Method LAKE – Room Average Contact

Summary:

This document describes the manner to treat it contact between solids (2D or 3D) in great transformations by a method of the type “mortar”. The formulation is very precise for contact pressures obtained, in particular when the grids are not coïncidents or that surfaces are curved.

This formulation is available in the order DEFI_CONTACT under the name ALGO_CONT=‘LAKE‘.

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1 Introduction

The finite element method is currently massively used in digital mechanics to simulate the problems of unilateral contact between two deformable objects. They are problems having a nonlinear boundary condition, namely to observe a condition of noninterpenetration for the field of displacement, a condition of sign for the normal pressure on the zone of contact and a condition of complementarity between these two fields. In the case general, such problems bring to the management of incompatible grids (i.e whose nodes of the zones of contact of the objects considered are not coïncidents).

The most used methods are based on algorithms of type node-segment in $2D$ and of node-facet type in $3D$ (in code_aster, it is the formulations DISCRETE and CONTINUOUS in mode INTEGRATION=' NOEUD'). These methods are known to fail the patchs tests in the case general of the incompatible grids and to lead to mathematical analyses under optimal. In addition, the academic medium was very productive in this field these fifteen last years. The most succeeded methods are the methods ***mortar*** adapted to the contac.

These methods consist in using a simple projection L^2 on a space joint, i.e., space of finite elements definite between the two objects. The various choices of joined spaces generate a whole family of methods of the mortar type. However these methods have important disadvantages when one considers their implementation in a generic way in an industrial code.

The method which one proposes here giftE digital and mathematical results of as good quality as those obtained by the methods mortar while keeping the aspects (locality, analytical definition of joined spaces, etc…) returning the implementation in *Code_hasster* easier by preserving the current architecture of the code

With this intention, one will define a simple condition of contact which consists in satisfying the conditions with contact on average on each macronutrient of a definite macro-grid in a suitable way (that it will be necessary to prepare thanks to the option 'DECOUPE_LAC' of the operator CREA_MAILLAGE).

2 Theoretical formulation of the method

2.1 The problem of contact

2.1.1 Strong formulation

Let us consider the problem of balance of two solids in balance on the figure 1 , noted by the index $l=1,2$.

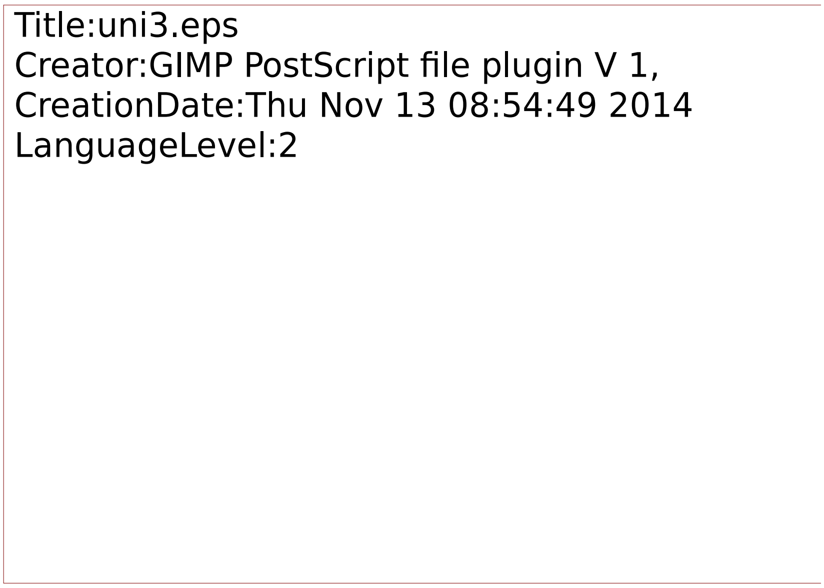


Figure 1: Solids in balance

The field of displacement is sought $u=(u_{1,}u_2):\overline{\Omega^1}\times\overline{\Omega^2}\rightarrow\mathbb{R}^d$ ($d=2$ or $d=3$) such as (1)-(8) are satisfied for $l=1,2$:

$$-\operatorname{div}\sigma(u_l)=f_l \text{ in } \Omega^l \quad (1)$$

$$\sigma(u_l)=A_l\varepsilon(u_l) \text{ in } \Omega^l \quad (2)$$

$$\sigma(u_l)n_l=F_l \text{ on } \Gamma_N^l \quad (3)$$

$$u_l=0 \text{ on } \Gamma_D^l \quad (4)$$

$$[u_N]\leqslant 0 \text{ on } \Gamma_C \quad (5)$$

$$\sigma_N\leqslant 0 \text{ on } \Gamma_C \quad (6)$$

$$\sigma_N[u_N]=0 \text{ on } \Gamma_C \quad (7)$$

$$\sigma_\tau=0 \text{ on } \Gamma_C \quad (8)$$

where $[v_N]=\sum_{l=1}^{2} v_l\cdot n_l$.

Equations (1) with (4) correspond to the mechanical problem considered (to simplify, one places oneself within the framework of a linear problem of elasticity) where A is tensor of fourth an elliptic, symmetrical order and with components in L^∞ coming from the law of behavior associated with the solid Ω^l with condition of Dirichlet on the edge Γ_D^l and condition of Neumann on the edge Γ_N^l . With these equations, the conditions of contact are added on Γ_C , namely a condition of noninterpenetration for the jump of displacement $[u_N]$ (5), a condition of sign for the normal pressure σ_N the zone of contact (6) and a condition of complementarity between these the last two fields (7). The last equation (8) comes owing to the fact that one considers the contact without friction.

2.1.2 Weak formulation

From the equations (1) with (8), one can introduce a weak formulation of the problem of contact in the form of the following variational inequation:

To find u such as $\forall v \in K$,

$$\sum_{l=1}^{2} \int_{\Omega^l} A_l \varepsilon(u_l) : \varepsilon(v_l - u_l) d\Omega^l \geqslant \sum_{l=1}^{2} \int_{\Omega^l} f_l \cdot (v_l - u_l) d\Omega^l + \int_{\Gamma_N^l} F_l \cdot (v_l - u_l)_N d\Gamma^l \quad (9)$$

where $K \subset V$ is the convex cone of acceptable displacements (K the condition of non-interp containséNétration). One thus use following spaces :

$$V_l = \{ v_l \in (H^1(\Omega^l))^d : v = 0 \text{ sur } \Gamma_D^l \}$$
$$V = V_1 \times V_2 \quad (10)$$
$$K = \{ v \in V : [v_N] \leq 0 \text{ sur } \Gamma_C \}$$

One will note thereafter:

$$a(u, v) = \sum_{l=1}^{2} \int_{\Omega^l} A_l \varepsilon(u_l) : \varepsilon(v_l) d\Omega^l \quad (11)$$

And:

$$l(v) = \sum_{l=1}^{2} \int_{\Omega^l} f_l \cdot v_l d\Omega^l + \int_{\Gamma_N^l} F_l \cdot v_l d\Gamma^l \quad (12)$$

One introduces the mixed problem are equivalent:

To find $u \in V$ and $\lambda \in M$ such as:

$$a(u, v) - b(\lambda, v) = l(v) \quad \forall v \in V$$
$$b(\mu - \lambda, u) \geq 0 \quad \forall \mu \in M \quad (13)$$

With:

$$b(\mu, v) = \int_{\Gamma_C} \mu [v_N] d\Gamma \quad (14)$$

And:

$$M = \left\{ \mu \in H^{-1/2}(\Gamma_C) : \int_{\Gamma_C} \mu \Psi d\Gamma \leq 0 \, \forall \Psi \in H^{-1/2}(\Gamma_C), \Psi \geq 0 \right\} \quad (15)$$

Note: *Problems (9) and (13) are well posed. There is thus existence and unicity of the solutions* (u, λ) *. Moreover, solutions* u *p roblème (13) and* u *p roblème (9) are identical.*

2.2 Problems discreteS

2.2.1 Inequation variational discrete

One considers the problem of discrete contact from variational the inequation point of view (discretization of the problem):

$$\text{To find } u^h \in K^h \text{ such as } \forall v^h \in K^h$$
$$a(u^h, v^h - u^h) \geqslant l(v^h - u^h) \quad (16)$$

where $K^h \subset V^h$ is the convex cone of acceptable displacements, a and l previously definite forms bilinear and linear.

To define a method, one must choose discrete spaces V_l^h and the unit K^h. That is to say T_l^h a regular triangulation of $\Omega^l \subset \mathbb{R}^d$, where $d=2,3$. One defines then:

$$V_l^h = \{ v_l^h \in (C(\overline{\Omega^l}))^d : v_l^h|_T \in P_k(T), \forall T \in T_l^h, v_l^h = 0 \text{ sur } \Gamma_D^l \} \quad (17)$$

where $k=1,2$, $V^h = V_1^h \times V_2^h$.

One proposes to use the following condition of contact:

$$K^h = \left\{ v^h \in V^h : \int_{T^m} [v_N^h] \, d\Gamma \leq 0 \quad \forall T^m \in T^M \right\} \quad (18)$$

where T^M is a macro-grid of elements T^m with to define according to the choice of V^h.

