			$1.2\dfrac{\tilde{k}}{L}-1.2\left(\dfrac{\Delta M^0_z}{L}\tilde{I}_y+\dfrac{\Delta M^0_y}{L}\tilde{I}_z\right)+\left(e_y+\dfrac{I_{yr2}}{2I_z}\right)\bar{M}^0_z-\left(e_z+\dfrac{I_{zr2}}{2I_y}\right)\bar{M}^0_y$	$\dfrac{-e_y N^0+L\bar{M}^0_z+M^0_{z1}}{10}-\dfrac{M^0_{z2}}{2}$	$\dfrac{-e_z N^0-L\bar{M}^0_y-M^0_{y1}}{10}+\dfrac{M^0_{y2}}{2}$	$\dfrac{-\tilde{k}+M^0_{y1}\tilde{I}_z+M^0_{z1}\tilde{I}_y}{10}$
5 θ_{y2}				$\dfrac{2LN°}{15}$		$\dfrac{2e_y LN^0}{15}-\dfrac{L\left(M^0_{z1}-3M^0_{z2}\right)}{30}$
6 θ_{z2}					$\dfrac{2LN°}{15}$	$\dfrac{2e_z LN^0}{15}+\dfrac{L\left(M^0_{y1}-3M^0_{y2}\right)}{30}$
7 $\theta_{x,x2}$			Sym			$\dfrac{4\tilde{k}L-L\tilde{I}_y\left(M^0_{z1}-3M^0_{z2}\right)}{30}-\dfrac{L\tilde{I}_z\left(M^0_{y1}-3M^0_{y2}\right)}{30}$

6 Loadings

Various types of loading available for the element `POU_D_TG` are:

Types or options	
`CHAR_MECA_FR1D1D`	loading broken down by actual values
`CHAR_MECA_FF1D1D`	loading broken down by function
`CHAR_MECA_PESA_R`	loading due to gravity
`CHAR_MECA_TEMP_R`	"thermal" loading
`CHAR_MECA_EPSI_R`	loading by imposition of a deformation (of standard thermal stratification)

The loadings are in the same way calculated that for the elements without warping [R3.08.01]. There is thus nothing in particular to the element `POU_D_TG`. The other types of loading described in [R3.08.01] are not available for this element.

With regard to warping, it is possible to give boundary conditions utilizing the degree of freedom GRX (what makes it possible to model constrained torsion: $GRX=0$), but on the other hand, nothing is designed to affect a loading of type bi--moment, whose physical interpretation is difficult to establish.

In connection with connection between elements, the transmission of warping is an open-ended question as the reference [feeding-bottle announces it 1]: the continuity of the variable GRX from one element to another (on which warping depends directly) depends by way of technology on the connection between the various beams (welding in the axis, in which case warping can be transmitted completely, connection by bracket,…).

For an assembled structure such as a lattice, it seems more reasonable to suppose than torsion is obstructed, therefore that warping is null at the ends. To determine the influence of this assumption, one will be able to refer to the test SSLL102 (beam of corner section) whose modelings C and D use the element `POU_D_TG,` with free torsion for modeling C, and torsion obstructed for modeling D [V3.01.102B].
It is noted that for the loading of inflection, the variation on displacement is weak (2.5%), but for a loading in torsion, one obtains for this section a side displacement not no one (discharge) from which the value differs notably according to the assumption taken:
$u_z=2.2\ 10^{-5}$ for free torsion and $u_z=2.62\ 10^{-5}$ for constrained torsion.
In the same way, rotation strongly varies:
$\theta_x=3,79\ 10^{-4}$ for free torsion and $\theta_x=6,39\ 10^{-4}$ for constrained torsion (`GRX` is null at the ends).

