Code_Aster

Titre : Estimateur d'erreur de ZHU-ZIENKIEWICZ
Responsable : DELMAS Josselin

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Estimator of error of ZHU-ZIENKIEWICZ

Summary:

One exposes in this document the method of estimate of the error of discretization suggested by ZHU - ZIENKIEWICZ.

This estimator is based on a continuous smoothing of the calculated constraints allowing to obtain a better precision on the nodal constraints compared to the methods standards.

Two successive versions of this estimator are described, corresponding each one to a different smoothing.

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1 Introduction

Research on estimators of error on the solutions obtained by calculations finite elements and their coupling with procedures of adaptive grid made these last years considerable great strides. The set aim is to mitigate the possible inadequacy of a modeling by adapting in an automatic way the grid the solution sought according to certain criteria (equal distribution of the error of discretization, minimization amongst nodes to reach a given precision, lower costs).

One introduces here an estimator of error of the type a posteriori within the framework of linear and homogeneous elasticity. Historically, this estimator, proposed by ZHU-ZIENKIEWICZ [bib1] in 1987, was largely used because of his facility of establishment in his the existing low costs and computer codes. Nevertheless, the bad reliability of this estimator for the elements of even degree was noted empirically (undervaluation of the error) and led the authors to a modification of their method in 1992 [bib2], [bib3] with digital checking of the asymptotic convergence of the estimator on all the types of elements.

Nevertheless, the scope of application of the version of 1992 being for the moment more reduced (see [3.2]), the two versions of this estimator were established in *Aster* and are the object of this note.

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2   Principle of the method

2.1   Equations to be solved

The solution is considered (u, s) of a linear elastic problem checking:

*   equilibrium equations:

$$\begin{cases} \mathbf{Lu} = \boldsymbol{q} & \text{dans } \Omega \\ \sigma_{ij} n_j = \bar{t}_i & \text{sur } \Gamma_t \end{cases}$$

with $L = {}^t BDB$ operator of elasticity

*   equations of compatibility:

$$\begin{cases} \boldsymbol{\varepsilon} = \mathbf{Bu} \\ \boldsymbol{u} = \bar{\boldsymbol{u}} & \text{sur } \Gamma_u \end{cases}$$

with $\Gamma = \Gamma_u \cup \Gamma_t$

*   the law of behavior:

$$\sigma = \boldsymbol{D} \boldsymbol{\varepsilon}$$

The problem discretized by finite elements consists in finding $(\boldsymbol{u}_h, \sigma_h)$ solution of:

$$\boldsymbol{u}_h = N \bar{\boldsymbol{u}}_h \qquad\qquad \text{éq 2.1-1}$$

checking $K \bar{\boldsymbol{u}}_h = \mathbf{f}$
with $\quad K = \int\limits_\Omega {}^t (BN) D (BN) d\Omega$

$$\mathbf{f} = \int\limits_\Omega {}^t Nq \, d\Omega + \int\limits_{G_t} {}^t N \bar{t} \, dG$$

where:

$\bar{\mathbf{u}}_h$ represent the nodal unknown factors of displacement
N associated functions of form

The constraints are calculated starting from displacements by the relation:

$$\sigma_h = DB\boldsymbol{u}_h \qquad\qquad \text{éq 2.1-2}$$

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2.2   Estimator of error and index of effectiveness

One notes $e = u - u_h$ the error on displacements

$e_s = s - s_h$ the error on the constraints

The standard of the energy of the error e is written:

$$\|\mathbf{e}\| = \left(\int_\Omega {}^t \mathbf{e} \mathbf{L} \mathbf{e} \, d\Omega \right)^{1/2}$$

in the case of elasticity

$$= \left(\int_\Omega {}^t \mathbf{e}_\sigma \mathbf{D}^{-1} \mathbf{e}_\sigma \, d\Omega \right)^{1/2}$$ éq 2.2-1

The total error above breaks up into a sum of elementary errors according to:

$$\|\mathbf{e}\|^2 = \sum_{i=1}^{N} \|\mathbf{e}\|_i^2$$

where N is the full number of elements.

$\|\mathbf{e}\|_i$ represent the local indicator of error on the element i.

