# Solution of a differential equation of the second order by the method of NIGAM

**Summary:**

We present in this document, a method of resolution of the linear differential equation of the second order obtained during the calculation of a spectrum of oscillator.

# 1 Introduction

During the calculation of a spectrum of oscillator, one is brought to solve a differential equation of the second order whose solution is an integral of DUHAMEL.

If this integral can be calculated exactly using the transform of LAPLACE for certain simple analytical functions (Dirac, Sine, Cosine, Heavyside,…) [bib1] it must be integrated numerically in the case general.

This document presents an effective method to solve this problem.

This method is put in work in *Code_Aster*, in the operator `CALC_FONCTION`, keyword factor `SPEC_OSCI`.

# 2 Analytical solution of the equation

During the calculation of the spectrum of oscillator of a accélérogramme [R4.05.03], one is brought to solve the linear differential equation of the second order:

$$\ddot{q}+2\xi\omega\dot{q}+\omega^2 q=-\alpha(t)$$

where
- $q(t)$ is relative displacement
- $\alpha(t)$ is the acceleration of the movement imposed on the base
- $\omega$ is the pulsation of the oscillator
- $\xi$ is the reduced damping of the oscillator

With initial conditions on $q$ and $\dot{q}$ .

The solution of this equation is written in the form:

$$q(t)=+\int_0^t h(t-\tau).\alpha(\tau)\,d\tau+q(0)\,g(t)+\dot{q}(0)\,h(t) \qquad \text{éq 2-1}$$

where $q(0)$ and $\dot{q}(0)$ are displacement and speed at the initial moment.

- Expression of $h(t)$ and $g(t)$ according to the value of reduced damping $\xi$ .
  - If $\xi<1$ (damping sub-critical):

$$h(t)=\frac{e^{-\xi\omega t}}{\omega\sqrt{1-x^2}}\sin\left(\omega t\sqrt{1-\xi^2}\right)$$
$$g(t)=e^{-\xi\omega t}\left[\cos\left(\omega t\sqrt{1-\xi^2}\right)+\frac{\xi}{\sqrt{1-\xi^2}}\sin\left(\omega t\sqrt{1-\xi^2}\right)\right] \qquad \text{éq 2-2}$$

- If $\xi=1$ (critical damping):

$$h(t)=te^{-\omega t}$$
$$g(t)=(1-\omega)e^{\omega t}$$

- If $\xi>1$ (supercritical damping):

$$h(t)=\frac{e^{-\xi\omega t}}{\omega\sqrt{\xi^2-1}}.sh\left(\omega t\sqrt{\xi^2-1}\right)$$
$$g(t)=e^{-\xi\omega t}\left[ch\left(\omega t\sqrt{\xi^2-1}\right)+\frac{\xi}{\sqrt{\xi^2-1}}sh\left(\omega t\sqrt{\xi^2-1}\right)\right]$$

# 3 Digital method

Digital method established in *Code_Aster* was proposed by NIGAM and JENNINGS [bib2] in the case of the damping sub-critical which corresponds to our initial seismic problem [R4.05.03].

By introducing the formulation [éq 2-2] in [éq 2-1] one is thus led to solve the differential equation:

$$\ddot{q}(t)+2\xi\omega\dot{q}(t)+\omega^2 q(t)=-\alpha(t)$$

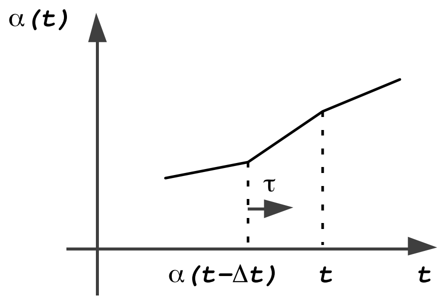
with worthless initial conditions, whose solution is written:

$$q(t)=\frac{1}{\omega_d}\int_0^t e^{-\xi\omega(t-\tau)}\sin\left[\omega_d(t-\tau)\right]\alpha(\tau)\,d\tau$$

with $\omega_d=\omega\sqrt{1-\xi^2}$

By supposing that $\alpha(t)$ vary linearly inside each interval $\Delta(t)$ , one can then write:

$$\alpha(\tau)=\alpha(t-\Delta t)+\frac{\tau}{\Delta t}\left[\alpha(t)-\alpha(t-\Delta t)\right] \text{ pour } \tau\in\left[0,\Delta t\right]$$



from where the equation to be solved (expressed in the new variable $\tau$ ):

$$\ddot{q}(t)+2\xi\omega\dot{q}(\tau)+\omega^2 q(\tau)=a+b\tau \text{ pour } \tau\in[0,\Delta t]$$

where
$$a=\alpha(t-\Delta t)$$
$$b=[\alpha(t)-\alpha(t-\Delta t)]/\Delta t$$

with the initial conditions:
$$q(0)=q(t-\Delta t)$$
$$\dot{q}(0)=\dot{q}(t-\Delta t)$$

The solution of this equation is the superposition of a particular solution and solutions of the homogeneous problem.

