Multifibre beams in great displacements

Summary:

This document presents the treatment of great multifibre displacements of the element of beam POU_D_TGM activated by the keyword DEFORMATION='GROT_GDEP'.
It makes it possible to combine at the same time non-linearity material (description of the fibre section) and geometrical non-linearity. The description of great displacements remains approximate and makes the assumption of small steps of time.
This element is adapted for the characterization of the ruin of structures lattice.

Note: the treatment of great displacements for the other elements of beam is described in [R3.08.01].

Contents

1 Position of the problem

In the field of the modeling of the elements of beams there exist a large number of theories to describe kinematics (Euler-Bernoulli, Timoshenko, Vlassov,…) and one counts naturally in *Code_Aster* a variation of these theories in as many model finite elements (POU_D_E, POU_D_T, POU_D_TG).
In addition to the assumptions on which they are founded, these models can be nouveau riches of a nonlinear behavior that it is of origin material (criterion of plasticity for the beams, behavior 1D being based on a description by fibre) or geometrical (great displacements and great three-dimensional rotations).

This document describes the element POU_D_TGM who is based on a model of Timoshenko[1] with constrained torsion of Vlassov[2]. This element associates a description by fibre of the section to profit from a nonlinear behavior 1D in traction and compression/inflection and a kinematics from the second order to allow a description in great displacements and great rotations[3].
One will concentrate on the geometrical aspect, the behavior multifibre being approached in detail in [4].

2 Notations

The notations used here correspond to those of [2] and [4].

3 Presentation of the approach

3.1 Introduction

The taking into account of great displacements, which they result from a rigid movement of body or from an unspecified transformation of the studied structure introduced an additional non-linearity (besides that introduced by behavioral non-linearity for example). This non-linearity results in the fact that initial configuration of the structure and final configuration (or “deformation”) cannot be confused more as it is usually the case for the treatment of the problems in small disturbances.
For finite elements of structures (*i.e.* beams, plates, hulls), it poses an additional problem: rotations are not any more one vectorial, one cannot thus any more transform them like vectors at the time of the passage of the local reference mark to the total reference mark.
Indeed, when vectors displacements are expressed, in a local reference mark of a finite element (reference mark attached to SEG2 beam in which all calculations are carried out to simplify the writing of the various terms), it is completely physical to transform them in a total reference mark attached to the structure. When one is interested in elements of structure, those also carry degrees of freedom of rotation; when these rotations are “small”, one can show that they are identified with vectors and thus to transform them like displacements.
That is not any more the case when rotations are “large” and that results in the not-commutation of rotations. For lack of a description taking into account this characteristic, the continuity of the degrees of freedom to the nodes of the elements is not then assured any more[5][6].

3.2 The element geometrically exact POU_D_T_GD

The finite element of beam POU_D_T_GD, already integrated into the code since many years[7], a beam of Timoshenko in great displacements with the direction models where the field of displacements used in the formulation (at the time of the passage 3D → beams) is written in an exact way. He takes in particular account of the exact operator rotation between two configurations of the element (i.e. one makes no simplifying assumption on displacements).
The behavior, only elastic, is always written in small deformations; this element had indeed been introduced to treat great “almost rigid” displacements of a structure in dynamics (movement of the spacers between the drivers[1] electric of an airline).
This element is formulated with a Total Lagrangian approach relatively complex, the exact treatment of great rotations requiring to call on the theory of the quaternions to update correctly displacements. The

1The spacers are used to guarantee a minimal spacing between the drivers of a beam behavior is directly formulated on the efforts generalized (without passing by the constraints) and really does not lend itself to an extension multifibre which by nature works on the constraints [2] .

3.3 The element multifibre POU_D_TGM

It is an element of beam of Timoshenko with warping of the transverse section and approach multifibre to give an account of the progression of plasticity in the section[8]. This element had a generic option already 'PETIT_REAC' (now removed) able to reactualize the geometry with each iteration of the algorithm of resolution by step. That made it possible to deal with problems in great displacements under the assumption of small rotations. However the geometrical absence of rigidity in the formulation makes in the case of convergence very difficult instability and this same for problems plans. Moreover, one does not exploit all the possibilities of the reactualization of the geometry (cf. 4.4.1).

3.4 A small history

It is as from the Seventies that the geometrical nonlinear analysis of the formed structures by beams developed with the use of the finite element method. The analysis initially went on plane structures[9], then on three-dimensional structures with work founders of Bathe in 1979[10]. It is indeed at that time that for the first time a formulation appears known as " *Updated Lagrangian* " (UL), i.e. a formulation which updates the geometry of the structure at each iteration of the algorithm of resolution, allowing to obtain a formulation simplified while remaining robust.
Thereafter by many work took as a starting point those of Bathe and enriched them, in particular with the treatment of the thorny problem of the great rotations highlighted by Argyris[11]. One can also quote work of Yang and McGuire[5][12] like those of Conci[6][13].
Another important point in nonlinear analysis of the structures is the need for estimating the residue well to get right results[14]. However in the case of the elements beams, the functions of geometrical interpolation are first order (the elements beams are not isoparametric). That involves errors in the calculation of the residue if the structure becomes deformed much and if enough elements are not used.
A solution obviously consists in increasing the number of finite elements especially for curved structures. However that can become expensive and of the authors thus techniques proposed known as of " *force recovery*[3] " to circumvent this problem and to allow to use one finite element by beam. Some as Conci for that separated movement from body rigid and movement involving a work of deformation not no one[6]. This approach caused a new formulation which one also describes as "Co-rotational" formulation[15].
Others undertook to correct the value of the residue by adding an additional term in the formulation[16][17].