The goal is to consider the error exact starting from the equation [éq 2.2-1] formulated in constraints. The basic idea of the method is to build a new noted stress field σ^* from σ_h and such as:

$$\mathbf{e}_\sigma \approx \mathbf{e}_\sigma^* = \sigma^* - \sigma_h$$

The estimator of error will be written then:

$$^0\|\mathbf{e}\| = \left(\int_\Omega {}^t \mathbf{e}_\sigma^* \mathbf{D}^{-1} \mathbf{e}_\sigma^* \, d\Omega \right)^{1/2}$$

The quality of the estimator is measured by the quantity θ, called index of effectiveness of the estimator:

$$\theta = \frac{^0\|e\|}{\|e\|}$$

An estimator of error is known as asymptotically exact if $\theta \rightarrow 1$ when $\|e\| \rightarrow 0$ (or when $h \rightarrow 0$), which wants to say that the estimated error will always converge towards the exact error when this one decreases.

In an obvious way, the reliability of $^0\|e\|$ depends on "quality" on σ^*.

The two versions of the estimator of ZHU-ZIENKIEWICZ are different on this level (see [§3]).

2.3 Construction of an estimator asymptotically exact

The characterization of such an estimator is provided by the following theorem (see [feeding-bottle 2]).

Theorem

If $\|\mathbf{e}^*\|=\|\mathbf{u}-\mathbf{u}^*\|$ is the standard of error of the rebuilt solution, then the estimator of error $^0\|e\|$ defined previously is asymptotically exact

if $\dfrac{\|\mathbf{e}^*\|}{\|\mathbf{e}\|}\to 0$ when $\|e\|\to 0$

This condition is carried out if the rate of convergence with h of $\|\mathbf{e}^*\|$ is higher than that of $\|e\|$. Typically, if it is supposed that the exact error of the approximation finite element converges in $\|\mathbf{e}\|=0\left(h^p\right)$

and the error of the solution rebuilt in

$$\|\mathbf{e}^*\|=0\left(h^{p+\alpha}\right) \text{ with } \alpha>0$$

then a simple calculation gives:

$$1-0\left(h^\alpha\right)\leq\theta\leq 1+0\left(h^\alpha\right)$$

and thus $\theta\to 1$ when $h\to 0$

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3 Construction of the stress field recomputed

3.1 Version 1987

The solution $\mathbf{u_h}$ resulting from the equation [éq 2.1-1] being C_0 on Ω (because of the choice of functions of form C_0), it follows that $\sigma_\mathbf{h}$ calculated by [éq 2.1-2] is discontinuous with the interfaces of the elements.

To get acceptable results on the nodal constraints, one generally resorts to an average with the nodes or a method of projection. It is this last method which is adopted here.

It is supposed that σ^* is interpolated by the same functions of form as $\mathbf{u_h}$, that is to say:

$$\sigma^* = N \bar{\sigma}^*$$
éq 3.1-1

and one carries out a total smoothing by least squares of $\sigma_\mathbf{h}$, which amounts minimizing the functional calculus $J(\tau) = \int_\Omega {}^t(\tau - \sigma_\mathbf{h})(\tau - \sigma_\mathbf{h}) d\Omega$ in the space generated by N.

By derivation, σ^* must check $\int_\Omega {}^t N(\sigma^* - \sigma_\mathbf{h}) d\Omega = 0$

by using the equation [éq 3.1-1], one obtains the linear system:

$$\mathbf{M}[\bar{\sigma}^*] = [\mathbf{b}]$$

with $\quad \mathbf{M} = \int_\Omega {}^t \mathbf{NN} d\Omega \quad$ and $\quad [\mathbf{b}] = \int_\Omega {}^t \mathbf{N} \sigma_\mathbf{h} d\Omega$

This total system is to be solved on each component of the tensor of the constraints. The matrix M is calculated and reversed only once.

3.2 Version 1992

The constraint of the field σ^* differ compared to the version 1987 in the following way:

one supposes σ^* polynomial of the same degree than displacements on the whole of the elements having an internal node top S jointly.

One notes $S_K = \bigcup_{S \in K} K$ this unit called patch.

