cyclic dynamic Under-structuring

Summary:

This report rests on the concepts of calculation per modal synthesis described in the document [R4.06.02].

We approach the methods of under - cyclic dynamic structuring. Completely dedicated under investigation of the structures to cyclic repetitivity, they benefit the best from the geometrical characteristics of the structure. The methods of CRAIG-BAMPTON and MAC NEAL, developed within this framework, are exposed.

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1 Introduction

In this document, we make the synthesis of the methods of cyclic dynamic under-structuring. We give a definition of the cyclic repetitivity (or cyclic symmetry) and we present the principal incidences of this property on the dynamic behavior of the structure (nodal circles and diameters, double modes). Then, we expose, in a rather detailed way, the two methods of cyclic dynamic under-structuring, implemented in *Code_Aster*. Improvements were made to the classical methods, by the taking into account of the presence of the nodes of the axis.

These methods suppose that the grid of the basic sector is such as its traces on the interfaces right-hand side and left are coïncidentes (compatible grids).

General notations:

ω_m	:	Maximum pulsation of a system ($rad.s^{-1}$)
M	:	Matrix of mass resulting from modeling finite elements
K	:	Matrix of rigidity resulting from modeling finite elements
q	:	Vector of the degrees of freedom resulting from modeling finite elements
f_{ext}	:	Vector of the forces external with the system
f_L	:	Vector of the bonding strengths applied to a substructure
Φ	:	Matrix containing the vectors of a base of projection organized in column
η	:	Vector of the generalized degrees of freedom
B	:	Matrix of extraction of the degrees of freedom of interface
L	:	Matrix of connection
T	:	Kinetic energy
U	:	Deformation energy
Id	:	Matrix identity
λ	:	Diagonal matrix of generalized rigidities
$R_e(\omega)$	:	Matrix of residual dynamic flexibility
$R_e(0)$	:	Matrix of residual static flexibility

Notations specific to the cyclic under-structuring:

N	=	many sectors
α	=	angle formed by the basic sector
β	=	dephasing AND element
Oz	=	cyclic axis of symmetry
θ	=	rotation of angle α and of axis Oz
$\mathrm{Re}(Z)$	=	real part of the complex Z
$\mathrm{Im}(Z)$	=	imaginary part of the complex Z
θ	=	matrix of passage of the nodes of right-hand side to the nodes of left
θ_a	=	matrix of change of sector for the nodes of the axis

Note:

The index	*D*	*is*	*relative*	*with the degrees of freedom of right-hand side*
“	*G*	*“*	*“*	*with the degrees of freedom of left*
“	*has*	*“*	*“*	*with the degrees of freedom of the axis*
“	*1*	*“*	*“*	*with the identified clean modes*
“	*2*	*“*	*“*	*with the unknown clean modes*

2 Cyclic repetitivity

2.1 Definition

It is said that a structure is with cyclic repetitivity of axis Oz, if there exists an angle $0<\alpha<\pi$ such that the structure is geometrically and mechanically invariant by rotation around Oz of this angle. If α is the smallest angle checking this property, then any angular portion of angle α structure is called “basic sector” (or “irreducible sector”).

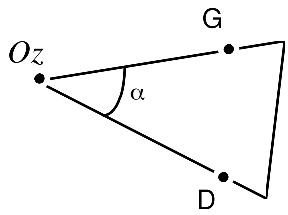
The total structure is then made up of N sectors:

$$N=\frac{2\Pi}{\alpha} \tag{1}$$

2.2 Wave propagation

One notes θ the rotation of axis Oz and of angle α defined in R^3.

Let us consider a basic sector of a structure with repetitivity of axis Oz, and two similar points of two contiguous sectors G and D :



One has the relation between the points G and D :

$$G=\theta(D) \tag{2}$$

It is noticed that the structure is left invariant by any rotation θ^m (with m entirety).

One can note that all rotations leaving the invariant structure (geometrically and mechanically) are of finished number:

$$\theta^m \quad m\in\{0,1,....N-1\} \tag{3}$$

Let us consider a scalar variable of state of the studied mechanical system U, and Z the associated complex:

$$U = \mathrm{Re}(Z) = \mathrm{Re}(U+jV) \tag{4}$$

It is possible to show, by the theory of the finished groups, the following relation for the points D and G [bib5]:

$$\exists \ m\in\{0,1,...,\frac{N}{2}\} \quad \text{tel que} \quad Z(G)=e^{jm\alpha}Z(D) \tag{5}$$

Note:

- *the quantities are expressed in the cylindrical reference mark (r, θ, z),*
- *for an axisymmetric structure (cyclic typical case of repetitivity), m index of FOURIER is called,*
- *in the case of a wave planes not deadened, $e^{jm\alpha}$ is complex dephasing between two contiguous sectors; the equation means that this dephasing can take only one finished number of known values,*
- *it is possible to limit the number of the values of m with the values ranging between 0 and $N/2$; indeed, it is shown that the wave associated with dephasing $N-m$ is identical to that associated with dephasing m, but progresses in opposite direction [bib5].*