3.5 Approach

The approach described here is based on an exhaustive bibliographical study and more particularly on the work of two theses on the nonlinear analysis of the formed structures by beams[18][19]. The element selected is based on a Lagrangian Formulation Brought up to date with each Iteration (FLAI)[4] and one supposes the rotations "moderated" by iteration so that one can simply bring back oneself if rotations are commutative until the second order (from where the term "moderated" in opposition to "small" *i.e.* first order).

The approach presented in detail in the following section can be broken up as follows:

1) As for all the elements of structures, one formulates assumptions on kinematics allowing to determine a three-dimensional field of displacements in any point of the section starting from the degrees of freedom considered.
2) One writes the principle of virtual work within the framework of a Lagrangian Formulation Brought up to date with each Iteration. One is in small deformations.

2There exist other approaches to introduce plasticity but with the power of the current computers, the approach multifibre seems most adequate
3literally to recover the force, i.e. to estimate work interns structure
4It is the French equivalent of term UL introduces higher

3) One introduces the field of displacement and the functions of interpolation (which interpolate the displacements generalized according to the unknown factors with the nodes) to obtain the forms of the tangent matrix which one will break up into several terms (each one returning to the various phenomena which one seeks to take into account).
4) The taking into account of great rotations of the structure rests on a modification of the field of initial displacement, by supposing the rotations moderated between two iterations.

4 Theoretical description of the element

4.1 Field of displacement and associated field of deformations

4.1.1 Assumptions on kinematics

The kinematics of adopted beam is the following one:

- The element of beam is supposed right, the constant section.
- The transverse section is indeformable in its plan but can be warped axially
- The shearing strains transverse are not neglected and involve a rotation of the section around its axes
- The deformations are small, but displacements and rotations can be large

One places oneself on the level of a finite element, i.e. of a segment with 2 nodes, a local reference mark with the element is defined by the axis of the segment and the main axes of geometrical inertia. The center O reference mark coincides with the node n°1 (which is the centre of gravity of the section), the center of torsion is also defined (or of shearing) C section, where the application of a shearing action does not generate work in torsion and vice versa.
It is easy to show[12] that if axial displacements are expressed starting from displacements of the point O and displacements in the plan of the section from those of the point C then all the efforts are uncoupled, i.e. the matrix of behavior (for the elements of beam, it is the matrix obtained after expression of the law Hooke and integration on the section) is diagonal. The restitution in the total reference mark is done then in two times: passage of all displacements in the local reference mark of center O then passage of the local reference mark to the total reference mark.

4.1.2 Expression of the field of displacement

For such a section, the displacement of a point P of X-coordinate x , coordinates (y, z) in the reference mark of center O express yourself as the sum of a term of translation and a term of rotation (assumption of indeformable cross-sections).
To take account of the warping of the sections, a nonuniform term of axial displacement is added. The field of displacement $\vec{\xi}$ is written then:

$$\vec{\xi}(x,y,z)=\begin{cases} u(x,y,z) &=& u_O(x)+\left(z\times\theta_y(x)\right)-\left(y\times\theta_z(x)\right)+\left(\omega(y,z)\times\theta_{x,x}(x)\right)\\ v(x,y,z) &=& v_C(x)-\left((z-z_C)\times\theta_x(x)\right)\\ w(x,y,z) &=& w_C(x)+\left((y-y_C)\times\theta_x(x)\right)\end{cases}$$

éq 4-1

Indices C refer to the center of torsion C who is not confused with the centre of gravity O in the case of nonbi--symmetrical sections. The passage of displacements in C with those in O is carried out by a simple change of reference mark. The function ω commonly called function of warping, and which depends only on the form of the section, makes it possible to describe nonuniform axial displacement. The degrees of freedom of the element are carried to the nodes and are interpolated in the length of the element. There are 14 degrees of freedom, that is to say 7 with each node which are: $(u, v, w, \theta_x, \theta_y, \theta_z, \theta_{x,x})$. One naturally finds 3 degrees of translation and 3 degrees of rotation to describe kinematics, moreover the rate of torsion $\theta_{x,x}$ is also taken as measures warping.

When rotations cannot be regarded as small any more, it is then necessary to enrich the field by displacement of a nonlinear term. Indeed, at the end of warping close and while placing itself in the reference mark of center O, the expression (éq 4-1) can be written vectoriellement like:

$$\vec{\xi}(x,y,z)=\vec{\xi}_O(x)+(\boldsymbol{R}(x)-\boldsymbol{I})\begin{pmatrix}0\\y\\z\end{pmatrix}$$ **éq 4-2**

This expression reveals the operator rotation $\boldsymbol{R}$. If P is the point of coordinates (y,z) then $\boldsymbol{R}$ allows to take along the vector $\overrightarrow{OP}$ in its final configuration $\overrightarrow{OP}'$. $\boldsymbol{R}$ depends on the antisymmetric matrix associated with the vector rotation $\vec{\theta}={}^t(\theta_x,\theta_y,\theta_z)$ who is an unknown factor of the problem as well as $\vec{\xi}_O$, one can even show that $\boldsymbol{R}$ is written like exponential of matrix[7] :

$$\boldsymbol{R}=e^{\theta}=\boldsymbol{I}+\boldsymbol{\theta}+\frac{(\theta)^2}{2!}+\cdots+\frac{(\theta)^p}{p!}+\cdots \text{ avec } \boldsymbol{\theta}=\begin{pmatrix}0&-\theta_z&\theta_y\\\theta_z&0&-\theta_x\\-\theta_y&\theta_x&0\end{pmatrix}$$ **éq 4-3**

In (éq 4-1), this operator was replaced by his development with the first order because rotations were small. To take into account great rotations, one of the possibilities[5] is to develop the operator rotation at least until the second order[19][16]. One speaks then about moderate rotations (*finite rotations* in English). It is the office plurality of these moderate rotations which will lead to great rotations.
The field of displacement is then enriched by a nonlinear, quadratic term in $\theta_x,\theta_y,\theta_z$. The development with the second order is obtained by developing the exponential one until the term in $\boldsymbol{\theta}^2$.