For each component of σ^*, one writes:

$$\sigma^*|_{S_K} = \mathbf{Pa_s}$$
éq 3.2-1

where $\mathbf{P}$ contains the suitable polynomial terms

$\mathbf{a_s}$ unknown coefficients of the corresponding students' rag processions

Example: $2D \qquad P1 \quad \mathbf{P} = [1, x, y] \quad \mathbf{a_s}{}^t = [a_1, a_2, a_3]$

$\qquad Q1 \quad \mathbf{P} = [1, x, y, xy] \qquad \mathbf{a_s} = {}^t[a_1, a_2, a_3, a_4]$

Determination of the coefficients of the polynomial $\mathbf{a_s}$ is done by minimizing the functional calculus:

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$$F(\mathbf{a}) = \sum_{i=1}^{N} \left(\boldsymbol{\sigma}_{\mathbf{h}}(x_i, y_i) - \boldsymbol{\sigma}^* \big|_{S_K}(x_i, y_i) \right)^2$$

$$= \sum_{i=1}^{N} \left(\boldsymbol{\sigma}_{\mathbf{h}}(x_i, y_i) - \mathbf{P}(x_i, y_i) \mathbf{a}_{\mathbf{s}} \right)^2$$

(discrete local smoothing of $\boldsymbol{\sigma}_{\mathbf{h}}$ by least squares)

where (x_i, y_i) are the coordinates of the points of GAUSS on S_K.

N is the full number of points of GAUSS on all the elements of the patch

The solution $\mathbf{a}_{\mathbf{s}}$ check:

$$\sum_{i=1}^{N} {}^t \mathbf{P}(x_i, y_i) \mathbf{P}(x_i, y_i) \mathbf{a}_{\mathbf{s}} = \sum_{i=1}^{N} {}^t \mathbf{P}(x_i, y_i) \boldsymbol{\sigma}_{\mathbf{h}}(x_i, y_i)$$

from where $\mathbf{a}_{\mathbf{s}} = \mathbf{A}^{-1} \mathbf{b}$ with $\mathbf{A} = \sum_{i=1}^{N} {}^t \mathbf{P}(x_i, y_i) \mathbf{P}(x_i, y_i)$

A can be very badly conditioned (in particular on the elements of high degree) and consequently, impossible to reverse in this form. To cure this problem, the authors [bib4] proposed a standardisation of the coordinates on each patch, which amounts carrying out the change of variables:
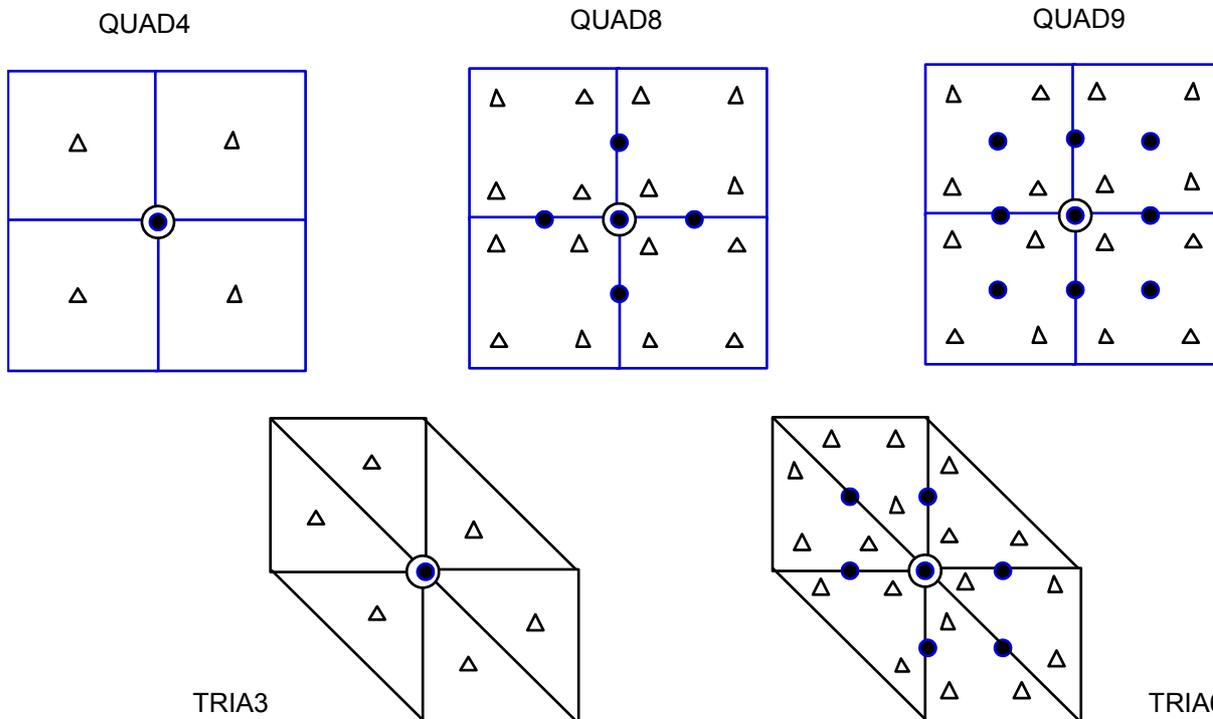
$$\bar{x} = -1 + 2 \frac{x - x_{min}}{x_{max} - x_{min}}$$

