# Seismic answer by spectral method

**Summary:**

The study of the answer of a structure under the effect of imposed movements of seismic type, with a single imposed movement (mono-support) or multiple (multi-support) is possible in transitory analysis (time history). One will refer to the note [R4.05.01].

For studies of dimensioning, one can be interested only in one estimate of the maximum efforts induced by the requests, to evaluate the safety margin with regulations of construction, without resorting to a transitory analysis.

The spectral method is based on the concept of spectrum of oscillator of a accélérogramme of earthquake. One details the method of development of this spectrum of answer available in the operator CALC_FONCTION [U4.32.04].

It is shown how this spectrum of oscillator can be used to evaluate one raising of the answer in relative displacement of a simple oscillator. This approach is justified if one does not wish to know the history of displacements and the efforts, while being limited to the analysis of the inertial effects.

The spectral method uses general notions of the method of modal recombination [R5.06.01].

One describes the various rules of combination usable to obtain one raising realistic but conservative maximum answer of the structure. These methods are available in the operator COMB_SISM_MODAL [U4.84.01].

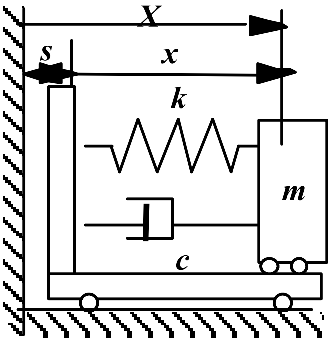
## Contents

# 1 Concept of spectrum of oscillator

The spectral method for the study of the answer of a structure under the effect of imposed movements of seismic type is based on the concept of spectrum of oscillator of a accélérogramme of earthquake.

## 1.1 Imposed movement defined by a accélérogramme A (T)

For an imposed movement $s$ of seismic type, one can deal with the problem in absolute displacement $X$ or in relative displacement $x$ such as: $X=x+s$ . The general equations of the movement of a simple oscillator are written then:




| Absolute movement | Relative movement |
| --- | --- |
| $m\ddot{X}+c\dot{X}+kX=c\dot{s}+ks$ | $m\ddot{x}+c\dot{x}+kx=-m\ddot{s}$ |

One retains the formulation from **relative movement** for two primary reasons:

- the seismic analysis of the structures uses the constraints induced by the inertial effects of the earthquake, constraints calculated starting from the structural deformations which are expressed starting from relative displacements;
- the characterization of the signal of excitation can be reduced in this case to the accélérogramme of the earthquake $\ddot{s}=A(t)$ , size provided directly by the seismographs. Signals of displacement $s$ and speed $\dot{s}$ are in general not available in the databases geotechnics.

For the determination of the response of a simple oscillator to an imposed movement and the notations conventional, one will refer to appendix 2 [R4.05.03 Appendix 2].

If the earthquake is defined by a accélérogramme $A(t)$ , absolute acceleration applied to the base, the reduced equation is in this case:

$$\ddot{x}+2\xi\ \omega_0\dot{x}+\omega_o^2 x=-\ddot{s}=-A(t) \qquad \text{éq 1.1-1}$$

The solution of this problem is the integral of DUHAMEL presented to appendix A [éq A3.3-1]:

$$x(t)=\frac{1}{\omega'_0}\int_0^t A(\tau)e^{-\xi\omega_0(t-\tau)}\sin\omega'_0(t-\tau)\,d\tau=f(A,\xi,\omega'_0) \qquad \text{éq 1.1-2}$$

$$\omega'_0=\omega_0\sqrt{1-\xi^2}$$

## 1.2 Spectrum of oscillator of a accélérogramme

The concept of spectrum of oscillator was introduced initially to compare between them the effects of different accélérogrammes. The spectrum of FOURIER of a signal $A(t)$ inform about its frequential contents. The response of a mechanical system to a movement imposed on the base depends largely on the dynamic characteristics of this system: Eigen frequencies and reduced damping $(\xi,\omega'_0)$. Appendix A details this aspect.

If one wishes to know the maximum value of the response of a simple oscillator to the parameters, $(A,\xi,\omega'_0)$ one must evaluate the integral of DUHAMEL which provides the answer of the oscillator [éq 1.1-2] to an excitation imposed on the base.

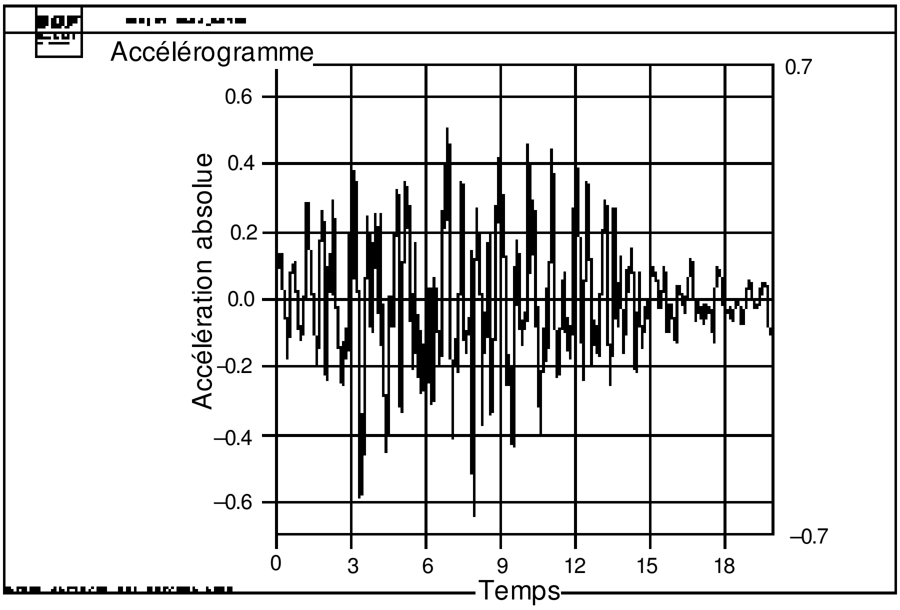



**Figure 1.2-a: Accélérogramme**

### 1.2.1 Spectrum of oscillator in relative displacement

From the integral of DUHAMEL, one can define the spectrum of oscillator of a accélérogramme $A(t)$ like the function of the maximum values of relative displacement $x(t)=f(A,\xi,\omega'_0)$ for each value of $(\xi,\omega'_0)$ by recalling that: $\omega'_0=\omega_0\sqrt{(1-\xi^2)}$.

$$Srox(A,\xi,\omega'_0)=|x(t)|_{max}$$

$$x(t)=\frac{1}{\omega'_0}\int_0^t A(\tau)e^{-\xi\omega_0(t-\tau)}\sin\omega'_0(t-\tau)\,d\tau=f(A,\xi,\omega'_0)$$

One notes, on the figure [Figure 1.2.1-a], that beyond a certain frequency ( $35\,Hz$ here), known as cut-off frequency of the spectrum, it does not have there significant dynamic amplification: relative displacement is null.

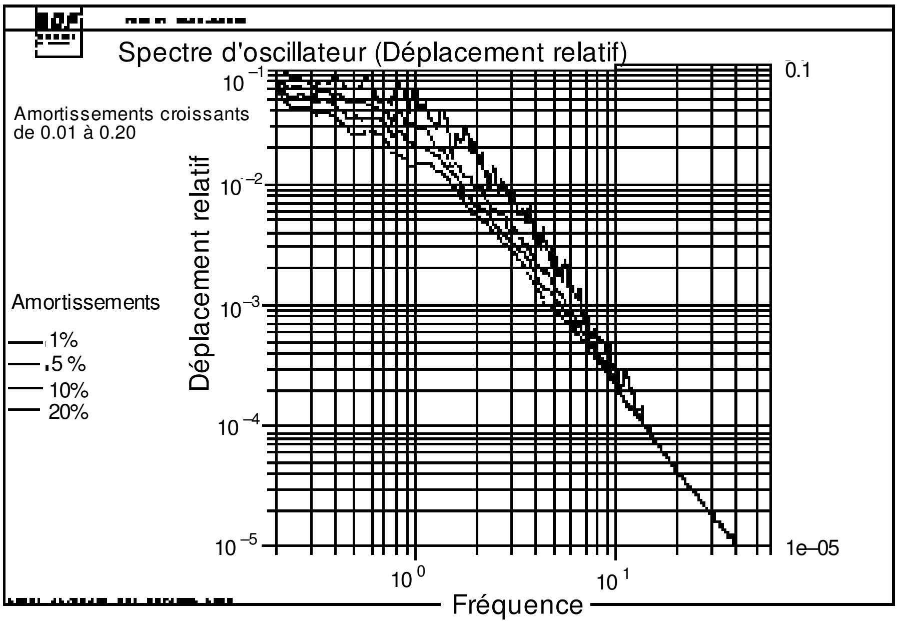



**Figure 1.2.1-a: Spectrum of oscillator in relative displacement**

### 1.2.2 Spectrum of oscillator in relative pseudovelocity

For structures with weak reduced damping $\xi < 0.2 = 20\%$, for which it is acceptable to assimilate $\omega_0$ and $\omega'_0$, one usually uses the spectrum of pseudovelocity defined by:

$$Sro\,\dot{x}(A,\xi,\omega_0) = \omega_0 Srox(A,\xi,\omega_0) = \omega_0 |x(t)|_{max}$$

The pseudonym speed is the value the speed which gives a value of the kinetic energy of the mass of the oscillator equal to that of the maximum deformation energy of the spring:

$$E_c = \frac{1}{2} m(\dot{x}(t))^2 = \frac{1}{2} m\left[Sro\dot{x}(A,\xi,\omega_0)\right]^2 = \frac{1}{2} m\,\omega_0^2 |x(t)|_{max}^2 = \frac{1}{2} k|x(t)|_{max}^2 = E_p$$

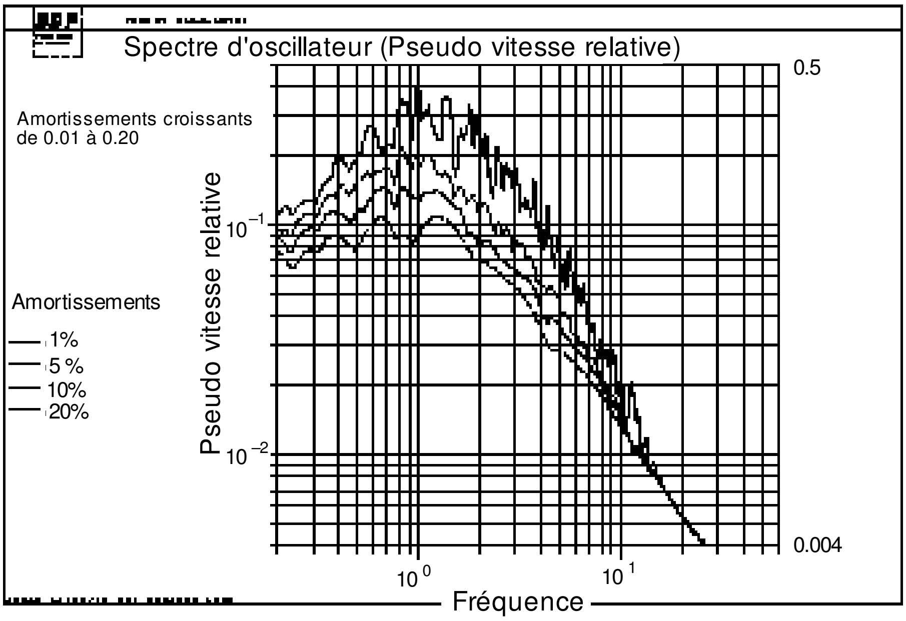



**Figure 1.2.2-a: Spectrum of oscillator in pseudonym relative speed**

### 1.2.3 Spectrum of oscillator in absolute pseudo-acceleration

In the same way for a weak reduced damping, one can define the spectrum of pseudo-acceleration defined by:

$$Sro\,\ddot{x}(A,\xi,\omega_0)=\omega_O^2 Srox(A,\xi,\omega_0)=\omega_0^2|x(t)|_{max}$$

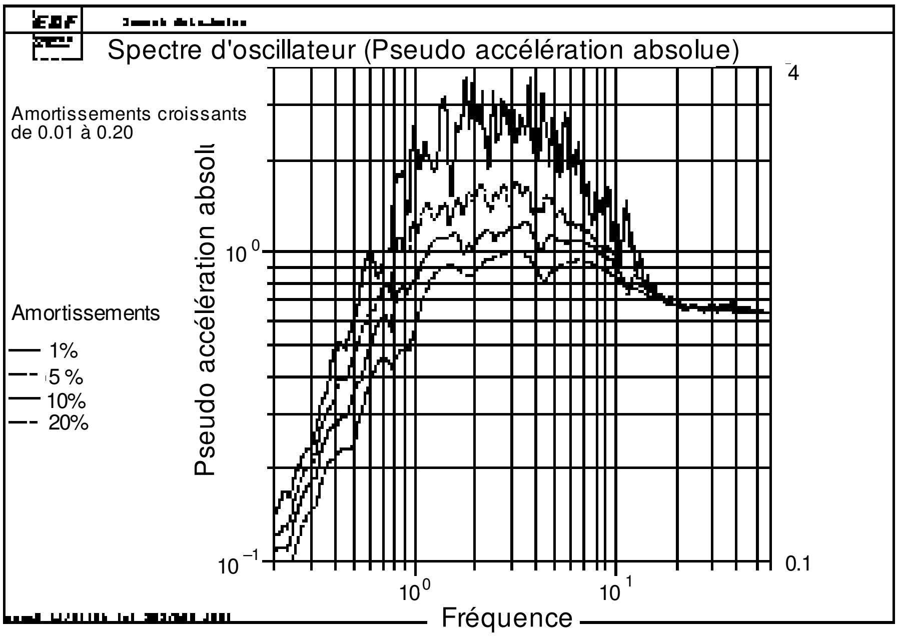



**Figure 1.2.3-a: Spectrum of oscillator in pseudonym absolute acceleration**

The interest of this spectrum of pseudo-acceleration lies in the fact that $Sro\,\ddot{x}(A,\xi,\omega_0)$ is a good approximation of the maximum of absolute acceleration $\ddot{X}(t)$ . Indeed, at the moment when relative displacement is maximum, relative speed is cancelled and the reduced equation is written $\ddot{x}+0+\omega_0^2 x_{max}=-\ddot{s}$ , which shows us that

$$|\ddot{X}|_{max}=|\ddot{x}+\ddot{s}|_{max}=|\omega_0^2 x_{max}|=\omega_O^2 Srox(A,\xi,\omega_0)=Sro\,\ddot{x}(A,\xi,\omega_0)$$

For this reason, this spectrum of oscillator is called **spectrum of absolute pseudo-acceleration**.