This condition of contact takes as a starting point the methods of the mortar type while preserving a local aspect. Indeed, the matrix of projection mortar is not any more one matrix full because the reverse with the basic matrix of space $P^0(T^M)$ is diagonal. This property ensures us that the matrices of projection mortar (produced matric enters the basic matrix reverses space $P^0(T^M)$ and the matrices of coupling between space mortar and spaces of approximation) are local. However, it is known that the simple condition of the type:

$$\int_{T^h \cap \Gamma_C} [v_N^h] \, d\Gamma \leq 0 \quad (19)$$

is not stable for all the elements usually used in the studies of engineering (in particular in 3D, where only the hexahedron with 27 nodes is stable naturally), and thus does not lead to optimal mathematical analyses. However, while widening the support of integration slightly, according to the cases, one manages to show results of optimal convergence and stability.

2.2.2 Discrete mixed formulation

One introduces the mixed problem are equivalent:

$$\text{Trouver } u^h \in V^h \text{ and } \lambda^h \in M^h \text{ such as}$$
$$a(u^h, v^h) - b(\lambda^h, v^h) = l(v^h) \quad \forall v^h \in V^h$$
$$b(\mu^h - \lambda^h, u^h) \geq 0 \quad \forall \mu^h \in M^h \quad (20)$$

hasvec:

$$M^h = \{ \mu^h \in X_1^h : \mu^h \leq 0 \text{ sur } \Gamma_C \} \quad (21)$$

Where:

$$X_1^h = \{ \mu^h \in L^2(\Gamma_C) : \mu^h|_{T^m} \in P^0(T^m), \forall T^m \in T^M \} \quad (22)$$

One thus obtains a method "mortar P^0" which will be automatically stabilized by the definition of the macro-grid T^M.

Note: *The condition of contact in* K^h *being fixed, the space of the multipliers of Lagrange will be always a subspace of* $P^0(T^m)$ *whatever the type of finite elements used within the space of approximation of displacements* V^h. *It is thus noticed that the order of the condition of contact remains fixed for any space of* *approximation considered (linear or quadratic), only the definition of the macro-grid varies to ensure the good mathematical properties of the method.*

2.3 Mathematical results

We point out here the principal mathematical results of the method LAKE. The fundamental assumption of the method defines the manner of building it *macro-grid* is the assumption of the internal degree of freedom:

Assumption of degree of freedom intern:
For any macronutrient $T^m \in T^M$, there exists a degree of freedom x_i of V_1^h such as $supp(\varphi_i) \subset T^m$, where φ_i is the basic function associated with x_i.

Note: *ON notes that there at the same time exists a minimal size for the macro-meshs making it possible to satisfy the assumption the preceding one and to guarantee the locality of the method. It is also noted that the choice of the grid traces used to define the macro-grid is free; here, grid 1 was chosen.*

One can now point out the two principal results of convergence of the method LAKE.

Theorem 1: *Are* u *and* u^h *solutions of the problems of continuous and discrete contact. It is supposed that* $u \in (H^s(\Omega^1))^d \times (H^s(\Omega^2))^d$ *with* $d=2,3$ *and* $3/2 < s \leq 2$ *(* $3/2 \leq s < 5/2$ *if* $k=2$ *). If the assumption internal degree of freedom is satisfied, then it exists a constant* $C>0$ *independent of* h_1*,* h_2 *(where* h_1 *and* h_2 *are the parameters of discretization of the solid* 1 *and of the solid* 2 *respectively) and* u*, such as:*

$$\|u-u^h\|_{1,\Omega^1,\Omega^2} \leq C(h_1^{s-1}+h_2^{s-1})\|u\|_{s,\Omega^1,\Omega^2} \quad (23)$$

Theorem 2: *Shear* (u,λ) *and* (u^h,λ^h) *solutions of the problems of continuous and discrete contact. It is supposed that* $u \in (H^s(\Omega^1))^d \times (H^s(\Omega^2))^d$ *with* $d=2,3$ *and* $3/2 < s \leq 2$ *(* $3/2 < s \leq 5/2$ *if* $k=2$ *). If the assumption internal DDL is satisfied, then it exists a constant* $C>0$ *independent of* h_1*,* h_2 *and* u*, such as:*

$$\|\lambda-\lambda^h\|_{1/2,*,\Gamma_C}+\|u-u^h\|_{1,\Omega^1,\Omega^2} \leq C(h_1^{s-1}+h_2^{s-1})\|u\|_{s,\Omega^1,\Omega^2} \quad (24)$$

where $\|.\|_{1/2,*,\Gamma_C}$ *is the dual standard of* $\|.\|_{1/2,\Gamma_C}$*.*

Note: *Dyears the continuation of this document one will adopt following convention: one will name surface slave the surface on which one will have defined the macro-meshs, i.e. airfoil the multipliers of Lagrange of contact in the case of the mixed formulation. Surface in opposite with the latter will be surface Master.*

2.4 Matric formulation

The following matrices are considered $C^E \in M_{e,k}(\mathbb{R})$, $C^M \in M_{m,k}(\mathbb{R})$ and $M^{LAC} \in M_k(\mathbb{R})$, where $M_{i,j}$ corresponds to the space of the rectangular matrices of size $i \times j$ and $M_i = M_{i,i}$ corresponds to the space of the matrices squares of size i, defined by:

$$\begin{aligned} C_{ij}^E &= \int_{T_j} (\varphi_N^E)_i \, d\Gamma \\ C_{ij}^M &= \int_{T_j} (\varphi_N^M)_i \, d\Gamma \\ M_{ij}^{LAC} &= \delta_{ij} |T^i| \end{aligned} \quad (25)$$

where them T^i are the supports of the basic functions of $P^0(T^M)$, them φ_N^E are products of the normal with basic functions associated with surface slave and them φ_N^M are products of the normal with basic functions associated with surface Master.

That is to say U the field of displacement and Λ the multiplier of Lagrange, the matric formulation of the problem is the following one:

$$\begin{bmatrix} K_{NN} & K_{NM} & K_{NE} & 0 \\ K_{MN} & K_{MM} & K_{ME} & C^M \\ K_{EN} & K_{EM} & K_{EE} & C^E \end{bmatrix} \times \begin{bmatrix} U_N \\ U_M \\ U_E \\ \Lambda \end{bmatrix} = F \tag{26}$$

With the following conditions:

$$\begin{aligned} M^{LAC^{-1}}\left({}^t C^E U_E + {}^t C^M U_M\right) \leqslant 0 \text{ in } \mathbb{R}^k, \\ M^{LAC} \Lambda \leqslant 0 \text{ in } \mathbb{R}^k, \\ M^{LAC^{-1}}\left({}^t C^E U_E + {}^t C^M U_M\right) \cdot M^{LAC} \Lambda = 0 \text{ in } \mathbb{R}. \end{aligned} \tag{27}$$

EN noticing that M^{LAC} is diagonal definite positive, one has equivalence with the following formulation:

$$\begin{bmatrix} K_{NN} & K_{NM} & K_{NE} & 0 \\ K_{MN} & K_{MM} & K_{ME} & C^M \\ K_{EN} & K_{EM} & K_{EE} & C^E \end{bmatrix} \times \begin{bmatrix} U_N \\ U_M \\ U_E \\ \Lambda \end{bmatrix} = F \tag{28}$$

With the following conditions:

$$\begin{aligned} {}^t C^E U_E + {}^t C^M U_M \leqslant 0 \text{ in } \mathbb{R}^k, \\ \Lambda \leqslant 0 \text{ in } \mathbb{R}^k, \\ {}^t C^E U_E + {}^t C^M U_M \cdot M^{LAC} \Lambda = 0 \text{ in } \mathbb{R}. \end{aligned} \tag{29}$$

The problem is solved (28) - (29) by using a strategy of the active constraints (set activates) coupled to an algorithm of Newton.

3 Implementation of method

This chapter implementation of the method "LAKE" (Local Average Contact) in *Code_hasster*. One will discuss more particularly the choices of implementation related to the new condition of contact by avoiding the creation of new types of mesh in *Code_hasster* and while forming part of the formalism of management of the already existing contact. It is thus advisable for this first version of implementation of the method to adopt the strategy of contact with a grid (one will define late elements of contact allowing to carry out elementary calculations of the contributions of contact) and an approach of generalized the Newton type.
The operators impacted by the method are:

- CREA_MAILLAGE;
- DEFI_CONTACT;
- STAT_NON_LINE.

The method suggested implies the following evolutions:

- A modification of the grid on the zone of contact slave (noninvasive in volume) in the operator CREA_MAILLAGE;
- A redefinition of Lagrangian of contact, implementation of space $P^0(T^M)$ in the operator DEFI_CONTACT;
- A new pairing of type "segment-segment" in 2D and of type "face-face" in 3D, new evaluation of the statutes of contact and new elementary calculation, impact in the operator STAT_NON_LINE and DYNA_NON_LINE.

3.1 Preparation of the grid

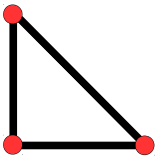
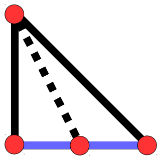
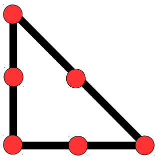
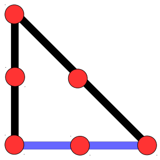
3.1.1 Principles of cutting

The theoretical base forces to us to satisfy itHypothèse degree of freedom interns (see §2.3). In a preoccupation of simplicity, a coherence between 2D and it 3D, and for not "déraffiner" the condition of contact, one will use the method of local refinement in the cases 2D and 3D. The keyword factor DECOUPE_LAC of the operator allows to prepare this macro-grid. In certain situations, the keyword will cut out the meshs of surfaces, in others, it will not be necessary. When an element of surface is cut out one thus arranges oneself for not *to propagate* this cutting in volume. This strategy makes it possible to limit the creation of too many meshs but can cause meshs of poor quality.
D E in order to not even modify the initial choices of the user, one will always propose a cutting ***conform*** : a triangle will be cut out in triangles, a hexahedron in hexahedrons, etc.