If N is even: $m=0$ and $m=N/2$ correspond to real modes:

$$\begin{aligned} m=0 \quad &\Rightarrow \forall D \quad U(\theta(D))=U(D) \\ m=N/2 \quad &\Rightarrow \forall D \quad U(\theta(D))=-U(D) \end{aligned} \tag{6}$$

All other values of m correspond to modes appearing per orthogonal pairs at a given frequency (one speaks then about degenerated modes):

$$U=\mathrm{Re}(Z) \quad et \quad V=\mathrm{Im}(Z) \tag{7}$$

If N is odd: $m=0$ corresponds to a real mode not degenerated:

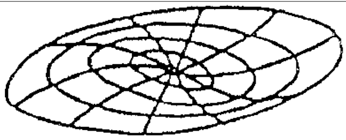
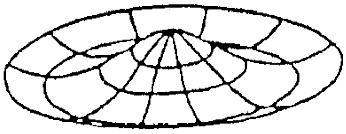
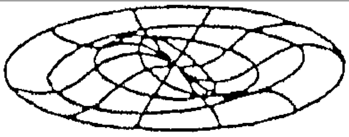
$$m=0 \quad \Rightarrow \quad \forall D \quad U(\theta(D))=U(D) \tag{8}$$

All other values of m correspond to degenerated modes appearing per orthogonal pairs:

$$U=\mathrm{Re}(Z) \quad et \quad V=\mathrm{Im}(Z) \tag{9}$$

2.3 Concept of diameters and nodal circles

The cyclic property of repetitivity, translated by the equation (5) allows to know a priori the pace of the clean modes of the structure, which strongly approaches what one can observe for axisymmetric structures. If one considers a clean mode of a structure with cyclic symmetry, all the sectors have the same deformation but with an amplitude function of their angular position, which one can translate by a dephasing between substructures. This mode can be classified starting from the number of diameters and nodal circles which characterize it. A nodal diameter (which is confused with a diameter only if the structure is axisymmetric) is a line of points of null movement passing by the axis of repetitivity; a nodal circle (which has the circular form only for the axisymmetric structures) is a line of points of null movement, it even with cyclic repetitivity. It is noted that it is the deformation of the mode of the substructure on which the mode of the complete structure is pressed which determines the number of circle (S) nodal (with). On the other hand, the number of diameter (S) nodal (with) is defined by dephasing between two consecutive sectors.

Deformation sector	Phase between sector	Deformation overall	Family
Inflection 1	NR sectors in phase		0 circle 0 diameter
Inflection 1	N/2 sectors in phase		0 circle 1 diameter
Inflection 2	NR sectors in phase		1 circle 0 diameter
Inflection 2	N/secteurs in phase		1 circle 1 diameter

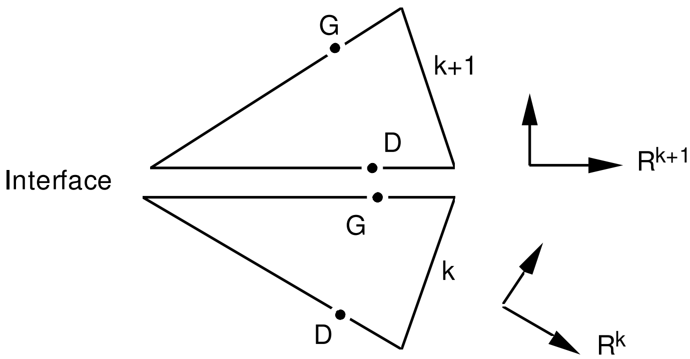
2.4 Boundary conditions

Generally, the implementation of the methods of under cyclic structuring with technique of reduction should not require particular treatment if the sector presents nodes being on the axis of rotation. One would be simply in a case where certain nodes belong simultaneously to the interfaces right-hand side and left. The taking into account of the boundary conditions in Code_Aster however forces to treat the two cases separately:

- on the one hand relations of continuity between the faces in glance, by excluding the axis
- in addition relations associated with the nodes carried by the axis of rotation, if there are some.

2.4.1 Equations of connection between the right faces and left – excluded axis

Let us consider a structure with cyclic repetitivity, and two successive basic sectors of this one:



The connections between sectors being regarded as perfect, there are the conditions between the sectors:

$$q_g^k = q_d^{k+1} \quad \text{Continuity of displacements} \tag{10}$$

$$f_{L_g}^k = -f_{L_d}^{k+1} \quad \text{Reciprocity of the efforts} \qquad (11)$$

The exhibitor indicates the number of the sector considered. The preceding conditions of connection are expressed in the total reference mark.

