Pre and Postprocessing for the thin hulls out of composite materials

Summary:

One extends the results of the theory of the elements of plates exposed in documentation [R3.07.03] to the case of multi-layer orthotropic materials. Documentation suggested gathers the thermal aspects and thermo-élasto-mechanics. The use of these materials is theoretically valid only in the case of a geometrical symmetry compared to the average layer of the plate. It is thus necessary that the coupling membrane-inflection is null.

Contents

1 Introduction

The modeling of the thermomechanical behavior by a theory of hulls of the structures made up of laminated composite materials present compared to the isotropic homogeneous case a certain number of characteristics:

- the coefficients intervening in the relations of linear behavior connecting the mechanical magnitudes and thermics defined on the average surface of the hull must be calculated starting from the space distribution in the thickness of various materials,
- the materials constitutive of the hull are in general orthotropic:
  - it is necessary to define, in each point of the average surface of the hull, a material direction fixing the reference mark in which the relations of behavior are described,
  - the form of the anisotropy produced on the total behavior of the hull can be unspecified,
- finally couplings between sizes characterizing of the symmetrical and antisymmetric phenomena compared to average surface can appear (coupling inflection - membrane, coupling temperature average average-gradient in the thickness). Into thermo_mecanic the results presented are however theoretically valid only when the coupling membrane-inflection is null,
- the analysis of the rupture or the damage of these structures requires to return on a level of description finer than that provided by the models of hulls: the criteria are formulated, layer by layer in the thickness, according to the “three-dimensional” constraints.

The preprocessing makes it possible to the user “to build” the sizes intervening in the theories of hulls starting from a simple space description of the distribution of the various materials (position, thickness, orientation).

Postprocessing intervenes once the layer, structural analysis completed to provide per layer, an evaluation of some criteria of rupture or damage.

The bias here is to specify pre and postprocessings so that they are independent, within the framework of the models of hull selected, of the type of element chosen by the user to calculate the structural analysis. Indeed, the digital difficulties of the calculation of the hulls and the representation of their geometry results in proposing according to the situations, several types of finite elements of hull or plate.

The note is divided into three parts. The first briefly points out the assumptions of the theory of hull used for thermomechanical calculations and the expressions of the coefficients homogenized to introduce. The second specifies the choices retained for the description of the orientation of materials compared to the elements like some notations. The last part details the application of these choices to the case of the hulls made up of homogeneous layers.

To allow the use of the options of calculations available in *Code_Aster*, it is thus necessary to define orders the pre one and postprocessing for composite materials laminated compatible with the existing orders.

2 Homogenized characteristics of a thin hull in thermoelasticity and thermics

2.1 Notations - Assumptions

The hull is made up various layers of orthotropic materials laid out average surface parallel to Σ (cf [Figure 2.1-a]).

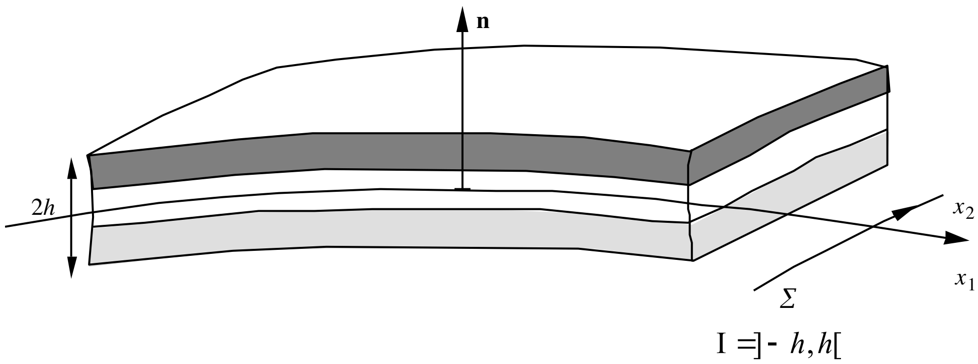

Figure 2.1-a

While noting (x_γ) coordinates (x_1, x_2) on Σ and x_3 the normal coordinate on the surface $x_3 \in]-h, h[$. one can define the various characteristics of materials intervening in thermics and thermo - elasticity. One will suppose moreover than it**one of the axes of orthotropism coincides with the normal n at the point (x_γ) with the hull Σ** .

- Conductivity: $k_{\alpha\beta}(x_\gamma, x_3)$ and $k_{33}(x_\gamma, x_3)$
- Voluminal heat: $\rho c(x_\gamma, x_3)$
- Dilation coefficients: $d_{\alpha\beta}(x_\gamma, x_3)$
- Elastic rigidity (plane constraint): $\Lambda_{\alpha\beta\lambda\mu}(x_\gamma, x_3)$
- Rigidity of shearing: $\Lambda_{\alpha 3\lambda 3}(x_\gamma, x_3)$
- Density: $\rho(x_\gamma, x_3)$

The Greek indices traverse $\{1,2\}$. The system (x_γ) necessarily does not correspond to the axes of orthotropism of materials in the tangent plan.

2.2 Thermics

One places oneself within the framework of the thermal model of hull describes in [R3.11.01] and [bib1]. A field of temperature "hull" is represented by the three fields $\left(T^m, T^s, T^i\right)$ defined on Σ in the following way in the thickness:

$$T\left(x_\gamma, x_3\right)=\sum_{j=1}^{3} T^j\left(x_\gamma\right)P_j\left(x_3\right)=T^m\left(x_\gamma\right)P_1\left(x_3\right)+T^s\left(x_\gamma\right)P_2\left(x_3\right)+T^i\left(x_\gamma\right)P_3\left(x_3\right) \quad (1)$$

Where them P_j are the polynomials of Lagrange:

$$I=\left]-h, h\right[: \quad P_1\left(x_3\right)=1-\left(x_3/h\right)^2$$
$$P_2\left(x_3\right)=\frac{x_3}{2h}\left(1+x_3/h\right) \quad (2)$$
$$P_3\left(x_3\right)=-\frac{x_3}{2h}\left(1-x_3/h\right)$$

The interpretation of the fields T^j is then the following one. First of all $T^m\left(x_\gamma\right)$ who represents the temperature on the average surface of the hull):

$$T^m\left(x_\gamma\right)=T\left(x_\gamma, 0\right) \quad (3)$$

Then the temperature on the upper surface of the hull:

$$T^s\left(x_\gamma\right)=T\left(x_\gamma, +h\right) \quad (4)$$

And the temperature on the lower surface of the hull:

$$T^i\left(x_\gamma\right)=T\left(x_\gamma, -h\right) \quad (5)$$

Thanks to the representation (1), the bilinear form is calculated K_S^T of $\left(T^m, T^s, T^i\right)\equiv T$ starting from the form of the 3D problem (indices ij take the values m, s, i):

$$K_\Sigma^T(T, \tau)=\int_\Sigma \left(A_{\alpha\beta}^{ji} . T^i_{,\alpha} . \tau^j_{,\beta}+B^{ji} . T^i . \tau^j\right) d\Sigma \quad (6)$$