6.1 Loadings distributed, options: `CHAR_MECA_FR1D1D` and `CHAR_MECA_FF1D1D`

The loadings are given under the keyword `FORCE_POUTRE`, that is to say by actual values in `AFFE_CHAR_MECA` (option `CHAR_MECA_FR1D1D`), that is to say by functions in `AFFE_CHAR_MECA_F` (option `CHAR_MECA_FF1D1D`). The loading is given only by forces distributed, not by moments distributed.
The second associated member with the loading distributed of traction and compression is:

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \text{ with } f_1=\int_0^1 f_{ext}(x)\left(1-\frac{x}{L}\right)dx$$

$$f_2=\int_0^1 f_{ext}(x)\,\frac{x}{L}\,dx$$

For a loading constant or varying linearly, one obtains:

$$F_{x_1} = L\left(\frac{n_1}{3}+\frac{n_2}{6}\right),$$
$$F_{x_2} = L\left(\frac{n_1}{6}+\frac{n_2}{3}\right).$$

n_1 and n_2 are the components of the axial loading as in points 1 and 2 coming from the data of the user replaced in the local reference mark.

If t_{y_1}, t_{y2}, t_{z1} and t_{z2} are those of the shearing action, one a:

$$\begin{aligned}
F_{y_1} &= L\left(\frac{7t_{y_1}}{20}+\frac{3t_{y_2}}{20}\right) & M_{z_1} &= L^2\left(\frac{t_{y_1}}{20}+\frac{t_{y_2}}{30}\right)\\
F_{y_2} &= L\left(\frac{3t_{y_1}}{20}+\frac{7t_{y_2}}{20}\right) & M_{z_2} &= -L^2\left(\frac{t_{y_1}}{30}+\frac{t_{y_2}}{20}\right),\\
F_{z_1} &= L\left(\frac{7t_{z_1}}{20}+\frac{3t_{z_2}}{20}\right), & M_{y_1} &= -L^2\left(\frac{t_{z_1}}{20}+\frac{t_{z_2}}{30}\right),\\
F_{z_2} &= L\left(\frac{3t_{z_1}}{20}+\frac{7t_{z_2}}{20}\right), & M_{y_2} &= L^2\left(\frac{t_{z_1}}{30}+\frac{t_{z_2}}{20}\right).
\end{aligned}$$

6.2 Loading of gravity, option: `CHAR_MECA_PESA_R`

The force of gravity is given by the module of acceleration $\mathbf{g}$ and a normalized vector $\mathbf{n}$ indicating the direction of the loading.

Remarks *(simplifying assumption)* :
The functions of form used for this calculation are those of the model Euler-Bernoulli.

The approach is similar to that used for the forces distributed, on condition that transforming initially the vector loading due to gravity in the local reference mark with the element. One obtains in the local reference mark of the beam:

$$F_{x_i} = \int_o^L \xi_i \rho S \;\; \mathbf{g}\cdot\mathbf{x}\, dx \qquad \left(\xi_1 = 1-\frac{x}{L}\ ,\ \ \xi_2 = \frac{x}{L}\right)$$

from where:

$$F_{x_1} = \rho\ \mathbf{g}\cdot\mathbf{x}\, L\left(\frac{S}{3}+\frac{S}{6}\right) \quad \text{au point 1,}$$
$$F_{x_2} = \rho\ \mathbf{g}\cdot\mathbf{x}\, L\left(\frac{S}{6}+\frac{S}{3}\right) \quad \text{au point 2}$$

Inflection in the plan (Gxz) **:**

$$F_{z_1} = \rho\mathbf{g}\cdot\mathbf{z}\, L\left(\frac{7S}{20}+\frac{3S}{20}\right)$$
$$M_{y_1} = -\rho\mathbf{g}\cdot\mathbf{z}\;\; L^2\left(\frac{S}{20}+\frac{S}{30}\right)$$
$$F_{z_2} = \rho\mathbf{g}\cdot\mathbf{z}\, L\left(\frac{3S}{20}+\frac{7S}{20}\right)$$
$$M_{y_2} = \rho\mathbf{g}\cdot\mathbf{z}\, L^2\left(\frac{S}{30}+\frac{S}{20}\right)$$