The asymptote of this high frequency spectrum (acceleration at worthless period) corresponds to the answer of a clean high frequency oscillator, i.e. very rigid. In this case, the mass tends to completely follow the imposed movement of the base. This asymptote thus corresponds to maximum acceleration $|A(t)|_{max}$ imposed movement (ground or anchor point of the oscillator). It is reached in practice starting from the cut-off frequency of the spectrum. For this reason, one says that a accélérogramme is fixed, for example, on $0.15\,g$ , when its maximum amplitude and its spectrum of oscillator of absolute pseudo-acceleration at worthless period are equal to $0.15\,g$ .

## 1.3 Determination of the spectrum of oscillator

Determination of the spectrum of oscillator of a accélérogramme $A(t)$ is available in the operator `CALC_FONCTION` [U6.34.04] with the keyword `SPEC_OSCI` : it is obtained by digital integration of the equation of DUHAMEL by the method of NIGAM [R5.05.01]. This order provides the spectrum of absolute pseudo-acceleration and, on request, the spectrum of pseudovelocity or the spectrum of relative displacement.

## 1.4 Representation and use of the spectra of oscillators

### 1.4.1 Representation tri-logarithmic curve

The spectra of answer of oscillator are usually represented by graphics tri-logarithmic curves which make it possible to read on only one graph the three sizes: relative displacement, relative pseudovelocity, absolute pseudo-acceleration.

This representation is obtained by tracing the spectrum of relative pseudovelocity $Sro\,\dot{x}$ in coordinates $\log-\log$ such as $\log Sro\,\dot{x}=f(\log\omega_0)$, on which one defers two graduations complementary to $\pm 45°$ if the scale of the graduations logarithmic curves is the same one on the two axes:

- a graduation logarithmic curve with $+45°$ to measure relative displacements $\log Sro\,x=\log\left(\frac{Sro\,\dot{x}}{\omega_0}\right)=\log Sro\,\dot{x}-\log\omega_0$
- a graduation logarithmic curve with $-45°$ to measure absolute accelerations $\log Sro\,\ddot{x}=\log(\omega_0\,Sro\,\dot{x})=\log Sro\,\dot{x}+\log\omega_0$

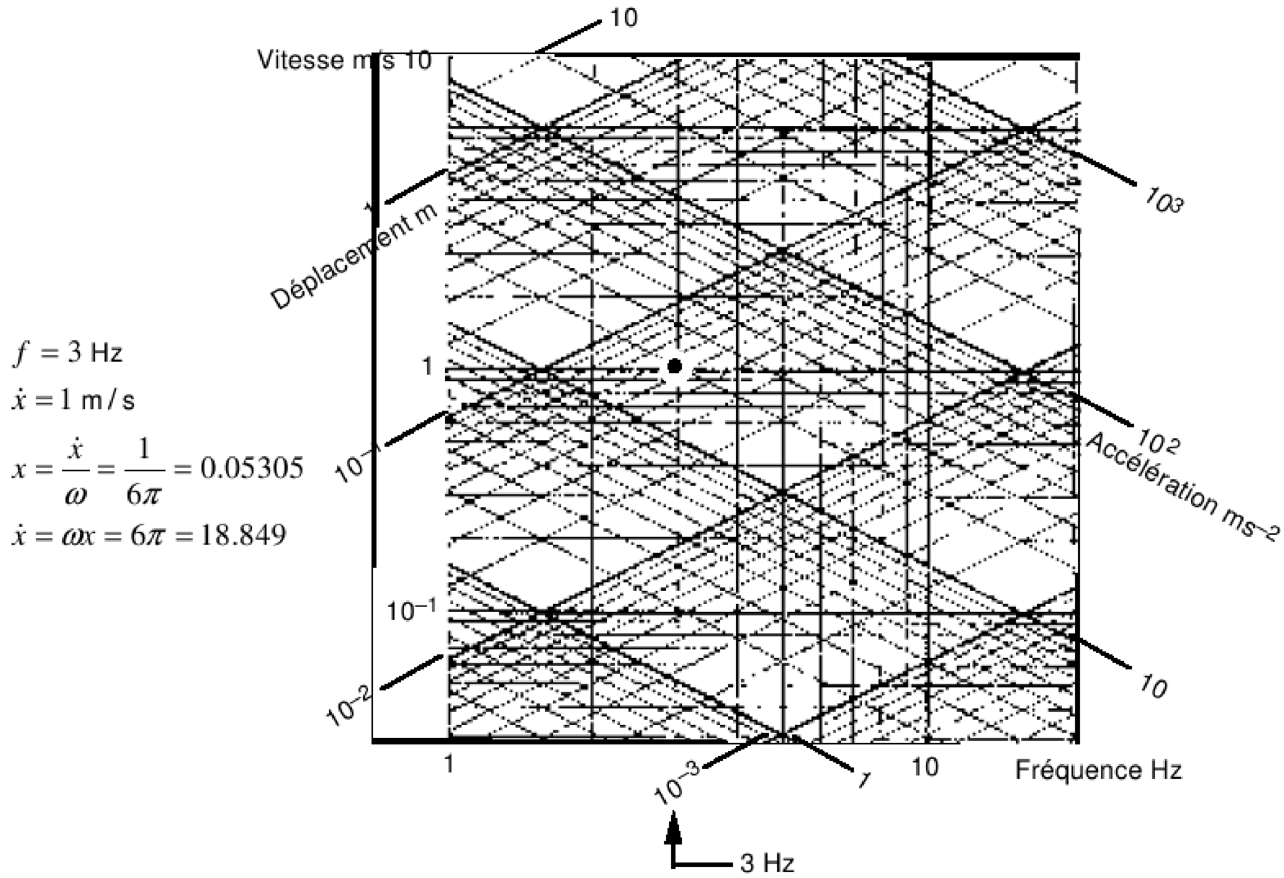



**Figure 1.4.1-a: Representation tri-logarithmic curve**

### 1.4.2 Use of the spectra of oscillators

To evaluate the maximum answer of a modal oscillator $(\omega_i, \xi_i)$ with a accélérogramme $A(t)$, one uses **the spectrum of absolute pseudo-acceleration**.

It is represented in *Code_Aster* by a tablecloth made up of several functions $Sro\,\ddot{x}=f(freq)$ with $\xi_n = cte$ .

One uses a linear interpolation on the damping reduced for $\xi_n < \xi_i < \xi_{n+1}$ because dynamic amplification with resonance for $\omega=\omega_0$ (either $\eta=1$ ) is equal to $\frac{x_m}{s_0}=\frac{1}{2\xi_i}$ [éq A2.2-3].

The variation of the module of the answer in the vicinity of resonance also justifies an interpolation logarithmic curve for $\omega_m < \omega_i < \omega_{m+1}$ . The spectrum of oscillator must be represented with a discretization in sufficiently fine frequency to limit the effects of the interpolation.

## 1.5 Spectra of oscillators used for studies

For the studies of industrial facilities, such as the nuclear power plants, the seismic analysis led to establish several models:

- a model of the civil engineer of design of the buildings to determine:
  - accidental requests for the calculation of the frameworks of these buildings;
  - movements imposed on the anchor points of the equipment (reactor vessel, supports of the networks of pipings, electrical equipment boxes.) at various levels of the buildings;
- models of study of checking of each equipment subjected to the imposed movements amplified by the dynamic behavior of the buildings.

### 1.5.1 Spectrum of ground of design and checking of the buildings

This stage, the equipment is known only like inertial overloads and one can admit that they do not bring any rigidity to the building. The structures in this case are subjected to a spectrum of ground.

The frequential contents of a spectrum of oscillator reflect that of the accélérogramme used and “are thus marked” by the properties of the ground instead of recording. To work out the spectrum of ground at the stage of the project, it is thus recommended to establish the spectra of oscillators for several accélérogrammes and to build a spectrum envelope which smoothes antiresonances.

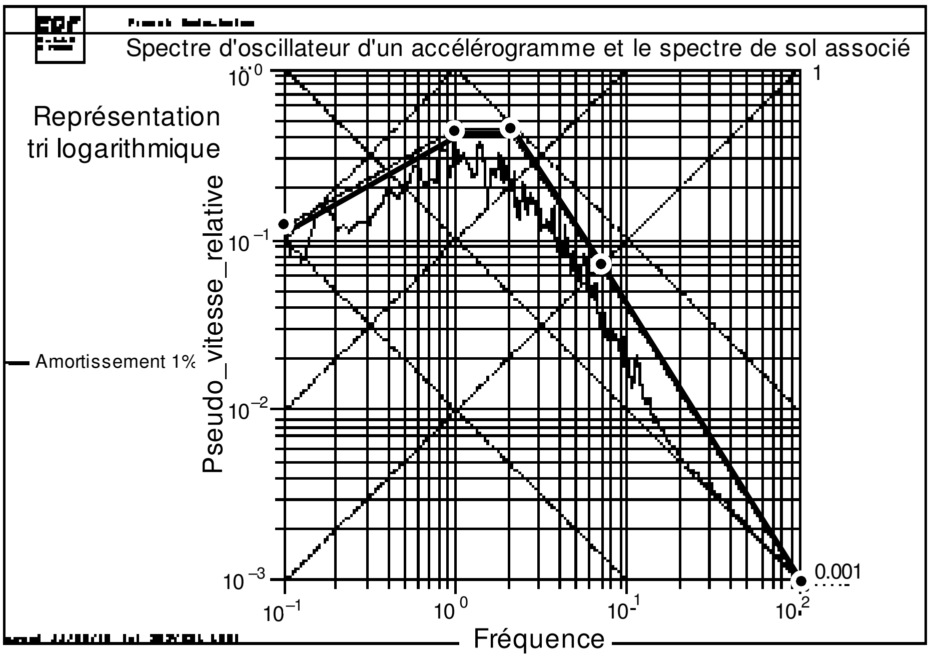



**Figure 1.5.1-a: Spectrum of ground for a project**

**Note:**

*In many cases, one does not know the rotation movement imposed by the earthquake, since the accélérogrammes of known earthquakes result from recordings of seismographs, sensors to a degree of freedom of translation.*

### 1.5.2 Spectrum of floor of checking of the equipment

The study of the dynamic behavior of the equipment subjected to the movements imposed by the structure support on the fulcrums is possible starting from the accélérogrammes of answer in these points, results of the transitory analysis of the behavior of the building: these accélérogrammes, known as of floor, makes it possible to build spectra of floors.

For a checking of the equipment, one can limit oneself to a spectral analysis starting from the spectra of floor and the differential displacements imposed on the supports.

The spectra of floor are representative of the dynamic amplification brought by the structure support: a smoothing of the spectrum can be useful to take into account uncertainty on the position of the Eigen frequencies of the building, but one will take care to preserve realistic margins, since the spectrum of ground is already one raising of the seismic request. The spectrum of oscillator must be represented with a discretization in sufficiently fine frequency “to collect” resonances of the structure.