With the usual convention of summation on the repeated indices). Où τ is a virtual field of temperature and where:

$$\begin{cases} A_{\alpha\beta}^{ij}=A_{\alpha\beta}^{ji}=A_{\beta\alpha}^{ij}, \qquad B^{ij}=B^{ji} \\ A_{\alpha\beta}^{ij}\left(x_\gamma\right)=\int_I k_{\alpha\beta}\left(x_\gamma, x_3\right)P_i\left(x_3\right)P_j\left(x_3\right)dx_3 \\ B^{ij}\left(x_\gamma\right)=\int_I k_{33}\left(x_\gamma, x_3\right)\frac{\partial P_i}{\partial x_3}\left(x_3\right)\frac{\partial P_j}{\partial x_3}\left(x_3\right)dx_3 \end{cases} \quad (7)$$

The bilinear form related to voluminal heat in the problem of evolution is written:

$$M(T,\tau)=\int_S C^{ij}.T^i.t^j$$
$$C^{ij}(x_g)=\int_I \rho c(x_g,x_3)P_i(x_3)P_i(x_3)dx_3 \tag{8}$$

2.3 Thermomechanical

One places oneself within the framework of the modeling of hull of Coils-Kirchhoff (thin hull) or Reissner-Mindlin (thick hull). In both cases, the sections are supposed to remain plane. Deformations of the tangent plan with Σ thus express themselves, in the thickness, using the tensors of deformations $E_{\alpha\beta}(x_\gamma)$, of variation of curve $K_{\alpha\beta}(x_\gamma)$ and of distortion $\gamma_\alpha(x_\gamma)$ surface [bib2]:

$$e_{\alpha\beta}(x_\gamma,x_3)=E_{\alpha\beta}(x_\gamma)+x_3K_{\alpha\beta}(x_\gamma) \quad \text{and} \quad e_{\alpha 3}(x_\gamma,x_3)=\frac{\gamma_\alpha(x_\gamma)}{2} \tag{9}$$

The material undergoing a local deformation of thermal origin given by ($T^{réf}$ is the temperature of reference):

$$e^{th}_{\alpha\beta}(x_\gamma,x_3)=\left(T(x_\gamma,x_3)-T^{réf}\right)d_{\alpha\beta}(x_\gamma,x_3) \tag{10}$$

The local stress field is given by the thermoelastic law in plane constraints:

$$s_{\alpha\beta}=L_{\alpha\beta\lambda\mu}\left(e_{\lambda\mu}-e^{th}_{\lambda\mu}\right) \tag{11}$$

Maybe with the preceding model for T :

$$s_{\alpha\beta}(x_\gamma,x_3)=L_{\alpha\beta\lambda\mu}(x_\gamma,x_3)\left[E_{\lambda\mu}(x_\gamma)+x_3K_{\lambda\mu}(x_\gamma)-e^{th}_{\lambda\mu}(x_\gamma,x_3)\right] \tag{12}$$

With:

$$e^{th}_{\lambda\mu}(x_\gamma,x_3)=\left(\sum_{j=1}^{3}T^j(x_\gamma).P_j(x_3)-T^{réf}\right)d_{\lambda\mu}(x_\gamma,x_3) \tag{13}$$

Generalized efforts (inflection $M^{\alpha\beta}$ and membrane $N^{\alpha\beta}$) are related to σ by:

$$\begin{cases} M^{\alpha\beta}(x_\gamma)=\int_I s^{\alpha\beta}(x_\gamma,x_3)x_3dx_3 \\ N^{\alpha\beta}(x_\gamma)=\int_I s^{\alpha\beta}(x_\gamma,x_3)dx_3 \end{cases} \tag{14}$$

So that the law of behavior of the hull is written at the point x_γ :

$$\begin{cases} M^{\alpha\beta}= & P^{\alpha\beta\lambda\mu}K_{\lambda\mu}+Q^{\alpha\beta\lambda\mu}E_{\lambda\mu}+M^{\alpha\beta}_{th} \\ N^{\alpha\beta}= & R^{\alpha\beta\lambda\mu}E_{\lambda\mu}+Q^{\alpha\beta\lambda\mu}K_{\lambda\mu}+N^{\alpha\beta}_{th} \end{cases} \tag{15}$$

Where:

$$\begin{cases} P^{\alpha\beta\lambda\mu}= & +\int_I L^{\alpha\beta\lambda\mu}(x_3)x_3^2dx_3 \\ Q^{\alpha\beta\lambda\mu}= & +\int_I L^{\alpha\beta\lambda\mu}(x_3)x_3dx_3 \\ R^{\alpha\beta\lambda\mu}= & +\int_I L^{\alpha\beta\lambda\mu}(x_3)dx_3 \\ N^{\alpha\beta}_{th}= & -\int_I L^{\alpha\beta\lambda\mu}e^{th}_{\lambda\mu}dx_3 \\ M^{\alpha\beta}_{th}= & -\int_I L^{\alpha\beta\lambda\mu}e^{th}_{\lambda\mu}x_3dx_3 \end{cases} \tag{16}$$

éq 2.3-5

When the temperature is calculated by the model of thermics, one can directly express the thermal efforts according to the three components $\left(T^m, T^s, T^i\right)$:

$$\begin{cases} M_{th}^{\alpha\beta} = -\left[\int_I L_{\alpha\beta\lambda\mu} d_{\lambda\mu}(x_3) P_j(x_3) x_3 \, dx_3\right]\left(T^j - T^{réf}\right) = DM_j^{\alpha\beta}\left(T^j - T^{réf}\right) \\ N_{th}^{\alpha\beta} = -\left[\int_I L_{\alpha\beta\lambda\mu} d_{\lambda\mu}(x_3) P_j(x_3) \, dx_3\right]\left(T^j - T^{réf}\right) = DN_j^{\alpha\beta}\left(T^j - T^{réf}\right) \end{cases} \quad (17)$$

Quantities DN and DM depend only on materials constitutive of the hull and their distribution.