3.1.2 The case 2D

In the case 2D, the interface of contact slave is "1D". The latter is with a grid is in SEG2 maybe in SEG3. Initially, it is noticed that it SEG3 associated with itself (macro-mesh) satisfied the assumption with the internal DDL, it will thus not be necessary to cut out the meshs having SEG3 in skin (TRIA6, QUAD8 and QUAD9). In the case SEG2, there exist two cuttings according to the type of the associated mesh of body. In the case TRIA3, one adds a node to the center of the mesh SEG2 and the two new elements of skin and the two new elements of body are rebuilt. In the case QUAD4. Since one wishes to carry out a cutting in conformity, one adds two nodes to in the mesh SEG2, and two nodes in the mesh QUAD4. One can then rebuild the three new elements of skin and them four new elements of body while informing information characterizing the macro-mesh. The results of these operations are presented in LE table below (the surface of contact is represented in blue).

Note: One will pay a special attention to the orientation of the new meshs created, with the conformity of the grid of calculation created like with the actualization of the groups of mesh. Knowing that one cannot bring up to date the groups of nodes. Indeed, it is not possible to determine if a new node returns in an already definite group of nodes or not.

	
TRIA3	TRIA3 afterwards DECOUPE_LAC
	
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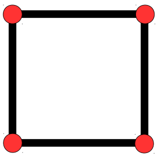
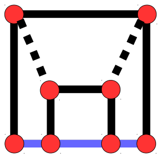
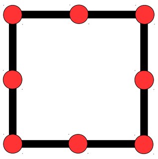
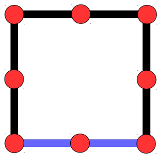
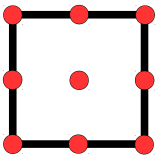
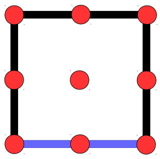
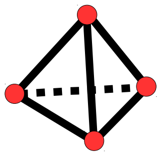
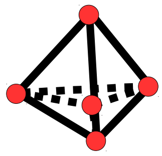
	
QUAD4	QUAD4 afterwards DECOUPE_LAC
	
QUAD8	QUAD8 afterwards DECOUPE_LAC (not of modifications)
	
QUAD9	QUAD9 afterwards DECOUPE_LAC (not of modifications)

3.1.3 The case 3D

In the case 3D, the interface of contact slave is 2D. The latter can be with a grid in TRIA3, TRIA6, QUAD4, QUAD8 and QUAD9. As in the case 2D, the mesh QUAD9 associated with it even satisfied the assumption with the internal DDL, one can thus directly inform the structures of data containing the relative information with the macro-grid without refining locally. In the other cases one will use two standard cuttings, for the tetrahedral cases, and the other for the hexahedral cases. These two cuttings are in conformity in the direction where one cuts out an element by generating the same type of elements. Results of these operations is presented in LE table below.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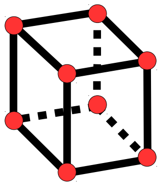
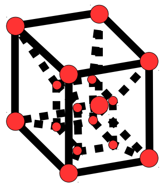
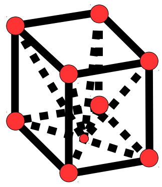
	
	HEXA8 afterwards DECOUPE_LAC (Version 1)
	
HEXA8	HEXA8 afterwards DECOUPE_LAC (Version 2)
HEXA20	Two versions (idem HEXA8)
HEXA27	HEXA27 afterwards DECOUPE_LAC (not of modifications)

3.1.4 Some remarks on cutting

Functionality DECOUPE_LAC of the operator is able to treat:

- various types of mesh inside the same zone of contact;
- the multi-zone of contact.

Concerning the complexity of calculations on the new grid one notices that one adds a certain number of nodes which will carry of DDLs and potentially of Lagrangian of contact. That is to say M the number of meshs of surface slave of the zone of contact considered, then for grids (of body) in:

- TRIA3 one adds M nodes and $2M$ meshs,
- QUAD4 one adds $4M$ nodes and $5M$ meshs,
- TETRA4 one adds M nodes and $4M$ meshs,
- TETRA10 one adds $5M$ nodes and $4M$ meshs,
- HEXA8 (version 1) one adds $8M$ nodes and $9M$ meshs,
- HEXA20 (version 1) one adds $28M$ nodes and $9M$ meshs.

Moreover the aspect ratio and the configuration of the generated meshs can possibly parasitize the postprocessing of certain data on surface slave (elements of skin and elements of body in contact).

3.2 Preparation of the contact

The preparation of the data of the contact is ensured by the operator DEFI_CONTACT. This Operator must take into account the new definition of Lagrangian of contact, i.e space $P^0(T^M)$. It is necessary to add for each macro-mesh of surface LAKE Lagrangian of contact (see Figure 2) knowing that it is not possible for ***Code_hasster*** to affect one degree of freedom with another physical entity that a node. One will present here one trick allowing to define space $P^0(T^M)$ locally on each mesh of calculation subject having an intelligent assembly of the matric system.

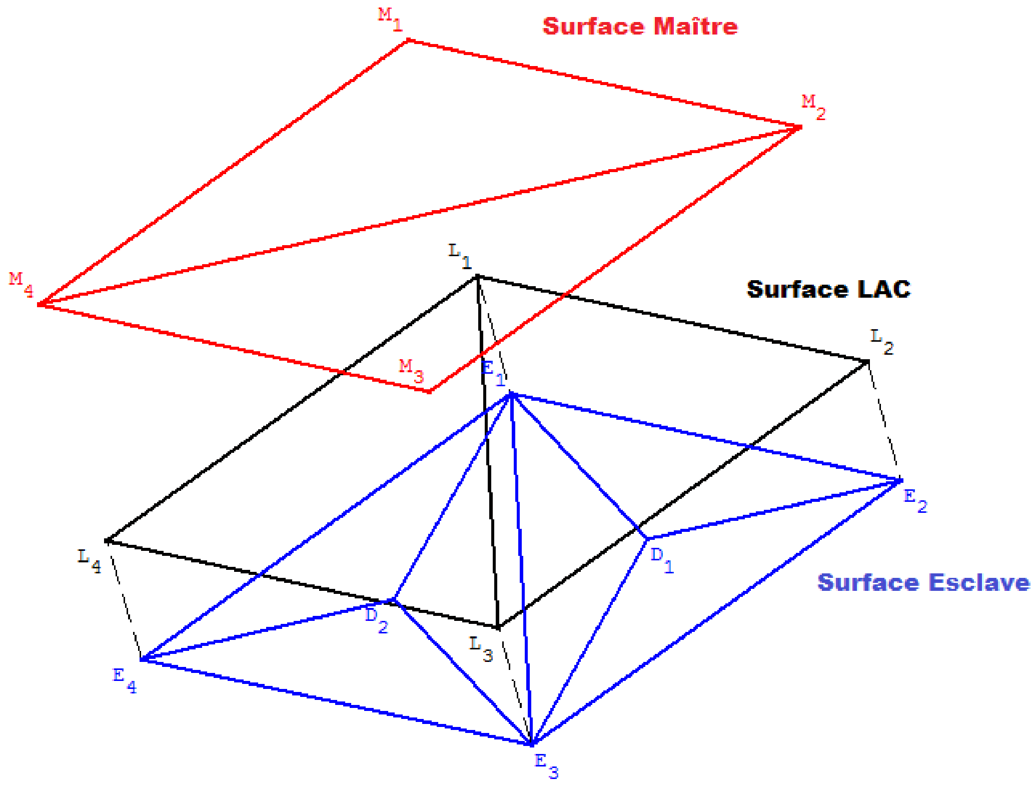


Figure 2: Three spaces of the zone of contact (Master, Esclave and LAKE)

3.2.1 Illustration of the technique on the case TRIA3

The trick lies in the fact that one knows for each mesh of the zone of contact slave of the grid given by `DECOUPE_LAC` which is the local number (in connectivity) of the internal node of the macro-mesh, to see Figure 3.

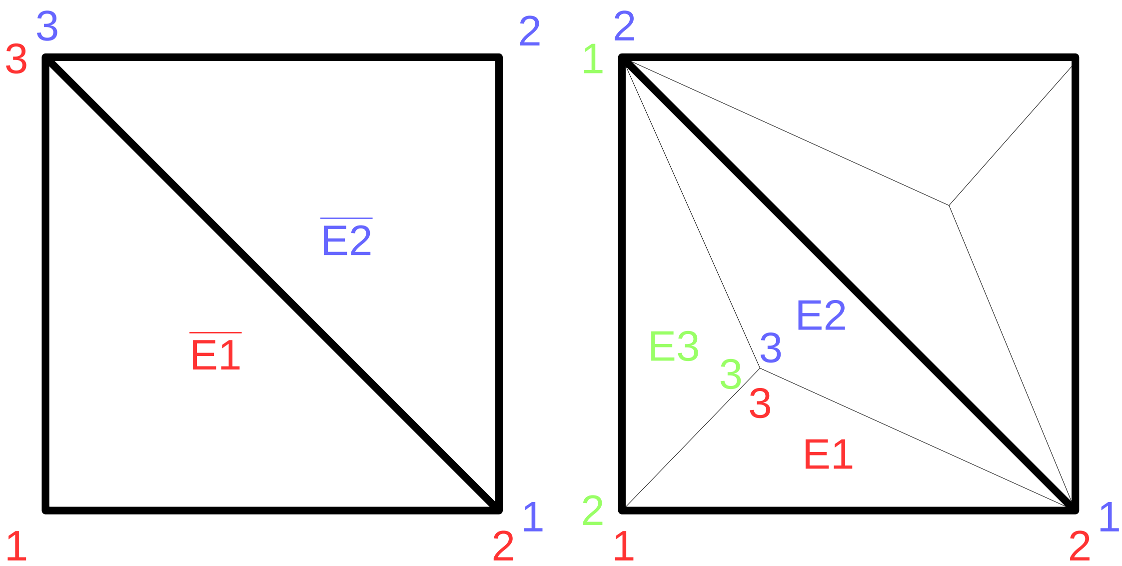


Figure 3: Local classification of MayLlage of calculation, case `TRIA3` **(obtained by** `DECOUPE_LAC`**)**

One thus knows on which node one must add it degree of freedom $LAGS\_C$ (see Figure 4), the local number of the node carrying it degree of freedom $LAGS\_C$ being fixed one can define only one new type of element allowing to take into account the localization of $LAGS\_C$.