By the relation (5) (wave propagation in the structure) and while posing: $\beta = m\alpha$, one a:

$$\begin{aligned} (q^{k+1})_{k+1} &= e^{j\beta}(q^k)_k \\ (f^{k+1})_{k+1} &= e^{j\beta}(f^k)_k \end{aligned} \qquad (12)$$

The index k mean that the quantity is expressed in the reference mark related to the sector k : R^k .

Equations of connection (12), written in the reference mark related to the sector k thus utilize the matrix of passage of the sector k with the sector $k+1$. This matrix is not other than the matrix of rotation of the degrees of freedom of right-hand side towards those of left, is the matrix of rotation of axis Oz and of angle α , noted θ .

We thus obtain the following system:

$$\begin{aligned} (q_g^k)_k &= e^{j\beta}\theta(q_d^k)_k \\ (f_{L_g}^k)_k &= -e^{j\beta}\theta(f_{L_d}^k)_k \end{aligned} \qquad (13)$$

Boundary conditions (13) allow to calculate the clean modes of the whole of the structure starting from one only basic sector.

2.4.2 Equations checked by the degrees of freedom carried by the axis

This formalization can be wide with the case of the nodes of the axis. We obtain then, for a given sector:

$$\begin{aligned} q_a &= e^{j\beta}\theta_a q_a \\ f_{L_a} &= -e^{j\beta}\theta_a f_{L_a} \end{aligned} \qquad (14)$$

The exponential one complexes being of module 1, the continuity of displacements of the axis can be also put in the more classical form

$$\theta_a q_a = e^{-j\beta} q_a \qquad (15)$$

It is about a problem to the eigenvalues, and centers it can admit displacements only if the couples $(q_a, e^{-j\beta})$ correspond to the clean vectors and eigenvalues of the matrix of rotation θ_a . Eigenvalues of θ_a are $(1, e^{j\alpha}, e^{-j\alpha})$, respectively associated with the clean vectors that are the axis of rotation, and with two axes axes orthogonal between them and orthogonal with the axis of rotation. Only values of β allowing to obtain displacement of the axis are thus:

- $\beta=0$, that is to say m=0. Displacements are done only in the direction of the axis of rotation.
- $\beta=\alpha$, that is to say m=1. Displacements are done in a normal direction with the axis of rotation.

In addition, the relations of balances can be also put in the form of a problem at the eigenvalues

$$\theta_a f_{L_a} = e^{-j(\beta+\pi)} f_{L_a} = e^{-j(\beta-\pi)} f_{L_a} \qquad (16)$$

As previously, this system admits solution not identically worthless only if $e^{-j(\beta-\pi)}$ is eigenvalue of θ_a, that is to say $\beta\in(\pi,\alpha+\pi,\pi-\alpha)$. For the cases m=0 and m=1, that amounts having $\alpha=\pi/2$ or $\alpha=\pi$. If the angular opening of the sector is different from π or $\pi/2$, that is to say a problem with two or four sectors, then one will necessarily have $f_{L_a}=0$ for the cases with 0 and 1 diameter.

3 Methods of cyclic under-structuring

3.1 Method of Craig-Bampton

One considers the problem with the eigenvalues of the total structure expressed on the basic sector. This last is thus subjected to the bonding strengths which are applied to him by the contiguous sectors. In addition, the basic sector checks the equations of connection (13). We thus have:

$$(K-\omega^2 M)q=f_L$$

$$q_g=e^{j\beta}\theta\, q_d \qquad (17)$$

$$f_{L_g}=-e^{j\beta}\theta\, f_{L_d}$$

We suppose that the base is made up of the dynamic clean modes of the basic sector embedded with its interfaces, noted φ, and of the constrained modes relating to the degrees of freedom of interfaces right and left, noted Ψ_d and Ψ_g.

Taking into account the fact that the only contribution to displacements of a degree of freedom of interface comes from the constrained mode corresponding, the transformation of RITZ can be written:

$$q=\begin{Bmatrix} q_i \\ q_d \\ q_g \end{Bmatrix}=\begin{bmatrix}\varphi & \Psi_d & \Psi_g\end{bmatrix}\begin{Bmatrix} \eta_i \\ q_d \\ q_g \end{Bmatrix}=\Phi\eta \qquad (18)$$

Consequently, by using the transformation of RITZ, the system of equations [éq 3.1-1] becomes:

$$(\overline{K}-\omega^2\overline{M})\begin{Bmatrix} \eta_i \\ q_d \\ q_g \end{Bmatrix}=\begin{bmatrix}\varphi & \Psi_d & \Psi_g\end{bmatrix}^T\begin{Bmatrix} 0 \\ f_{L_d} \\ f_{L_g} \end{Bmatrix}$$

$$q_g=e^{j\beta}\theta q_d \qquad (19)$$
$$f_{L_g}=-e^{j\beta}\theta\, f_{L_d}$$

The surmounted matrices of a bar are projections of the matrices finite elements on the basis of modal basic sector (generalized matrices).