Inflection in the plan (Gxy) **:**

$$F_{y_1}=\rho\,\mathbf{g}\cdot\mathbf{y}\,L\left(\frac{7S}{20}+\frac{3S}{20}\right)$$
$$M_{z_1}=\rho\,\mathbf{g}\cdot\mathbf{y}\quad L^2\left(\frac{S}{20}+\frac{S}{30}\right)$$
$$F_{y_2}=\rho\,\mathbf{g}\cdot\mathbf{y}\,L\left(\frac{3S}{20}+\frac{7S}{20}\right)$$
$$M_{z_2}=-\rho\,\mathbf{g}\cdot\mathbf{y}\,L^2\left(\frac{S}{30}+\frac{S}{20}\right)$$

6.3 Thermal loading, option: CHAR_MECA_TEMP_R

To obtain this loading, it is necessary to calculate axial displacements induced by the difference in temperature $T-T_{\text{référence}}$:

$$u_1 = -L\alpha\left(T-T_{\text{référence}}\right)$$
$$u_2 = L\alpha\left(T-T_{\text{référence}}\right)$$

(α : thermal dilation coefficient)

Then, one calculates simply the forces induced by $\mathbf{F} = \mathbf{K}\,\mathbf{u}$.

Like K is the matrix of local rigidity to the element, one must then carry out a change of reference mark to obtain the values of the components of the loading in the total reference mark.

6.4 Loading by imposed deformation, option: CHAR_MECA_EPSI_R

One calculates as for the elements `POU_D_T` the loading starting from a state of deformation (this option was developed to take into account the thermal stratification in pipings). The model takes into account only one work in traction and compression and pure inflection (not of shearing action, not torque).

The deformation is given by the user using the keyword `PRE_EPSI` in `AFFE_CHAR_MECA`. While being given $\frac{\partial u}{\partial x}$, $\frac{\partial \theta_y}{\partial x}$ and $\frac{\partial \theta_z}{\partial x}$ on the beam, one obtains the second elementary member associated with this loading:

with node 1:

$$F_{x_1} = E S_1 \frac{\partial u}{\partial x} \ ,$$
$$M_{y_1} = E I_{y_1} \frac{\partial \theta_y}{\partial x} \ ,$$
$$M_{z_1} = E I_{z_1} \frac{\partial \theta_z}{\partial x} \ ,$$

with node 2:

$$F_{x_2} = E S_2 \frac{\partial u}{\partial x} \ ,$$
$$M_{y_2} = E I_{y_2} \frac{\partial \theta_y}{\partial x} \ ,$$
$$M_{z_2} = E I_{z_2} \frac{\partial \theta_z}{\partial x}$$

7 Torque of the efforts - nodal Forces and reactions

7.1 Options available

Various options of postprocessing available for the element POU_D_TG are:

Types or options	
EFGE_ELNO	torque of the efforts to the 2 nodes of each element
SIEF_ELGA	field of efforts necessary to the calculation of the nodal forces (option "FORC_NODA") and of the reactions (option "REAC_NODA").
FORC_NODA	nodal forces expressed in the total reference mark
REAC_NODA	nodal reactions

7.2 The torque of the efforts

7.2.1 Generalized efforts, option: EFGE_ELNO

One seeks to calculate with the two nodes of each element "beam" constituting the grid of the studied structure, the efforts exerted on the element "beam" by the rest of the structure. The values are given in the local base of each element. By integrating the equilibrium equations, one obtains the efforts in the local reference mark of the element:

$$\mathbf{R}_{LOC} = K^e_{LOC} u_{LOC} + M^e_{LOC} \ddot{u}_{LOC} - f^e_{LOC}$$

where:

$$\mathbf{R}_{LOC} = \left(-N^1, -V^1_Y, -V^1_Z, -M^1_T, -M^1_Y, -M^1_Z, -M^1_\omega, N^2, V^2_Y, V^2_Z, M^2_T, M^2_Y, M^2_Z, M^2_\omega\right)$$