**Note:**

*Techniques of direct determination of the spectra of floors, starting from the spectrum of ground and modes of the structure were developed [bib1], but are not currently available in Code_Aster.*

# 2 Seismic answer by modal recombination

## 2.1 Recalls of the formulation

The spectral method of seismic analysis is based on the formulation of the transitory dynamic response by modal recombination presented in the documents “Methods of RITZ in dynamics linear and nonlinear” [R5.06.01] and “Analyzes seismic by direct method or modal recombination” [R4.05.01].

Let us summarize the principles of the approach detailed in the note [R4.05.01] for a structure represented in form discretized by the matric system:

$$\mathbf{M}\ddot{\mathbf{U}}+\mathbf{C}\dot{\mathbf{U}}+\mathbf{K}\,\mathbf{U}=\mathbf{F}(t) \qquad \text{éq 2.1-1}$$

**Notations moving absolute**

$\mathbf{U}$ *represent all the components of the movement (internal degrees of freedom of structure and degrees of freedom subjected to an imposed movement): one separates them in the form* $\mathbf{U}=\begin{pmatrix}\mathbf{X}\\ \mathbf{s}\end{pmatrix}$ .

*The operators describing the structure become:* $\mathbf{K}=\begin{bmatrix}\mathbf{k}_{xx} & \mathbf{k}_{xs}\\ \mathbf{k}_{sx} & \mathbf{k}_{ss}\end{bmatrix}$ $\mathbf{C}=\begin{bmatrix}\mathbf{c}_{xx} & \mathbf{c}_{xs}\\ \mathbf{c}_{sx} & \mathbf{c}_{ss}\end{bmatrix}$

$\mathbf{M}=\begin{bmatrix}\mathbf{m}_{xx} & \mathbf{m}_{xs}\\ \mathbf{m}_{sx} & \mathbf{m}_{ss}\end{bmatrix}$

The problem moving relative of the structure compared to the supports with the decomposition absolute Movement = relative Movement + Movement of training led to introduce the change of variable $\mathbf{U}=\mathbf{u}+\mathbf{E}$ .

**Assumption**

*It is supposed that no force of excitation is applied to the degrees of freedom of structure, which reduces the second member* $\mathbf{F}(t)$ *, with the same partition with* $\mathbf{F}=\begin{pmatrix}\mathbf{0}_x\\ \mathbf{r}_s\end{pmatrix}$

The equation 2.1-1 becomes then:

$$\begin{bmatrix}\mathbf{m}_{xx} & \mathbf{m}_{xs}\\ \mathbf{m}_{sx} & \mathbf{m}_{ss}\end{bmatrix}\begin{bmatrix}\ddot{\mathbf{X}}\\ \ddot{\mathbf{s}}\end{bmatrix}+\begin{bmatrix}\mathbf{c}_{xx} & \mathbf{c}_{xs}\\ \mathbf{c}_{sx} & \mathbf{c}_{ss}\end{bmatrix}\begin{bmatrix}\dot{\mathbf{X}}\\ \dot{\mathbf{s}}\end{bmatrix}+\begin{bmatrix}\mathbf{k}_{xx} & \mathbf{k}_{xs}\\ \mathbf{k}_{sx} & \mathbf{k}_{ss}\end{bmatrix}\begin{bmatrix}\mathbf{X}\\ \mathbf{s}\end{bmatrix}=\begin{bmatrix}\mathbf{0}_x\\ \mathbf{r}_s\end{bmatrix}$$ **éq. 2.1-2**

The first system of equations extracts from [éq. 2.1-2]:

$$\mathbf{m}_{xx}\ddot{\mathbf{X}}+\mathbf{c}_{xx}\dot{\mathbf{X}}+\mathbf{k}_{xx}\mathbf{X}=-\mathbf{m}_{xs}\ddot{\mathbf{s}}-\mathbf{c}_{xs}\dot{\mathbf{s}}-\mathbf{k}_{xs}\mathbf{s}$$ **éq. 2.1-3**

the determination of the answer of the internal degrees of freedom of structure allows, while the second:

$$\mathbf{m}_{sx}\ddot{\mathbf{X}}+\mathbf{c}_{sx}\dot{\mathbf{X}}+\mathbf{k}_{sx}\mathbf{X}+\mathbf{m}_{ss}\ddot{\mathbf{s}}+\mathbf{c}_{ss}\dot{\mathbf{s}}+\mathbf{k}_{ss}\mathbf{s}=r$$

the determination of the reaction allows $r(t)$ between the structure and its supports.

**Notations moving relative**

The vector of the movement is broken up $\mathbf{U}=\begin{pmatrix}\mathbf{X}\\ \mathbf{s}\end{pmatrix}$ in the sum of two vectors $\mathbf{u}$ and $\mathbf{E}$ with:

$\mathbf{E}=\begin{pmatrix}\mathbf{e}_{xs}\mathbf{s}_x\\ \mathbf{s}_x\end{pmatrix}$ the static movement of deformation of the structure under the effect of the displacements imposed on the supports, which one calls movement of training,

and $\mathbf{u}=\begin{pmatrix}\mathbf{x}_s\\ \mathbf{0}_s\end{pmatrix}$ the movement of residual deformation of the structure compared to the preceding deformation to obtain the absolute deformation $\mathbf{U}=\begin{pmatrix}\mathbf{X}\\ \mathbf{s}\end{pmatrix}$.

The vector $\mathbf{e}_{xs}\mathbf{s}_s$ is obtained by carrying out the static raising, on the internal degrees of freedom of the structure, of the displacements imposed on the supports is (by using the first line of equation 2.1-2 and by eliminating the dynamic components): $\mathbf{e}_{xs}\mathbf{s}_s=-\mathbf{k}_{xx}^{-1}\mathbf{k}_{xs}\mathbf{s}_s$ , that is to say $\mathbf{e}_{xx}=-\mathbf{k}_{xx}^{-1}\mathbf{k}_{xs}$ .

The passage of the absolute movement to the relative movement can be also written by introducing the operator of passage $\mathbf{\Psi}$ :

$$\mathbf{U}=\begin{pmatrix}\mathbf{X}\\ \mathbf{s}\end{pmatrix}=\mathbf{u}+\mathbf{E}=\begin{pmatrix}\mathbf{x}_s\\ \mathbf{0}_s\end{pmatrix}+\begin{pmatrix}\mathbf{e}_{xs}\mathbf{s}_s\\ \mathbf{s}_s\end{pmatrix}=\mathbf{\Psi}\begin{pmatrix}\mathbf{x}_s\\ \mathbf{s}_s\end{pmatrix} \text{ with } \mathbf{\Psi}=\begin{bmatrix}\mathbf{I}_{xx} & \mathbf{e}_{xs}\\ \mathbf{0}_{sx} & \mathbf{I}_{ss}\end{bmatrix}$$

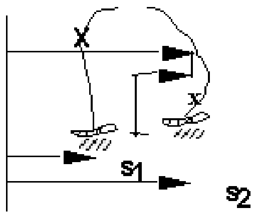
The system [éq. 2.2.1-1] takes the general shape then:

$$\mathbf{M}\mathbf{\Psi}\begin{pmatrix}\ddot{\mathbf{x}}_x\\ \ddot{\mathbf{s}}_s\end{pmatrix}+C\mathbf{\Psi}\begin{pmatrix}\dot{\mathbf{x}}_x\\ \dot{\mathbf{s}}_s\end{pmatrix}+\mathbf{K}\mathbf{\Psi}\begin{pmatrix}\mathbf{x}_x\\ \mathbf{s}_s\end{pmatrix}=\begin{pmatrix}\mathbf{0}_x\\ \mathbf{r}_s\end{pmatrix}$$ éq 2.1-4

### 2.1.1 Multiple imposed movement: multi-support

This situation corresponds to a discrete number of points of connection of the structure in supports subjected to different imposed displacements.
We interest initially in the quasi-static answer $\mathbf{X}^{qs}$ degrees of freedom of structure. Then, $\mathbf{e}_{xs}=\boldsymbol{\varphi}_s$, where the matrix $\boldsymbol{\varphi}_s$ indicate the matrix of the static modes reduced to the degrees of freedom of structure.
One thus obtains: $\mathbf{X}^{qs}=\boldsymbol{\varphi}_s \mathbf{s}$

The matrix $\boldsymbol{\varphi}_s$ gather $6\,n_{appuis}$ static modes for the models of structures and 3 times the number of supports for the models of continuous mediums. Each static mode $\boldsymbol{\varphi}_s=-\mathbf{k}_{xx}^{-1}\mathbf{k}_{xs_j}$ is a mode of fastener, corresponding to a unit displacement imposed on a component of support, the other worthless components, and being produced by the operator `MODE_STATIQUE` [28].
The change of reference mark can then be expressed by:




$$\mathbf{U}=\left(\frac{\mathbf{X}}{\mathbf{s}}\right)=\left(\frac{\mathbf{x}_s}{\mathbf{0}_s}\right)+\left(\frac{\boldsymbol{\varphi}_s\mathbf{s}_s}{\mathbf{s}_s}\right)$$

The absolute answer is written then in the form: $\mathbf{U}=\begin{bmatrix}\mathbf{I}_{xx} & \boldsymbol{\varphi}_s\\ \mathbf{0}_{sx} & \mathbf{I}_{ss}\end{bmatrix}\begin{bmatrix}\mathbf{x}_x\\ \mathbf{s}_s\end{bmatrix}$

where $\mathbf{I}$ indicate the matrix identity, $\mathbf{x}_x$ the vector of relative displacements of the structure compared to the supports, and $\begin{bmatrix}\mathbf{I}_{xx} & \boldsymbol{\varphi}_s\\ \mathbf{0}_{sx} & \mathbf{I}_{ss}\end{bmatrix}=\boldsymbol{\Psi}$ is the matrix of passage of the absolute movement to the relative movement.
The system [éq. 2.1-3] becomes as follows:

$$\begin{bmatrix}\mathbf{m}_{xx} & \mathbf{m}_{xs}\\ \mathbf{m}_{sx} & \mathbf{m}_{ss}\end{bmatrix}\begin{bmatrix}\mathbf{I}_{xx} & \boldsymbol{\varphi}_s\\ \mathbf{0}_{sx} & \mathbf{I}_{ss}\end{bmatrix}\begin{bmatrix}\ddot{\mathbf{x}}_x\\ \ddot{\mathbf{s}}_s\end{bmatrix}+\begin{bmatrix}\mathbf{c}_{xx} & \mathbf{c}_{xs}\\ \mathbf{c}_{sx} & \mathbf{c}_{ss}\end{bmatrix}\begin{bmatrix}\mathbf{I}_{xx} & \boldsymbol{\varphi}_s\\ \mathbf{0}_{sx} & \mathbf{I}_{ss}\end{bmatrix}\begin{bmatrix}\dot{\mathbf{x}}_x\\ \dot{\mathbf{s}}_s\end{bmatrix}+\begin{bmatrix}\mathbf{k}_{xx} & \mathbf{k}_{xs}\\ \mathbf{k}_{sx} & \mathbf{k}_{ss}\end{bmatrix}\begin{bmatrix}\mathbf{I}_{xx} & \boldsymbol{\varphi}_s\\ \mathbf{0}_{sx} & \mathbf{I}_{ss}\end{bmatrix}\begin{bmatrix}\mathbf{x}_x\\ \mathbf{s}_s\end{bmatrix}=\begin{bmatrix}\mathbf{0}_x\\ \mathbf{r}_s\end{bmatrix} \quad \text{éq 2.1.1-1}$$

The first equation of this system is written:

$$\mathbf{m}_{xx}\ddot{\mathbf{x}}_x+\mathbf{c}_{xx}\dot{\mathbf{x}}_x+\mathbf{k}_{xx}\mathbf{x}_x=-\left(\mathbf{m}_{xx}\boldsymbol{\varphi}_s+\mathbf{m}_{xs}\right)\ddot{\mathbf{s}}_s-\left(\mathbf{c}_{xx}\boldsymbol{\varphi}_s+\mathbf{c}_{xs}\right)\dot{\mathbf{s}}_s-\left(\mathbf{k}_{xx}\boldsymbol{\varphi}_s+\mathbf{k}_{xs}\right)\mathbf{s}_s$$

The diagonalisation of the term of rigidity is acquired:

$$\begin{bmatrix}\mathbf{k}_{xx} & \mathbf{k}_{xs}\\ \mathbf{k}_{sx} & \mathbf{k}_{ss}\end{bmatrix}\begin{bmatrix}\boldsymbol{\varphi}_s\\ \mathbf{I}_{ss}\end{bmatrix}=\begin{bmatrix}\mathbf{0}_x\\ \mathbf{r}_s\end{bmatrix}$$

Concerning the terms of damping, decoupling is acquired only if damping is proportional to rigidity, usually allowed assumption.