One can show that the constrained modes are orthogonal with the normal modes with respect to the matrix of rigidity [bib5]. Thus, the corresponding products are worthless.

Let us adopt the following notations:

- m : index relating to the clean modes of the sector,
- d : index relating to the constrained modes of the right interface,

g : index relating to the constrained modes of the left interface.

One can thus write these matrices in the form:

$$\overline{K}=\begin{bmatrix}\overline{K_{mm}} & 0 & 0\\ 0 & \overline{K_{dd}} & \overline{K_{dg}}\\ 0 & \overline{K_{gd}} & \overline{K_{gg}}\end{bmatrix}\qquad \overline{M}=\begin{bmatrix}\overline{M_{mm}} & \overline{M_{md}} & \overline{M_{mg}}\\ \overline{M_{dm}} & \overline{M_{dd}} & \overline{M_{dg}}\\ \overline{M_{gm}} & \overline{M_{gd}} & \overline{M_{gg}}\end{bmatrix} \tag{20}$$

Taking into account their definition, the constrained modes check:

$$\Psi_d=\begin{bmatrix}\Psi_{di}\\ \Psi_{dd}\\ \Psi_{dg}\end{bmatrix}=\begin{bmatrix}\Psi_{di}\\ Id\\ 0\end{bmatrix}\qquad \Psi_g=\begin{bmatrix}\Psi_{gi}\\ \Psi_{gd}\\ \Psi_{gg}\end{bmatrix}=\begin{bmatrix}\Psi_{gi}\\ 0\\ Id\end{bmatrix} \tag{21}$$

The second member of the matric equation (19) becomes:

$$\begin{bmatrix}\varphi_i^T & 0 & 0\\ \Psi_{di}^T & Id & 0\\ \Psi_{gi}^T & 0 & Id\end{bmatrix}\begin{Bmatrix}0\\ f_{L_d}\\ f_{L_g}\end{Bmatrix}=\begin{Bmatrix}0\\ f_{L_d}\\ f_{L_g}\end{Bmatrix} \tag{22}$$

By taking account of these notations, let us develop the matric equation checked by the basic sector:

$$\begin{aligned}&\overline{K_{mm}}\eta_i-\omega^2(\overline{M_{mm}}\eta_i+\overline{M_{md}}q_d+\overline{M_{mg}}q_g)=0\\ &\overline{K_{dd}}q_d+\overline{K_{dg}}q_g-\omega^2(\overline{M_{dm}}\eta_i+\overline{M_{dd}}q_d+\overline{M_{dg}}q_g)=f_{L_d}\\ &\overline{K_{gd}}q_d+\overline{K_{gg}}q_g-\omega^2(\overline{M_{gm}}\eta_i+\overline{M_{gd}}q_d+\overline{M_{gg}}q_g)=f_{L_g}\\ &q_g=e^{j\beta}\theta q_d\\ &f_{L_g}=-e^{j\beta}\theta f_{L_d}\end{aligned} \tag{23}$$

Let us introduce the two last equations of this system into the three first:

$$\begin{aligned}&(\overline{K_{mm}}-\omega^2\overline{M_{mm}})\eta_i-\omega^2(\overline{M_{md}}+e^{j\beta}\overline{M_{mg}}\theta)q_d=0\\ &(\overline{K_{dd}}+e^{j\beta}\overline{K_{dg}}\theta)q_d-\omega^2\left(\overline{M_{dm}}\eta_i+(\overline{M_{dd}}+e^{j\beta}\overline{M_{dg}}\theta)q_d\right)=f_{L_d}\\ &(\overline{K_{gd}}+e^{j\beta}\overline{K_{gg}}\theta)q_d-\omega^2\left(\overline{M_{gm}}\eta_i+(\overline{M_{gd}}+e^{j\beta}\overline{M_{gg}}\theta)q_d\right)=-e^{j\beta}\theta f_{L_d}\end{aligned} \tag{24}$$