Note:

When the provision of materials is symmetrical compared to Σ *, certain integrals, being nap of odd terms, are cancelled:* $Q^{\alpha\beta\lambda\mu}=0$ *,* $DM_1^{\alpha\beta}=DM_3^{\alpha\beta}=0$ *and* $DN_2^{\alpha\beta}=0$ *.*

The efforts cutting-edges and stresses shear transverse are obtained by writing of the local equilibrium equations without voluminal force:

$$\sigma^{ij}_{,j}=0 \text{ where } \{i, j\} \in \{1,2,3\} \quad (18)$$

What makes it possible to write the shearing action:

$$V^{\alpha}(x_\gamma) = M^{\alpha\beta}_{,\beta}(x_\gamma) \quad (19)$$

And the shear stress:

$$\sigma^{\alpha 3}(x_\gamma, x_3) = -\int_{-h}^{x_3} \sigma^{\alpha\beta}_{,\beta}(x_\gamma, z) \, dz \quad (20)$$

By using the fact that $\sigma^{\alpha 3}(x_{\gamma,+h}) = \sigma^{\alpha 3}(x_{\gamma,-h}) = 0$. The role of the preprocessing is to calculate the various sizes A , B , C , P , Q , R , DM and DN , starting from the description of the material (number, orientation and thickness of the various layers, local characteristics ρc , k , ρ , L and d).

3 Reference mark in the tangent plan with the hull. Matric notation

3.1 Reference mark

One considers the total reference mark of the structure (X, Y, Z). In the case of the laminated composites the orientation of full-course is defined compared to a direction of reference $\boldsymbol{e}_{réf}$ in the tangent plan (T).

This vector $\boldsymbol{e}_{réf}$ is determined by the projection of a vector $\boldsymbol{X}_1$, given by the user under the keyword ANGL_REP of AFFE_CARA_ELEM, on the tangent level (T) in an unspecified point of the hull.

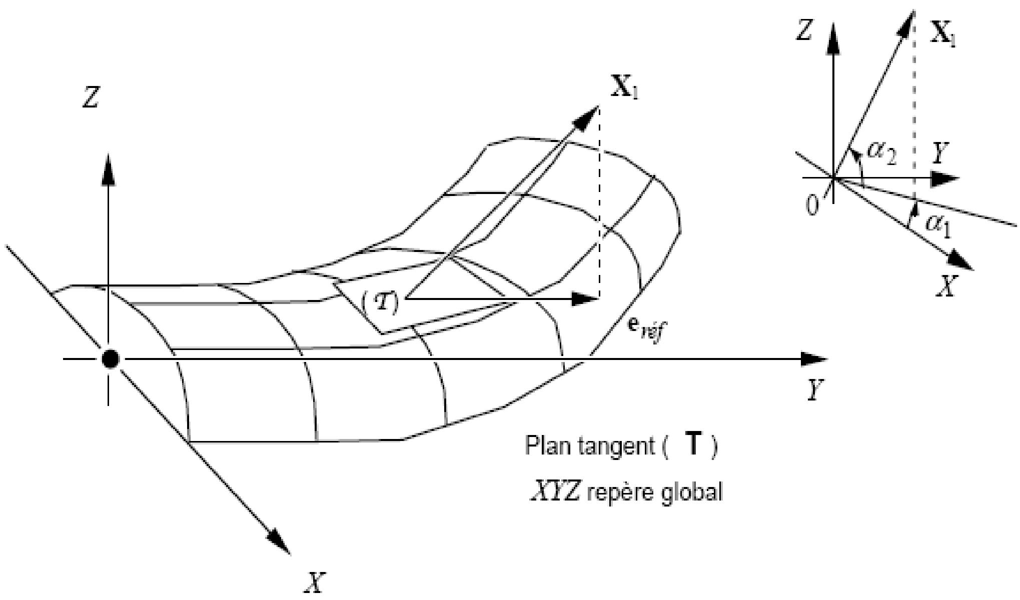

Figure 3.1-1: Definition of the local reference mark

The vector $\boldsymbol{X}_1$ is defined by the user by two directed angles:

α_1 : enter OX and $\boldsymbol{X}_{1\,\text{proj}(X,\,Y)}$
α_2 : enter $\boldsymbol{X}_{1\,\text{proj}(X,\,Y)}$ and X_1

α_1 : fact of passing from the direction $0\mathrm{X}$ with projection in the plan XOY vector $\boldsymbol{X}_1$.
α_2 : fact of passing from this projection to X_1 itself: to see figure 3.1-1.

Whenever in a given zone of the hull, T is orthogonal with $\boldsymbol{X}_1$, the user will have to define another vector (in practice for certain meshs).

For a finite element of type facets planes, contained in the tangent plan T, the orthonormal reference mark is defined $\left(\boldsymbol{V}_1, \boldsymbol{V}_2\right)$ room with the element using the classification of the tops. For example for the triangle:

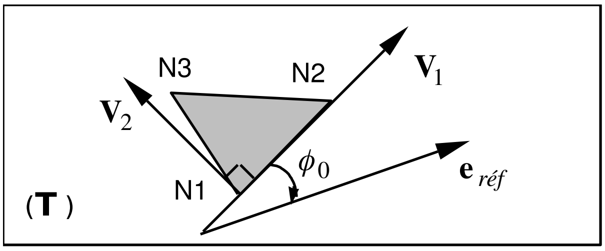

Figure 3.1-b: Local reference mark of the element $\left(\boldsymbol{V}_1, \boldsymbol{V}_2\right)$

The directed angle $j_0=\left(\boldsymbol{V}_1, \boldsymbol{e}_{réf}\right)$ allows to pass from the local reference mark to the element to the reference mark of reference.

3.2 Matric notation

In thermics as into thermomechanical, the programming of the elements requires to express the operators of elasticity and conduction in the local reference mark of the finite element $\left(\boldsymbol{V}_1, \boldsymbol{V}_2\right)$. One is used simplifying the representation of the tensorial sizes as follows.

3.2.1 Thermics

One represents the tensorial sizes in the reference mark $\left(\boldsymbol{V}_1, \boldsymbol{V}_2\right)$:

$$A^{ij}_{\alpha\beta}\left((i,j)\in(m,s,i)^2 \text{ et } (\alpha,\beta)\in(1,2)^2\right) \tag{21}$$

In a vectorial form with six vectors by taking account of symmetries [§2.2]:

$$\boldsymbol{A}^{ij}=\begin{pmatrix} A^{ij}_{11} \\ A^{ij}_{22} \\ A^{ij}_{12} \end{pmatrix}=\int_{-h}^{h} P_i\left(x_3\right).P_j\left(x_3\right).\begin{pmatrix} k_{11} \\ k_{22} \\ k_{12} \end{pmatrix} dx_3 \tag{22}$$

Where $\boldsymbol{k}=\begin{pmatrix} k_{11} \\ k_{22} \\ k_{12} \end{pmatrix}$ indicate the thermal vector conductivity built using the tensor $\begin{pmatrix} k_{\alpha\beta} & & 0 \\ & & 0 \\ 0 & 0 & k_{33} \end{pmatrix}$ (cf [§2.1]), and of $P_i\left(x_3\right)$, polynomials of Lagrange in the thickness. One makes in the same way for $\boldsymbol{B}^{ij}$ and $\boldsymbol{C}^{ij}$.