4.1.3 Associated field of deformations

In addition let us calculate the deformations of Green-Lagrange of the field of displacement (éq 4-1), we use right now the assumption of the small deformations, indeed consider the complete expression of the deformations of Green-Lagrange (the indeformable sections being supposed in their plan, the half of the terms is already worthless):

$$\boldsymbol{F}=\boldsymbol{I}+\frac{\partial\vec{\xi}}{\partial\vec{x}}=\boldsymbol{I}+\nabla\vec{\xi} \quad \text{(tensor gradient of the transformation)}$$ **éq 4-4**

$$\boldsymbol{E}=\frac{1}{2}({}^t\boldsymbol{F}\boldsymbol{F}-\boldsymbol{I})=\begin{cases}E_{xx}&=&u_{,x}+\frac{1}{2}(u_{,x}{}^2+v_{,x}{}^2+w_{,x}{}^2)\\2E_{xy}&=&(u_{,y}+v_{,x})+(u_{,x}u_{,y}+w_{,x}w_{,y})\\2E_{xz}&=&(u_{,z}+w_{,x})+(u_{,x}u_{,z}+v_{,x}v_{,z})\end{cases}$$ **éq 4-5**

Small deformations one from of being deduced that $|u_{,x}|\ll 1$ and consequently $|u_{,x}u_{,y}|\ll|u_{,y}|$ and $|u_{,x}u_{,z}|\ll|u_{,z}|$. One can thus simplify the expression supplements $\boldsymbol{E}$ by neglecting the quadratic terms in $|u_{,i}|$ where $i=(x,y,z)$. In [12][16] these terms are not neglected, they lead then to a form of the more complex tangent matrix and an improved convergence. On the other hand they do not modify the precision of the results.

By injecting the field of displacement (éq 4-1) in this simplified expression one obtains finally the following form $\boldsymbol{E}=\boldsymbol{\epsilon}+\boldsymbol{\eta}$:

5There is the different one, to see [20] for a state of the art

$$\epsilon=\begin{cases}\epsilon_{xx} &= u_{O,x}(x)+\left(z\times\theta_{y,x}(x)\right)-\left(y\times\theta_{z,x}(x)\right)+\left(\omega(y,z)\times\theta_{x,xx}(x)\right)\\ 2\epsilon_{xy} &= \left(v_{C,x}(x)-\theta_z\right)+\left(\omega_{,y}-(z-z_C)\times\theta_{x,x}(x)\right)\\ 2\epsilon_{xz} &= \left(w_{C,x}(x)+\theta_y\right)+\left(\omega_{,z}+(y-y_C)\times\theta_{x,x}(x)\right)\end{cases}\quad \text{éq 4-6}$$

$$\eta=\begin{cases}\eta_{xx} &= \frac{1}{2}\Big(v_{C,x}{}^2+w_{C,x}{}^2+\left((y-y_C)^2+(z-z_C)^2\right)\theta_{x,x}{}^2\\ &\quad +2(y-y_C)\theta_{x,x}w_{C,x}-2(z-z_C)\theta_{x,x}v_{C,x}\Big)\\ 2\eta_{xy} &= \theta_x\left(w_{C,x}+(y-y_C)\theta_{x,x}(x)\right)\\ 2\eta_{xz} &= -\theta_x\left(v_{C,x}-(z-z_C)\theta_{x,x}(x)\right)\end{cases}\quad \text{éq 4-7}$$

where ϵ are the linearized deformations and η deformations known as nonlinear (resulting from the quadratic terms in the deformations of Green-Lagrange). If one then injects the nonlinear field of displacements for the treatment of great rotations, a nonlinear term η_{gr} additional appears.

4.2 Principle of virtual work

4.2.1 Assumptions used

To write the principle of virtual work (PTV), one will make the assumption of a Lagrangian Formulation Brought up to date with each Iteration (FLAI) [10] , i.e. the geometry of the studied structure will be constantly updating. Locally that results in the position of the nodes of the element which changes with each iteration of the algorithm of Newton [14] , allowing to calculate a new matrix of passage of the local reference mark to the total reference mark.

That is to say an element in initial configuration C_0. One can describe the deformations of the element starting from 3 configurations, the initial configuration[6] therefore, like 2 other configurations: C_1 who indicates a known deformed configuration but not inevitably balances some and C_2 who indicates an unknown deformed configuration near to C_1. The last known configuration is taken as reference, one will write the PTV in this configuration.
To be located in the algorithm of resolution, one can for example imagine that C_0 indicate the configuration at the beginning of the steps of time $(t=0)$. Moreover one already carried out a first iteration, one thus determined C_1 who is not in balance. One will thus carry out a new iteration, while choosing C_1 like reference and by writing the PTV in an unknown configuration and so on… until finding a configuration in balance and then passing to the step of next time.

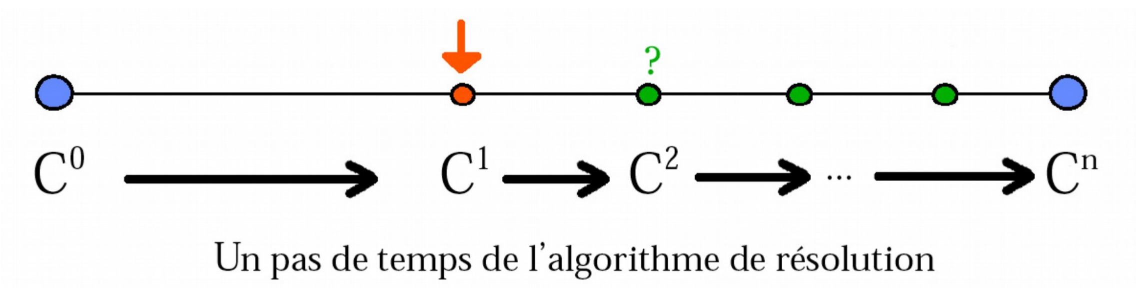

Illustration 4.1: Various configurations of reference.