The association of the two last equations makes it possible to eliminate the terms from the bonding strengths. One leads then to a problem with the eigenvalues final which one can put in the form:

$$\left(\tilde{K}(\beta)-\omega^2\tilde{M}(\beta)\right)\tilde{q}=0 \tag{25}$$

With: $\tilde{q}=\begin{Bmatrix}\eta_i\\ q_d\end{Bmatrix}$

$$\tilde{K}=\begin{bmatrix}\overline{K_{mm}} & 0\\ 0 & \overline{K_{dd}}+e^{j\beta}\overline{K_{dg}}\theta+e^{-j\beta}\theta^T\overline{K_{gd}}+\theta^T\overline{K_{gg}}\theta\end{bmatrix} \tag{26}$$

$$\tilde{M}=\begin{bmatrix} \overline{M_{mm}} & \overline{M_{md}}+e^{j\beta}\overline{M_{mg}}\theta \\ \overline{M_{dm}}+e^{-j\beta}\theta^T\overline{M_{gm}} & \overline{M_{dd}}+e^{j\beta}\overline{M_{dg}}\theta+e^{-j\beta}\theta^T\overline{M_{gd}}+\theta^T\overline{M_{gg}}\theta \end{bmatrix} \quad (27)$$

The matrices of mass and rigidity of the final problem are square. The eigenvalues solutions are thus real. In addition, the problem is of reduced size.

The resolution of the problem to the complex eigenvalues (25) allows to determine the complex generalized coordinates of the clean modes of the total structure. The complex values of displacements of the basic sector in the total mode are given, starting from the generalized coordinates, by the following formula:

$$q'=\begin{bmatrix}\varphi & \Psi_d+e^{j\beta}\theta\Psi_d\end{bmatrix}\tilde{q} \quad (28)$$

To determine the actual values of displacements, it is necessary to distinguish three cases according to the values dephasing AND element:

Case n° 1 : $\beta=0$:

Displacements q' given by the formula (28) are then with actual values. All the sectors deformed even and vibrate in phase. There is then only one real clean mode:

$$q=\mathrm{Re}(q')=q' \quad (29)$$

Case n°2 : $0<\beta<(N+1)/2$:

The displacements provided by the formula (28) are with complex values. With each one of these complex modes two orthogonal degenerated real modes correspond:

$$q_1=\mathrm{Re}(q') \qquad q_2=\mathrm{Im}(q') \quad (30)$$

Case n°3 : $\beta=N/2$ (=> N is even):

The displacements provided by (28) are then with complex values. There is $N/2$ nodal diameters, two contiguous sectors vibrate then in opposition of phase. Each complex mode is at the origin of only one real mode:

$$q=\mathrm{Re}(q')=-\mathrm{Im}(q') \quad (31)$$

3.2 Method of Mac Neal

One considers the problem with the eigenvalues of the total structure expressed on the basic sector. This last is thus subjected to the bonding strengths which are applied to him by the contiguous sectors. In addition, the basic sector checks the equations of connection (13). We thus have:

$$\begin{gathered}(K-\omega^2M)q=f_L\\ q_g=e^{j\beta}\theta q_d\\ f_{L_g}=-e^{j\beta}\theta f_{L_d}\end{gathered} \quad (32)$$

The modal base used to reduce dimensions of the problem to be solved, is a modal base with free interfaces including of the dynamic modes and the modes of fastener relating to the degrees of freedom of the interfaces right and left. Let us suppose that the degrees of freedom of the basic sector are ordered in the following way:

$$q = \begin{Bmatrix} q_i \\ q_d \\ q_g \end{Bmatrix} \begin{matrix} \text{degrés de liberté internes} \\ \text{degres de liberté de l'interface droite} \\ \text{degrés de liberté de l'interface gauche} \end{matrix} \quad (33)$$

Are B_d and B_g, rectangular matrices of extraction such as:

$$q_d = B_d q \quad \text{et} \quad q_g = B_g q \quad (34)$$

The boundary condition on displacements becomes with these notations:

$$B_g q = e^{j\beta} \theta B_d q \quad \Rightarrow B_{dg} q = 0$$
$$\text{with } B_{dg} = e^{j\beta} \theta B_d - B_g \quad (35)$$

For the forces, the boundary condition becomes:

$$f_L = B_g^T f_{L_g} + B_d^T f_{L_d} \quad \Rightarrow \quad f_L = \left(B_g^T - e^{-j\theta} B_d^T \theta^T \right) f_{L_g} = -B_{dg}^T f_{L_g} \quad (36)$$

Let us regard as base, for the transformation of RITZ, the whole of the dynamic clean modes of the basic sector, by distinguishing the identified modes and the unknown modes:

$$q = [\varphi_1 \quad \varphi_2] \begin{Bmatrix} \eta_1 \\ \eta_2 \end{Bmatrix} \quad (37)$$

where index 1 (resp. 2) refers to the known modes (resp. unknown). In the continuation, we will suppose that the clean modes are normalized with the unit modal mass.

While replacing q by its expression according to the clean modes, and while multiplying on the left by transposed of the matrix of the modes, matric equations (32) and (35) become:

$$\begin{aligned} (\lambda_1 - \omega^2 Id)\eta_1 &= \varphi_1^T f_L \\ (\lambda_2 - \omega^2 Id)\eta_2 &= \varphi_2^T f_L \\ B_{dg}\varphi_1 \eta_1 + B_{dg}\varphi_2 \eta_2 &= 0 \end{aligned} \quad (38)$$

where λ is the matrix of generalized rigidities (the generalized masses are unit).