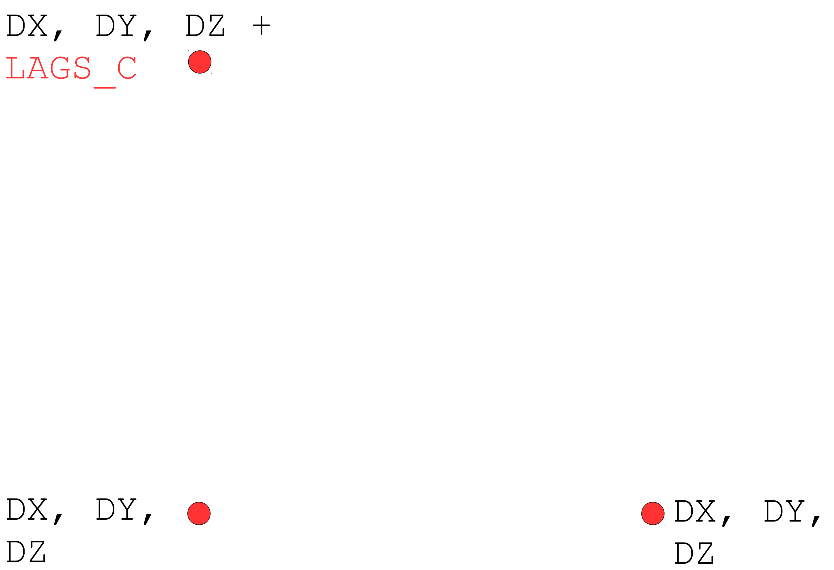


Figure 4: Attribution of the degrees of freedom on each new mesh of contact slave `TRIA3`

Each degree of freedom $LAGS\_C$ is community property only with three elements, the three elements composing the macro-mesh. Since the basic function P^0 associated with the multiplier of Lagrange of contact does not depend on the geometrical tops of the macro-mesh, one can cut out elementary calculation on the macro-mesh in elementary calculation on each mesh composing it. During the assembly, one summons the contributions of each mesh subjacent with the macro-mesh. Thus while allotting to each degree of freedom $LAGS\_C$ a basic function locally constant on the element, one obtains well that them degrees of freedom $LAGS\_C$ correspond to space $P^0(T^M)$.

3.2.2 Illustration of the difficulties on the case `QUAD4`

The case of an interface of contact slave with a grid in **QUAD4** allows to highlight all the difficulties which this trick can generate:

- Various local localizations of the node associated with degree of freedom $LAGS\_C$;
- Reduction of degree of freedom $LAGS\_C$, coming owing to the fact that one does not have one degree of freedom $LAGS\_C$ commun run with all the meshs subjacent with the macro-mesh.

The second difficulty comes from the fact cutting "conforms" in six HEXA8, to see figure 5.

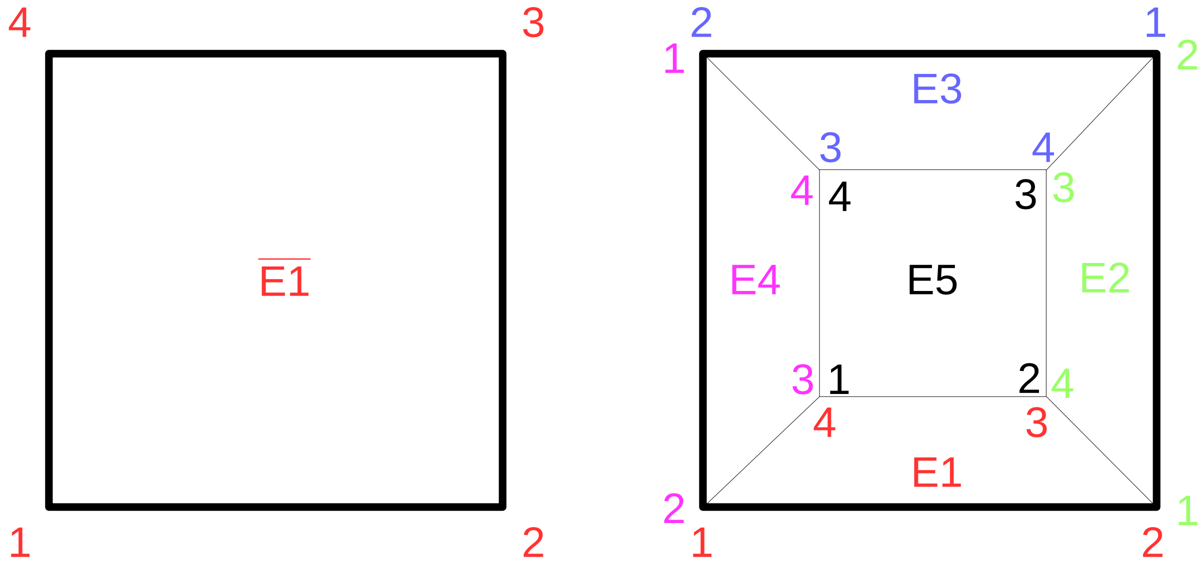


Figure 5: Local classification of the grid of calculation, case QUAD4 (obtained by DECOUPE_LAC)

Not having one degree of freedom $LAGS\_C$ commun run with all the meshs subjacent with the macro-mesh, one must create locally four $LAGS\_C$ carried by each degree of freedom addition. The first difficulty then is noticed. There are two kinds of meshs subjacent with the macro-mesh, to see figures 6 and 7 :

- Four meshs carrying two degrees of freedom $LAGS\_C$;
- U mesh carrying them four degrees of freedom $LAGS\_C$.

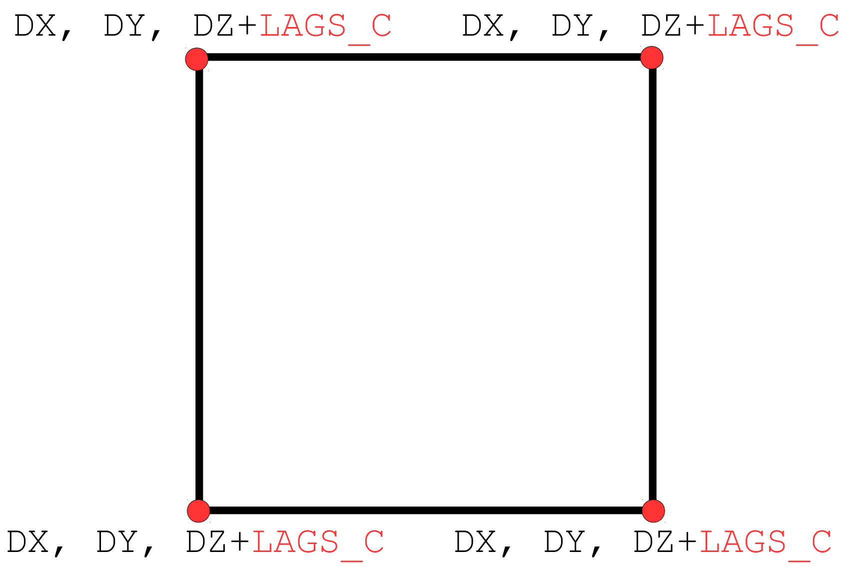


Figure 6: Attribution of the degrees of freedom on the new meshs of contact slave QUAD4 (E5 type)

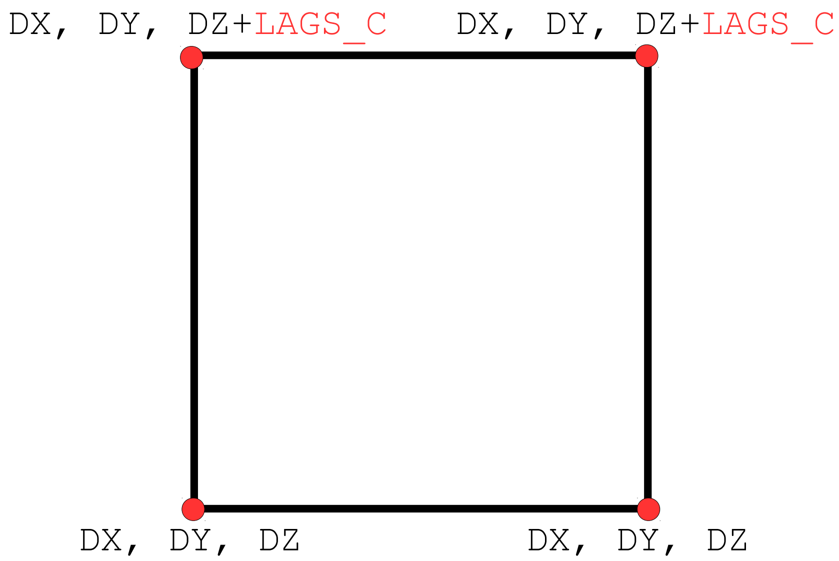


Figure 7: Attribution of the degrees of freedom on the new meshs of contact slave QUAD4 (type E1 E2 E3 E4)

It is necessary, during the assignment of elements of contact "late" on the e-mailof contact slaves in DEFI_CONTACT, knowledge if the mesh is of type E5 or E1, E2, E3, E4. One will thus buckle on the macro-meshs then on the subjacent meshs knowing that they are ordered in a known way at the time of the update of the group of mesh defining surface slave of the zone of contact considered (for example E1, E2, E3, E4 and E5).
It will be treated in STAT_NON_LINE by the use of an intelligent assembly. During the assembly, one will have to condense the contributions in single $LAGS\_C$, in order to impose well the condition of contact on average local on each macro-mesh T^m.