One can pass a first remark: supports of the elements of beam in *Code_Aster* are segments with two nodes (SEG2), consequently their functions of geometrical interpolation are linear. One will be able

6it is determined by the data of the grid by the user never with these elements to thus reproduce the “last known configuration exactly”, that represents an approximation and it will thus be necessary to take care to have a sufficiently fine grid for the problems where the curve of the elements becomes considerable. The solution which consists in correcting the residue or to use a Co-rotational formulation was not established because she would have asked profound changes of the code[7].

The FLAI present of many advantages, for example one will be able to confuse tensor constraints of Cauchy (expressed on the final configuration with the preceding iteration) and tensor of the constraints of Piola-Kirchoff of second species (expressed on the initial configuration with the current iteration) because between the two iterations, the geometry will have been brought up to date:

$$\boldsymbol{S}=det(\boldsymbol{F})\boldsymbol{F}^{-1}\boldsymbol{\sigma}\ {}^t\boldsymbol{F}^{-1}=\sigma \quad \text{because} \quad \boldsymbol{F}=\boldsymbol{I} \qquad \textbf{éq 4-8}$$

That means that in the configuration of reference (that in particular which makes it possible to express the virtual deformations) the initial displacements already acquired by the element will be always worthless because the last calculated configuration will be taken as reference. Only the initial state of stresses in a given configuration will be in general nonnull.

With regard to the behavior, he is written in an incremental way and he is expressed in Lagrangian quantities thanks to the tensor of Piola-Kirchhoff $\mathrm{II}\,\boldsymbol{S}$ and with the deformations of Green-Lagrange $\boldsymbol{E}$. The small deformations make it possible to linearize the law of behavior and consequently:

$$\Delta \boldsymbol{S}=\boldsymbol{C}:\Delta \boldsymbol{E}\approx \boldsymbol{C}:\Delta \boldsymbol{\epsilon} \qquad \textbf{éq 4-9}$$

For elements of beams, one makes the assumption of plane constraints in the section. That amounts supposing that the beam consists of longitudinal fibres working in traction and compression.
This assumption ($\sigma_{yy}=\sigma_{zz}=\sigma_{yz}=0$) allows to express the law of behavior in the case of the beams in the form:

$$\boldsymbol{C}:\Delta \boldsymbol{\epsilon}=\begin{cases} E\,\delta\,\epsilon_{xx} \\ 2\mathrm{G}\,\delta\,\epsilon_{xy} \\ 2\mathrm{G}\,\delta\,\epsilon_{xz} \end{cases} \qquad \textbf{éq 4-10}$$

Finally the assumption of a field of displacement with moderate rotations will be licit to obtain great rotations between step and for the total loading. Indeed, the FLAI makes that one will summon at the beginning of a step and each iteration the increases in displacements and constraints calculated. The total increase in displacement and rotation on a step could consequently be large.

4.2.2 Writing of the PTV and differentiation

The principle of virtual work which is the weak formulation of the laws of the mechanics of the continuous mediums writes:

$$\int_{C_2} \boldsymbol{\sigma}:\boldsymbol{\epsilon}^* dV=\int_{C_2} \vec{f}.\vec{\xi}^* dV+\int_{\partial C_2} \vec{t}.\vec{\xi}^* dS \qquad \textbf{éq 4-11}$$

where all the quantities are expressed on the unknown deformed configuration, $\boldsymbol{\sigma}$ is the tensor of the constraints of Cauchy. Moreover $\vec{\xi}^*$ indicate a field of virtual displacement kinematically acceptable within the meaning of the boundary conditions.

To solve such a problem, one must initially change configuration thanks to a change of variable. The principle of virtual work writes in Lagrangian quantities (in the configuration of reference thus and not

7The simple solution of the term of correction in the residue requires the safeguard of the tangent matrix to the preceding iteration, which is not possible simply with current architecture.

the deformed configuration) is expressed using the virtual deformations of Green-Lagrange $\boldsymbol{E}^*$ and of the tensor of the constraints of Piola-Kirchhoff of second species (PKII) $\boldsymbol{S}$.

$$W_{int}=\int_{C_1}\boldsymbol{S}:\boldsymbol{E}^*\,dV=\int_{C_1}\vec{f}.\vec{\xi}^*\,dV+\int_{\partial C_1}\vec{t}.\vec{\xi}^*\,dS \qquad \text{éq 4-12}$$

where $\vec{f}$ represent the density of force voluminal, $\vec{t}$ density of force surface. Thereafter, one will note W_{ext} the member of right-hand side, and it will be supposed that the loading is conservative and not follower so that the work of the external forces is constant on a step of time.

The tensor of constraints PKII (éq 4-8) express the same quantity as the tensor of the constraints of Cauchy but by taking as reference the initial configuration. The tensor of the virtual deformations of Green-Lagrange is written:

$$\begin{cases} & \boldsymbol{E}^* &=& \frac{1}{2}\left({}^t\boldsymbol{F}^*\boldsymbol{F}+{}^t\boldsymbol{F}\boldsymbol{F}^*\right) \\ \text{avec} & \boldsymbol{F} &=& \boldsymbol{I}+\frac{\partial\vec{\xi}^*}{\partial\vec{x}} \\ \text{et} & \boldsymbol{F}^* &=& \frac{\partial\vec{\xi}^*}{\partial\vec{x}} \end{cases} \qquad \text{éq 4-13}$$

To solve balance one must check (éq 4-12) for any virtual displacement kinematically acceptable with zero $\vec{\xi}^*$. One applies the method of Newton to the functional calculus $W_{int}-W_{ext}$ and for this reason one differentiates the functional calculus compared to the unknown factor $\vec{\xi}$, that makes it possible to transform the resolution of a nonlinear problem (W_{int} does not depend linearly on $\vec{\xi}$) in a continuation of resolution of linear systems.