One can thus draw a formulation from it from η_2 :

$$\eta_2 = (\lambda_2 - \omega^2 Id)^{-1} \varphi_2^T f_L \quad (39)$$

Consequently, one can eliminate η_2 system of equations (38). One then obtains the problem with the eigenvalues according to:

$$\begin{aligned} (\lambda_1 - \omega^2 Id)\eta_1 + \varphi_1^T B_{dg}^T f_{Lg} &= 0 \\ B_{dg}\varphi_1 \eta_1 - B_{dg}\varphi_2 (\lambda_2 - \omega^2 Id)^{-1} \varphi_2^T B_{dg}^T f_{Lg} &= 0 \end{aligned} \quad (40)$$

The final system to solve can be written:

$$(\tilde{K} - \omega^2 \tilde{M})\tilde{q} = 0 \quad (41)$$

With:

$$\tilde{q} = \begin{Bmatrix} \eta_1 \\ f_{L_g} \end{Bmatrix} \quad (42)$$

The forms of the matrices of rigidity and mass are:

$$\tilde{K}=\begin{bmatrix} \lambda_1 & \varphi_1 B_{dg}^T \\ B_{dg}\varphi_1 & -B_{dg} R_e(\omega) B_{dg}^T \end{bmatrix} \qquad \tilde{M}=\begin{bmatrix} Id & 0 \\ 0 & 0 \end{bmatrix} \tag{43}$$

The matrix $\left[R_e(\omega)\right]$ is the matrix of residual dynamic flexibility of the not identified modes:

$$R_e(\omega)=\Phi_2(\lambda_2-\omega^2 Id)^{-1}\Phi_2^T \tag{44}$$

One approximates residual dynamic flexibility by his static contribution, by taking of to account the modes of fastener. Then, the formula of restitution which makes it possible to calculate the complex values of displacements starting from the generalized coordinates of the solutions modes of (41) is the following one:

$$q'=\left[\varphi_1 \quad -R_e(0) B_{dg}^T\right]\tilde{q} \tag{45}$$

The actual values of displacements are determined, as for the method of Craig - Bampton, by the relations (29), (30) and (31).

3.3 Taking into account of the nodes of the axis - Method Craig & Bampton

From an algorithmic point of view, the modes of interface associated with the degrees of freedom carried by the nodes with the axis are taken into account only for the cases m = 0 and m = 1, which are the only cases which can present nonworthless movements of axis (c.f. Section 2.4).

NB: It is important to note that calculations taking of account the movements of the axis can be realized only with one method of reduction of the type Craig & Bampton. The approach with the method of Mac Neal is not implemented.

One supposes in this paragraph that the degrees of freedom carried by the nodes of the axis, as well as the nodes of interfaces right-hand side and left, were blocked for the calculation of the dynamic modes of the basic sector and were the object of calculations of constrained modes.

The base of projection is thus made up of the dynamic clean modes of the basic sector embedded with its interfaces, noted φ, and of the constrained modes relating to the degrees of freedom of interfaces right, left and centers, noted Ψ_d, Ψ_g and Ψ_a.

As we saw with the section 2.4, if m is equal to or higher than 2, the displacement of the nodes of the axis is null. The taking into account of the nodes of the axis thus has direction only if m=0 or m=1. In practice, to limit the occupation memory and the number of operations, the matrices are assembled by taking account of the DDL of the axis only in these two cases.

The problem with the eigenvalues of the total structure and the equations of connection, expressed on this basis are worth then:

$$\begin{gathered} (\overline{K}-\omega^2\overline{M})\begin{Bmatrix} \eta_i \\ q_d \\ q_g \\ q_a \end{Bmatrix}=\left[\varphi \quad \Psi_d \quad \Psi_g \quad \Psi_a\right]\begin{Bmatrix} 0 \\ f_{L_d} \\ f_{L_g} \\ f_{L_d} \end{Bmatrix} \\ q_g=e^{j\beta}\theta q_d \quad \text{et} \quad q_a=e^{j\beta}\theta q_a, \\ f_{L_g}=-e^{j\beta}\theta f_{L_g} \quad \text{et} \quad f_{L_a}=-e^{j\beta}\theta f_{L_a} \end{gathered} \tag{46}$$