3.2.3 LE case SORTED6

This case is similar to the case TRIA3, to see figure 8.

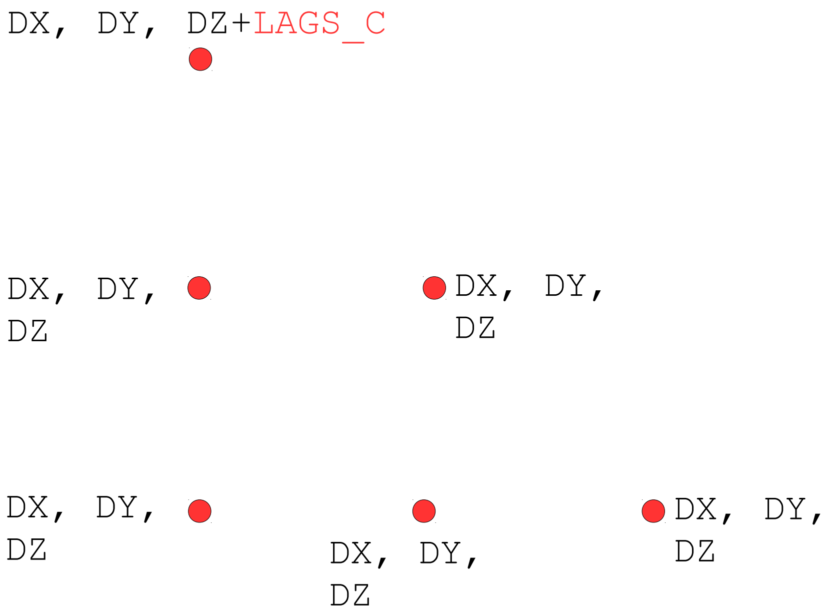


Figure 8: Attribution of the degrees of freedom on the new meshs of contact slave TRIA6

3.2.4 LE case QUAD8

This case is similar to the case QUAD4, however one do not affect degree of freedom $LAGS\_C$ with the nodes mediums (see figures 9 and 10).

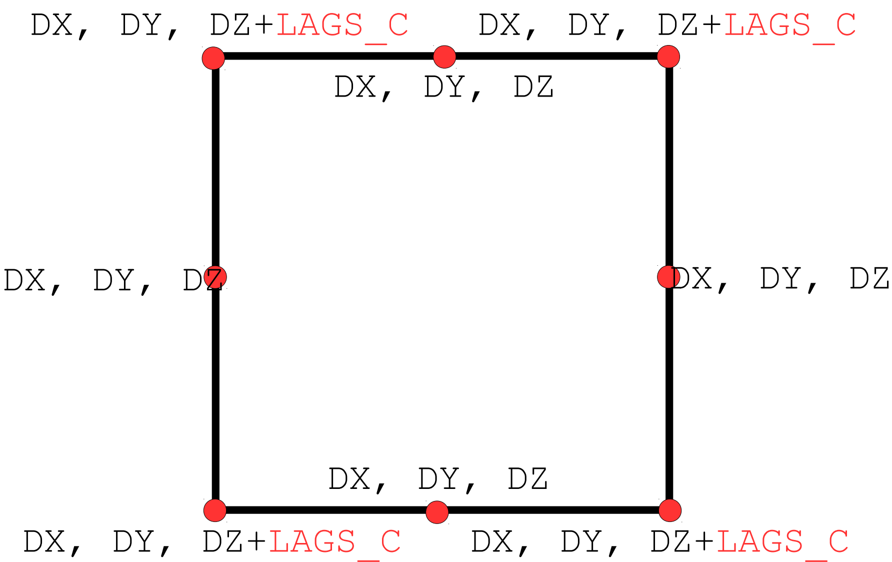


Figure 9: Attribution of the degrees of freedom on the new meshs of contact slave QUAD8 (E5 type)

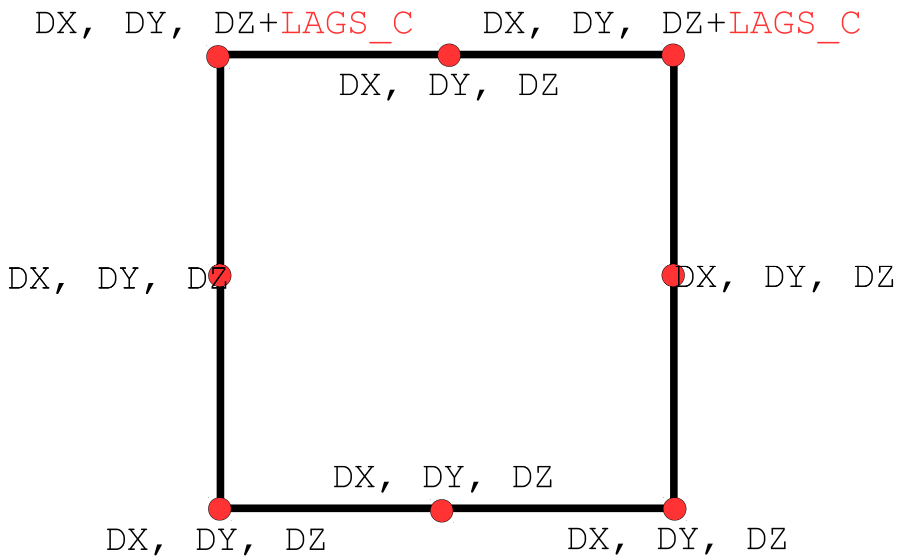


Figure 10: Attribution of the degrees of freedom on the new meshs of contact slave QUAD8 (type E1 E2 E3 E4)

3.2.5 case SEG2

One must differentiate this case in two under-CAS, the first for SEG2 resulting from TRIA3 and the second for those resulting from QUAD4. These cases are similar to the case QUAD4, indeed, as one must preserve the orientation of the meshs one cannot freely order the nodes in the connectivity of the grid resulting from the operation DECOUPE_LAC . For SEG2/TRIA3, there exist two types of meshs (see figureS 11 and 12),

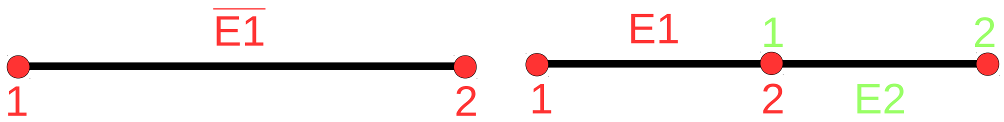


Figure 11: Local classification of the grid of calculation, case SEG2/TRIA3 (obtained by DECOUPE_LAC)

Figure 12: Attribution of the degrees of freedom on the meshs of contact slave SEG2/TRIA3

For SEG2/QUAD4, there exist three types of meshs (vto oir Figures 13 and 14).

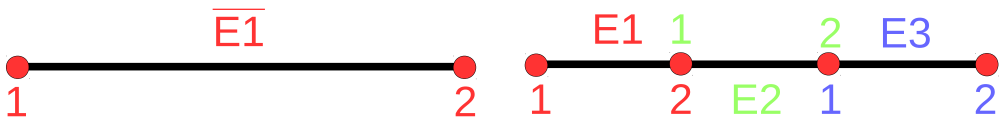


Figure 13: Local classification of the grid of calculation, case SEG2/QUAD4 (obtained by DECOUPE_LAC)

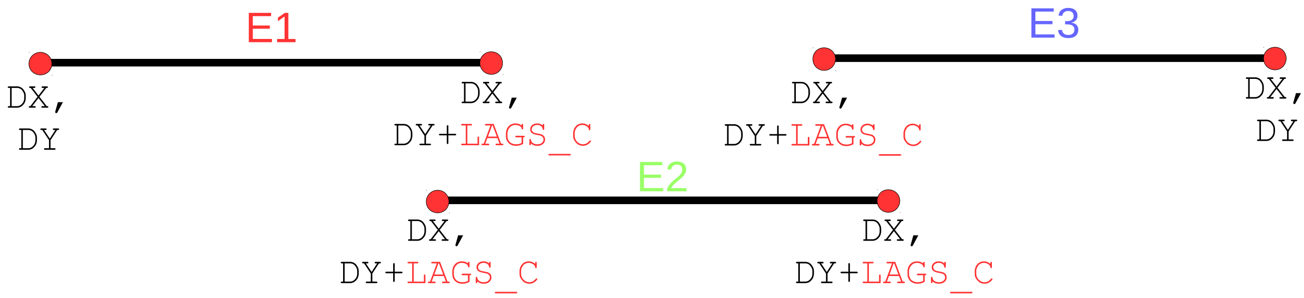


Figure 14: Attribution of the degrees of freedom on the meshs of contact slave SEG2/QUAD4

3.2.6 S case SEG3 or QUAD9

It was already noticed that these two types of mesh correspond directly to macro-meshs (the meshs and the macro-meshs are confused). In this case, there is thus one element single to affect on the meshs of contact SEG3 and QUAD9, to see Figures 15 and 16.

