Vibroacoustic elements

Summary :

The elements described here make it possible to carry out calculations of the frequencies and clean modes of a structure coupled to a fluid. They also allow the acoustic calculation of answer.

After the formulation of the problem of coupling fluid-structure, this document describes the approach followed to implement in *Code_Aster* new finite elements.

Contents

1 Introduction

The vibratory behavior of a structure is often modified if this one is in the presence of a fluid: it is what is called the vibroacoustic coupling. One distinguishes the cases from coupling in two categories: either the fluid is infinite (it is the case of the immersed structures), or the fluid is contained in a limited medium (it is the case of the tanks more or less filled with fluid).

The finite elements described here make it possible to solve the problems of coupling with a fluid of finished size.

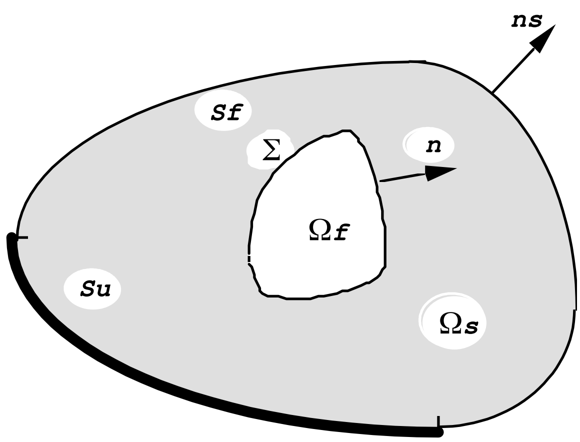
General notations:

P :	instantaneous total pressure in a point of the fluid,
p_0 :	pressure at rest,
p :	acoustic pressure,
ρ_t :	instantaneous total density in a point of the fluid,
ρ_0 :	density of the fluid at rest,
ρ :	acoustic density,
ρ_S :	density of the structure,
$\mathbf{u}_f$:	acoustic displacement,
$\mathbf{u}$:	displacement of the structure,
Φ :	gradient of acoustic displacements,
ω , f :	pulsation, frequency,
c :	speed of sound in the fluid,
λ , k :	number, wavelength of wave,
σ :	tensor of the constraints of the structure,
ε :	tensor of the structural deformations,
C :	tensor of elasticity of the structure,
T :	tensor of the constraints of the fluid.

2 Vibroacoustic coupling

2.1 Presentation

That is to say an elastic structure defined in a field Ω_s who vibrates in the presence of a true fluid, nonheavy, compressible, in isentropic evolution defined in a field Ω_f. One indicates by $\Sigma=\Sigma_f\cap\Sigma_S$, their common surface. Σ_f being the edge of the field Ω_f, and Σ_s being the edge of the field Ω_s. One notes n, the normal external with the fluid field Ω_f.



At a given moment, the state of the fluid is defined by its field of pressure P and that of the structure by its field of displacement $\mathbf{U}$.
It is considered that the coupled system is subjected to small disturbances around its state of balance where the fluid and the structure are at rest. As follows:

$$P=p_0+p \text{ and } \mathbf{U}=\mathbf{u}\left(\mathbf{u}_0=\mathbf{0}\right) \quad (1)$$

The problem of interaction fluid-structure then consists in solving two problems simultaneously:

- one in the structure subjected, on Σ, with a field of pressure p imposed by the fluid,
- the other in the fluid subjected to a field of displacement $\mathbf{u}$ wall Σ.

Note:

> *It is imperative that the normal external with the fluid field is always directed in the same direction. It is strongly advised to keep the convention of orientation of the structure towards the fluid for all modelings of interface fluid-structure.*

2.2 Formulation of the vibroacoustic problem

2.2.1 Description of the structure

Assumption:

The structure is homogeneous and obeys the laws of linear elasticity.

Taking into account this assumption, one can write the various following equations controlling the state of the structure [bib2].

2.2.1.1 Conservation equation of the momentum

The conservation equation of the momentum is written, in the absence of voluminal forces others that inertial forces:

$$\sigma_{ij,j}-\rho_s\frac{d^2 u_i}{dt^2}=0 \quad (2)$$

where : ρ_s is the density of the structure,
$\mathbf{u}$ is displacement,
σ_{ij} is the tensor of the constraints.

2.2.1.2 Relation of compatibility

One establishes the relation of compatibility classqiue on the tensor of the deformations:

$$\varepsilon_{kl}=\frac{1}{2}(u_{k,l}+u_{l,k}) \quad (3)$$

where ε_{kl} is the tensor of the deformations.