One can thus write the matrices in the form:

$$\overline{K}=\begin{bmatrix}\overline{K_{mm}} & 0 & 0 & 0\\ 0 & \overline{K_{dd}} & \overline{K_{dg}} & \overline{K_{da}}\\ 0 & \overline{K_{gd}} & \overline{K_{gg}} & \overline{K_{ga}}\\ 0 & \overline{K_{ad}} & \overline{K_{ag}} & \overline{K_{aa}}\end{bmatrix} \quad \overline{M}=\begin{bmatrix}\overline{M_{mm}} & \overline{M_{md}} & \overline{M_{mg}} & \overline{M_{ma}}\\ \overline{M_{dm}} & \overline{M_{dd}} & \overline{M_{dg}} & \overline{M_{da}}\\ \overline{M_{gm}} & \overline{M_{gd}} & \overline{M_{gg}} & \overline{M_{ga}}\\ \overline{M_{am}} & \overline{M_{ad}} & \overline{M_{ag}} & \overline{M_{aa}}\end{bmatrix} \tag{47}$$

Taking into account their definition, the constrained modes check:

$$\Psi_d=\begin{bmatrix}\Psi_{di}\\ \Psi_{dd}\\ \Psi_{dg}\\ \Psi_{da}\end{bmatrix}=\begin{bmatrix}\Psi_{di}\\ Id\\ 0\\ 0\end{bmatrix} \quad \Psi_g=\begin{bmatrix}\Psi_{gi}\\ \Psi_{gd}\\ \Psi_{gg}\\ \Psi_{ga}\end{bmatrix}=\begin{bmatrix}\Psi_{gi}\\ 0\\ Id\\ 0\end{bmatrix} \quad \Psi_a=\begin{bmatrix}\Psi_{ai}\\ \Psi_{ad}\\ \Psi_{ag}\\ \Psi_{aa}\end{bmatrix}=\begin{bmatrix}\Psi_{ai}\\ 0\\ 0\\ Id\end{bmatrix} \tag{48}$$

The second member of the matric equation (46) becomes:

$$\begin{bmatrix}\varphi_i & 0 & 0 & 0\\ \Psi_{di} & Id & 0 & 0\\ \Psi_{gi} & 0 & Id & 0\\ \Psi_{ai} & 0 & 0 & Id\end{bmatrix}\begin{pmatrix}0\\ f_{L_d}\\ f_{L_g}\\ f_{L_a}\end{pmatrix}=\begin{pmatrix}0\\ f_{L_d}\\ f_{L_g}\\ f_{L_a}\end{pmatrix} \tag{49}$$

The taking into account of the equations of connection is done by projection. One introduces

$$\tilde{q}=\begin{pmatrix}\eta\\ q_d\\ q_a\end{pmatrix} \tag{50}$$

and the projector is considered $\tilde{P}$ defined by

$$\tilde{P}=\begin{bmatrix}Id & 0 & 0\\ 0 & Id & 0\\ 0 & e^{j\beta}\theta & 0\\ 0 & 0 & e^{j\beta}\theta_a\end{bmatrix} \tag{51}$$

The problem project on $\tilde{P}$ becomes

$$\tilde{P^H}\left(\overline{K}-\omega^2\overline{M}\right)\tilde{P}\tilde{q}=\tilde{P^H} f \ , \tag{52}$$

This problem naturally checks the equations of connection kinematics, since which one has

$$\tilde{P}\tilde{q}=\begin{pmatrix}\eta\\ q_d\\ e^{j\beta}\theta q_d\\ e^{j\beta}\theta_a q_a\end{pmatrix}=\begin{pmatrix}\eta\\ q_d\\ q_g\\ q_a\end{pmatrix} \tag{53}$$

and the efforts of connections check

$$\tilde{P^H} f=\begin{pmatrix}0\\ f_{L_d}+e^{-j\beta}\theta^H f_{L_g}\\ e^{-j\beta}\theta_a^H f_{L_a}\end{pmatrix}=\begin{pmatrix}0\\ 0\\ \tilde{f}_{L_a}\end{pmatrix} \tag{54}$$

If the angular opening of the sector is lower than π, and where one has strictly more than two sectors, one in addition showed that $f_{L_a}=0$ (section 2.4.2).

One leads then to a problem with the eigenvalues final which one can put in the form:

$$(\tilde{K}-\omega^2\tilde{M})\tilde{q}=0 \quad (55)$$

With:

$$\tilde{K}=\begin{bmatrix} \overline{K_{mm}} & 0 & 0 \\ 0 & \overline{K_{dd}}+\overline{K_{dg}}\theta e^{j\beta}+e^{-j\beta}\theta^T\overline{K_{gd}}+\theta^T\overline{K_{gg}}\theta & \overline{K_{da}}\theta_a e^{j\beta}+\theta^T\overline{K_{ga}}\theta_a \\ 0 & e^{-j\beta}\theta_a^T\overline{K_{ad}}+\theta_a\overline{K_{ag}}\theta & \theta_a^T\overline{K_{aa}}\theta_a \end{bmatrix} \quad (56)$$