This makes it possible well to uncouple the system [éq 2.1.1-1]:

$$\mathbf{m}_{xx}\ddot{\mathbf{x}}_x+\mathbf{c}_{xx}\dot{\mathbf{x}}_x+\mathbf{k}_{xx}\mathbf{x}_x=\mathbf{g}_x(t) \qquad \text{éq 2.1.1-2}$$

with $\mathbf{g}_x(t)=-\mathbf{m}_{xx}\left(\boldsymbol{\varphi}_s+\mathbf{m}_{xx}^{-1}\mathbf{m}_{xs}\right)\ddot{\mathbf{s}}_s=-\mathbf{m}_{xx}\ddot{\mathbf{x}}^{qs}-\mathbf{m}_{xs}\ddot{\mathbf{s}}_s$

The loading are equivalent $\mathbf{g}_x(t)$ is due contrary to the sum of the acceleration of the supports and acceleration relating to the static modes.
This formulation must be interpreted like the decomposition of the movement of the structure in a movement of training corresponding to an instantaneous static deformation (differential displacement of the supports) and a relative movement corresponding to the inertial effects around this new static deformation.

This interpretation is in conformity with the ranking of the requests defined by the rules of construction (ASME, RCC-M):

- the constraints induced by the relative movement are, as for the statical stresses, of the primary constraints (effects of inertia),
- constraints induced by differential displacements of the supports which are they classified in secondary constraints.

The answer of the degrees of freedom of structure being thus determined, the second equation of the system [éq. 2.1.1. - 1] makes it possible to obtain the reaction $\mathbf{s}(t)$ .

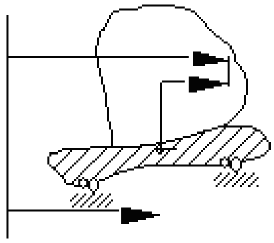
### 2.1.2 Single imposed movement: mono-support

The movement of training corresponds then to a movement of noted rigid body $\begin{bmatrix}\boldsymbol{\varphi}_R\\ \mathbf{s}_R\end{bmatrix}$. The matrix $\boldsymbol{\varphi}_R$ indicate the matrix gathering the rigid modes of body of the reduced structure to the degrees of freedom of structure.
The structure is subjected to this total movement with an acceleration $\gamma(t)$, to which one superimposes the relative movement of the degrees of freedom of structure: absolute acceleration can thus be written in the form:

$$\ddot{\mathbf{U}}=\begin{bmatrix}\ddot{\mathbf{X}}\\ \ddot{\mathbf{s}}\end{bmatrix}=\begin{bmatrix}\ddot{\mathbf{x}}_x\\ \mathbf{0}_s\end{bmatrix}+\begin{bmatrix}\boldsymbol{\varphi}_R\\ \mathbf{s}_R\end{bmatrix}\gamma(t)$$

The rigid part of displacement checks the relation: $\begin{bmatrix}\mathbf{k}_{xx} & \mathbf{k}_{xs}\\ \mathbf{k}_{sx} & \mathbf{k}_{ss}\end{bmatrix}\begin{bmatrix}\boldsymbol{\varphi}_R\\ \mathbf{s}_R\end{bmatrix}=\mathbf{0}$

One from of deduced that: $\boldsymbol{\varphi}_R=\boldsymbol{\varphi}_S\mathbf{s}_R$



$$\mathbf{U}=\left(\frac{\mathbf{X}}{\mathbf{s}}\right)=\left(\frac{\mathbf{x}_s}{\mathbf{0}_s}\right)+\left(\frac{\boldsymbol{\varphi}_R\mathbf{s}_x}{\mathbf{s}_s}\right) \text{ and } \boldsymbol{\Psi}=\begin{bmatrix}\mathbf{I}_{xx} & \boldsymbol{\varphi}_R\\ \mathbf{0}_{sx} & \mathbf{I}_{ss}\end{bmatrix}$$

The second member of the equation [éq. 2.1.1-2] can then be written in the form:

$$\mathbf{g_x}(t)=-\mathbf{m_{xx}}\left(\boldsymbol{\varphi}_\mathbf{s}+\mathbf{m_{xx}^{-1}}\mathbf{m_{xs}}\right)\ddot{\mathbf{s}}_{xx}=-\left(\mathbf{m_{xx}}\boldsymbol{\varphi}_\mathbf{R}+\mathbf{m_{xs}}\mathbf{s}_R\right)\gamma(t)$$ **éq. 2.1.2-1**

This writing highlights well which the seismic loading depends only on overall acceleration and inertia associated with the rigid modes of body.

### 2.1.3 Summary

The equations [éq 2.1.1-2] and [éq 2.1.2-1] lead to the general form (written without index for more clearness):

$$\mathbf{m}\ddot{\mathbf{z}}+\mathbf{c}\dot{\mathbf{z}}+\mathbf{k}\mathbf{z}=-\mathbf{m}\left(\boldsymbol{\varphi}_\mathbf{X}+\mathbf{m^{-1}}\mathbf{m_{xs}}\right)\ddot{\mathbf{s}}=-\mathbf{m}\mathbf{O}\ddot{\mathbf{s}}$$ éq 2.1.3-1

Terms $\mathbf{m_{xs}}$ correspond under the terms of coupling of the matrix of mass with the degrees of freedom of support: this fraction of the total mass is very weak and it is justified to neglect it. Let us recall that this term is indeed null for the models of structures whose matrix of mass is diagonal: models masses - arises, models with elements of the type “lumped farmhouse”.

In this case, the simplified formulas are obtained:

- •mono-support: $\mathbf{O}=\boldsymbol{\varphi}_\mathbf{R}$ where $\boldsymbol{\varphi}_\mathbf{R}$ are the six modes of solid body,
- •multi-support: $\mathbf{O}=\boldsymbol{\varphi}_s$ where $\boldsymbol{\varphi}_s$ are them $6n$ supports modes of fastener.

The second member $-\mathbf{m}\mathbf{O}$ is built by the operator `CALC_CHAR_SEISME` [U4.63.01].

## 2.2 Answer on modal basis

### 2.2.1 Temporal answer of a modal oscillator

If the studied structure is represented by its spectrum of low frequency real clean modes $\varphi$ in embedded base, solution of $\left(K-M\omega^2\right)\varphi=0$ or of $\left(k-m\omega^2\right)\varphi=0$ one can introduce a new transformation $\mathbf{x}=\boldsymbol{\varphi}\,\mathbf{q}$ and the system of equations [éq 2.1.3-1] is written, by using the matrix of modal factors of participation $\mathbf{P}$ :

$$\ddot{\mathbf{q}}+\frac{\boldsymbol{\phi}^\mathbf{T}\mathbf{c}\boldsymbol{\phi}}{\boldsymbol{\phi}^\mathbf{T}\mathbf{m}\boldsymbol{\phi}}\dot{\mathbf{q}}+\omega^2\mathbf{q}=-\frac{\boldsymbol{\phi}^\mathbf{T}\mathbf{m}\mathbf{O}}{\boldsymbol{\phi}^\mathbf{T}\mathbf{m}\boldsymbol{\phi}}\ddot{\mathbf{s}}=-\mathbf{P}\ddot{\mathbf{s}}$$ éq 2.2.1-1

**Assumption:**

*For industrial studies concerned with the seismic analysis by spectral method, one limits oneself to the case of damping proportional, known as of RAYLEIGH, for which one can diagonaliser the term* $\frac{\boldsymbol{\varphi}^\mathbf{T}\mathbf{c}\boldsymbol{\varphi}}{\boldsymbol{\varphi}^\mathbf{T}\mathbf{m}\boldsymbol{\varphi}}=2\xi\boldsymbol{\omega}$ *. Damping is then represented by a modal damping* $\xi_i$ *possibly different for each clean mode [R4.05.01].*

Each clean mode, characterized by the parameters $\left(\omega_i,\xi_i\right)$ is compared to a simple oscillator whose behavior is represented in the case general by:

$$\ddot{q}_i+2\xi_i\omega_i\dot{q}_i+\omega^2 q_i=-\left(\mathbf{P}\ddot{\mathbf{s}}\right)_i$$ éq 2.2.1-2

Let us recall that them $\ddot{\mathbf{s}}$ are accelerations of training.

### 2.2.2 Modal factor of participation in mono-support

When the movement of training is single, [éq 2.2.1-2] becomes:

$$\ddot{q}_i+2\xi_i\omega_i\dot{q}_i+\omega^2 q_i=-p_i\ddot{s}$$ éq 2.2.2-1

with

$$p_i=\frac{\boldsymbol{\varphi}_i^T.\mathbf{m}.\mathbf{O}}{\boldsymbol{\varphi}_i^T.\mathbf{m}.\boldsymbol{\varphi}_i}=\frac{\boldsymbol{\varphi}^T.\mathbf{m}.\boldsymbol{\varphi}_R}{\mu_i}$$ éq 2.2.2-2

where $\mu_i$ is the generalized modal mass, which depends on the standardisation of the clean mode. Let us state some properties of the factors of modal participation $p_i$ in the case of rigid of translation, but extensible modes with the modes of rotation.

- A mode $\varphi_{RX}$, that we will note $\delta_X$, to recall that components in the direction $X$ are unit, belongs to the space of dimension $N$ degrees of freedom of which them $N$ clean modes constitute a base in which $\boldsymbol{\delta}_\mathbf{X}=\sum_i \alpha_i \boldsymbol{\varphi}$.
  From the properties of orthogonality of the clean modes $\boldsymbol{\varphi}_i^T.\mathbf{m}.\boldsymbol{\varphi}_j=\mu_i\delta_{ij}$, the coefficients are identified $\alpha_i$ with the factors of modal participation $p_{iX}$ in the direction $X$ and

$$\boldsymbol{\delta}_\mathbf{X}=\sum_{i=1,\dots,N} p_{iX}\boldsymbol{\varphi}_i$$ éq 2.2.2-3

- Moreover $\boldsymbol{\delta}_\mathbf{X}^\mathbf{T}.\mathbf{m}.\boldsymbol{\delta}_\mathbf{X}=m_T$ total mass of the structure trained in the movement $\boldsymbol{\delta}_\mathbf{X}$, which leads to, having noted $N$ the full number of degrees of freedom of the system, equal to the number of modes:

$$\boldsymbol{\delta}_\mathbf{X}^\mathbf{T}.\mathbf{m}.\boldsymbol{\delta}_\mathbf{X}=\sum_{ij} p_{iX}\,p_{jX}\,\boldsymbol{\varphi}_j^T.\mathbf{m}.\boldsymbol{\varphi}_i=\sum_{i=1,\dots,N} p_{iX}^2\mu_i \quad \text{and} \quad m_T=\sum_{i=1,\dots,N} p_{iX}^2\mu_i \quad \text{or} \quad \frac{\sum_{i=1,\dots,N} p_{iX}^2\mu_i}{m_T}=1$$ éq 2.2.2-4

The modal parameter $p_{iX}$ depends on the standard of the clean mode and is accessible, for each clean mode, in the concept result of the type `mode_meca` under the name `FACT_PARTICI_DX` ; in the same way $p_{iX}^2\mu_i$, independent of the standard, is accessible under the name of `MASS_EFFE_UN_DX`.

### 2.2.3 Modal factor of participation in multi-support

For a multiple imposed movement, [éq 2.2.1-2] becomes:

$$\ddot{q}_i+2\xi_i\omega_i\dot{q}_i+\omega^2 q_i=-\sum_j p_{ij}\ddot{s}_j$$ éq 2.2.3-1

with

$$p_{ij}=\frac{\boldsymbol{\varphi}_i^T.\mathbf{m}.\mathbf{O}}{\boldsymbol{\varphi}_i^T.\mathbf{m}.\boldsymbol{\varphi}_i}=\frac{\boldsymbol{\varphi}_i^T.\mathbf{m}.\boldsymbol{\varphi}_{s_j}}{\mu_i}$$ éq 2.2.3-2

where $\mu_i$ is the generalized modal mass, which depends on the standardisation on the clean mode and them $p_{ij}$ can be regarded as factors of participation relating to the mode $i$ and with a direction $j$ of imposed movement of a support.

As previously, one can establish [bib4] the two properties:

$$\boldsymbol{\varphi}_{sj}=\sum_i p_{ij}\boldsymbol{\varphi}_i \quad \text{and} \quad \boldsymbol{\varphi}_{sj}^T.\mathbf{m}.\boldsymbol{\varphi}_{sj}=\sum_{i=1,\dots,N} p_{ij}^2\mu_i$$ éq 2.2.3-3

One makes, at this stage, no assumption of dependence between the various terms $p_{ij}$. Let us recall that the components $\ddot{s}_j$ express the acceleration of training applied to a direction of support $j$.
Factors of participation $p_{ij}$ are not built independently and seem only intermediate variables in the order `COMB_SISM_MODAL` [U4.84.01].

# 3 Seismic answer by spectral method

The spectral method is an approximate technique of evaluation of the maximum of the answer of the structure starting from the maxima of answer of each modal oscillator read on the spectrum of oscillator of the excitation.