2.2.1.3 Law of behavior in isotropic linear elasticity

Assumption: the solid is elastic linear, therefore:

$$\sigma_{ij}=C_{ijkl}\varepsilon_{kl} \quad (4)$$

with the moduli of elasticity C_{ijkl} checking the identities: $C_{ijkl}=C_{klij}=C_{jikl}=C_{ijlk}$
C being the tensor of elasticity.

2.2.2 Description of the fluid

Assumption: the fluid obeys the laws of linear acoustics.

2.2.2.1 Conservation equation of the momentum

The conservation equation of the momentum is written, in the absence of sources:

$$T_{ij,j}-\rho_0\frac{d^2 u_{fi}}{dt^2}=0 \quad (5)$$

where: T_{kl} is the tensor of the constraints in the fluid,
ρ_0 is the density of the fluid in a natural state,
x is the field of displacement of a particle of fluid.

2.2.2.2 Conservation equation of the mass

With the first order and in the absence of acoustic sources, the conservation equation of the mass is expressed by the relation:

$$\frac{\partial \rho}{\partial t}+\rho_0 \operatorname{div}\left(\frac{\partial \mathbf{u}_f}{\partial t}\right)=0 \quad (6)$$

2.2.2.3 Law of behavior

Assumption: the fluid obeys the laws of linear acoustics, therefore:

$$T_{ij}=-p\,\delta_{ij} \qquad (7)$$

The fluid is supposed in evolution barotrope (the pressure p is, for the fluid given, a function known only density):

$$p=\rho\,c_0^2 \qquad (8)$$

where c_0 is the speed of sound in the fluid at rest.

2.2.2.4 Equation of propagation of the waves or equation of Helmholtz

One deduces it by combination from the conservation equations from the mass (6) and of the momentum (5) written in harmonic mode, with the pulsation ω :

$$\Delta\,p+k^2\,p=0 \qquad (9)$$

where $k=\omega/c$ is the number of wave.

2.2.3 Description of the interaction fluid-structure

With the interface fluid-structure (Σ), the fluid being nonviscous, it does not adhere to the wall. One thus writes:

- the continuity of the normal constraints:

$$\sigma_{ij}\cdot n_i=T_{ij}\cdot n_i=-\,p\,\delta_{ij}\cdot n_i \qquad (10)$$

- the continuity normal speeds:

$$\frac{du_i}{dt}\cdot n_i=\frac{dx_i}{dt}\cdot n_i \qquad (11)$$

2.2.4 Formulation of the coupled problem

Ultimately, the formulation of the problem of vibroacoustic in terms of displacements for the structure and pressure in the fluid led to the equations of the harmonic problem (P) in the solid medium:

$$C_{ijkl}\cdot u_{k,lj}+\omega^2\rho_S\,u_i=0 \text{ dans } \Omega_S \qquad (12)$$

The equation of propagation the waves years fluid environment:

$$\Delta\,p+k^2\,p=0 \text{ dans } \Omega_f \qquad (13)$$

With the two equations of coupling fluid-structure. In constraints:

$$C_{ijkl}\cdot u_{k,l}\cdot n_i=-\,p\,\delta_{ij}\cdot n_i \text{ sur } \Sigma \qquad (14)$$

And in speeds:

$$u_i\cdot n_i=\frac{1}{\rho_0\,\omega^2}\frac{\partial\,p}{\partial\,n} \text{ sur } \Sigma \qquad (15)$$

2.3 Variational equations associated with the problem

One solves the problem coupled by using the finite element method starting from the weak formulation of the problem.

2.3.1 Variational equations associated with the structure

That is to say $\delta\,u$, kinematically acceptable in Ω_S , the equation (12) can be written in the integral form:

$$\int_{\Omega_S}[C_{ijkl}\cdot u_{k,li}\,\delta\,u_i+\omega^2\,\rho_S\,u_i\,\delta\,u_i]\,dV=0 \qquad (16)$$

After integration by parts, one obtains the weak formulation:

$$\int_{\Omega_S}[C_{ijkl}\cdot u_{k,l}\,\delta u_{i,j}-\omega^2\rho_S u_i\,\delta u_i]dV-\int_\Sigma C_{ijkl}\,\delta u_i\,\delta u_{k,l}\cdot n_i^S\,dS=0 \tag{17}$$

By taking of account the boundary condition (14):

$$\int_{\Omega_S}[C_{ijkl}\cdot u_{k,l}\,\delta u_{i,j}-\omega^2\rho_S u_i\,\delta u_i]dV-\int_\Sigma p\,\delta u_i\cdot n_i\,dS=0 \tag{18}$$