$$\tilde{M}=\begin{bmatrix} \overline{M_{mm}} & \overline{M_{md}}+\overline{M_{mg}}\theta e^{j\beta} & \overline{M_{ma}}\theta_a e^{j\beta} \\ \overline{M_{dm}}+e^{-j\beta}\theta^T\overline{M_{gm}} & \overline{M_{dd}}+\overline{M_{dg}}\theta e^{j\beta}+e^{-j\beta}\theta^T\overline{M_{gd}}+\theta^T\overline{M_{gg}}\theta & \overline{M_{da}}\theta_a e^{j\beta}+\theta^T\overline{M_{ga}}\theta_a \\ e^{-j\beta}\theta_a^T\overline{M_{am}} & e^{-j\beta}\theta_a^T\overline{M_{ad}}+\theta_a^T\overline{M_{ag}}\theta & \theta_a^T\overline{M_{aa}}\theta_a \end{bmatrix} \quad (57)$$

One restores modal complex displacements by the following formula:

$$q'=\begin{bmatrix}\varphi & \Psi_d+e^{j\beta}\theta\Psi_d & \Psi_a\end{bmatrix}\tilde{q} \quad (58)$$

NB: Typical case of the problem with two and four sectors

The formulation suggested remains licit in the typical case of the problem with two sectors, taking into account orthogonality enters q_a and f_{L_a} in the case of cyclic symmetry. Indeed, equations of connection (14) lead to

$$q_a^H f_{L_a}=(e^{j\beta}\theta_a q_a)^H(-e^{j\beta}\theta_a f_{L_a})=-q_a^H f_{L_a}=0 \quad (59)$$

In addition, solutions of (15) are also solution of the quadratic problem

$$\tilde{q}=\underset{\tilde{q}_0}{\mathrm{ArgMin}}\left(\tilde{q}_0^H\tilde{P}^H(\overline{K}-\omega^2\overline{M})\tilde{P}\tilde{q}_0-\tilde{q}_0^H\tilde{P}^H f_0\right) \quad (60)$$

However, for the solutions searched, checking the conditions of connection between face and on the level of the axis, one has, according to (59)

$$\tilde{q}_0^H\tilde{P}^H f_0=\begin{Bmatrix}\eta_0^H & q_{0\mathrm{d}}^H & q_{0\mathrm{a}}^H\end{Bmatrix}\begin{Bmatrix}0\\0\\f_{L_{0a}}\end{Bmatrix}=0 \quad (61)$$

The problem (60) can then put itself in the form

$$\tilde{q}=\underset{\tilde{q}_0}{\mathrm{ArgMin}}\left(\tilde{q}_0^H\tilde{P}^H(\overline{K}-\omega^2\overline{M})\tilde{P}\tilde{q}_0\right) \quad (62)$$

Taking into account the convexity of the problem (62), solutions of the problem (60) thus check also

$$\tilde{P}^H(\overline{K}-\omega^2\overline{M})\tilde{P}\tilde{q}=0 \quad (63)$$

Foot-note: the relation (59) also state that the projector defined in the relation (51) is not the only acceptable one. The term $e^{j\beta}\theta_a$ can be replaced by Id. This choice was not made to preserve a classical form at the reduced problem, similar to that obtained when it is not necessary to separate the terms carried by the axis and those carried by the interfaces right-hand side and left.

In addition, eigenvalues of the term $e^{j\beta}\theta_a$ being very of module 1, the conditioning of the total problem is not affected by this change.

4 Implementation in *Code_Aster*

The treatment of the basic sector is identical to that of the substructures in the classical under-structuring. It utilizes the operators CALC_MODES [U4.52.02], DEFI_INTERF_DYNA [U4.64.01] and DEFI_BASE_MODALE [U4.64.02].

The clean modes of the structure with cyclic symmetry are calculated by the operator MODE_ITER_CYCL [U4.52.05] according to the base of projection of the basic sector previously definite and amongst sectors of the complete structure.

The restitution of the results on physical basis is identical to the classical under-structuring. It utilizes the operator REST_SOUS_STRUC [U4.63.32] and possibly the operator DEFI_SQUELETTE [U4.24.01].

5 Conclusion

The principles of under structuring make it possible to expose the transformation of RITZ and the modal recombination to lead to the modal synthesis which integrates these two techniques. The rules of liaisonnement between substructures are clarified.

Two methods were developed in *Code_Aster* : that of Craig-Bampton and that of Mac Neal. We present, here, their characteristics, as well in the definition of the initial modal base, as in its exploitation.

After having exposed the definition of a structure to cyclic symmetry and the properties which result from this, we presented the methods of cyclic under-structuring put in work in *Code_Aster*. They appear very interesting for the calculation of the clean modes of a structure with cyclic symmetry, such as the rotors of the revolving machines of which they benefit fully from the geometrical and mechanical characteristics.

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7 Description of the versions of the document

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