$$\bar{y} = -1 + 2 \frac{y - y_{min}}{y_{max} - y_{min}}$$

where $x_{min}, x_{max}, y_{min}, y_{max}$ represent the values minimum and maximum of x and y on the patch.

This method notably improves conditioning of A and removes completely the preceding problem.

Once $\mathbf{a}_{\mathbf{s}}$ determined, the nodal values are deduced according to the equation [éq 3.2-1] only on the internal nodes with the patch, except in the case of patchs having nodes of edge.

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Patchs interns:

QUAD4   QUAD8   QUAD9

TRIA3   TRIA6

△   points of GAUSS where the constraints are calculated σ_h according to the equation [éq 2.1-2]

●   nodes of calculation of σ^* σ^*

◉   internal top defining the patch

Patchs edges:
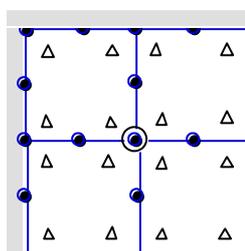
The nodal values with the nodes mediums belonging to 2 patchs are realised, in the same way for the internal nodes in the case of the QUAD9.

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Note:

> *In the case of finite elements of different type, the choice of P in the equation [éq 3.2 -1] is delicate (problems of validity of a_s if space is too rich, loss super - convergence if it is not it enough). A thorough study seems essential.*
>
> *This is why the estimator $ZZ2$ is limited for the moment to grids comprising one type of element. This restriction does not exist for $ZZ1$.*

The authors showed numerically [bib3] that with this choice of σ^*, their estimator was asymptotically exact for elastic materials of which the characteristics are independent of the field and for all the types of elements and that the rates of convergence with h of $\|e^*\|$ were improved compared to the previous model (especially for the elements of degree 2: to see case test SSLV110 Manuel de Validation), from where a better estimate of the error.

One will find an illustration of these rates of convergence in the reference [feeding-bottle 5].

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4 Establishment in *Code_Aster* and current limits of use

4.1 Establishment in *Code_Aster*

The two preceding estimators are established in *Code_Aster* in the ordering of postprocessing `CALC_ERREUR` [U4.81.04]. They are activated starting from options (`ERZ1_ELEM` for `ZZ1` and `ERZ2_ELEM` for `ZZ2`) and enrich a structure of data `RESULT`.

Moreover, the calculation of the stress field smoothed by one or the other of the methods described with [paragraph 3] can be separately started calculation of estimate of the error (option `SIZ1_NOEU` for `ZZ1` and `SIZ2_NOEU` for `ZZ2`).

The estimator of error provides:

- a field by element comprising 3 components:

  - the estimate of the relative error on the element,
  - the estimate of the absolute error on the element,
  - the standard of the energy of the calculated solution σ_h.

- exit-listing comprising same information at the total level (on all the structure)

All the fields obtained are displayable via the order `IMPR_RESU`.

4.2 Operational limits

The theoretical framework is homogeneous linear elasticity

For `ZZ1`, modeling 2D (forced and plane deformations, axisymmetric) and `3D` are allowed whereas for `ZZ1`, only modeling 2D (forced and plane deformations, axisymmetric) are allowed.

Types of elements: triangles with 3 and 6 nodes,
                       quadrangles with 4.8 and 9 nodes.

For `ZZ2`, the grid should comprise one type of elements.

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6   Description of the versions of the document

Version Aster	Author (S) Organization (S)	Description of the modifications
02/06/09	*X. DESROCHES* (EDF/IMA/MMN)	Initial text