2.3.2 Variational equation associated with the equation of the fluid

That is to say δp , kinematically acceptable in Ω_f. One writes in variational form the equation (13):

$$\int_{\Omega_f}[\Delta\,p\,\delta\,p+k^2\,p\,\delta\,p]\,dV=0 \tag{19}$$

After integration by parts, one obtains:

$$\int_{\Omega_f}\mathrm{div}(\delta\,p\cdot\nabla\,p)\,dV-\int_{\Omega_f}\nabla\,p\cdot\nabla\,\delta\,p\,dV+k^2\int_{\Omega_f}p\,\delta\,p\,dV=0 \tag{20}$$

Then:

$$\int_{\Sigma_f}\delta\,p\cdot\frac{dp}{\partial n}dS-\int_{\Omega_f}\nabla\,p\cdot\nabla\,\delta\,p\,dV+k^2\int_{\Omega_f}p\,\delta\,p\,dV=0 \tag{21}$$

Maybe, if one takes into account the boundary condition (15):

$$\frac{1}{\rho_0\omega^2}\int_{\Omega_f}[\nabla\,p\cdot\nabla\,\delta\,p-k^2\,p\cdot\delta\,p]\,dV-\int_\Sigma u_n\cdot\delta\,p\,dS=0 \tag{22}$$

2.4 Discretization by finite elements

The approximation by finished parts of the complete problem led to the following system:

$$\begin{bmatrix}\mathbf{K}-\omega^2\mathbf{M} & -\mathbf{C}\\ -\mathbf{C}^T & \dfrac{\mathbf{H}}{\rho_0\omega^2}-\dfrac{\mathbf{Q}}{\rho_0 c^2}\end{bmatrix}\begin{bmatrix}\mathbf{U}\\ p\end{bmatrix}=0 \tag{23}$$

In a more classical form:

$$\begin{bmatrix}\mathbf{K} & -\mathbf{C}\\ 0 & \mathbf{H}\end{bmatrix}\begin{bmatrix}\mathbf{U}\\ p\end{bmatrix}-\omega^2\begin{bmatrix}\mathbf{M} & 0\\ \rho_0\mathbf{C}^T & \dfrac{\mathbf{Q}}{c^2}\end{bmatrix}\begin{bmatrix}\mathbf{U}\\ p\end{bmatrix}=0 \tag{24}$$

$\mathbf{K}$ and $\mathbf{M}$ are the matrices of stiffness and mass of the structure. $\mathbf{H}$ is the matrix of “rigidity” of the fluid obtained starting from the bilinear form:

$$\mathbf{H}\rightarrow\int_{\Omega_f}\nabla\,p\cdot\nabla\,\delta\,p\,dV \tag{25}$$

And $\mathbf{Q}$ are the matrix “masses” fluid obtained by:

$$\mathbf{Q}\rightarrow\int_{\Omega_f}p\cdot\delta\,p\,dV \tag{26}$$

$\mathbf{C}$ is the matrix of coupling obtained starting from the following bilinear form:

$$\mathbf{C}\rightarrow\int_{\Sigma_f}p\cdot u_n\,dS \tag{27}$$

The choice of the formulation led to a nonsymmetrical matric system.

2.5 Choice of an additional variable for the description of the fluid

2.5.1 Formulation of the new problem

To obtain a symmetrical problem, one associates with the variable of pressure, an additional variable. This new variable is, that is to say the potential of displacement of the fluid Φ such as:

$$\mathbf{u}_f=\mathbf{grad}\,\Phi \quad (28)$$

See [bib1], [bib4], [bib5]. That is to say the variable π [bib3] such as:

$$\pi=-\rho\cdot\Phi \quad (29)$$

The variable π allows to directly take into account the fluids with variable density. However, it does not represent anything physically. This is why, the potential of displacements is preferred to him. Displacement is thus replaced $\mathbf{u}_f$ fluid by $\mathbf{grad}\,\Phi$ in the equations of the problem (P) [§2.2.4]. One thus obtains the new problem to be solved (P') :

$$C_{ijkl}\cdot u_{k,lj}+\omega^2\rho_S u_i=0 \text{ dans } \Omega_S \quad (30)$$

For the fluid:

$$\rho_0\omega^2\Delta\Phi+k^2 p=0 \text{ dans } \Omega_f \text{ and } p=\rho\omega^2\Phi \text{ dans } \Omega_f \quad (31)$$