One thus solves:

$$W_{ext}=W_{int}^1+\int_{C_1}\left(\Delta\boldsymbol{S}:\boldsymbol{E}^*+\boldsymbol{S}:\Delta\boldsymbol{E}^*\right)dV \qquad \text{éq 4-14}$$

where W_{int}^1 represent the work already balanced in configuration 1. The principle is to balance this work with that of the internal forces W_{ext} using successive iterations.

The first term under the integral models material rigidity. Indeed, as the law of behavior is linearized and as the configuration of reference is the last known configuration (thus $\vec{\xi}=\vec{0}$ [8]), one can write while using (éq 4-9) and (éq 4-13):

$$\boldsymbol{E}^*=\boldsymbol{\epsilon}^*=\frac{1}{2}\left({}^t\boldsymbol{F}^*+\boldsymbol{F}\right) \qquad \text{éq 4-15}$$

$$\int_{C_1}\Delta\boldsymbol{S}:\boldsymbol{E}^*\,dV=\int_{C_1}\Delta\boldsymbol{\epsilon}:\boldsymbol{C}:\boldsymbol{\epsilon}^*\,dV \qquad \text{éq 4-16}$$

The FLAI allows in fact here to free itself from an additional term which couple nonlinear deformations virtual and linear deformations and that one calls term of initial displacements (these displacements are the displacements already undergone by our structure compared to its configuration of reference and which are always worthless here). In fact thanks to the FLAI, one off-sets the change of geometry and

8With the risk to repeat itself, it is false in practice, because the functions of form of the elements beam are linear. In [17], one does not make this simplification and one uses the additional term to correct the residue.

the non-linearity which results from this in the change of room-total reference mark suitable for the finite element method.
The second term under the integral models the geometrical rigidity of the structure. It is thanks to this term that one will be able to translate the effects of the second order and to converge quickly towards the solution. This term indeed directly utilizes the nonlinear deformations since according to (éq 4-13), $\Delta \boldsymbol{E}^{*}=\Delta \boldsymbol{\eta}^{*}$. If the field of displacement contains moreover of the nonlinear terms for the taking into account of rotations moderated then of course, the nonlinear deformations contain also an additional term.

It remains to be noticed that tensor of the constraints of Cauchy and PKII are identified because initial configuration and finale are confused following the reactualization.

4.2.3 Assessment

One thus has:

$$\Delta W=\int_{C_1}\left(\Delta \boldsymbol{\epsilon}:\boldsymbol{C}:\boldsymbol{\epsilon}^{*}+\boldsymbol{\sigma}:\Delta \boldsymbol{\eta}^{*}\right)dV=W_{ext}-W_{int}^{1} \qquad \textbf{éq 4-17}$$

Physically, all occurs like Ci the residue $W_{ext}-W_{int}^{1}$ iteration after iteration by a work is dissipated ΔW partly due to the deformation of the material (material rigidity) and partly due to great displacements of the structure (geometrical rigidity). Geometrical rigidity has a direction since the structure undergoes great displacements.

- The term of material rigidity is a classical term which is already present in the current formulation of the elements `POU_D_TGM` . It can possibly translate through the approach multifibre non-linearity material (plasticity for example).
- The geometrical term of rigidity that one usually finds in great displacements is new (in the configuration of reactualized reference, there exists a state of stress in general not no one). It thus translates geometrical non-linearity and it is grace inter alia in this term omitted in the formulation `PETIT_REAC` that convergence will be faster even finally possible in the case of instabilities.
- The second term in the member of right-hand side is the work of the internal forces to the preceding iteration, it is its exact calculation which guarantees a convergence towards a result right, this is why, one will propose an improvement in his calculation.

That in the geometrical term of rigidity the unknown factor intervenes through the virtual deformations, these last should well be noted indeed depend linearly on the unknown factors.

4.3 Determination of the tangent matrix

4.3.1 Recall on the discretization

From now on it is a question of determining the tangent matrix which is the combination of the various terms referred to above. For that it is necessary to utilize the discretization in finite elements, in particular the functions of interpolation then to clarify the integrals of volume which intervene in (éq 4-17).
For elements of structure like the beams, this discretization is carried out in two times: on the one hand integration on the sections located at the points of Gauss and on the other hand integration in the length. As one chose a behaviour multifibre for the treatment of plasticity, integration in the section is made thanks to fibres for the axial behavior (normal effort, bending moments) and using laws of behavior in efforts generalized for the rest of the efforts [4].
One calculates integrals on the section in each point of Gauss (of the element of beam) because they are the quantities in these points which will then make it possible to integrate numerically in the length (using formulas of squaring of Gauss). The generalized deformations are then expressed using a matrix of functions of interpolation. Digital integration then makes it possible to determine a tangent matrix: one will thus seek here only to express the matrix after integration on the section because it is

its expression which will be necessary for the implementation in the code, digital integration being transparent.