## 3.1 Spectral response of a modal oscillator in mono-support

The maximum answer in relative displacement of a modal oscillator $(\omega_i, \xi_i)$ for a direction $X$ is given by reading on a spectrum of oscillator of absolute pseudo-acceleration confer [§1.4.2] the value of absolute pseudo-acceleration $a_{iX} = Sro\, \ddot{x}_X(A, \xi_i, \omega_i)$ and while dividing by $\omega_i^2$, from where:

$$q_{iXmax} = p_{iX} \frac{SRO\, \ddot{x}_X(A, \xi_i, \omega_i)}{\omega_i^2} = p_{iX} \frac{a_{iX}}{\omega_i^2}$$ éq 3.1-1

The contribution of this oscillator to the relative displacement of the structure for the component $x^k$ depends on the factor on participation and component $\varphi_i^k$ in physical space:

$$x_{iXmax}^k = \phi_i^k q_{iXmax} = \phi_i^k p_{iX} \frac{a_{iX}}{\omega_i^2}$$ éq 3.1-2

and the contribution to the pseudonym absolute acceleration $\ddot{x}^k$ is the same $\ddot{x}_{iXmax}^k = \phi_i^k p_{iX} a_{iX}$.

## 3.2 Spectral response of a modal oscillator in multi-support

One proceeds in the same manner to determine, starting from the value read $\ddot{S}_{jX}$ on the spectrum of oscillator of absolute pseudo-acceleration associated with $\ddot{s}_j$, the contribution of the support $j$ in the direction $X$:

$$q_{iXmax\,j} = p_{ijX} \frac{SRO\, \ddot{s}_j(A, \xi_i, \omega_i)}{\omega_i^2} = \frac{\ddot{S}_{jXX}}{\omega_i^2}$$ éq 3.2-1

The expression of the contribution of this oscillator to the relative displacement of the structure for the component $x^k$ in physical space and for an imposed movement $j$ becomes:

$$x_{iXmax\,j}^k = \phi_i^k q_{iXmax\,j} = \phi_i^k p_{ijX} \frac{\ddot{S}_{jX}}{\omega_i^2}$$ éq 3.2-2

## 3.3 Generalization with other sizes

**Note:**

*The spectral method of analysis is strictly limited to the sizes depending linearly on displacements in linear elasticity: generalized strains, stresses, efforts, nodal forces, reactions of supports.*

*In particular it cannot apply to equivalent sizes of deformation or constraints (Von Mises).*

For each size $R^k$, component of a field by elements, it is possible to calculate the modal component $r_i^k$ associated with the clean mode $\varphi_i$ what leads to:

$$R^k_{iXmax}=r^k_i\,q_{iXmax}=r^k_i\,p_{iX}\frac{a_{iX}}{\omega_i^2}$$ éq 3.3-1

or

$$R^k_{iXmax\,j}=r^k_i\,q_{iXmax\,j}=r^k_i\,p_{ijX}\frac{\ddot{S}_{jX}}{\omega_i^2}$$ éq 3.3-2

# 4 Rules of combination of the modal answers

To evaluate one raising of the answer $R$ structure, one must now combine the modal answers $R^k_{imax}$ defined previously. Several levels of combination are necessary:

- combination of the clean modes selected $\varphi_i$, for $i=1,\dots,N_r\leqslant N$,
- static correction by pseudo-mode,
- effect of the excitations different applied to groups from supports,
- combination according to the directions of excitation earthquake.

## 4.1 Direction of the earthquake and directional answer

Various considerations result in separately studying the seismic behavior according to each direction of space:

- for the study of a building on a ground, the accélérogramme of the vertically imposed movement is different from that describing the horizontal movement, itself different following two orthogonal directions from space;
- for the study of equipment, the spectra of floor differ significantly according to the three directions from space, since they integrate the participations of various modes of the building (inflection of floors, inflection or torsion of the framework.).

This resulted in establishing a directional modal answer $R_x$ starting from spectra of oscillator different and factors of modal participation established in each direction $X$ representing one of the directions of the reference mark `TOTAL` of definition of the grid ( $X,Y,Z$ ) or an explicitly definite particular direction by the user.

## 4.2 Choice of the modes suitable to combine

To represent correctly the modes of deformation likely to be excited by the imposed movement, it would be necessary to know all the clean modes of frequency lower than the cut-off frequency of the spectrum, beyond which there is no significant dynamic amplification. This condition can prove to be difficult to fill for the complex structures having a large number of clean modes.

Size of the modal base necessary ( $N_r$ modes) must thus be evaluated to make sure that no mode having an important contribution in the internal efforts and the constraints was omitted in each studied direction.

### 4.2.1 Expression of the modal deformation energy

The deformation energy associated with each clean mode $U_i=\frac{1}{2}\mathbf{x}^{\mathbf{T}}_{\mathbf{i\,max}}.\mathbf{k}.\mathbf{x}_{\mathbf{i\,max}}$ can be expressed for a particular direction:

$$U_{iX}=\frac{1}{2}\left(p_i\frac{a_{iX}}{\omega_i^2}\right)^2\boldsymbol{\varphi}_i^T.\mathbf{k}.\boldsymbol{\varphi}_i=\frac{1}{2}\left(p_{iX}\frac{a_{iX}}{\omega_i^2}\right)^2\omega_i^2\,\mu_i=\frac{1}{2}\frac{a_{iX}^2}{\omega_i^2}p_{iX}^2\,\mu_i$$ éq 4.2.1-1

This expression corresponds to an excitation mono-support and can extend to the case from the multi-support.

The ranking of the modes with decreasing deformation energies makes it possible not to retain systematically, for a general study of the structure, modes which do not produce significant deformations. On the other hand, for the study of the effect of the requests in a particular zone of the structure, it will be necessary to use the "local" modes which can be detected by an analysis of the distribution of the deformation energy on groups of mesh.

Let us note that one does not have an estimate of the total deformation energy to quantify the mistake made by being unaware of certain modes.

### 4.2.2 Expression of the modal kinetic energy

The kinetic energy associated with each clean mode is written $V_i=\frac{1}{2}\dot{\mathbf{x}}^{\mathbf{T}}_{\mathbf{i\,max}}.\mathbf{m}.\dot{\mathbf{x}}_{\mathbf{i\,max}}$ who gives:

$$V_{iX}=\frac{1}{2}\left(p_{iX}\frac{a_{iX}}{\omega_i}\right)^2\boldsymbol{\varphi}_i^T.\mathbf{m}.\boldsymbol{\varphi}_i=\frac{1}{2}\frac{a_{iX}^2}{\omega_i^2}p_{iX}^2\mu_i \qquad \text{éq 4.2.2-1}$$

The expression [éq 4.2.2-1] utilized effective modal mass $p_{iX}^2\mu_i$ defined in [§2.2], which makes it possible to state the criterion of office plurality of the unit effective modal masses [éq 2.2.2-4].

**Criterion of office plurality of the effective modal masses**

*The quality of a modal base, from the point of view of the representation of the inertial properties of the structure, is evaluated by cumulating, for this direction, the unit effective modal masses of the modes available. A threshold of admissibility of 95% of the total mass is usually allowed. The same criterion can apply an excitation multi-support partially in the case of with* $N_r$ *modes while comparing* $\boldsymbol{\varphi}_{sj}^T.\mathbf{m}.\boldsymbol{\varphi}_{si}$ *and* $\sum_{i=1}^{N_r}p_{ij}^2\mu_i$ .

*The sum of the effective modal masses is worth in fact the total mass which works on the selected modal basis. In other words, this working total mass is worth the total mass minus the contributions in mass which are carried by embedded degrees of freedom (which thus do not work on the modal basis). Thus, for example, on a system with 1 degree of freedom mass-arises with a mass* $M1$ *at the top and another mass* $M2$ *at the level to erase it, then the working mass will be worth* $M1$ *and total mass* $M1+M2$ *. Consequently, the unit effective modal mass for the only mode of the system will be worth* $M1/(M1+M2)$ *. The total office plurality will thus have the same value and, according to the ratio in* $M1$ *and* $M2$ *, one will not be able inevitably to thus reach 90% of the total mass* $(M1+M2)$ *, even by considering all the modes (there is only one only mode on this example). In practice, more the model with the finite elements will be so and realistic, more the difference between the working mass and the total mass will be weak.*

**Estimate of the mistake made with an incomplete modal base**

The criterion of office plurality of the effective modal masses cannot always be satisfied. Indeed one limits oneself in general to a modal base of $N_r$ clean modes with $N_r\ll N$ degrees of freedom. For rigid foundations, the spectrum of the Eigen frequencies necessary usually exceeds the cut-off frequency of the spectrum of oscillator.

From the expression [éq 4.2.2-1], one can write the total kinetic energy in the form:

$$V_X=\sum_{i=1}^{N_r}V_{iX}+\sum_{i=N_r+1}^{N}V_{iX}$$

who allows to express the absolute error from [éq 3.1-1]:

$$2\Delta V_X = \sum_{N_r+1}^{N} V_{iX} = \sum_{N_r+1}^{N} \frac{a_{iX}^2}{\omega_i^2} p_{iX}^2 \mu_i \leq \frac{a_{(N_r+1)X}^2}{\omega_i^2} \sum_{N_r+1}^{N} p_{iX}^2 \mu_i$$

while noting $a_{(n+1)X}=SRO\,\ddot{x}_X(A,\xi_{min},\omega_{N_r+1})$ the value read on the spectrum of absolute pseudo-acceleration for $\omega_n \leq \omega_{n+1}$ and the modal damping weakest $\xi_{min}$ likely to give the raising amplitude. If the maximum frequency of the base $f_n$ exceed the cut-off frequency, then $a_{(n+1)X}=a_{nX}=|A(t)_{max}|$ . This gives one raising of the absolute error:

$$\Delta V_X = \frac{1}{2}\frac{a_{(n+1)X}^2}{\omega_i^2}\sum_{N_r+1}^{N} p_{iX}^2 \mu_i = \frac{1}{2}\frac{a_{(N_r+1)X}^2}{\omega_i^2}\left(m_T - \sum_{1}^{N_r} p_{iX}^2 \mu_i\right) \qquad \text{éq 4.2.2-2}$$

### 4.2.3 Conclusion

The sizes allowing for choice of the modes necessary to each analysis are available in *Code_Aster* (operator `POST_ELEM` with the options `MASS_INER`, `ENER_POT` and `ENER_CIN` and modal parameters `FACT_PARTICI_DX` and `MASS_EFFE_UN_DX` in the concept result of the type `mode_meca`).

No criterion of automatic admissibility is currently programmed and the sizes $\boldsymbol{\varphi}_{Sj}^T.\mathbf{m}.\boldsymbol{\varphi}_{Sj}$ and $\sum_{1}^{N_r} p_{ij}^2 \mu_i$ , necessary to the checking of the criterion for an excitation multi-support, are not printed.

## 4.3 Static correction by pseudo-mode

### 4.3.1 Mono-support

The evaluation of one raising of the answer to a seismic excitation requires, as suggests it the preceding analysis, a correction by a term representing the static contribution of the neglected clean modes.

If one subjects the structure to a static constant acceleration in the direction $X$ , the linear elastic answer $\boldsymbol{\varphi}_{aX}$ is solution of $\mathbf{k}.\boldsymbol{\varphi}_{aX}=\mathbf{m}.\delta_X \mathbf{I}$ , without dynamic amplification. The field of elastic displacements $\boldsymbol{\varphi}_{aX}$ nodes of the structure subjected to a constant acceleration in each direction is produced by the operator `MODE_STATIQUE` [U4.52.14] with the keyword `PSEUDO_MODE`.

By breaking up this deformation on the basis as of clean modes, one obtains (cf [§2.2.2]):

$$\mathbf{k}.\boldsymbol{\varphi}_{aX}=\mathbf{m}.\sum_{i}^{N} p_{IX}\boldsymbol{\varphi}_i \quad \text{from where} \quad \boldsymbol{\varphi}_{aX}=\mathbf{k}^{-1}.\mathbf{m}.\sum_{i=1}^{N} p_{iX}\boldsymbol{\varphi}_i=\sum_{i=1}^{N}\frac{p_{iX}}{\omega_i^2}\boldsymbol{\varphi}_i$$

This makes it possible to introduce a pseudo-mode $\boldsymbol{\varphi}_{cX}$ , for each direction, while withdrawing from the quasi-static mode $\boldsymbol{\varphi}_{aX}$ static contributions of $N_r$ modes used $\boldsymbol{\varphi}_i$ in low reserve:

$$\boldsymbol{\varphi}_{cX}=\boldsymbol{\varphi}_{aX}-\sum_{i=1}^{N_r}\frac{p_{iX}}{\omega_i^2}\boldsymbol{\varphi}_i \qquad \text{éq 4.3.1-1}$$

The expression [éq 4.3.1-1] is homologous with the term $\left(m_T - \sum_{i=1}^{N_r} p_{iX}^2 \mu_i\right)$ [éq 4.2.2-2] and the pseudo-mode makes it possible to introduce a correction of the static effects of the neglected modes.