And two equations of coupling:

$$C_{ijkl}\cdot u_{k,l}\cdot n_i=-\rho_0\omega^2\Phi\,\delta_{ij}\cdot n_j \text{ sur } \Sigma \text{ and } u_i\cdot n_i=\frac{\partial\Phi}{\partial n} \text{ sur } \Sigma \quad (32)$$

2.5.2 Variational formulation associated with the problem (P')

One applies to the equation (31) the formula of Green:

$$\int_{\Omega_f}[\rho_0\omega^2\Delta\Phi+k^2 p]\Psi\,dV=0 \quad \forall\Psi\, c.a. \quad (33)$$

After integration by parts:

$$\Leftrightarrow\int_{\Omega_f}[k^2 p\Psi-\rho_0\omega^2 grad\,\Psi\, grad\,\Phi]dV+\int_{\Sigma_f}\Psi\rho_0\omega^2\frac{\partial\Phi}{\partial n}dS=0 \quad \forall\Psi\, c.a. \quad (34)$$

Maybe, if one takes into account the boundary condition (32a):

$$\int_{\Omega_f}\frac{p\Psi}{\rho_0 c^2}dV-\int_{\Omega_f} grad\,\Phi\, grad\,\Psi\, dV+\int_\Sigma \Psi u_n\, dS=0 \quad \forall\Psi\, c.a. \quad (35)$$

Moreover, one writes in weak form the equation (31b) some is q kinematically acceptable:

$$\int_{\Omega_f}(p-\rho_0\omega^2\Phi)q\,dV=0 \quad \forall q\, c.a. \quad (36)$$

And:

$$\int_{\Omega_f}\frac{pq}{\rho_0 c^2}dV-\omega^2\int_{\Omega_f}\frac{\Phi q}{c^2}dV=0 \quad \forall q\, c.a. \quad (37)$$

By summoning the equations (35) and (37), one obtains the variational equation associated with the fluid:

$$\int_{\Omega_f}\frac{p\,q}{\rho_0 c^2}dV-\rho_0\omega^2\left[\int_{\Omega_f}\frac{\Phi q+p\Psi}{\rho_0 c^2}dV-\int_{\Omega_f}\frac{grad\,\Phi\, grad\,\Psi}{\rho_0 c^2}dV+\int\Psi u_n\, dS\right]=0 \quad \forall(q,\Psi)c.a. \quad (38)$$

2.6 Discretization by finite elements

While proceeding with the same approach as that used in [§2.3], one is led to the following matric system:

$$\begin{bmatrix} \mathbf{K} & 0 & 0 \\ 0 & \frac{\mathbf{M}_f}{\rho_0 c^2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ p \\ \Phi \end{bmatrix} - \omega^2 \begin{bmatrix} \mathbf{M} & 0 & \rho_0 \mathbf{M}_\Sigma \\ 0 & 0 & \frac{\mathbf{M}_{fl}}{c^2} \\ \rho_0 \mathbf{M}_\Sigma^T & \frac{\mathbf{M}_{fl}^T}{c^2} & \rho_0 \mathbf{H} \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ p \\ \Phi \end{bmatrix} = 0 \quad (39)$$

$\mathbf{K}$ and $\mathbf{M}$ being respectively matrices of stiffness and mass of the structure. $\mathbf{M}_\Sigma$ is the matrix of coupling fluid-structure obtained starting from the bilinear form:

$$\mathbf{M}_\Sigma \rightarrow \int_\Sigma \Phi u \, dS \quad (40)$$

$\mathbf{M}_f$, $\mathbf{M}_{fl}$, $\mathbf{H}$ being fluid matrices, respectively obtained starting from the bilinear forms:

$$\begin{aligned} \mathbf{M}_f &\rightarrow \int_{\Omega_f} p^2 dV \\ \mathbf{M}_{fl} &\rightarrow \int_{\Omega_f} p \Phi \, dV \\ \mathbf{H} &\rightarrow \int_{\Omega_f} (grad \, \Phi)^2 dV \end{aligned} \quad (41)$$

To obtain a matrix of rigorously worthless mass and fluid rigidity, by convention, it is enough to put $\rho_0 = c = 0$ (either coefficients `RHO` and `CELE_R` in `DEFI_MATERIAU`).