4.3.2 Expression of material rigidity

We can clarify the term of material rigidity, by clarifying the law of behavior initially (éq 4-10) then by replacing the linearized deformations $\boldsymbol{\epsilon}$ by their expression (éq 4-6) and finally while integrating on the section, thus preserving only one integral length.

$$\begin{aligned}\int_{C_1}\boldsymbol{\Delta\epsilon}:\boldsymbol{C}:\boldsymbol{\epsilon}^* dV &= \int_{C_1}\left(E\,\delta\,\epsilon_{xx}\,\epsilon_{xx}^*+2\mathrm{G}\,\delta\,\epsilon_{xy}\,\epsilon_{xy}^*+2\mathrm{G}\,\delta\,\epsilon_{xz}\,\epsilon_{xz}^*\right)dV\\ &= \int_{L_1}\left({}^t(\Delta\,D_s)\boldsymbol{K_s}\,D_s^*\right)dl\end{aligned} \quad \text{éq 4-18}$$

where one a: $D_s=\boldsymbol{B}\,U$, $\boldsymbol{B}$ being the matrix of interpolation of the generalized deformations $\left(u_{O,x},(v_{C,x}-\theta_z),(w_{C,x}+\theta_y),\theta_{x,x},\theta_{y,x},\theta_{z,x},\theta_{x,xx}\right)$ and U the vector of the nodal unknown factors. $\boldsymbol{K_s}$ is a behavioral matrix which translates at the same time the characteristics of material and the geometrical characteristics of the section (from where the appearance of the constants $S_y^{'}$, $S_z^{'}$, J and I_ω referring respectively to the reduced sections, the constant of torsion and the constant of warping):

$$\boldsymbol{K_s}=\begin{pmatrix}\int_S E\,ds & 0 & 0 & 0 & \int_S Ez\,ds & -\int_S Ey\,ds & 0\\ & GS_y^{'} & 0 & 0 & 0 & 0 & 0\\ & & GS_z^{'} & 0 & 0 & 0 & 0\\ & & & GJ & 0 & 0 & 0\\ & & & & \int_S Ez^2\,ds & -\int_S Eyz\,ds & 0\\ & sym & & & & \int_S Ey^2\,ds & 0\\ & & & & & & EI_\omega\end{pmatrix} \quad \text{éq 4-19}$$

$$\boldsymbol{B}=\begin{pmatrix}-\frac{1}{L} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{L} & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & \phi_1 & 0 & 0 & 0 & \phi_2 & 0 & 0 & \phi_3 & 0 & 0 & 0 & \phi_4 & 0\\ 0 & 0 & \psi_1 & 0 & \psi_2 & 0 & 0 & 0 & 0 & \psi_3 & 0 & \psi_4 & 0 & 0\\ 0 & 0 & 0 & N_{1,x} & 0 & 0 & N_{2,x} & 0 & 0 & 0 & N_{3,x} & 0 & 0 & N_{4,x}\\ 0 & 0 & \xi_{5,x} & 0 & \xi_{6,x} & 0 & 0 & 0 & 0 & \xi_{7,x} & 0 & \xi_{8,x} & 0 & 0\\ 0 & -\xi_{5,x} & 0 & 0 & 0 & \xi_{6,x} & 0 & 0 & -\xi_{7,x} & 0 & 0 & 0 & \xi_{8,x} & 0\\ 0 & 0 & 0 & N_{1,xx} & 0 & 0 & N_{2,xx} & 0 & 0 & 0 & N_{3,xx} & 0 & 0 & N_{4,xx}\end{pmatrix}$$

éq 4-20

$$\begin{aligned}\phi_1 &= \xi_{1,x}+\xi_5 & \psi_1 &= \xi_{1,x}-\xi_5\\ \phi_2 &= -\xi_{2,x}+\xi_6 & \psi_2 &= \xi_{2,x}-\xi_6\\ \phi_3 &= \xi_{3,x}+\xi_7 & \psi_3 &= \xi_{3,x}-\xi_7\\ \phi_4 &= -\xi_{4,x}+\xi_8 & \psi_4 &= \xi_{4,x}-\xi_8\end{aligned} \quad \text{éq 4-21}$$

The integrals of surface which still appear in the expression of $\boldsymbol{K}_s$ refer to the terms calculated thanks to the grid of the transverse fibre section. Except in the elastic case, the extra-diagonal terms are in general not null because the Young modulus E is not then homogeneous any more in the section. Expressions of the functions of interpolations $\xi_{i=(1,\dots,8)}$, $N_{j=(1,\dots,4)}$ are given in [2].

One can notice that the approach chosen to determine the matrix of material rigidity is not that adopted in *Code_Aster* for the elements of beam of Euler `POU_D_E` and of Timoshenko `POU_D_T`. Indeed, in our work, the approach is derived from the laws of the mechanics of the continuous mediums 3D and one gives oneself for that a field of displacement 3D, which enables us to utilize great displacements by writing the PTV in Lagrangian quantities.
To formulate the elements of beam of Euler and Timoshenko in assumption of the small elastic disturbances, the approach used in *Code_Aster*[1] consist in directly writing the basic principle of dynamics on elementary sections of beam, then to write of it a weak formulation using functions tests, and finally introducing laws of behavior in generalized efforts (of the type $N=ES\epsilon$). One realizes whereas by choosing the functions tests well, it is possible to express the work developed in the field test only according to the unknown factors with the nodes, that without having at any time discretized the unknown factors.
Thus when one works in small elastic disturbances, it can be more interesting to use these exact elements rather than the element presented in this work which numerically integrates the matrix of rigidity by squaring of Gauss.

4.3.3 Expression of geometrical rigidity

Same manner, one can clarify geometrical rigidity. For that, each component $\Delta\eta^*_{xi}$ of $\boldsymbol{\Delta\eta}^*$ can put itself in the form of a quadratic pseudo-form *i.e.* ${}^tC\,L_{xi}\,C^*$ where L_{xi} is a matrix which does not depend on displacements and C a vector dependent on the generalized unknown factors and their derivative: ${}^tC=\left(\theta_x, v_{C,x}, w_{C,x}, \theta_{x,x}\right)$. With obvious notations, one from of deduced:

$$\begin{aligned}\int_{C_1}\boldsymbol{\sigma}:\boldsymbol{\Delta\eta}^*\,dV &= \int_{L_1}\left({}^t\Delta C\underbrace{\left(\int_{S_1}\left(L_{xx}\sigma_{xx}+2L_{xy}\sigma_{xy}+2L_{xz}\sigma_{xz}\right)dS\right)}_{\boldsymbol{R}_s}C^*\right)dl \\ &= \int_{L_1}\left({}^t(\Delta C)\,\boldsymbol{R}_s\,C^*\right)dl\end{aligned} \quad \textbf{éq 4-22}$$