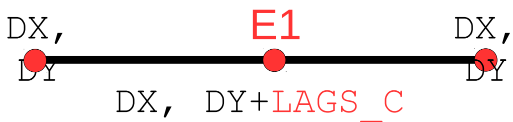


Figure 15: Attribution of the degrees of freedom on the meshs of contact slave SEG3

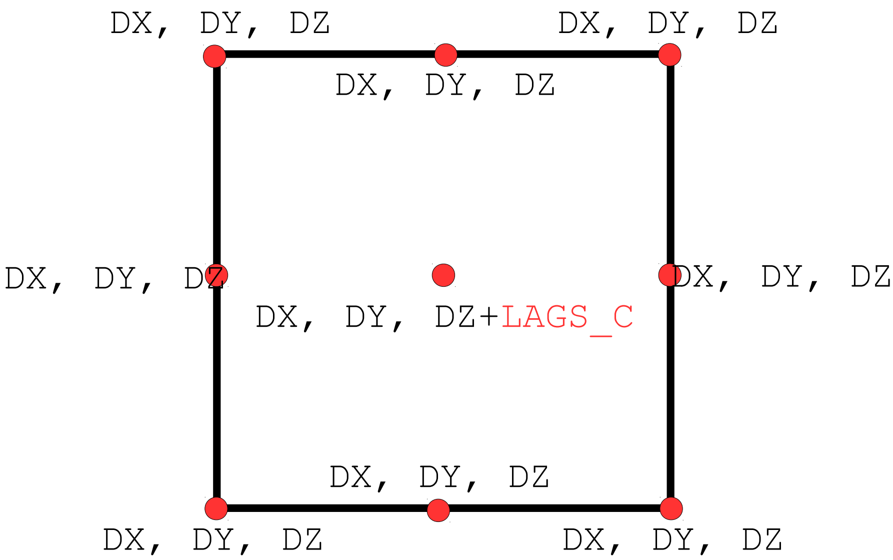


Figure 16: Attribution of the degrees of freedom on the meshs of contact slave QUAD9

3.3 Resolution of the problem of contact

There is Tkings principal impacts:

- One must to have a pairing of type "segment-segment" in 2D and of type "face-face" in 3D;
- One must calculate matrices of contact;
- One must develop an assembly of the intelligent total system (management of the second difficulty related to the trick in DEFI_CONTACT, to see §3.2.2).

The first two points in the case of require a tool of intersection of meshs effective and robust right meshs (linear and quadratic) and in the case of curved meshs (quadratic).

3.3.1 Algorithm of resolution of non-linearities related to the contact

One chooses the use of an algorithm of generalized the Newton type, managing geometrical non-linearity, the strategy of the type forced active for the non-linearity of contact and it not linearity of behavior inside the same loop of Newton. This algorithmic framework is already present in *Code_hasster* (see [R5.03.52]). It will be thus sufficient to replace all internal bricks of the algorithm to implement the new method. One poses:

$$U=\begin{bmatrix}U_N\\U_M\\U_E\end{bmatrix}\ ;\ K=\begin{bmatrix}K_{NN} & K_{NM} & K_{NE}\\K_{MN} & K_{MM} & K_{ME}\\K_{EN} & K_{EM} & K_{EE}\end{bmatrix}\ ;\ C=\begin{bmatrix}0\\C^M\\C^E\end{bmatrix} \quad (30)$$

By taking of account “integrated game” initial $\overline{G_0}$ (see the definition with the §), the problem is equivalent to:

$$\begin{cases}[K \quad C]\times\begin{bmatrix}U\\\Lambda\end{bmatrix}=F\\ {}^tCU\leqslant\overline{G_0}\ \text{dans}\ \mathbb{R}^k\\ \Lambda\leqslant 0\ \text{dans}\ \mathbb{R}^k\\ {}^tCU\cdot\Lambda=0\ \text{dans}\ \mathbb{R}\end{cases} \quad (31)$$

The following function is introduced:

$$A(U,\Lambda)=\Lambda+\max(0,\rho(CU-\overline{G_0})-\Lambda) \quad (32)$$

One can then show that is equivalent to:

$$[K \quad C]\times\begin{bmatrix}U\\\Lambda\end{bmatrix}=F \text{ and } A(U,\Lambda)=0 \quad (33)$$

The resolution of the non-linearity of contact returns then in search of active constraints. The contact is active when the statute of the point is active, which one will note $S\Lambda_i=1$. The total algorithm is obtained according to:

- Initial pairing: calculation of $\overline{G_0}$
- Buckle on Lagrange $i=1..dim(\Lambda)$
  - If $\overline{G_{0i}}\leqslant 0$ then the point is not in contact, therefore $\Lambda_i^0=0$
  - If not, the point is in contact, therefore $\Lambda_i^0=1$
- Buckle Newton $k=1$
  - As long as { $S\Lambda^k\neq S\Lambda^{k+1},\forall k$ and $\|r\|_2\geqslant tole$ (residue of balance of Newton)
    - Calculation of the matrices of rigidity and the second elementary members
    - Calculation of the matrices and the second elementary members of contact
    - Buckle on Lagrange $i=1..dim(\Lambda)$
      - If $S\Lambda_i^k=1$ then $\begin{bmatrix}0 & C_k^E & 0\\ {}^tC_k^E & 0 & {}^tC_k^M\\ 0 & C_k^M & 0\end{bmatrix}$

- If not $\begin{bmatrix} 0 & 0 & 0 \\ 0 & Id & 0 \\ 0 & 0 & 0 \end{bmatrix}$

- Assembly of the total system and resolution:
  - $\begin{bmatrix} J_k^K & C_k & 0 \\ {}^t C_k & 0 & 0 \\ 0 & 0 & Id \end{bmatrix} \times \begin{bmatrix} \Delta U_k \\ \Delta \Lambda_k^a \\ \Delta \Lambda_k^{na} \end{bmatrix} = \begin{bmatrix} r_k \\ \overline{G_k} \\ -\Delta \Lambda_{k-1} \end{bmatrix}$.
- Update unknown factors:
  - $U_{k+1} = U_k + \Delta U_k$ and $\Lambda_{k+1} = \Lambda_k + \Delta \Lambda_k$
- Pairing on the brought up to date geometry calculation of $\overline{G_{k+1}}$
- Evaluation new statutes:
  - Buckle on Lagrange $i = 1..dim(\Lambda)$
    - If $\Lambda_{k+1i} + \rho G_{k+1i} \leqslant 0$ then the point is in contact, therefore $S\Lambda_i^{k+1} = 1$
    - If not, the point is not in contact, therefore $S\Lambda_i^{k+1} = 0$
- Calculation of the residue of Newton r_{k+1}

3.3.2 Pairing

The goal of this phase is to manage the geometrical non-linearity of the problem and that resulting from possible great displacements. For each mesh of contact slave, one seeks the meshs Masters likely to communicate with the mesh slave. One must also at the time of this operation calculate the average “game” between two surfaces for each macro-mesh. With this intention, one will use four tools:

- Research tools of complexity linear using the information of vicinity on the level of the edges;
- A tool for projection of two meshs on the same datum-line (parametric space of reference of the mesh slave);
- A tool of intersection of plane meshs;
- A computational tool of average distance enters a triangle and a surface (case 3D) or between two segments (case 2D).

3.3.2.1 Algorithm of linear search for complexity

One hasdapte with the case of the contact algorithm PANG of calculation of matrix of projection mortar for the decomposition of field. This algorithm is of linear complexity since one found a couple of starting meshs. It uses a research restricted on the close meshs main Masters of the meshs intersecting a mesh slave. Moreover it uses information of vicinity on the level them meshs slaves to update the list of the couples of starting meshs.
ItT algorithm allows an important time-saver compared to a pairing of the type forces rough (i.e double loop of master-slave pairing). However it misses robustness. Indeed, in the case of zone of pairing not having only one related component, it is put in failure. One then does not detect only one of the zones of pairing and the algorithm of contact is consequently put in failure. One proposes a more robust but more expensive version of the initial algorithm by associating to him a recursive search for couple of starting meshs by rough force.

3.3.2.2 Tool for projection

This tool is already available in ***Code_hasster***. Indeed, one of the tools of the pairing of the method CONTINUOUS (see [R5.03.52]) carry out the orthogonal projection of a point of real space in the parametric space of a given mesh. One will thus be able to project any mesh Master in the parametric space of the mesh slave in order to carry out the test of intersection. Two important points are noticed:

- One limits oneself to case where the projection of the mesh Master in parametric space slave is ***convex*** ;
- Since one carries out projection in the parametric space of the mesh Master, one will purely carry out only one intersection $1D$ or $2D$.

Moreover, another routine is available in ***Code_hasster***: projection under an imposed direction. It is noted that this algorithm of projection will be useful for the tool “game” average.

3.3.2.3 Tool average “game”

One introduces this tool into the case 2D describes in Figure 17. The grid slave consists of two meshs ($E1$, $E2$) and thus one macro-mesh associated with one degree of freedom $LAGS\_C$; the main grid also consists of two meshs ($M1$, $M2$).