While placing itself in the reference mark of the element $(\boldsymbol{V}_1, \boldsymbol{V}_2)$, the matrix of passage is used $\boldsymbol{P}_k^{(m)}$ tensor of conductivity $k=\begin{pmatrix} k_{11} \\ k_{22} \\ k_{12} \end{pmatrix}$ of $(\boldsymbol{V}_1, \boldsymbol{V}_2)$ towards the reference mark associated with $\boldsymbol{e}_{réf}$ [bib3]:

$$\boldsymbol{P}_k^{(m)}=\begin{bmatrix} C^2 & S^2 & 2CS \\ S^2 & C^2 & -2CS \\ -CS & CS & C^2-S^2 \end{bmatrix} \text{ where } \begin{matrix} C=\cos(\varphi_0) \\ S=\sin(\varphi_0) \end{matrix} \tag{23}$$

It results from it that the matrix from passage $(\boldsymbol{P}_{\boldsymbol{k}}^{\boldsymbol{m}})^{-1}$ tensor of conductivity of the reference mark associated with $\boldsymbol{e}_{réf}$ towards $(\boldsymbol{V}_1, V_2)$ is given by:

$$\boldsymbol{P}_{k^{(m)^{-1}}}=\begin{bmatrix} C^2 & S^2 & -2CS \\ S^2 & C^2 & 2CS \\ CS & -CS & C^2-S^2 \end{bmatrix} \text{ where } \begin{matrix} C=\cos(\varphi_0) \\ S=\sin(\varphi_0) \end{matrix} \tag{24}$$

3.2.2 Thermomechanics

One also represents in a vectorial form in the reference mark $(\boldsymbol{V}_1, \boldsymbol{V}_2)$:

- on the one hand, normal constraints σ_{11}, σ_{22} , shearing σ_{12} in the plan and transverse shearing σ_{13} and σ_{23} :

$$\sigma=\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix} \text{ and } \tau=\begin{pmatrix} \sigma_{13} \\ \sigma_{23} \end{pmatrix} \tag{25}$$

- in addition, corresponding deformations:

$$\varepsilon=\begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{pmatrix},\ \frac{1}{2}\gamma=\begin{pmatrix} \varepsilon_{13} \\ \varepsilon_{23} \end{pmatrix} \text{ and } \gamma_{12}=2\varepsilon_{12} \tag{26}$$

who break up with the generalized deformations of membrane $\boldsymbol{E}$ and of inflection $\boldsymbol{K}$:

$$\varepsilon(x_3)=\varepsilon(u)(x_3)-\varepsilon^{th}(x_3) \text{ with } \varepsilon(u)(x_3)=\boldsymbol{E}+x_3\boldsymbol{K} \tag{27}$$

And for the thermal part:

$$\varepsilon^{\text{th}}(x_3)=\boldsymbol{d}(x_3)\left(T_{(x_3)}-T^{réf}\right) \tag{28}$$

For an ordinate $x_3 \in]-h, h[$, and:

$$\boldsymbol{E}=\begin{pmatrix} E_{11} \\ E_{22} \\ E_{11} \end{pmatrix}, \quad \boldsymbol{K}=\begin{pmatrix} K_{11} \\ K_{22} \\ K_{11} \end{pmatrix} \text{ and } \boldsymbol{d}=\begin{pmatrix} d_{11} \\ d_{22} \\ d_{11} \end{pmatrix} \tag{29}$$

Where $\boldsymbol{d}$ is the vector associated with the dilation coefficients thermal. The vector forced σ is obtained using the matrix of rigidity (3 X 3):

$$\boldsymbol{\sigma}=\boldsymbol{R}.\left(\varepsilon(u)-\varepsilon^{th}\right) \tag{30}$$

With $\boldsymbol{R}$, opposite of the matrix of flexibility (see [§4.3]). While placing itself in the reference mark of the element $\left(\boldsymbol{V}_1, \boldsymbol{V}_2\right)$, the matrix of passage is used $\boldsymbol{P}^{(m)}$ tensor of deformations $\varepsilon=\begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{pmatrix}$ of $\left(\boldsymbol{V}_1, \boldsymbol{V}_2\right)$ towards the reference mark associated with $\boldsymbol{e}_{réf}$ [bib3]:

$$\boldsymbol{P}^{(m)}=\begin{bmatrix} C^2 & S^2 & CS \\ S^2 & C^2 & -CS \\ -2CS & 2CS & C^2-S^2 \end{bmatrix} \text{ where } \begin{matrix} C=\cos(\varphi_0) \\ S=\sin(\varphi_0) \end{matrix} \tag{31}$$

While placing itself in the reference mark of the element $\left(\boldsymbol{V}_1, \boldsymbol{V}_2\right)$, the matrix of passage is used $\boldsymbol{P}_2^{(\boldsymbol{m})}$ tensor of deformations $\frac{1}{2}\gamma=\begin{pmatrix} \varepsilon_{13} \\ \varepsilon_{23} \end{pmatrix}$ of $\left(\boldsymbol{V}_1, \boldsymbol{V}_2\right)$ towards the reference mark associated with $\boldsymbol{e}_{réf}$:

$$P_2^{(m)}=\begin{bmatrix} C & S \\ -S & C \end{bmatrix} \text{ where } \begin{matrix} C=\cos(\varphi_0) \\ S=\sin(\varphi_0) \end{matrix} \tag{32}$$

In the same way, while placing itself in the reference mark of the element $\left(\boldsymbol{V}_1, \boldsymbol{V}_2\right)$, the matrix of passage $\boldsymbol{P}_s^{(m)}$ tensor of constraints $\sigma=\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix}$ of $\left(\boldsymbol{V}_1, \boldsymbol{V}_2\right)$ towards the reference mark associated with $\boldsymbol{e}_{réf}$ is worth:

$$\boldsymbol{P}_s^{(m)}=\begin{bmatrix} C^2 & S^2 & 2CS \\ S^2 & C^2 & -2CS \\ -CS & CS & C^2-S^2 \end{bmatrix} \text{ where } \begin{matrix} C=\cos(\varphi_0) \\ S=\sin(\varphi_0) \end{matrix} \tag{33}$$

It results from it that the form of the matrix of passage of the reference mark associated with $\boldsymbol{e}_{réf}$ towards the reference mark of the element $\left(\boldsymbol{V}_1, \boldsymbol{V}_2\right)$ for the constraints above is such as:

$$\boldsymbol{P}_{\sigma}^{(m)^{-1}}=\boldsymbol{P}_{\sigma}^{(m)}(-\varphi_0)={}^t\boldsymbol{P}^{(m)} \tag{34}$$

This property will be particularly useful in the continuation of the talk.