The contribution of the static pseudo-mode of correction is the value read on the spectrum of absolute pseudo-acceleration for an infinite pulsation, said ZPA (zero period acceleration). In practice however, as it is not always easy to respect Critère of office plurality of the effective modal masses for the base of $N_r$ selected clean modes, it is allowed to take the value: $a_{(N_r+1)X} = SRO\,\ddot{x}_X(A, \xi_{min}, \omega_{N_r})$ for the pulsation $\omega_{N_r+1} \geq \omega_{N_r}$ mode following the last of the base of clean modes selected, and for the modal damping weakest $\xi_{min}$. This is considered to raise this contribution in order to get a conservative result.
If the modal base were correctly selected, i.e. if it includes all the modes until the cut-off frequency of the seismic signal, the keyword FREQ_COUP of the operator COMB_SISM_MODAL offer the possibility of specifying the value of the frequency where the asymptote (value of ZPA) starts. One thus ensures oneself not to raise the answer unduly.

The correction to be brought to relative displacements and the sizes which result some (generalized efforts, constraints, reactions of supports) in excitation mono-support is then:

$$\mathbf{x}_{cX}^k = a_{(N_r+1)X} \cdot \boldsymbol{\varphi}_{cX}^k$$

in accordance with the conditions of estimate of the error cf [§ 4.2.2].

For the evaluation of the correction of absolute acceleration, one obtains:

$$\ddot{\mathbf{x}}_{cX}^k = a_{(N_r+1)X} \cdot \left(\boldsymbol{\delta}_{\mathbf{X}} - \sum_{i=1}^{N_r} p_{iX} \boldsymbol{\varphi}_i\right)^k$$

### 4.3.2 Multi-support

In excitation multi-support, the formulation of the pseudo-mode and its contribution take again the preceding principle.
The field of elastic displacements $\boldsymbol{\phi}_{ajX} = \mathbf{k}^{-1} . \mathbf{m} . \boldsymbol{\phi}_{SjX}$ nodes of the structure subjected to a unit acceleration of the support $j$ in the direction $X$ is produced by the operator MODE_STATIQUE [U4.52.14] with the keyword PSEUDO_MODE.

One defines, in the same way that for the mono-support, $a_{(N_{rj}+1)jX}$ like the "ZPA" with the support $j$ in the direction $X$. It data, by default, by the value read on the SRO for the last mode of the modal base. For a quite selected base, it must correspond in practice to the asymptotic value of acceleration. One can adjust it, if necessary, by the keyword FREQ_COUP.
The correction to be brought to relative displacements and the sizes which result some writes then, for the support $j$ in the direction $X$ :

$$\boldsymbol{x}_{cjX} = \boldsymbol{\phi}_{cjX}\, a_{(N_{rj}+1)jX} \quad \text{with} \quad \boldsymbol{\phi}_{cjX} = \boldsymbol{\phi}_{ajX} - \sum_{i=1}^{N_r} \frac{P_{ijX} \boldsymbol{\phi}_i}{\omega_i^2}$$

For absolute acceleration, the correction is written:

$$\ddot{\boldsymbol{x}}_{cjX} = \left(\boldsymbol{\phi}_{SjX} - \sum_{i=1}^{N_r} \boldsymbol{\phi}_i P_{ijX}\right) a_{(N_{rj}+1)jX}$$

## 4.4 General information on the rules of combination

The rules of combination or office plurality of the various components, modal or directional, are multiple and more or less complex to implement.

One presents the methods "natural" from the point of view of their aptitude required one raising realistic requests induced in a structure represented by a base of real clean modes resulting from a model in linear elasticity, raising estimated without transitory analysis for a size of component $G^k$, that one will name $G^k_{max}$. For the continuation the suffix $max$ indicate the estimate of the maximum value reached during the seismic excitation, by being unaware of the moment when it was reached and the index $r$ applies to clean modes, pseudo-modes, directions of supports…

**Note:**

*Whatever the method of combination used, the value of a component obtained by combination cannot be used as data to calculate a new size: for example, the calculation of one raising of a differential displacement between two points must be calculated mode by mode, then combined.*

### 4.4.1 Arithmetic combination

$$G^k_{max} = \sum_r G^k_{r\max}$$

It is not usable for the directional answers since the spectral method disregards moments when the maximum values are reached in two directions or for two different modes. No relation of phase, and thus of sign, exists between the contributions to combine. It is thus available only in the case multi-support, for the office plurality of the modal directional answers of support and for the office plurality of differential displacements.

### 4.4.2 Combination in absolute value

$$G^k_{max} = \sum_r \left| G^k_{r\max} \right|$$

In an obvious way, it can provide an upper limit, since it supposes that all the contributions reach their maximum at the same moment with the same sign. Too much penalizing, it is available, but unusable industrially.

### 4.4.3 Simple quadratic combination

This method is also known under denomination SRSS (Public garden Root of Sum of Squares).

$$G^k_{max} = \sqrt{\sum_r \left( G^k_{r\max} \right)^2}$$

**Assumption:**

***assumption** who justifies this method of combination can state himself:*
*the probable maximum of the energy stored in the structure is the sum of the probable maxima of the energy stored on each mode and each directional component of the earthquake, i.e., with respect to energy, the clean modes and the components of the earthquake are uncoupled. It is similar to the rule of addition of the Gaussian random variables and to worthless average.*

The validity of this assumption, which will be discussed for each typical case of use of this method of combination, is not established and various proposals were presented to obtain a better approximation whenever it is put at fault cf [§ 3.4.1.2].

In addition, one will be able to refer to [bib3] for a criticism of this approach, in particular of his aptitude to consider a maximum probable of the deformations and constraints, but the alternative approach that it evokes was the object of any development in *Code_Aster*.

## 4.5 Establishment of the directional answer in mono-support

The directional answer, previously definite, is obtained by simple quadratic combination of two terms which we will discuss:

$$R_X=\sqrt{R_m^2+\left(R_{qs}+R_c\right)^2}$$

with:

$R_m$ dynamic answer combined of $N_r$ modal oscillators,

$R_{qs}$ quasi-static combined answer of the modal oscillators (=0 except for the combination of the Gupta type),

$R_c$ contribution of the static correction of the neglected modes (pseudo-mode).

The assumptions justifying the method of quadratic combination simple, on this level, do not seem to have to be reconsiderations [bib1].

To simplify the notations, one notes $R_m$ instead of $R_{mX}$ ,…

### 4.5.1 Combined answer of the modal oscillators

The answer of the structure $R_m$ , in a direction of earthquake, is obtained by one of the possible combinations of the contributions of each clean mode taken into account for this direction. The number of possible methods proves simply the difficulty in releasing a sufficient justification to guarantee a conservative and realistic estimate. If the simple quadratic combination (SRSS or CQS) is evoked by all, one will retain [bib1] that it is often put at fault and one will prefer the complete quadratic combination to him (CQC). The other methods are available for possible comparisons.

#### 4.5.1.1 Somme of the absolute values

This combination corresponds to an assumption of complete dependence of the oscillators associated with each mode specific and led to a systematic overvaluation with the answer:

$$R_m=\sum_{i=1}^{N_r}|R_i|$$

#### 4.5.1.2 Simple quadratic combination (CQS)

By considering that the contribution of each modal oscillator is an independent random variable, an estimate of the maximum answer, for the component of displacement $x_{max}^k$ , can be obtained by simple quadratic combination of the contributions of each mode from where, for an excitation mono-support:

$$x_{max}^k=\sqrt{\sum_{i=1}^{N_r}\left(x_{i\,max}^k\right)^2}=\sqrt{\sum_{i=1}^{N_r}\left(\varphi_i^k\;p_i q_{i\,max}\right)^2} \qquad \text{éq 4.5.1.2 - 1}$$

Generally, for any size $R_i$ associated with a modal oscillator $(\omega_i,\xi_i)$ :

$$R_m=\sqrt{\sum_{i=1}^{N_r}R_i^2}$$

It constitutes a good approximation of reality when the spectrum of oscillator defining the earthquake is to broad waveband, and when the clean modes of the structure are quite separate from/to each other and are located inside or in the vicinity of this band. It is in particular put at fault if clean modes are at frequencies close or for modes far away from the peak to excitation [bib2]. The other methods of combination of the modal answers try to correct this point.

#### 4.5.1.3 Quadratic combination supplements (CQC)

The quadratic combination supplements (established by DER KIUREGHIAN [bib5]) makes a correction to the preceding rule by introducing coefficients of correlation depending on depreciation and the distances between close clean modes:

$$R_m=\sqrt{\sum_{i_1}\sum_{i_2}\rho_{i_1 i_2}R_{i_1}R_{i_2}}$$

with the coefficient of correlation:

$$\rho_{ij}=\frac{8\sqrt{\xi_i\xi_j\omega_i\omega_j}(\xi_i\omega_i+\xi_j\omega_j)\omega_i\omega_j}{(\omega_i^2-\omega_j^2)^2+4\xi_i\xi_j\omega_i\omega_j(\omega_i^2+\omega_j^2)+4(\xi_i^2+\xi_j^2)\omega_i^2\omega_j^2}$$

or by introducing the report of pulsation or frequencies between two modes $\eta=\omega_j/\omega_i$ :

$$\rho_{ij}=\frac{8\eta\sqrt{\xi_i\xi_j\eta}(\xi_i+\xi_j\eta)}{(1-\eta^2)^2+4\eta\xi_i\xi_j(1+\eta^2)+4\eta^2(\xi_i^2+\xi_j^2)}$$

and for $\xi$ constant:

$$\rho_{ij}=\frac{8\eta\xi^2(1+\eta)\sqrt{\eta}}{(1-\eta^2)^2+4\eta\xi^2(1+\eta^2)+8\eta^2\xi^2}$$

#### 4.5.1.4 Combination of ROSENBLUETH

This rule (proposed by E. ROSENBLUETH and J. ELORDY [bib6]) introduced a correlation between modes, different from that of method CQC. The answers of the oscillators are combined by double nap (Double Sum Combination):

$$R_m=\sqrt{\sum_{i_1}\sum_{i_2}\rho_{i_1 i_2}R_{i_1}R_{i_2}}$$

It requires an additional data, the duration $s$ “strong” phase of the earthquake. The coefficient of correlation is then:

$$\rho_{ij}=\left(1+\left(\frac{\omega'_i-\omega'_j}{\xi'_i\omega_i+\xi'_j\omega_j}\right)^2\right)^{-1}$$ where $\omega'_i=\omega_i\sqrt{1-\xi'^2_i}$ and $\xi'^2_i=\xi_i+\frac{2}{s\omega_i}$

#### 4.5.1.5 Combination with rule of the 10%

The close modes (of which the frequencies different from less than 10%) are initially combined by summation of the absolute values. The values resulting from this first combination are then combined quadratically (simple quadratic combination). This method was proposed by American regulation U.S. Nuclear Regulatory Commission (Regulatory Guides 1.92 - February 1976) to attenuate the conservatism of the method of nap of the absolute values. It remains at fault for structures with a dense frequency spectrum clean and should not be used any more.

#### 4.5.1.6 Combination according to Gupta

GUPTA [NRC1.92], to take into account the correlations between modes due to the quasi-static part of the answer, introduced the rigid factor of answer, which varies from 0 to 1 the correlation between the modal answers of intermediate frequencies enters $f_1$ and $f_2$, two frequencies to be determined (typically of $2\ Hz$ with $20\,Hz$ ).

GUPTA breaks up each modal answer $R_r$ in a dynamic part $R_r^p$ and a quasi-static part $R_r^{qs}$ : $R_r^{qs}=\alpha_r R_r$ and $R_r^p=\sqrt{1-\alpha_r^2}R_r$

Thus, for each mode $r$ , the rigid factor of answer is affected $\alpha_r$ with the modal answer $R_r$ :

- $\alpha_r=0$ for $f\le f_1$ and $\alpha_r=1$ for $f\ge f_2$
- $\alpha_r$ is estimated for the frequency $f_r$ according to the following formula:

$$\alpha_r=\frac{\ln f_r/f_1}{\ln f_2/f_1}$$

The dynamic combined answer of the modal oscillators is carried out according to the combination 'CQC':

$$R_d=\sqrt{\sum_{r_1}\sum_{r_2}\rho_{r_1 r_2}R_{r_1}^p R_{r_2}^p}$$

The quasi-static combined answer of the modal oscillators is carried out according to an algebraic combination:

$$R_{qs}=\sum_{i=1}^{N_r} R_r^{qs}$$

This combination according to GUPTA is available only in the case mono-support.

### 4.5.2 Contribution of the static correction of the neglected modes

The contribution of the pseudo-mode cf [§4.3.1] can be combined quadratically because independence with the contributions of the modes of vibration is not disputed.

## 4.6 Establishment of the directional answer in multi-support

### 4.6.1 Calculation of the total answer

The order of the combinations to be carried out differs according to whether the excitations of the supports (or the groups of supports) can be regarded as correlated or décorrélées between them.

