Estimator of error in residue

Summary

The estimator of error in residue allows to estimate the error of discretization due to the finite element method on the elements of a grid 2D or 3D. It is an explicit estimator of error utilizing the residues of the equilibrium equations and the jumps of the normal constraints to the interfaces, contrary to the estimator of Zhu - Zienkiewicz, which uses a technique of smoothing of the constraints a posteriori [R4.10.01] and [bib5].

This estimator is established in *Code_Aster* in elastoplasticity 2D and 3D.

1 Introduction

The estimator of error in residue was developed in 1993 by Bernardi-Métivet-Verfurth [bib1]. It is an explicit estimator of error utilizing the residues of the equilibrium equations (from where its name). It applies to elliptic problems (Fish, Stokes, or linear elasticity) in dimension 2 or 3. These problems are supposed to be discretized by finite elements associated with a regular triangulation.

Historically, the first estimator of explicit error relating to the unbalances is due to Babuska and Rheinbolt [bib2] for the problems 1D with linear elements. Gago extended this estimator to the 2D and added to the formulas the jumps of traction to the interfaces of the elements [bib3] and [bib4]. New estimators were proposed then, in whom the defects of surface traction at the borders of the field were also taken into account thus that an improvement of the estimate of the jumps inter - elements giving of the more reliable results.

One is interested here in the estimator in residue applied to the case of linear elasticity. The set aim is, at the conclusion of an elastic design, to determine the map of error on the grid in sight to possibly adapt this one (by refinement and/or déraffinement) or simply for information. The adaptation can be done by chaining with the software of cutting Lobster.

2 Formulation of the estimator in residue

That is to say Ω open of R^n, $n=2$ or 3, of border Γ, and T a regular triangulation of Ω.

In linear elasticity, the continuous problem is written:

to find (u,σ) such as:

$$\begin{cases} div\,\sigma + f = 0 & \text{dans } \Omega \\ u = u_D & \text{sur } \Gamma_D \\ \sigma.\mathbf{n} = g_N & \text{sur } \Gamma_N \end{cases}$$

Γ_D is the border of Dirichlet of the grid
u_D is the displacement imposed on this border
Γ_N is the border of Neumann
$\mathbf{n}$ the unit normal with Γ_N

g_N is the loading applied to this border; it can be continuous or discretized.
f is force of a voluminal type (gravity, rotation); it can be continuous or discretized.

σ_h is the constraint obtained by the resolution of the discrete problem:

$$\begin{cases} div\,\sigma_h + f = 0 & \text{dans } \Omega \\ u_h = u_D & \text{sur } \Gamma_D \\ \sigma_h.\boldsymbol{n} = g_N & \text{sur } \Gamma_N \end{cases}$$

with the relation $\sigma_h = \mathbf{DB}\, u_h$ where:

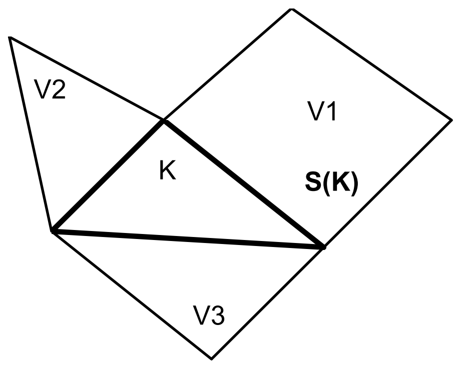
$\boldsymbol{D}$ is the matrix of Hooke
$\boldsymbol{B}$ is the linearized operator of the deformations

If K indicate an element running of the grid, the estimator of error (noted $\eta(\Omega)$) is defined as being the quadratic average of the site indicators of error, noted $\eta(K)$:

$$\eta(\Omega) = \left[\sum_{K \in T} \eta(K)^2 \right]^{1/2}$$

The indicator by local residue

The indicator is composed of three terms; the first represents the residue of the equilibrium equation on each mesh, the second term the jump of the normal constraints on the interfaces, the third term the difference between the normal constraints and the loading imposed on Γ_N if the element intersects Γ_N .



K :	Elément courant où l'on souhaite calculer l'erreur,
V1 à V3 :	Eléments ayant un bord commun avec l'élément courant,
S(K) :	Ensemble des bords de l'élément courant ayant des voisins.

Figure 2-a: Internal elements with a grid

- the first term of the estimator is the standard L^2 residue of the equilibrium equation on the mesh K, multiplied by h_K who is, either the diameter of the circle circumscribed for a triangular finite element, or the maximum diagonal for a quadrangle,
- the second term is the integral, on $S(K)$ defined [Figure 2-a], of the jumps of normal constraints integrated on each edge F element which has a neighbor, and multiplied by the root of h_F , which is the length of the edge F,

Γ_N : Frontière de Neumann

g_N : Force appliquée sur la frontière de Neumann

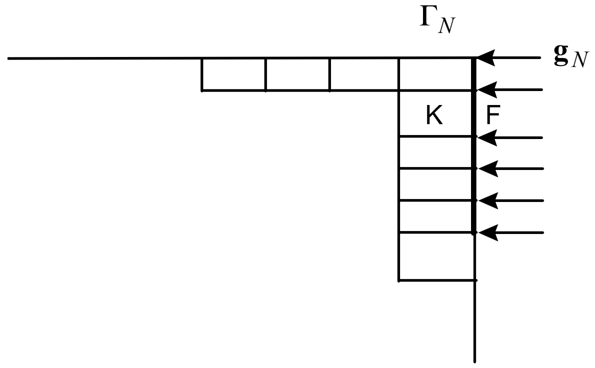


Figure 2-b: Elements located on the border of a grid

- the third term is the integral, on the intersection of each edge F edges ∂K current element K with the border of Neumann Γ_N, jumps between the normal constraints of the element and the force of Neumann g_N, multiplied by the root of h_F, length of the edge F.