4 Hulls made up of homogeneous layers

4.1 Description of the layers

One considers the hull made up of a stacking of N_{couch} layers (parallel with the tangent plan) in the thickness $]-h, h[$ constituted each one of one of M_{mater} orthotropic homogeneous materials (laminated hull [Figure 4.1-a]).

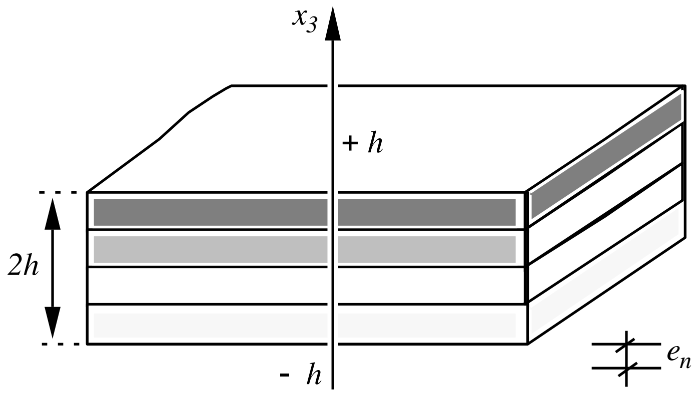

Figure 4.1-a

A layer n is defined by:

- its thickness e_n with the ordinates of the interfaces lower and higher:

$$x_3^{n-1} = -h + \sum_{j=1}^{n-1} e_j \text{ and } x_3^n = x_3^{n-1} + e_n \quad (35)$$

- the constitutive material m, and its physical characteristics,
- the angle ϕ_n first direction of orthotropism (noted L) in the tangent plan ($\boldsymbol{T}$) compared to the direction of reference $\boldsymbol{e}_{réf}$ (see figure 4.1-1).

Note:

In the case of a layer made up of fibres in a resin matrix, the first direction of orthotropism corresponds to the direction of fibres.

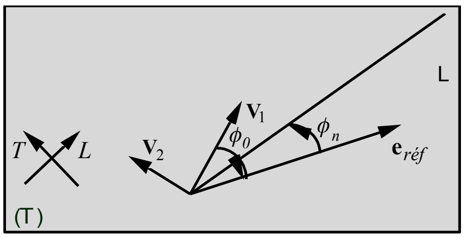

Figure 4.1-1: Definition of the reference mark on an orthotropic layer

4.2 Thermics

The expression of the vectors $A^{ij}\left((i,j)\in\{m,s,i\}^2, i\leq j\right)$ defined in [the §3.2.1] is obtained starting from conductivities k_m material m constituting the layers n . In the cases of orthotropism (L, T) material m , the coefficients of conductivity are:

$$k_{(L,T)}=\begin{pmatrix}k_L\\k_T\\0\end{pmatrix} \quad (36)$$

In the case of a transverse isotropic material, the coefficient k_{33} is equal to k_T . To have the expression of A^{ij} in the reference mark of the element (V_1, V_2) one must apply following rotation, of the reference mark of orthotropism towards the reference mark of the element, as clarified with [the §3]:

$$k^{(m)}=\begin{pmatrix}k_{11}\\k_{22}\\k_{12}\end{pmatrix}=\begin{pmatrix}C^2 & S^2\\S^2 & C^2\\CS & -CS\end{pmatrix}\begin{pmatrix}k_L\\k_T\end{pmatrix}_{(L,T)} \quad \text{with} \quad \begin{matrix}C=\cos(\varphi_i+\varphi_0)\\S=\sin(\varphi_i+\varphi_0)\end{matrix} \quad (37)$$

Vectors A^{ij} can then express itself by integration in the thickness of the contributions of layer:

$$A^{ij}=\sum_{n=1}^{N_{couch}}\int_{x_3^{n-1}}^{x_3^n} P_i(x_3)\cdot P_j(x_3)\cdot k_{(m)}\cdot dx_3 \quad (38)$$

Terms $B^{ij}\left((i,j)\in\{2,3\}^2, i\leq j\right)$ are:

$$B^{ij}=\sum_{n=1}^{N_{couch}}\int_{x_3^{n-1}}^{x_3^n}\frac{\partial P_i(x_3)}{\partial x_3}\cdot\frac{\partial P_j(x_3)}{\partial x_3}\cdot k_{33(m)}\cdot dx_3 \quad (39)$$

In the same way for C^{ij} :

$$C^{ij}=\sum_{n=1}^{N_{couch}}\int_{x_3^{n-1}}^{x_3^n} P_i(x_3)\cdot P_j(x_3)\cdot \rho C_{(m)}\cdot dx_3 \quad (40)$$

4.3 Thermomechanics

4.3.1 Relation of behavior

In the case of the laminated hulls, it is shown that the relation between the deformations $\boldsymbol{\varepsilon}$ and the constraint $\boldsymbol{\sigma}$ in the layer n depends on the constants of orthotropic material m . For the elastic coefficients, one has $E_{\mathrm{LL}}^{(m)}$, $E_{\mathrm{TT}}^{(m)}$, $\nu_{\mathrm{LT}}^{(m)}$, $G_{\mathrm{LT}}^{(m)}$, $G_{\mathrm{LZ}}^{(m)}$ and $G_{\mathrm{TZ}}^{(m)}$ and dilation coefficients $d_{¿}^{(m)}$ and $d_{\mathrm{TT}}^{(m)}$. In the axes of orthotropism (L,T) material m , the matrix of flexibility $\boldsymbol{S}$ express yourself by:

$$\boldsymbol{S}_{(m)}\Big|_{(L,T)}=\begin{bmatrix}\frac{1}{E_{\mathrm{LL}}} & -\frac{\nu_{LT}}{E_{TT}} & 0\\ -\frac{\nu_{TL}}{E_{TT}} & \frac{1}{E_{TT}} & 0\\ 0 & 0 & \frac{1}{G_{\mathrm{LT}}}\end{bmatrix}_{(m)} \quad \text{with} \quad \frac{\nu_{LT}}{E_{LL}}=\frac{\nu_{TL}}{E_{TT}} \tag{41}$$