#### 4.6.1.1 Case with groups of décorrélés supports

The sequence of the combinations can be stated as follows, in the order:
- for each direction $X$ , for each mode $i$ , for each group of supports:
    1) calculation of the directional answers of supports modal (combination intra-group on the supports): considering that the supports of the same group are correlated, one proposes a summation of the algebraic values;
- for each direction $X$ , for each group of supports $j$ :
- calculation of the combined answer of the modal oscillators (combination on the modes);
- calculation of the pseudo-mode $R_{Xj}^c$ (static correction of the neglected modes);
- calculation of the movement of training $R_{Xj}^e$ ;
- calculation of the directional answer of support: $R_{Xj}=\sqrt{R_{Xj}^{m\,2}+R_{Xj}^{c\ 2}+R_{Xj}^{e\ 2}}$
- for each direction (combination joint committee on the groups of supports): calculation of the directional answer, according to a quadratic summation CQS because the groups are supposed to be décorrélés between them;
- calculation of the total answer (combination on the directions).

#### 4.6.1.2 Case with correlated supports

If all the supports are correlated between them (when one cannot display of groups of décorrélés supports), one proposes the following diagram, which has the advantage of being able to establish a parallel between the treatment of the case mono-support treated like such and the case mono-support treated like a typical case of the case multi-support. The sequence of the combinations can be summarized as follows:

- for each direction $X$ , for each mode $i$ :

- calculation of the modal directional answers (combination on the supports): the supports being supposed correlated between them, a summation of the algebraic values or absolute values are proposed;
- for each direction $X$ :
- calculation of the combined answer $R_X^m$ modal oscillators (combination on the modes);
- calculation of the pseudo-mode $R_X^c$ (static correction of the modes neglected, algebraic summation on the supports of the pseudo-modes of support $R_{Xj}^c$ );
- calculation of the movement of training $R_{Xj}^e$ (algebraic summation on the supports of the movements of training of support $R_{Xj}^e$ ) ;
- calculation of the directional answer: $R_X=\sqrt{R_X^{m2}+R_X^{c\,2}+R_X^{e\,2}}$ ,
- calculation of the total answer (combination on the directions).

### 4.6.2 Separate calculation of the components primary education and secondary of the answer

Each component is the object of a similar separate treatment. This approach is adapted to postprocessings RCC-M in force for the seismic analysis of pipings [§ 4.9]:

#### 4.6.2.1 Primary education component $R_{IX}$ (inertial answer)

The order of the combinations to be carried out differs according to whether the excitations of the supports (or the groups of supports) can be regarded as correlated or décorrélées between them.

- groups of décorrélés supports:

•for each imposed movement $\ddot{s}_j$ , calculation of the directional answers of support (office plurality on the modes):

$$R_{IjX}=\sqrt{R_j^{m2}+R_j^{c2}}$$

$R_j^m$ answer of support combined of the modal oscillators (office plurality on the modes)

$R_j^c$ contribution of the static correction of the neglected modes (pseudo-mode of support)

•combination of the answers $R_{IjX}$ (office plurality on the supports).

- correlated supports:

•for each imposed movement $\ddot{s}_j$ , calculation of the modal directional answers (office plurality on the supports):

$$R_{IiX}=\sqrt{R_i^{m2}+R^{c2}}$$

$R_i^m$ combined modal answer of the modal oscillators (office plurality on the supports)

$R^c$ contribution of the static correction of the neglected modes (pseudo-mode)

•combination of the answers $R_{IiX}$ (office plurality on the modes).

#### 4.6.2.2 Secondary component $R_{II}$ (quasi-static answer)

•combination of the answers $R_{ej}$

### 4.6.3 Office plurality on the modes

The selection criterion of the method of combination of the contributions of the modes is the same one as for an excitation mono-support and one will use method CQC preferentially.

### 4.6.4 Contribution of the pseudo-mode

The corrective term by pseudo-mode cf [§4.3.2] can be combined quadratically.

### 4.6.5 Contribution of the movements of training

The movement of training of the structure not being uniform, one can add a term with the calculation of the directional answer. This is not necessary if one chooses to regard this static contribution as a specific loading case inducing of the secondary constraints. This term is defined starting from the maximum relative displacement which cannot be known starting from the only spectra of absolute pseudo-acceleration of the supports.

$$R_{ej}=\varphi_{Sj}\delta_{j\,max}$$

$\varphi_{Sj}$ static mode for the support $j$

$\delta_{j\,max}$ maximum relative displacement of the support $j$ compared to a support of reference (for which $\delta_{j\,max}=0$ )

### 4.6.6 Office plurality on the supports

This stage is obligatory, but the choice of the method of combination of the directional answers remains very open. Indeed, the assumption of independence of $\ddot{s}_j$ strongly depends on the clean modes of the structure support of the studied equipment. An analysis of the studied system is necessary to possibly gather the supports by groups: for example, for a piping connecting two buildings, either all the supports are regarded as correlated between them, or one can display groups décorrélés between them (the group of the supports of the building 1, that of the building 2 and, finally, that of the intermediate stanchions), whose supports inside each group are correlated.

Office plurality intra-group

The excitations with the supports of the same group being supposed correlated between them, the office plurality is carried out algebraically according to the linear combination defined by: $R_X=\sum R_{jX}$

Office plurality joint committee

The groups of supports being made up so that they are décorrélés between them, the office plurality joint committee is carried out by simple quadratic combination.

### 4.6.7 Equivalence mono-support multi-support

A calculation relative to excitations with the identical supports all can be carried out according to two different ways, theoretically equivalent:

- calculation mono-support;
- calculation multi-support with excitations identical to the supports.

One draws attention to the fact that, if the results are well strictly equal in the absence of static correction via the pseudo-mode, they can present light differences when the static correction is taken into account, because the respective treatments of this static correction are then not strictly equivalent.

## 4.7 Combination of the directional answers

Two rules of combination of the directional answers are available.

### 4.7.1 Quadratic combination

This combination corresponds to the assumption of strict independence of the answers in each direction cf [§ 3.3.3]. Let us recall that this rule of combination does not have any geometrical meaning, although the three directions of analysis are orthogonal.

$$R=\sqrt{R_X^2+R_Y^2+R_Z^2}$$

The assumptions justifying the method of quadratic combination simple, on this level, do not seem to have to be reconsiderations [bib3], but this method is not used.

### 4.7.2 Combination of NEWMARK

This rule of empirical combination is most usually used and in general leads to estimates slightly stronger than the preceding one. It supposes that when one of the directional answers is maximum, the others are with most equal to $4/10$ their respective maximum contributions. For each direction $i(X, Y, Z)$, the 8 values are calculated:

$$R_l=\pm R_X \pm 0{,}4\, R_Y \pm 0{,}4\, R_Z$$

what leads, by circular shift, with 24 values and $R=\max\ (R_l)$

## 4.8 Warning on the combinations

Several remarks are essential to warn the user on the way of using the methods of combination and the sizes combined in a note of study.

**Notice 1:**

*If one wishes to use arithmetic combinations (direction) and quadratic combinations (modes), the quadratic office pluralities must be always carried out in the last.*

**Notice 2:**

*Any quadratic combination applies only to the sizes for which, in instantaneous values, the office plurality has the direction of a sum: combination of the components of displacement, or effort generalized or of constraint of each clean mode.*
*The modal or directional quadratic combination cannot thus apply to intensities of constraint (constraint principal, of Von Mises, Tresca).*

**Notice 3:**

*The results of a combination, whatever the rule of office plurality, should not be used as data to calculate other sizes: for example, a differential displacement between two points (or a deformation) can be calculated only starting from modal differential displacements that one combines then.*
*A fortiori the generalized efforts and the constraints can be calculated only mode by mode before any combination and not starting from inertias deduced from the fields of acceleration obtained by combination of modal accelerations.*

## 4.9 Lawful practices

### 4.9.1 Partition of the primary and secondary components of the answer

The various supports of a line of piping can be animated different movements. The same section of piping can be distributed on different buildings, different levels or equipment. It thus undergoes a multiple excitation. This results in two types of loading [§ 2.1.2]:

- an excitation whose frequential contents vary from one support to another and who constitutes a primary education loading according to classification RCC-M,
- Seismic Differential Displacements (DDS) inducing a state of stress by displacements imposed on the supports and classified like secondary.

The generalized moments resulting from these 2 loadings intervene separately in the inequations of dimensioning RCC-M and on several levels.
For a deepened postprocessing RCC-M, It is thus necessary to have the components primary education and secondary of the seismic answer.
In a more general way, method of combination of the answers of supports can differ according to whether one treats the case of the inertial or differential components. Moreover, the number of supports concerned with these two summations can not be equal. One is often brought to impose overall differential movements even for supports associated with spectra different users. In addition, of the DDS formulated in rotation are sometimes to consider. They cannot be associated with an inertial loading (limited to the translations).

*Code_Aster* thus propose two treatments:

- Determination of the total answer:
- Contributions inertial and statics of training are cumulated during the calculation of the directional answers of support [§4.6].
- Partition of the primary and secondary components of the total answer:
  The two preceding contributions are not any more cumulated during the calculation of the directional answers and are the object of 2 independent treatments:
- The inertial component is obtained by removing the term of training $R_{ej}$ in the calculation of the total answer [§ 4.6].
- The static component is given by combining the terms of training defined under the keyword `DEPL_MULT_APPUI`. The methods of combination of these loading cases DDS are indicated in the keyword `COMB_DEPL_APPUI`.

### 4.9.2 Method of the spectrum envelope

Even if pipings are subjected to a multiple seismic excitation, the common practice is to be reduced to the calculation of a structure mono-supported while preserving the loading cases DDS. This simplified approach implies to define a single spectrum by direction for all the supports of piping. For each direction, one adopts a spectrum then “wraps” various spectra with the supports. Spectra retained for the horizontal directions $X$ and $Y$ are identical.
In almost the whole of the cases, this method is generating of “margin of dimensioning”.

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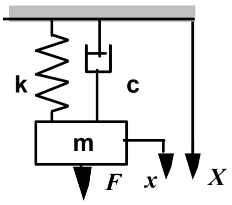
# 6 Description of the versions of the document

| **Version Aster** | **Author (S) Organization (S)** | **Description of the modifications** |
|---|---|---|
| 5 | *J.R. LEVESQUE, L. VIVAN, D. SELIGMANN, EDF*R & D /AMA | Initial text |
| 06/04/11 | *J.R. LEVESQUE, L. VIVAN, Y. PONS EDF*R & D /AMA | |
| 9.4 | S.AUDEBERT EDF-R&D/AMA | |
| 10 | S.AUDEBERT EDF-R&D/AMA | Taking into account of card REX 12005: modification of the office plurality §4.6.6 intra-group |
| 11 | S.AUDEBERT EDF-R&D/AMA | Taking into account of card REX 17054: introduction of the method of modal recombination according to GUPTA |

# Annexe 1 Transitory answer of a deadened simple oscillator

## A1.1 Forced vibration of a system to a degree of freedom in translation

For a simple oscillator of rigidity $k$, of mass $m$ and of viscous damping $c$, the equation of the movement is form:




$$m\ddot{X} + c\dot{X} + kX = 0$$

for which the traditional notations are:

- The own pulsation of the not-deadened system: $\omega_0 = \sqrt{\frac{k}{m}}$ ;
- critical damping: $c_{\text{critique}} = 2\,m\,\omega_0$ ;
- reduced damping, expressed as a percentage critical damping): $\xi = \frac{c}{c_{\text{critique}}} = \frac{c}{2\,m\,\omega_0}$ ;
- the own pulsation of the deadened system: $\omega_0{}' = \omega_0\sqrt{(1-\xi^2)}$ ;
- the static deflection for a force $F_0$ : $\delta_{st} = \frac{F_0}{k}$ ;
- the reduced frequency: $\eta = \frac{\omega}{\omega_0}$ ;
- the reduced equation of the system: $\ddot{X} + 2\xi\omega_0\dot{X} + \omega_0^2 X = 0$ .