2.7 Calculations of acoustic answer

2.7.1 Speed imposed on the fluid

On a part Σ_v fluid border Σ_f, one can impose a limiting condition standard normal speed v_0. The term of edge of the fluid is written then:

$$-\rho_0 \omega^2 \int_{\Sigma_f} \Psi \cdot u_n \, ds = -\rho_0 \omega^2 \int_{\Sigma_f - \Sigma_z} \Psi \cdot u_n \, ds + i \omega \rho_0 \int_{\Sigma_v} \Psi \, v_0 \, ds \quad (42)$$

2.7.2 Impedance imposed on a wall of the fluid

On a part Σ_z fluid border Σ_f, one can impose a limiting condition of standard impedance Z :

$$p = Z \, v_n \quad (43)$$

where v_n is the outgoing normal speed of the fluid. By deferring this condition in the equation translating the conservation of the momentum (5) and by taking account of the law of behavior of the fluid (7), one a:

$$grad \; p + \frac{\rho_0}{z} \dot{p} = 0 \quad (44)$$

To preserve the symmetry of the system, one expresses the preceding equation according to the potential of displacement of the fluid Φ, one a:

$$grad \, \ddot{\Phi} + \frac{\rho_0}{z} \frac{\partial^3 \phi}{\partial t^3} = 0 \quad (45)$$

The term of edge of the fluid is written then:

$$\rho_0 \omega^2 \int_{\Sigma_f} \Psi \cdot \frac{\partial \Psi}{\partial n} ds = \rho_0 \omega^2 \int_{\Sigma_f - \Sigma_z} \Psi \cdot \frac{\partial \Psi}{\partial n} ds + i \omega^3 \int_{\Sigma_v} \frac{\rho_0^2}{z} \Phi \Psi \, ds \quad (46)$$

Ultimately, to impose an impedance on a wall of the fluid amounts introducing into the system a term of damping.

2.7.3 Discretization by finite elements

If one imposes limiting conditions of standard imposed speed or impedance of wall imposed on the fluid, one is led to solve the following matric system:

$$\begin{bmatrix} \mathbf{K} & 0 & 0 \\ 0 & \dfrac{\mathbf{M}_f}{\rho_0 c^2} & 0 \\ 0 & 0 & 0 \end{bmatrix} - \omega^2 \begin{bmatrix} \mathbf{M} & 0 & \rho_0 \mathbf{M}_\Sigma \\ 0 & 0 & \dfrac{\mathbf{M}_{fl}}{c^2} \\ \rho_0 \mathbf{M}_\Sigma^T & \dfrac{\mathbf{M}_{fl}^T}{c^2} & \rho_0 \mathbf{H} \end{bmatrix} + i\omega^3 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \dfrac{\rho_0^2}{Z}\mathbf{Q} \end{bmatrix} = \begin{bmatrix} 0 \\ i\omega \\ \mathbf{V} \end{bmatrix} \quad (47)$$

$\mathbf{Q}$ being the matrix obtained starting from the bilinear form $\int_{\Sigma_z} \Phi^2 \, dS$ and $\mathbf{V}$, the vector obtained from $\int_{\Sigma_V} \rho_0 v_0 \Phi \, dS$.

3 Integration in *Aster*

The elements described previously belong, for the fluid part, with modeling `'3D_FLUIDE'` phenomenon `MECHANICS` and, for the interface fluid-structure, with modeling `'FLUI_STRU'`same phenomenon.

They lead to voluminal or surface elements, for the fluid part, in pressure-potential of displacement and with surface elements for the interface fluid-structure in potential of displacement of fluid-displacement of the structure.

3.1 Buckling of Euler, geometrical stiffness

One considers the problem of buckling of Euler of an elastic solid in interaction with a fluid field, modelled with the vibroacoustic assumption. In *Code_Aster*, one models fluctuating fields (displacements of the solid, pressures fluid: (u, p, Φ)). The buckling of Euler is thus treated by the research of the cancellation of the Eigen frequencies of vibrations of the coupled system, by including the presence of the geometrical stiffness associated at the static state prestressed in the solid, controlled by a parameter of loading criticizes to determine.
It is admitted initially that the geometrical stiffness associated at the initial static state in the fluid is negligible. One thus proposes to define a matrix of worthless geometrical stiffness in the fluid field and on the interfaces fluid/structure. However this method of analysis neglects the fact that the fluid pressure is a following loading for the solid, which thus introduces a nonlinear term, which often has notable effects on the critical loads of buckling. It is thus preferable to deal with the problem by a nonlinear dynamic analysis associated with an analysis of stability, by controlling the mechanical loading: to consult the U2.06.11 document.

4 Bibliography

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5 Description of the versions of the document

Version Aster	Author (S) Organization (S)	Description of the modifications
2.3	Fe.Waeckel EDF/DER/EP/AMV	Initial text