Rigidity $\boldsymbol{\Lambda}_{(m)}=\boldsymbol{S}_{(m)}^{-1}$ being:

$$\boldsymbol{\Lambda}_{(m)}\Big|_{(L,T)}=\begin{bmatrix}\frac{E_{\mathrm{LL}}}{1-\nu_{TL}\cdot\nu_{\mathrm{LT}}} & \frac{\nu_{TL}\cdot E_{\mathrm{LL}}}{1-\nu_{TL}\cdot\nu_{\mathrm{LT}}} & 0\\ \frac{\nu_{LT}\cdot E_{TT}}{1-\nu_{TL}\cdot\nu_{\mathrm{LT}}} & \frac{E_{TT}}{1-\nu_{TL}\cdot\nu_{\mathrm{LT}}} & 0\\ 0 & 0 & G_{\mathrm{LT}}\end{bmatrix}_{(m)} \tag{42}$$

Rigidity in transverse shearing is expressed for its part in the following way:

$$\boldsymbol{\Lambda}_{\tau(m)}\Big|_{(L,T)}=\begin{bmatrix}G_{LZ} & 0\\ 0 & G_{TZ}\end{bmatrix}_{(m)} \tag{43}$$

While placing itself in the reference mark of the element $(\boldsymbol{V}_1,\boldsymbol{V}_2)$, the matrix of passage is used $\boldsymbol{P}^{(m)}$ tensor of deformations defined in [§3] of $(\boldsymbol{V}_1,\boldsymbol{V}_2)$ towards the reference mark of orthotropism:

$$\boldsymbol{P}^{(m)}=\begin{bmatrix}C^2 & S^2 & 2CS\\ S^2 & C^2 & -2CS\\ -CS & CS & C^2-S^2\end{bmatrix} \quad \text{where} \quad \begin{matrix}C=\cos(\varphi_i+\varphi_0)\\ S=\sin(\varphi_i+\varphi_0)\end{matrix} \tag{44}$$

In the same way the vector dilation is expressed in the reference mark $(\boldsymbol{V}_1,\boldsymbol{V}_2)$:

$$\boldsymbol{d}^{(m)}=\begin{pmatrix}d_{11}\\ d_{22}\\ d_{12}\end{pmatrix}=\boldsymbol{P}^{m^{-1}}\begin{pmatrix}d_{LL}\\ d_{TT}\\ 0\end{pmatrix}_{(L,T)}=\begin{pmatrix}C^2 & S^2\\ S^2 & C^2\\ 2CS & -2CS\end{pmatrix}\begin{pmatrix}d_{LL}\\ d_{TT}\end{pmatrix}_{(L,T)} \tag{45}$$

One thus has in the layer n (material: m), in x_3 :

$$s_{(n)}=\boldsymbol{P}_s^{(m)^{-1}}.\Lambda\big|_{(L,T)}.\boldsymbol{P}^{(m)}.(e(u)-e^{th})={}^{T}\boldsymbol{P}^{(m)}.\Lambda\big|_{(L,T)}.\boldsymbol{P}^{(m)}.(e(u)-e^{th})=\boldsymbol{\Lambda}_{(\boldsymbol{m})}(e(u)-e^{th}) \tag{46}$$

With:

$$\varepsilon(u)=\begin{pmatrix}E_{11}\\E_{22}\\E_{12}\end{pmatrix}+x_3\begin{pmatrix}K_{11}\\K_{22}\\K_{12}\end{pmatrix} \text{ and } \varepsilon^{th}=\begin{pmatrix}d_{11}\\d_{22}\\d_{12}\end{pmatrix}.\left(T\left(x_3\right)-T^{réf}\right) \quad (47)$$

Note:

In the code, one chose to carry out the passage of the reference mark of orthotropism to the reference mark of the element in two stages. A first stage relates to the passage of the reference mark of orthotropism to the reference mark defined by `ANGL_REP` . Data of `DEFI_MATERIAU` are thus transformed at the time of this first passage. One treats then equivalent material as one would do it with classical elements of plates.
The treatment of thermal dilation is made in the form of a contribution to the second member of the matric equation solve resulting from the principle of virtual work. This contribution is written:

$$\sigma^{th(n)}=-^{T}\boldsymbol{P}^{(m)}.\boldsymbol{\Lambda}|_{(L,T)}.\begin{Bmatrix}d_{LL}\Delta T\\d_{TT}\Delta T\\0\end{Bmatrix}.$$

4.3.2 Transverse shearing

Rigidity in transverse shearing of each layer is written in the reference mark $(\boldsymbol{V}_1,\boldsymbol{V}_2)$ in the same way that dilation:

$$\boldsymbol{\Lambda}_{t(m)}|_{(V_1,V_2)}={}^{t}\boldsymbol{P}_2^{(m)}.\boldsymbol{\Lambda}_{t(m)}.\boldsymbol{P}_2^{(m)} \quad (48)$$

With $\boldsymbol{P}_2^{(m)}=\begin{bmatrix}C & S\\-S & C\end{bmatrix}$ the vectorial matrix of passage of $(\boldsymbol{V}_1,\boldsymbol{V}_2)$ towards the reference mark of orthotropism. Rigidity in transverse shearing total of the hull $[\boldsymbol{R}_c]$ calculated so as to be equal to that given by the law of three-dimensional elasticity [bib2], the matrix $[\boldsymbol{R}_c]$ is defined so that the surface density of transverse energy of shearing U_2 obtained for a three-dimensional distribution of the constraints σ_{13} and σ_{23} that is to say identical to that associated with the model of plate of Reissner-Mindlin noted U_2 . With the final one:

$$U_1=\frac{1}{2}\int_{-h}^{h}\langle\tau\rangle\left[\boldsymbol{\Lambda}_{\tau(m)}\right]^{-1}\{\tau\}d_3 \quad (49)$$

And:

$$U_2=\frac{1}{2}\boldsymbol{V}\left[\boldsymbol{R}_c\right]^{-1}\boldsymbol{V}=\frac{1}{2}\left(\int_{-h}^{h}\{\tau\}d_3\right)\left[\boldsymbol{H}_c\right]^{-1}\frac{1}{2}\left(\int_{-h}^{h}\{\tau\}d_3\right) \quad (50)$$

With $\langle\tau\rangle=\langle\sigma_{13}\sigma_{23}\rangle$. With the equilibrium equations:

$$\begin{cases}\sigma_{13}=-\int_{-h}^{x_3}\left(\sigma_{11,1}+\sigma_{12,2}\right)d_3\\\sigma_{23}=-\int_{-h}^{x_3}\left(\sigma_{12,1}+s_{22,2}\right)d_3\end{cases} \quad (51)$$

And conditions:

$$0=\sigma_{13}=\sigma_{23} \text{ for } x_3=\pm h \quad (52)$$

Plane constraints σ_{11} , σ_{22} and σ_{12} express themselves according to the efforts resulting by making the assumption from pure inflection and absence from coupling membrane/inflection. It results from it that:

$$\sigma\left(x_3\right)=x_3.\boldsymbol{\Lambda}_{(m)}(x_3)\boldsymbol{P}^{-1}.\boldsymbol{M} \text{ and } \boldsymbol{A}\left(x_3\right)=\boldsymbol{\Lambda}_{(m)}(x_3)\boldsymbol{P}^{-1} \tag{53}$$