The answer **total** with a harmonic excitation of the form $F(t) = F_0\cos(\omega t)$ is the sum:

•of a free answer $X_1(t)$ deadened oscillatory general solution where $X_{l0}$ and $\varphi_0$ are determined by the initial conditions:

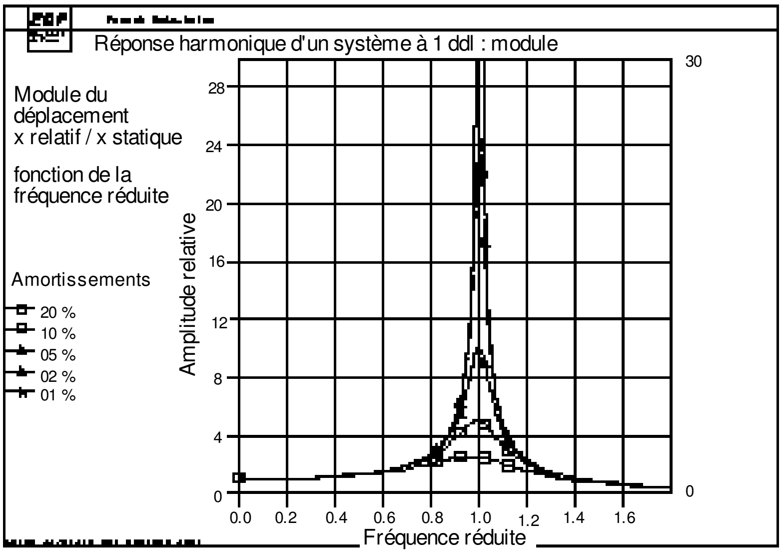
$$X_1(t) = X_0 e^{-\xi\omega_0 t}\cos(\omega'_0 t + \varphi_0)$$

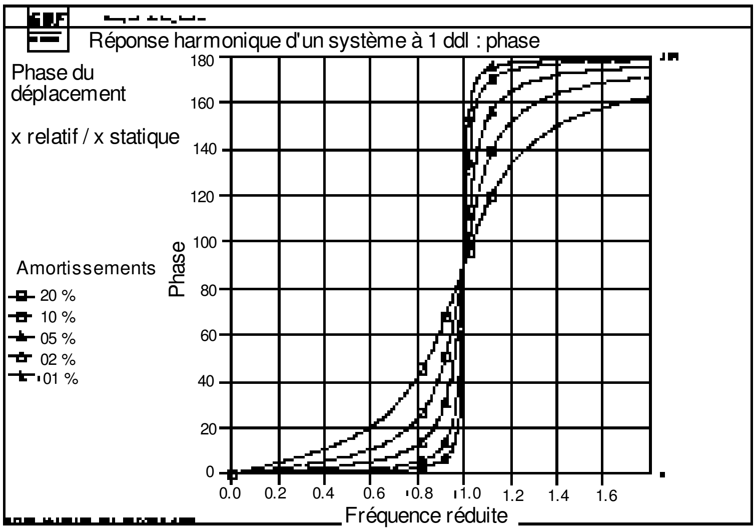
•of a forced answer $X_f(t)$ permanent particular solution $X_f(t) = X_{f0}\cos(\omega t - \varphi)$

$$X_{f0} = \frac{F_0}{\sqrt{(k-m\omega^2)^2 + (c\omega)^2}}\ \varphi = arctg\left(\frac{c\omega}{k-m\omega^2}\right) \qquad \text{éq A1.1-1}$$

who is written in reduced form:

$$\frac{X_{f0}}{\delta_{st}} = \frac{k X_{f0}}{F_0} = \frac{1}{\sqrt{(1-\eta^2)^2 + (2\xi\eta)^2}}\ \varphi = arctg\left(\frac{2\xi\eta}{1-\eta^2}\right) \qquad \text{éq A1.1-2}$$




**A1.1-a figure: Answer of an oscillator in imposed force (module and phase)**

The answer to a harmonic excitation of the form $F(t)=F_0 e^{j\omega t}$ is written with a forced answer permanent particular solution $X_f(t)=X_{f0} e^{(j\omega t-\varphi)}$

$$X_{f0}=\frac{F_0}{\sqrt{(k-m\omega^2)^2+(c\omega)^2}}\ \varphi=\mathrm{arctg}\left(\frac{c\omega}{k-m\omega^2}\right) \quad \text{éq A1.1-3}$$

who is written in reduced form:

$$\frac{k X_{f0}}{F_0}=\frac{1}{1-\eta^2+j2\xi\eta}\equiv H(j\omega)\varphi=\mathrm{arctg}\left(\frac{2\xi\eta}{1-\eta^2}\right) \quad \text{éq A1.1-4}$$

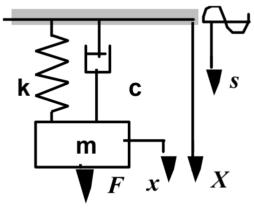
where $H(j\omega)$ is the harmonic answer complexes of a simple oscillator:

$$H(j\omega)=\frac{1}{\sqrt{(1-\eta^2)^2+(2\xi\eta)^2}}$$

# Annexe 2Movement imposed of a system on a degree of freedom in translation

## A2.1 Absolute movement of a system to a degree of freedom

For a simple oscillator of rigidity $k$, of mass $m$ and of viscous damping $c$, the equation of the movement **absolute** is form:




$$m\ddot{X}+c(\dot{X}-\dot{s})+k(X-s)=0$$
$$m\ddot{X}+c\dot{X}+kX=ks+c\dot{s}$$
$$\ddot{X}+2\xi\omega_0\dot{X}+\omega_0^2X=\omega_0^2 s+2\xi\omega_0\dot{s}$$

The answer **forced** with a harmonic imposed movement of the form $s(t)=s_0\cos(\omega t)$ is form $X_m(t)=X_{m0}\cos(\omega t-\varphi_1-\varphi_2)$ nap of two terms of answer, permanent particular solutions:

•term induced by the excitation in displacement $X_{d0}\cos(\omega t-\varphi_d)$

$$X_{d0}=\frac{k\,s_0}{\sqrt{(k-m\omega^2)^2+(c\omega)^2}}\qquad \varphi_d=\mathrm{arctg}\left(\frac{c\omega}{k-m\omega^2}\right)$$

•term induced by the excitation of speed $X_{v0}\cos(\omega t-\varphi_v)$

$$X_{v0}=\frac{\omega c\,s_0}{\sqrt{(k-m\omega^2)^2+(c\omega)^2}}\varphi_v=\mathrm{arctg}\left(\frac{c\omega}{k-m\omega^2}\right)$$

what leads to a total forced answer:

$$X_m(t)=X_m\cos(\omega t-\varphi_1-\varphi_2)\equiv s_0\sqrt{\frac{k^2+(c\omega)^2}{[(k-m\omega^2)^2+(c\omega)^2]}}\cos(\omega t-\varphi_1-\varphi_2)$$

from where the reduced form of the absolute amplitude:

$$\frac{X_m}{s_0}=\sqrt{\frac{1+(2\xi\eta)^2}{[(1-\hat{e}ta^2)^2+(2\xi\eta)^2]}}\qquad \varphi_1=\mathrm{arctg}\left(\frac{2\xi\eta}{1-\eta^2}\right)\qquad \varphi_2=\mathrm{arctg}\left(\frac{1}{2\xi\eta}\right)$$

If the movement imposed on the base is expressed in complex form $s(t)=\Re(s_0e^{j\omega t})$, the relative amplitude or transmissibility can be written starting from the harmonic answer complexes of a simple oscillator $H(j\omega)$

$$\frac{X_m}{s_0}=\sqrt{1+(2\xi\eta)^2}|H(j\omega)| \qquad \text{éq A2.1-1}$$

## A2.2 Movement relative of a system to a degree of freedom

The problem of the answer to an imposed movement can be treated in relative displacement of the mass compared to the base while posing $x = X - s$

The equation of the movement **relative** for a harmonic imposed movement of the form $s(t) = s_0 \cos(\omega t)$ is then of the form $m\ddot{x} + c\dot{x} + kx = -m\ddot{s}$ or in reduced form:

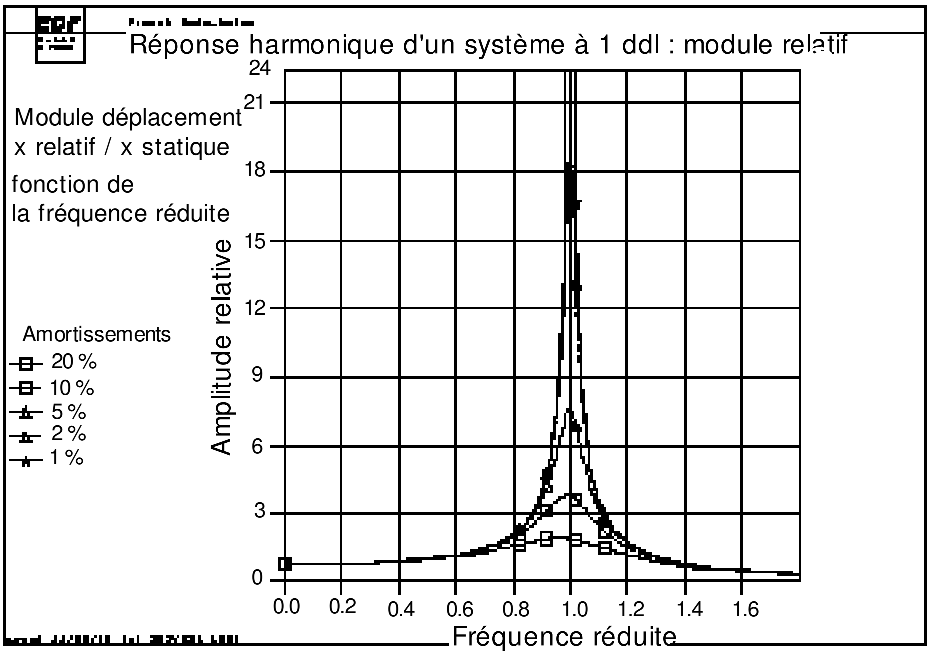
$$\ddot{x} + 2\xi\omega_0\dot{x} + \omega_0^2 x = -\ddot{s} = \omega^2 s_0 \cos(\omega t)$$ éq A2.2-1

The answer **forced** relative is then, for a permanent solution $x_{m0}\cos(\omega t - \varphi)$ ,

$$x_{m0} = \frac{m\omega^2 s_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} \qquad \varphi = \mathrm{arctg}\left(\frac{c\omega}{k - m\omega^2}\right)$$ éq A2.2-2

who is written in reduced form:

$$\frac{x_{m0}}{s_0} = \frac{\eta^2}{\sqrt{(1-\eta^2)^2 + (2\xi\eta)^2}}$$ éq A2.2-3




**A2.2-a figure: Answer of an oscillator moving imposed (module of relative displacement)**

# Annexe 3Movement imposed not periodical of a system on a degree of freedom

With the problem dealt previously was limited to a periodic imposed movement. For a nonperiodic excitation, of variable amplitude with time, being exerted for one finished length of time, one considers the answer to a series of impulses.

## A3.1 Impulse response

The simplest form is the unit impulse force, which applied to a rest mass before the application of the impulse ( $x=\dot{x}=0$ for $t<0$ or $t=0^-$ ) can be written:

$$\tilde{f}=\lim_{\Delta t\to 0}\int_t^{t+\Delta t} F\,dt=F\cdot dt=1=m\dot{X}(t=0)-m\dot{X}(t=0^-)=m\dot{X}_0$$

The initial conditions are then noted $X(t=0)=X_0$ and $\dot{X}(t=0)=\dot{X}_0=\frac{1}{m}$

The general equation of the response in free vibration of a system to a degree of freedom:

$$X_l(t)=e^{-\xi\omega_0 t}\left(X_0\cos\omega'_0 t+\frac{\dot{X}_0+\xi\omega_0 x_0}{\omega'_0}\sin\omega'_0 t\right)$$

becomes the impulse response then $g(t)$ of a system to a degree of freedom

$$X_l(t)=g(t)=\frac{e^{-\xi\omega_0 t}}{m\omega'_0}\sin\omega'_0 t \quad \text{éq A3.1-1}$$

For a nonunit impulse $\tilde{F}=F\cdot\Delta t$ , initial speed is $\dot{X}_0=\frac{F}{m}$ and the answer becomes:

$$X_l(t)=\frac{\tilde{F}e^{-\xi\omega_0 t}}{m\omega'_0}\sin\omega'_0 t=\tilde{F}g(t) \quad \text{éq A3.1-2}$$

If the impulse force is applied to one moment $\tau$ unspecified, the answer is:

$$X_l(t)=\tilde{F}g(t-\tau)$$

## A3.2 Answer in unspecified forced vibration

The force of excitation $F(t)$ can be broken up into a series of impulses of variable amplitude $F(\tau)$ applied to the moment $\tau$ during a time $\tau$ . If $\Delta\tau\to 0$ , the answer to one moment $t$ is obtained by:

$$X(t)=\int_0^t F(\tau)g(t-\tau)d\tau$$

and while replacing by the expression of the impulse response [éq A.3-2], one obtains the equation of convolution for a system at rest at moment 0 of the form:

$$X(t)=\frac{1}{m\omega'_0}\int_0^t F(\tau)e^{-\xi\omega_0(t-\tau)}\sin\omega'_0(t-\tau)d\tau \quad \text{éq A3.2-1}$$

known under the name of integral of DUHAMEL.

## A3.3 Answer moving unspecified imposed

For an analysis moving relative represented by [éq A2.2-1]:

$$\ddot{x}+2\xi\,\omega_0\,\dot{x}+\omega_0^2 x=-\ddot{s}=\omega^2 s_0\cos(\omega t)$$

the integral of DUHAMEL becomes:

$$x(t)=\frac{1}{\omega'_0}\int_0^t \ddot{s}(\tau)e^{-\xi\omega_0(t-\tau)}\sin\omega'_0(t-\tau)\,d\tau \qquad \text{éq A3.3-1}$$