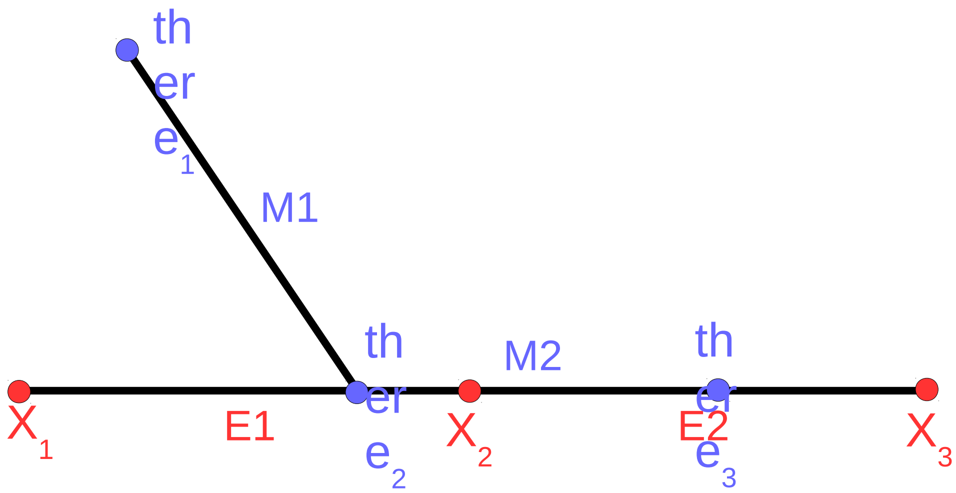


Figure 17: Configuration 2D studied, in blue the meshs M1 Masters and m2, in red the meshs slaves E1 and E2

By combining the tools of projection and intersection, one obtains the meshs of contact described in Figure 18. One will describe the tool of average game on the first mesh of contact.

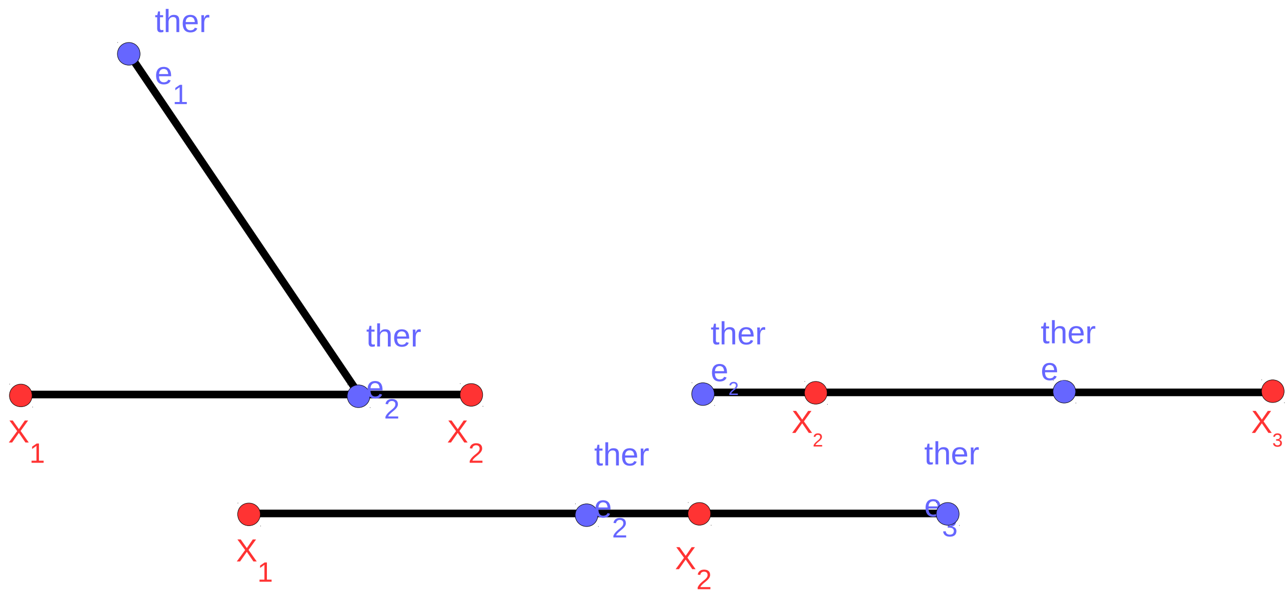


Figure 18: Meshs of contact obtained by the first two operations of pairing

If one carries out the operations projection on the mesh slave and intersection, one obtains projected points Y_1 and Y_2 within the space of reference of the mesh slave $E1$, respectively P_1 and P_2, and the intersection (segment magenta). The result of these operations is described in Figure 19.

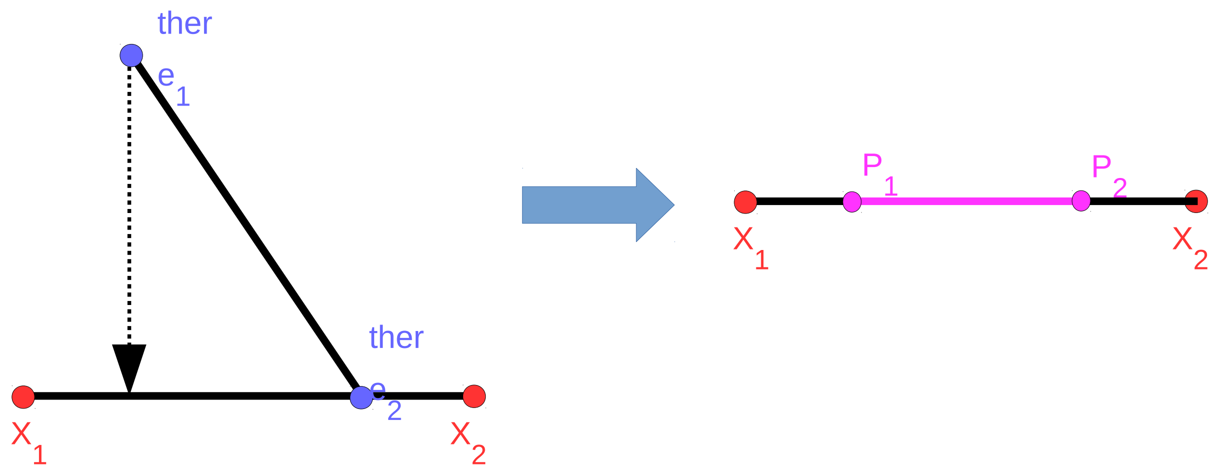


Figure 19: Projection and intersection within the space of reference of the mesh slave $E1$

In regarding the intersection as a pseudonym element, one can bring back the points of Gauss resulting from a formula of squaring within the space of reference of the mesh slave. One considers here by simplicity a formula of squaring with one not Gauss Pg ; its associated weight is the weight of the intersection. One turns over then in real space to be able to estimate the distance with the points of Gauss. These operations are described on Lhas Figure 20.

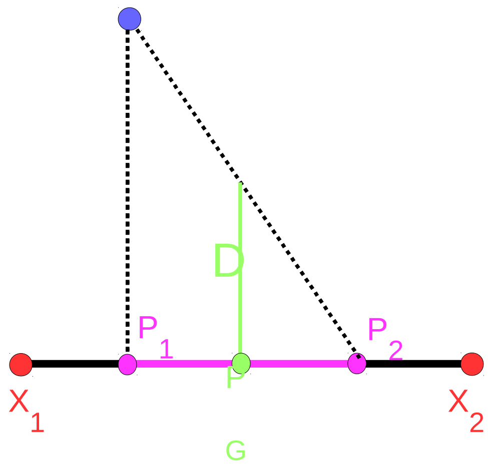


Figure 20: Game at the point of point of Gauss, d

One adds the following contribution in the component of the vector "integrated game", $\overline{G}$, corresponding to degree of freedom $LAGS\_C$ mesh of contact:

$$\overline{G}_{LAGS\_C}=\overline{G}_{LAGS\_C}+\sum_{i=1}^{N} d_i \omega_i \| (\partial_1 \sigma \times \partial_2 \sigma)(Pg_i) \|_2 \quad (34)$$

where d_i is the game signed with the i-ème not of Gauss (Pg_i), ω_i is the weight associated with the i-ème not Gauss within the space of reference with the mesh Master mesh of contact, σ is the transformation of passage element of reference real element and N is the number of points of Gauss defined on the intersection.

By iteration on the meshs of contact, one obtains the vector $\overline{G}$ whose dimension corresponds to the number of degrees of freedom $LAGS\_C$; in our example there is not only one component in the vector "game" integrated.

In the case three-dimensional, a tool for convex triangulation of polygon is necessary. Indeed, it is easier to calculate it $\overline{G}$ by pieces on a set of triangles (simpler definition of the formulas of squaring than on an unspecified polygon).

To finish the operation it is enough to divide each component of the vector "game" integrated by the weight of intersection found on the associated macro-mesh.