Where $\boldsymbol{P}$ is the matrix of rigidity of inflection of the whole of multi-layer defined by the first equation of (16). These calculations, as well as the following are to be carried out in a single reference mark. One chooses in *Code_Aster* the intrinsic reference mark with the element. It is thus necessary to transform matrix A in this reference mark. One has then:

$$\left\{\tau\left(x_3\right)\right\}=\boldsymbol{D}_1\left(x_3\right)\boldsymbol{V}+\boldsymbol{D}_2\left(x_3\right)\left\{\lambda\right\} \tag{54}$$

With:

$$\boldsymbol{V}=\langle M_{11,1}+M_{12,2};M_{12,1}+M_{22,2}\rangle \tag{55}$$

And:

$$\langle\lambda\rangle=\langle M_{11,1}-M_{12,2};M_{12,1}-M_{22,2};M_{22,1};M_{11,2}\rangle \tag{56}$$

And:

$$\begin{aligned}\boldsymbol{D}_1&=\int_{-x_3}^{h}-\frac{z}{2}\begin{bmatrix}A_{11}+A_{33} & A_{13}+A_{32}\\ A_{31}+A_{23} & A_{22}+A_{33}\end{bmatrix}dz\\ \boldsymbol{D}_2&=\int_{-x_3}^{h}-\frac{z}{2}\begin{bmatrix}A_{11}-A_{33} & A_{13}-A_{32} & 2\mathrm{A}_{12} & 2\mathrm{A}_{31}\\ A_{31}-A_{23} & A_{33}-A_{22} & 2\mathrm{A}_{32} & 2\mathrm{A}_{21}\end{bmatrix}dz\end{aligned} \tag{57}$$

U_1 is thus written:

$$U_1=\frac{1}{2}\langle\boldsymbol{V}|\lambda\rangle\begin{bmatrix}\boldsymbol{C}_{11} & \boldsymbol{C}_{12}\\ \boldsymbol{C}_{12^T} & \boldsymbol{C}_{22}\end{bmatrix}\left\{\frac{\boldsymbol{V}}{\lambda}\right\} \tag{58}$$

With:

$$\begin{aligned}\underset{2\times2}{\mathbf{C}_{11}}&=\int_{-h}^{h}\mathbf{D}_{1^T}\boldsymbol{\Lambda}_{\tau(m)}{}^{-1}\mathbf{D}_1 d_3\\ \underset{2\times4}{\mathbf{C}_{12}}&=\int_{-h}^{h}\mathbf{D}_{1^T}\boldsymbol{\Lambda}_{\tau(m)}{}^{-1}\mathbf{D}_2 d_3\\ \underset{4\times4}{\mathbf{C}_{22}}&=\int_{-h}^{h}\mathbf{D}_{2^T}\boldsymbol{\Lambda}_{\tau(m)}{}^{-1}\mathbf{D}_2 d_3\end{aligned} \tag{59}$$

From where final displacements:

$$U_1=U_2\Leftrightarrow\langle\boldsymbol{V}|\lambda\rangle\begin{bmatrix}\mathbf{C}_{11}-\mathbf{H}_c{}^{-1} & \mathbf{C}_{12}\\ \mathbf{C}_{12^T} & \mathbf{C}_{22}\end{bmatrix}\left\{\frac{\boldsymbol{V}}{\lambda}\right\}=0 \quad \forall\boldsymbol{V},\{\lambda\} \tag{60}$$

The solution is thus suggested $\boldsymbol{H}_c=\boldsymbol{C}_{11^{-1}}$. The coefficients of transverse correction of shearing correspond to the report of the terms of $\boldsymbol{H}_c$ with the integral on the thickness of the laminate of the terms of $\boldsymbol{\Lambda}_{\tau(m)}$.

4.3.3 Generalized efforts

The generalized efforts put in a vectorial form are obtained by integration in the thickness of the hull by summoning the contributions of the layers (thickness $\boldsymbol{e}_n=x_3^n-x_3^{n-1}$):

$$\mathbf{M}=\begin{pmatrix}M_{11}\\M_{22}\\M_{12}\end{pmatrix}=\int_l \sigma\cdot x_3\cdot dx_3=\sum_{n=1}^{N_{couch}}\int_{x_3^{n-1}}^{x_3^n}\sigma_{(n)}\cdot x_3\cdot dx_3$$

$$\mathbf{N}=\begin{pmatrix}N_{11}\\N_{22}\\N_{12}\end{pmatrix}=\int_l \sigma\cdot dx_3=\sum_{n=1}^{N_{couch}}\int_{x_3^{n-1}}^{x_3^n}\sigma_{(n)}\cdot dx_3 \tag{61}$$

If one expresses like previously (with m material of the layer n):

$$\boldsymbol{\sigma}_{(n)}=\boldsymbol{\Lambda}_{(m)}\cdot\left(\mathbf{E}+x_3\cdot\mathbf{K}-\mathbf{d}_{(\mathbf{m})}\left(T\left(x_3\right)-T^{réf}\right)\right) \tag{62}$$

One can note the efforts generalized in the form:

$$\begin{aligned}\mathbf{M}-\mathbf{M}^{\boldsymbol{th}}&=\mathbf{P}\cdot\mathbf{K}+\mathbf{Q}\cdot\mathbf{E}\\ \mathbf{N}-\mathbf{N}^{th}&=\mathbf{Q}\cdot\mathbf{K}+\mathbf{R}\cdot\mathbf{E}\end{aligned} \tag{63}$$

With $\boldsymbol{P},\boldsymbol{Q},\boldsymbol{R}$ matrices 3 X 3 being expressed by:

$$\begin{aligned}\mathbf{P}&=\sum_{n=1}^{N_{couch}}\boldsymbol{\Lambda}_{(m)}\int_{x_3^{n-1}}^{x_3^n}x_3^2\cdot dx_3 &&=\sum_{n=1}^{N_{couch}}\boldsymbol{\Lambda}_{(m)}\cdot\frac{1}{3}\cdot\left(\left(x_3^n\right)^3-\left(x_3^{n-1}\right)^3\right)\\ \mathbf{Q}&=\sum_{n=1}^{N_{couch}}\boldsymbol{\Lambda}_{(m)}\int_{x_3^{n-1}}^{x_3^n}x_3\cdot dx_3 &&=\sum_{n=1}^{N_{couch}}\boldsymbol{\Lambda}_{(m)}\cdot\frac{1}{2}\cdot\left(\left(x_3^n\right)^2\cdot\left(x_3^{n-1}\right)^2\right)\\ \mathbf{R}&=\sum_{n=1}^{N_{couch}}\boldsymbol{\Lambda}_{(m)}\cdot\left(x_3^n-x_3^{n-1}\right) &&=\sum_{n=1}^{N_{couch}}\boldsymbol{\Lambda}_{(m)}\cdot e_n\end{aligned} \tag{64}$$