Integration over the length is also carried out by interpolating the generalized unknown factors, the unknown factors becoming the degrees of freedom to the nodes. Thus one replaces C by $\boldsymbol{B}_\sigma U$. Below, one gives the form of the matrix of interpolation $\boldsymbol{B}_\sigma$ and of the matrix $\boldsymbol{R}_s$:

$$\boldsymbol{R}_s=\begin{pmatrix} 0 & -T_z & T_y & \frac{1}{2}K_{,x} \\ & N_x & 0 & -M_y+z_C N_x \\ & & N_x & -M_z-y_C N_x \\ sym & & & K \end{pmatrix} \quad \textbf{éq 4-23}$$

$$B_\sigma=\begin{pmatrix} 0 & 0 & 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & 0 & N_3 & 0 & 0 & N_4 \\ 0 & \xi_{1,x} & 0 & 0 & 0 & -\xi_{2,x} & 0 & 0 & \xi_{3,x} & 0 & 0 & 0 & -\xi_{4,x} & 0 \\ 0 & 0 & \xi_{1,x} & 0 & \xi_{2,x} & 0 & 0 & 0 & 0 & \xi_{3,x} & 0 & \xi_{4,x} & 0 & 0 \\ 0 & 0 & 0 & N_{1,x} & 0 & 0 & N_{2,x} & 0 & 0 & 0 & N_{3,x} & 0 & 0 & N_{4,x} \end{pmatrix}$$

éq 4-24

In the geometrical matrix of rigidity intervene the generalized efforts (normal effort N, efforts cutting-edges T_y and T_z, bending moments M_y and M_z) as well as geometrical characteristics of the section. The coefficient K coefficient of Wagner is called and is written:

$$K=N_x\left(\frac{I_y+I_z}{S}+y_C^2+z_C^2\right)+M_y\left(\frac{I_{zr2}}{I_y}-2z_C\right)-M_z\left(\frac{I_{yr2}}{I_z}-2y_C\right)+M_\omega\frac{I_{\omega r2}}{I_\omega}$$

éq 4-25

The noncommonplace characteristics of section are:

$$\begin{aligned} I_{yr2} &= \int_S y(y^2+z^2)dS \\ I_{zr2} &= \int_S z(y^2+z^2)dS \\ I_{\omega r2} &= \int_S \omega(y,z)(y^2+z^2)dS \end{aligned}$$

éq 4-26

These characteristics is in general nonworthless for the sections not having double symmetry, it is the case for example angles equipping the supports lattice.
The last term in the coefficient of Wagner is not taken into account in the code because one does not have in general the expression of the function of warping ω to calculate $I_{\omega r2}$.

4.3.4 Form of the matrix of correction for the taking into account of great rotations of the structure

To take into account great rotations of the structure, it is necessary to express the matrix of correction made to geometrical rigidity by the addition of the nonlinear field of deformation $\Delta\boldsymbol{\eta}_{gr}$. The detailed approach of derivation is given in [19] or in [16]. One does not present calculations here, because they are long and tiresome. Other authors obtained the same matrix of correction starting from consideration on the nature of the bending moments and the torque[6][5].
It is interesting to notice that the only modification made by the taking into account of great rotations relates to an addition with the tangent matrix. The residue is unchanged, i.e. without the taking into account of great rotations, one is not likely to get false results[9] but rather to encounter difficulties of convergence when with certain problems are dealt.
Indeed, the problem of great rotations within the meaning of the size is not strictly speaking an obstacle: for problems plans, it is absolutely not necessary to use the matrix of correction (more exactly its contribution cancels itself). The true problem lies in the transformation of rotations during a change of reference mark. However when rotations are infinitesimal, one can replace them by their development with the first order and then they are identified with vectors, which makes it possible to transform them like displacements at the time of the passage of the local reference mark to the total reference mark. On the other hand when they are large, it is not possible any more, but one is likely to encounter problems of convergence only for precise cases where the continuity of rotations to the nodes is lost.
Thus the problem arises for structures presenting a coupling inflection-torsion and finite elements not colinéaires.

9except of course if one uses the tangent matrix directly to make a search for mode of buckling

The matrix of tangent rigidity is thus enriched by a term of correction $\boldsymbol{K}_c$ following the taking into account of rotations moderated in the expression of displacements:

$$\boldsymbol{K_c}=\begin{pmatrix} [0]_{3\times3} & & & & \\ & -[k_c^1]_{3\times3} & & (\mathbf{0}) & \\ & & [0]_{4\times4} & & \\ & (\mathbf{0}) & & [k_c^2]_{3\times3} & \\ & & & & 0 \end{pmatrix} \quad \text{éq 4-27}$$

$$\text{with} \quad [k_c^i]_{3\times3}=\frac{1}{2}\begin{pmatrix} 0 & -M_z^i & M_y^i \\ -M_z^i & 0 & 0 \\ M_y^i & 0 & 0 \end{pmatrix} \quad \text{éq 4-28}$$

4.4 Calculation of the residue

To supplement the model, one presents in the continuation two improvements suggested by work of theses [18] and [19] to improve the precision of the results.
The calculation of the residue, and more particularly of the work of the internal forces W_{int}^1 is carried out by digital integration after having integrated the law of behavior:

$$W_{int}^1=\int_{C_1}\boldsymbol{\sigma}:\boldsymbol{\epsilon}\,dV=\left(\int_{L_1}{}^tQ\,\boldsymbol{B}\,U\,dl\right) \quad \text{éq 4-29}$$

Q is a vector containing the 7 generalized efforts and $\boldsymbol{B}$ the matrix of interpolation of the generalized deformations already given in (éq 4-20) and which makes it possible to express the increments of deformations associated with nodal displacements U.
The improvements suggested touch respectively with the expression of the law of behaviour in torsion thus with Q and with the calculation of the linearized deformations $\boldsymbol{\epsilon}$. They must allow a better precision but also an acceleration of convergence.