There is thus the following formula for the estimator in residue:

$$\eta(K)=h_K\|f+div\,\sigma_h\|_{L^2(K)}+\frac{1}{2}\sum_{F\in S(K)} h_F^{1/2}\|[\sigma_h.n]\|_{L^2(F)}+\sum_{F\subset\partial K\subset\Gamma_N} h_F^{1/2}\|g_N-\sigma_h.n\|_{L^2(F)} \quad \text{éq 2-1}$$

For the choice of the various terms of [éq 2-1], one returns to [bib1].

3 Properties of the estimator in residue

One notes $\eta_{EX}(K)$ the exact error $\|u-u_h\|_{H^1(K)}$ on the element K (unknown factor a priori)

and $\eta_{EX}(\Omega)$ the total exact error $\|u-u_h\|_{H^1(\Omega)}$

There are then the following properties ([bib1]):

- some is the element K, the elementary error $\eta(K)$ is raised by the exact site error (multiplied by a constant independent of the triangulation),

  that is to say $\forall K \quad \eta(K)\leq C_1\times\eta_{EX}(K)$

- the exact total error is raised by the error considered total $\eta(\Omega)$ (multiplied by a constant independent of T)

  that is to say $\eta_{EX}(\Omega)\leq C_2\times\eta(\Omega)$

Constants $C_1 \text{ et } C_2$ depend a priori on the type of finite element and boundary conditions of the problem. Kelly and Gago [bib3] proposed in 2D a constant C_2 depending only on the degree p polynomial of interpolation used:

$$C_2 = \left(\frac{1}{24\,p^2}\right)^{1/2} \text{soit} \quad C_2 = \frac{1}{2\text{p}\sqrt{6}} \text{ pour les TRIA3 et QUAD4 (degré 1)}$$

$$C_2 = \frac{1}{4\text{p}\sqrt{6}} \text{ pour les TRIA6 et QUAD8 (degré 2)}$$

For the 3D, one does not have evaluation of the constant. One can nevertheless say that the error considered total over-estimates the total exact error in all the cases. This result is not inevitably true at the local level.

4 Establishment in *Aster*

The estimator in residue is established in 2D and 3D on all the types of elements. He is calculated by the order `CALC_ERREUR` by activating the option `'ERME_ELEM'`.

This option calculates on each element:

- the absolute error $\eta(K)$ (see [éq 2-1]),
- the standard of the tensor of the constraints $\|\sigma_h\|_{L^2(K)}$ who is used to normalize the absolute error,
- the relative error $\eta_{rel}(K) = 100 \times \dfrac{\eta(K)}{\sqrt{\eta(K)^2 + \|\sigma_h\|^2_{L^2(K)}}}$.

Foot-note:

This definition of the relative error implies that in the zones where the constraints are very low, the relative error can be important and nonsignificant. It is then the absolute error which it is necessary to consider.

It also calculates at the total level:

- the absolute error $\eta(\Omega) = \left[\sum_{K \in T} \eta(K)^2\right]^{1/2}$,
- the total standard of the tensor of the constraints $\|\sigma_h\|_{L^2(\Omega)} = \left[\sum_{K \in T} \|\sigma_h\|^2_{L^2(K)}\right]^{1/2}$,
- the relative error $\eta_{rel}(\Omega) = 100 \times \dfrac{\eta(\Omega)}{\sqrt{\eta(\Omega)^2 + \|\sigma_h\|^2_{L^2(\Omega)}}}$.

According to the expression [éq 2-1], one sees that to calculate the indicator of error on the mesh K, one must know:

- possible loadings f on K and g_N on $\partial K \cap \Gamma_N$ (or their discretization f_h and g_{Nh}),
- quantities h_K , h_F and $\mathbf{n}$ dependent on the geometry of the element,
- the stress field σ_h ,
- the list of the neighbors of K to recover the constraints on these elements, necessary to the calculation of the 2ème term of [éq 2-1].

1 and 2 can be calculated or recovered easily.

3 must be calculated as a preliminary by one of the options `'SIGM_ELNO'` or `'SIEF_ELNO'`.

In the contrary case, an error message fatal is transmitted.

4 requires the calculation of a particular connectivity mesh-meshs, besides standard connectivity mesh-nodes. This new object is stored in the structure of data of type grid.

For the detail of the establishment in *Aster*, to see [bib6].
For the validation of the estimator, to see [bib7].

5 Bibliography

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6 Description of the versions of the document

Version Aster	Author (S) Organization (S)	Description of the modifications
07/04/09	*X.DESROCHES (EDF-R&D/AMA)*	