Shearing action $\boldsymbol{V}$ is obtained by derivation of the moment [§4.3.2]. The generalized efforts of thermal origin are calculated directly:

$$\begin{aligned}\boldsymbol{M}^{th}&=\sum_{n=1}^{N_{couch}}\boldsymbol{\Lambda}_{(m)}.\int_{x_3^{n-1}}^{x_3^n}x_3.\left(T\left(x_3\right)-T^{réf}\right).\boldsymbol{d}_{(m)}.dx_3\\ \boldsymbol{N}^{th}&=\sum_{n=1}^{N_{couch}}\boldsymbol{\Lambda}_{(m)}.\int_{x_3^{n-1}}^{x_3^n}\left(T\left(x_3\right)-T^{réf}\right).\boldsymbol{d}_{(m)}.dx_3\end{aligned} \tag{65}$$

4.3.4 Localization of the constraints (postprocessing)

Conversely, following a calculation by finite element and of obtaining the deformations $\boldsymbol{E}$ and variations of curve $\boldsymbol{K}$, one can then calculate the stress field $\sigma_{(n)}\left(n=1,N_{couch}\right)$ in each layer of the element.

It is necessary to calculate in each layer (n) , the matrix $\boldsymbol{\Lambda}_{(m)}$ and terms $\left(T\left(x_3\right)-T^{réf}\right).\boldsymbol{d}_{(m)}$ (cf [§3.2]) ($m=mat_n$ represent the characteristics material of the layer n).

Constraints $\sigma_{\alpha\beta}$ with an ordinate $x_3\in\left]x_3^{n-1},x_3^n\right[$ in the layer (n) are then:

$$\sigma_{(n)}\left(x_3\right)=\boldsymbol{\Lambda}_{(m)}.\left[\boldsymbol{E}+x_3.\boldsymbol{K}-\boldsymbol{d}_{(m)}\left(T\left(x_3\right)-T_{réf}\right)\right] \tag{66}$$

And transverse shearing:

$$\tau_{(n)}(x_3) = D_1(x_3) . \boldsymbol{V} + D_2(x_3) . \lambda \quad (67)$$

Note:

In the code postprocessings of the elements of plates are generally defined in the reference mark associated with `ANGL_REP`. The constraints in the intrinsic reference mark of the element are thus brought back in the reference mark of the variety. One a:

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix}_{eref} = \begin{pmatrix} C^2 & S^2 & +2CS \\ S^2 & C^2 & -2CS \\ -CS & +CS & C^2-S^2 \end{pmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix}_n$$

where $C = \cos(\varphi_0)$, $S = \sin(\varphi_0)$ $(\text{cf.}[\S 4.1])$

where φ_0 *is the angle enters* $\boldsymbol{V}_1$ *and* $\boldsymbol{e}_{réf}$

4.3.5 Calculation of the criteria of rupture in the layers (postprocessing)

The limiting values of breaking stresses depend on material of the layer, the direction and the direction of the request (for a group of elements corresponding to the same field material):

$$\text{mat}_n \left| \begin{array}{ll} X : \text{limite en traction dans le sens L} & \text{(1ère direction orthotropie : sens des fibres)} \\ X' : \text{limite en compression dans le sens L} & \text{(1ère direction orthotropie : sens des fibres)} \\ Y : \text{limite en traction dans le sens T} & \text{(2ème direction orthogonale à la 1ère)} \\ Y' : \text{limite en compression dans le sens T} & \text{(2ème direction orthogonale à la 1ère)} \\ S : \text{limite en cisaillement dans le sens LT} & \end{array} \right.$$

It is necessary to calculate the constraints in the reference mark of the layer (defined by the axes of orthotropism) starting from the constraints in the reference mark of the element. The angle enters $\mathbf{V}_1$ and $\boldsymbol{e}_{réf}$ is φ_0 , and that enters $\boldsymbol{e}_{réf}$ and locates it orthotropism is φ_n :

$$\begin{pmatrix}\sigma_L\\ \sigma_T\\ \sigma_{LT}\end{pmatrix}_n=\begin{pmatrix}C^2 & S^2 & +2CS\\ S^2 & C^2 & -2CS\\ -CS & +CS & C^2-S^2\end{pmatrix}\begin{pmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{pmatrix}_n \quad \text{where} \quad \begin{matrix}C=\cos(\phi_0)\\ S=\sin(\phi_0)\end{matrix} \tag{68}$$

For the maximum criterion of constraint, the five following criteria are calculated by layer:

$$\text{For } n=1, N-couch \text{ one has } \begin{matrix}\dfrac{\sigma_{L_{(n)}}}{X_{(mat_n)}}\left(\text{si } \sigma_{L_{(n)}}>0\right) & \dfrac{\sigma_{L_{(n)}}}{X'_{(mat_n)}}\left(\text{si}\,\sigma_{L_{(n)}}<0\right)\\ \dfrac{\sigma_{T_{(n)}}}{Y_{(mat_n)}}\left(\text{si}\,\sigma_{T_{(n)}}>0\right) & \dfrac{\sigma_{T_{(n)}}}{Y'_{(mat_n)}}\left(\text{si}\,\sigma_{T_{(n)}}<0\right)\\ \dfrac{|\sigma_{LT_{(n)}}|}{S_{(mat_n)}} & \end{matrix} \tag{69}$$

The criterion of Tsai-Hill is written in each layer in the following way:

$$C_{TH}=\frac{\sigma^2_{L(n)}}{X^2_{(mat_n)}}-\frac{\sigma_{L(n)}.\,\sigma_{T(n)}}{X^2_{(mat_n)}}+\frac{\sigma^2_{T(n)}}{Y^2_{(mat_n)}}+\frac{\sigma^2_{LT(n)}}{S^2_{(mat_n)}} \tag{70}$$

The material is broken when $C_{TH}\geq 1$. Values X and Y are replaced by X' and Y' when constraints $\left(\sigma_{L(n)}\ ,\ \sigma_{T(n)}\right)$ corresponding are negative.

5 Bibliography

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6 Description of the versions of the document

Version Aster	**Author (S)** **Organization (S)**	**Description of the modifications**
6,3	P. MASSIN, F. NAGOT, F. VOLDOIRE EDF- R&D/AMA	Initial text
7,4	P. MASSIN, J.M.PROIX- R&D/AMA	