4.4.1 Calculation of the increment of deformation

On the one hand, it appears interesting to take account of the update with each iteration of the geometry, indeed, one can then calculate more astutely the increment of deformation since the preceding step. Instead of calculating the increment of deformation starting from the increment of displacement of the step, one can calculate the deformation due to the displacement of the last iteration of Newton and cumulate it with each iteration. As displacement between two iterations tends towards zero, one is more precise.
Indeed if δu_i indicate the increment of consecutive displacement to the iteration i algorithm of resolution by step and if $\epsilon(u)$ indicate the increment of generalized deformation associated with an increment of displacement u then with the iteration n , one proceeds as follows:

$$\Delta U=\sum_{i=0}^{n}\delta u_i \quad \text{éq 4-30}$$

$$\text{instead of writing:} \quad \Delta\epsilon=\epsilon(\Delta U) \quad \text{éq 4-31}$$

$$\text{one will write rather:} \quad \Delta\epsilon=\sum_{i=0}^{n}\epsilon(\delta u_i) \quad \text{éq 4-32}$$

That requires a little more memory because it is necessary to be able to store the vector of the increments of deformation $\epsilon(\delta u_i)$, however this technique of calculation is more relevant when the geometry is updating with each iteration.

4.4.2 Law of behaviour in torsion

In addition, one chooses a more complete law of behavior in the expression of the torsion of a beam. Indeed [19] showed that the new expression which translates the influence of warping on the normal constraint, makes it possible to in the case of get corner results in perfect adequacy with a modeling hull. One thus adds at the torque, the moment says “Moment of Wagner”. Its calculation utilizes the coefficient of Wagner K already quoted in (éq 4-25).

$$M_x=(GJ+K)\theta_{x,x} \quad \textbf{éq 4-33}$$

5 Implementation in Code_Aster

5.1 Use

One reaches the treatment of great displacements for the multifibre beams POU_D_TGM by selecting the deformation 'GROT_GDEP' under BEHAVIOR.
All the nonlinear behaviors 1D are available[21].

5.2 Development

5.2.1 Matrix of correction

The form of the matrix of correction due to the taking into account of moderate rotations is given in (éq 4-27). It intervenes at the end of the calculation of tangent rigidity, since it is appeared as a term very integrated. The only obstacle relates to the bending moments which constitute this term very integrated, indeed, it acts of the bending moments with the nodes of the element, whereas one knows them only at the points of Gauss (following the integration of the law of behavior).
It is thus is possible to use the calculation of W^1_{int} to extract the moments from them or then to interpolate the values with the nodes using a polynomial of order 2. It is this last solution which was chosen, because at the beginning of each step of time, it is not possible to extract these values starting from work, because the latter is not calculated (at the beginning of each step of time, only a matrix of tangent rigidity of speed is assembled for the phase of prediction).
Once the interpolated values, one is thus satisfied to correct the geometrical matrix of rigidity. That is done just after the end of the loop on the points of Gauss.

5.2.2 Calculation of the deformations improved

As one announced, this calculation is relevant only when one updates the geometry of the structure at each iteration.
To be able to carry out this new calculation, it is necessary to be able to store the increment of deformation in each point of Gauss and after each iteration (it is a vector of dimension 7). This information is stored in the field specific to the elements of structure STRX_ELGA.
This field is also used to store the generalized efforts thatR the data of the only stress field by fibres cannot be enough to go with the efforts cutting-edges, the torque and the bi--moment.

6 Validation

The validation of the new element is carried out in 3 times. Initially one compares the results got on CAS-tests having analytical solutions (primarily plans), then on academic CAS-tests (3D) and finally on experimental results.

6.1 analytical CAS-tests

6.1.1 Beam in buckling (SSNL502)

This test represents a calculation of stability of a beam comforts subjected to a compressive force at an end. It makes it possible to validate modeling in the nonlinear quasi-static field in great displacements and in the presence of instability (Buckling of Euler).

6.1.2 Plate cantilever subjected to one moment (SSNV138)

This test constitutes the quasi-static calculation of an elastic plate embedded on a side and subjected to one bending moment at the other side, leading to great displacements in the plan.

6.2 academic CAS-tests

6.2.1 Great displacements of the arc with aperture 45° (SSNL136)

In this test, one studies an arc plan of opening 45° embedded at an end and subjected to a bending stress perpendicular to his plan at the other end. This test, very severe, makes it possible to validate the element of beam multifibre in the geometrical nonlinear field of great displacements and great rotations.

6.2.2 Discharge of a blade-square (SSNL133)

An L-shaped structure made up of two slim beams of mean rectangular section is subjected to a force at an end and is embedded at the other end. The field of the test is that of nonlinear elastic mechanics in great displacements and great rotations, with instability of the standard discharge of beam.

6.2.3 Elastoplastic ruin of the gantry of Lee (SSNL134)

The purpose of this test is to validate the nonlinear possibilities simultaneously material and geometrical of the multifibre element of beam. The element is implemented on a CAS-test usually treated in the literature with regard to the elastic behavior because it presents an answer complexes with *snap-back* and of *snap-through:* it is the gantry of Lee. One supposes here an elastoplastic behavior of the gantry, which makes it possible to test the good integration of the law of behavior of the multifibre elements but also the correct treatment of great displacements.

6.3 Experimental validation

6.3.1 Console MEKELEC (SSNL135)

MEKELEC indicates a structure carried out in 1991 starting from a console of pylon P4T. Dimensions and the profiles used were adapted to the realization of tests. One wished at the time establishing a base of precise experimental results in order to validate the results of simulations.
The validation consists in comparing the experimental and digital loads of ruin of 3 cases of loading.

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