3.3.2.4 Management of the quadratic grids

In the quadratic case curves, one cannot approach the geometry of the mesh Master in parametric space slave. For to simplify the algorithm, one will use one simple linearization in the place of decomposition of the mesh in linear elements usually used in the literature of the methods mortar for the contact. The geometry slave is as for it approximate in an exact way by its parametric space. Contrary to the current approach consisting in using an auxiliary plan of intersection (definite starting from surface slave), one chose to carry out all the operations of projection and intersection in parametric space slave. This choice allows at the same time the use of a criterion of tolerance of pairing independent of modeling and to purely carry out only intersections $2D$ or $1D$. By using this approach and by controlling the quality of the grid, one prevents the appearance of the severe pathological cases which lead on fatal errors or a not-convergence of the algorithm of generalized Newton. Moreover, contrary to the classical methods mortar $P2/P2$ the method LAKE is not confronted with the problem of sign with respect to the basic functions of the space of the multipliers of Lagrange. The attention will have to thus be mainly carried on the level of the quality of the grid, is more particularly of the grid of surface slave. Indeed, the algorithmic choices implemented in this first version of the implementation of the method do not take into account the nonconvex case of intersections. If the meshs slaves do not respect this constraint, one obtains nonrelevant intersections, one observes consequently aberrant coefficients of intersections (superior with one) what leads to a failure of convergence of the algorithm of resolution of the problem.

3.3.3 Calculation of the matrices of contact

The calculation of the elementary matrices of contact is directed by the field of statutes $S\Lambda$. If the constraint associated with the macronutrient T is active, then one must calculate the following contributions:

$$C_i^{M_{elem}}=\int_T \varphi_i^M \vec{n}_m \, d\Gamma=\sum_{M_c}\int_{M_m\cap M_e}\varphi_i^M \vec{n}_m \, d\Gamma$$
$$C_i^{E_{elem}}=\int_T \varphi_i^E \vec{n}_e \, d\Gamma=\sum_{M_c}\int_{M_m\cap M_e}\varphi_i^E \vec{n}_e \, d\Gamma \quad (35)$$

where φ^E and φ^M are the functions of bases slaves and Masters, $\vec{n}_e$ and $\vec{n}_m$ respective normals, M_c the whole of the meshs of contact, and M_E the whole of the meshs escalves. If the constraint of contact associated with the macronutrient T then there is inactive are no conditions of contact to respect on this macronutrient and one fixes at zero it degree of freedom $LAGS\_C$ associated. Since the macro-mesh T do not exist as an element, one will calculate “under-elementary” matrices on the level of the meshs of contact M_c (association of a mesh of contact slave and Master).

3.3.3.1 Calculation of the elementary matrices of contact

One calculates in fact the elementary matrices of contact $C^{M_{elem}}$ corresponding to the integrals under the loop on the meshs of contact (see equations 35). For each mesh of contact, one must calculate the following matrices:

$$C_i^{M_{elem}}=\int_{M_m\cap M_e}\varphi_i^M \vec{n}_m d\Gamma$$
$$C_i^{E_{elem}}=\int_{M_m\cap M_e}\varphi_i^E \vec{n}_e d\Gamma \quad (36)$$

where M_e is the mesh slave of the mesh of contact and M_m is the mesh Master of the mesh of contact. Since the selected condition of contact corresponds to a null game only on average, one must proceed to an projection-intersection to define the support of integration. With this intention, one re-uses the tools presented in the preceding section. While following the diagram of the tool integrated game, one manages to obtain the coordinates and weight of a formula of squaring in the parametric space of the mesh slave of the mesh of contact. While buckling on degrees of freedom main displacements, one can then calculate the coefficients of the matrix $C_i^{E_{elem}}$. Then by using projection on the parametric space of the mesh Master of the mesh of contact one then obtains the support of integration in space reference of the mesh Master, which makes it possible to calculate (by defining a diagram of integration) the coefficients of the matrix $C_i^{M_{elem}}$.

3.3.3.2 Choice of the normal

The question of the choice of the normal was not tackled yet. This choice not being single, one proposes to use in our case two types of normal. Either one evaluates the normal at each point of main Gauss and slave, or one uses a field of smoothed normals (keyword `LISSAGE=' OUI'`) calculated on each deformed configuration (main and slave). These two choices make it possible to avoid the multiple problem of definition of the normal at the top of the elements when curved geometries are considered. In the quadratic case, these two choices are relatively similar, on the contrary in the case of the linear grids, the field of smoothed normal makes it possible to regularize the field of the normals and to better approach the curved geometries, to see Figure 21.

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LanguageLevel:2

Figure 21: Various choices of field of normals

3.3.4 Intelligent assembly

We present here a track to manage the second difficulty related to the trick used in `DEFI_CONTACT`. One is located in the case `HEXA8` ; one wants to calculate the matrix C^E . It is supposed that the grid slave is only composed of a macro-mesh, there are thus five meshs on surface slave (to simplify we will speak only about the contributions slaves), eight degrees of freedom of displacement, four Lagrangian of contact “data processing”, but only one Lagrangian with the mathematical direction. Figure 22 represent the grid slave considered.

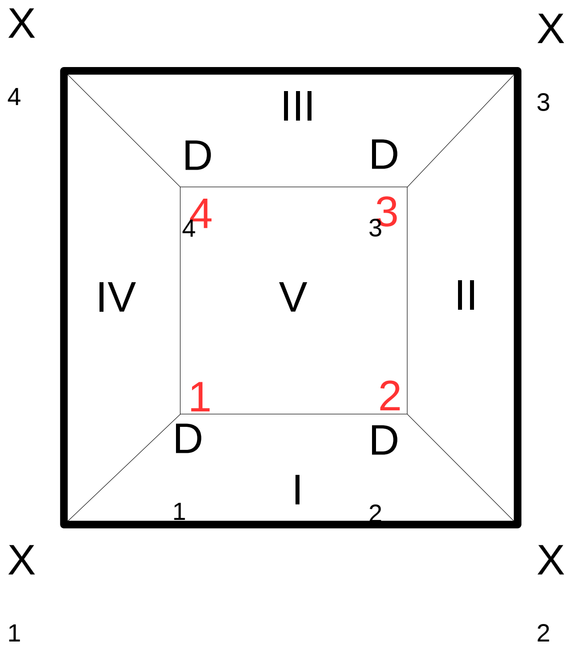

Figure 22: Grid slave, in black (x_1 ,…, x_4 , d_1 , …, d_4) degrees of freedom of displacement, in red (1,…, 4) degrees of freedom of Lagrange "data processing".

The elementary matrices are the following ones:

- Element I:

$$\begin{bmatrix} a_I & b_I & c_I & d_I \\ a_I & b_I & c_I & d_I \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ d_2 \\ d_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad (37)$$

- Element II:

$$\begin{bmatrix} a_{II} & b_{II} & c_{II} & d_{II} \\ a_{II} & b_{II} & c_{II} & d_{II} \end{bmatrix} \times \begin{bmatrix} x_2 \\ x_3 \\ d_3 \\ d_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad (38)$$

- Element III:

$$\begin{bmatrix} a_{III} & b_{III} & c_{III} & d_{III} \\ a_{III} & b_{III} & c_{III} & d_{II} \end{bmatrix} I \times \begin{bmatrix} x_3 \\ x_4 \\ d_4 \\ d_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \quad (39)$$

- Element IV:

$$\begin{bmatrix} a_{IV} & b_{IV} & c_{IV} & d_{IV} \\ a_{IV} & b_{IV} & c_{IV} & d_{IV} \end{bmatrix} \times \begin{bmatrix} x_4 \\ x_1 \\ d_1 \\ d_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \quad (40)$$

- Element V:

$$\begin{bmatrix} a_V & b_V & c_V & d_V \\ a_V & b_V & c_V & d_V \\ a_V & b_V & c_V & d_V \\ a_V & b_V & c_V & d_V \end{bmatrix} \times \begin{bmatrix} x_4 \\ x_1 \\ d_1 \\ d_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \quad (41)$$

A classical assembly would give us the total matrix C^E following:

$$\begin{bmatrix} a_I+b_{IV} & b_I & 0 & a_{IV} & a_V+c_I+c_{IV} & b_V+d_I & c_V & d_V+d_{IV} \\ a_I & a_{II}+b_I & b_{II} & 0 & a_V+c_I & b_V+d_I+d_{II} & c_V+c_{II} & d_V \\ 0 & a_{II} & b_{II}+a_{III} & b_{III} & a_V & b_V+d_{II} & c_V+c_{II}+d_{III} & d_V+c_{III} \\ b_{IV} & 0 & a_{III} & b_{III}+a_{IV} & a_V+c_{IV} & b_V & c_V+d_{III} & d_V+c_{III}+d_{IV} \end{bmatrix} \quad (42)$$

It is noticed easily that this matrix does not correspond to the coupling of displacements with the functions $P^0(T^M)$. One must find a technique of assembly compatible with the assembly used in ***Code_hasster*** (and which should not be completely incompatible with parallelism) allowing to obtain the following matrix of coupling:

$$^t\boldsymbol{DEPL} \times \begin{bmatrix} a_I+b_{IV} \\ b_I+a_{II} \\ b_{II}+a_{III} \\ b_{III}+a_{IV} \\ a_V+c_I+c_{IV} \\ b_V+d_I+d_I \\ c_V+d_{III}+c_{II} \\ d_V+c_{III}+d_{IV} \end{bmatrix} = LAG_C \quad (43)$$

One wants during the elementary assembly to thus multiply the contributions by $\frac{1}{2}$ in the case of elements of the type E1, E2, E3 and E4 (elementary matrices (37), (38), (39) and (40)) and by $\frac{1}{4}$ in the case as of elements of the type E5 (matrix elementary (41)). Then for each macronutrient, one summons the corresponding lines of the total matrix of the system and one keeps only one Lagrangian in the system corresponding to the mathematical condition of contact. Three Lagrangian additional data processing are fixed equal to the physical multiplier of Lagrange.

3.3.5 Postprocessing

All variables of resolution of the contact, statutes, multipliers of Lagrange, “game”, are only relevant on the level of the macro-mesh, the field at exit is named `CONT_ELEM`.

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