$\mathbf{K}^{\mathbf{e}}_{LOC}$ elementary matrix of rigidity of the element beam,

$\mathbf{M}^{\mathbf{e}}_{LOC}$ elementary matrix of mass of the element beam,

$\mathbf{f}^{\mathbf{e}}_{LOC}$ vector of the efforts "distributed" on the element beam,

$\mathbf{u}_{LOC}$ vector "degree of freedom" limited to the element beam,

$\ddot{\mathbf{u}}_{LOC}$ vector "acceleration" limited to the element beam.

One changes then the signs of the efforts to node 1.
Indeed, by taking for example the case of the traction and compression, one shows [R3.08.01] that efforts in the element (option EFGE_ELNO) are obtained by:

$$\begin{bmatrix} -N(o) \\ N(L) \end{bmatrix} = [K] \begin{bmatrix} u(o) \\ u(L) \end{bmatrix} - \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

7.2.2 Generalized efforts, option: SIEF_ELGA

The option "SIEF_ELGA" is established for reasons of compatibility with other options. It is used only for calculation of the nodal forces. It produces fields of efforts by elements. It is calculated by:

$$\mathbf{R}_{LOC} = \mathbf{K}^{\mathbf{e}}_{LOC} \mathbf{u}_{LOC}$$

7.3 Calculation of the nodal forces and the reactions

7.3.1 Nodal forces, option: FORC_NODA

This option calculates a vector of nodal forces on all the structure, expressed in total reference mark. It produces a field with the nodes in the order CALC_CHAMP by assembly of the elementary terms.

For this calculation, the principle of virtual work is used and one writes [R5.03.01]:

$$\mathbf{F} = \mathbf{Q}^{\mathbf{T}} \sigma$$

where $\mathbf{Q}^{\mathbf{T}}$ symbolically represent the matrix associated with the operator divergence. For an element, one writes the agricultural work of virtual deformations:

$$\left(\mathbf{Q}^{\mathbf{T}}\sigma\right)u^{*}=\int_{\Omega}\sigma\left(\mathbf{u}\right)\varepsilon\left(u^{*}\right)\quad\forall u^{*}\ \textbf{kinematically acceptable}$$

For the elements of beam, one calculates simply the nodal forces by assembly of the elementary nodal forces calculated by the option SIEF_ELGA, which is expressed by:

$$\left[\mathbf{F}_{LOC}\right]=\left[\mathbf{K}_{LOC}\right]\left[\mathbf{U}_{LOC}\right]$$

7.3.2 Nodal reactions, option: REAC_NODA

This option, called by CALC_CHAMP, allows to obtain the reactions R with the supports, expressed in the total reference mark, starting from the nodal forces $\mathbf{F}$ by:

$$\mathbf{R}=\mathbf{F}-\mathbf{F}^{char}+\mathbf{F}^{iner}$$

$\mathbf{F}^{char}$ et $\mathbf{F}^{iner}$ being nodal forces respectively associated with the loadings given (specific and distributed) and with the efforts with inertia.

8 Bibliography

[1] J.L. BATOZ, G. DHATT. "Modeling of the structures by finite elements" - HERMES.

[2] V. OF TOWN OF GOYET. "Nonlinear static analysis by the finite element method of the formed space structures by beams with nonsymmetrical sections" Thesis of the University of Liege. 1989.

[3] J. SLIMI. "Simulation of ruin of pylon" Report SERAM N°14.033, ENSAM June 1993.

9 Description of the versions of the document

Version Aster	Author (S) Organization (S)	Description of the modifications
07/04/09	***J.L. FLEJOU*** , *J.M. PROIX (EDF-R&D/AMA)*	
2/22/2013	***J.L. FLEJOU***	Addition remarks on `POU_D_TGM`.