- a particular solution: $q_p(t)=-\dfrac{a}{\omega^2}+\dfrac{2\xi b}{\omega^3}-\dfrac{b}{\omega^2}\tau$
- solutions of the homogeneous problem: $q_h(t)=e^{-\xi\omega\tau}[C_1.\cos(\omega_d\tau)+C_2.\sin(\omega_d\tau)]$

Consequently: $q(\tau)=e^{-\xi\omega\tau}[C_1.\cos(\omega_d\tau)+C_2.\sin(\omega_d\tau)]-\dfrac{a}{\omega^2}+2\dfrac{\xi b}{\omega^3}-\dfrac{b.\tau}{w^2}$

and while deriving $q$ (compared to $t$ ) one a:

$$\dot{q}(\tau)=(-\xi\omega)\,e^{-\xi\omega\tau}(C_1\cos\omega_d\tau+C_2\sin\omega_d\tau)+e^{-\xi\omega\tau}(-C_1\omega_d\sin\omega_d\tau+C_2\omega_d\cos\omega_d\tau)-\frac{b}{\omega^2}$$

Coefficients $C_1$ and $C_2$ are then determined by the initial conditions at the beginning of the interval (it is - with-to say for $\tau=0$ ).

$$C_1=q(t-\Delta t)+\frac{a}{\omega^2}-\frac{2\xi b}{\omega^3}$$
$$C_2=\frac{1}{\omega_d}\left[\dot{q}(t-\Delta t)+\xi\omega q(t-\Delta t)+\frac{\xi a}{\omega}-\frac{2\xi^2-1}{\omega^2}b\right]$$

and while deferring $C_1$ and $C_2$ in the expression of $q$ and $\dot{q}$ one obtains the matric equality for $\tau=\Delta t$ :

$$\begin{Bmatrix}q(t)\\\dot{q}(t)\end{Bmatrix}=A(\xi,\omega,\Delta t)\begin{Bmatrix}q(t-\Delta t)\\\dot{q}(t-\Delta t)\end{Bmatrix}+B(\xi,\omega,\Delta t)\begin{Bmatrix}\alpha(t-\Delta t)\\\alpha(t)\end{Bmatrix}$$

# 4 Coefficients of matrices A and B of the system to be solved

**Matrix** $A$ **:**

$$a_{11}=e^{-\xi\omega\Delta t}\left[\frac{\xi}{\sqrt{1-\xi^2}}\sin\left(\omega_d\Delta t\right)+\cos\left(\omega_d\Delta t\right)\right]$$

$$a_{12}=\frac{e^{\xi\omega\Delta t}}{\omega_d}\sin\left(\omega_d\Delta t\right)$$

$$a_{21}=-\frac{\omega}{\sqrt{1-\xi^2}}e^{-\xi\omega\Delta t}\sin\left(\omega_d\Delta t\right)$$

$$a_{22}=e^{-\xi\omega\Delta t}\left[\cos\left(\omega_d\Delta t\right)-\frac{\xi}{\sqrt{1-\xi^2}}\sin\left(\omega_d\Delta t\right)\right]$$

**Matrix** $B$ **:**

$$b_{11}=e^{-\xi\omega\Delta t}\left[\left(\frac{2\xi^2-1}{\omega^2\Delta t}+\frac{\xi}{\omega}\right).\frac{\sin\left(\omega^d\Delta t\right)}{\omega^d}+\left(\frac{2\xi}{\omega^3\Delta t}+\frac{1}{\omega^2}\right)\cos\left(\omega^d\Delta t\right)\right]-\frac{2\xi}{\omega^3\Delta t}$$

$$b_{12}=e^{-\xi\omega\Delta t}\left[\frac{2\xi^2-1}{\omega^2\Delta t}.\frac{\sin\left(\omega^d\Delta t\right)}{\omega^d}+\frac{2\xi}{\omega^3\Delta t}.\cos\left(\omega^d\Delta t\right)\right]-\frac{1}{\omega^2}+\frac{2\xi}{\omega^3\Delta t}$$

$$b_{21}=e^{-\xi\omega\Delta t}\left[\left(\frac{2\xi^2-1}{\omega^2\Delta t}+\frac{\xi}{\omega}\right).\left(\cos\left(\omega_d\Delta t\right)-\frac{\xi}{\sqrt{1-\xi^2}}\sin\left(\omega^d\Delta t\right)\right)-\right.$$
$$\left.\left(\frac{2\xi}{\omega^3\Delta t}+\frac{1}{\omega^2}\right).\left(\omega_d\sin\left(\omega_d\Delta t\right)+\xi\omega\cos\left(\omega_d\Delta t\right)\right)\right]+\frac{1}{\omega^2\Delta t}$$

$$b_{22}=-e^{-\xi\omega\Delta t}\left[\left(\frac{2\xi^2-1}{\omega^2\Delta t}\right).\left(\cos\left(\omega_d\Delta t\right)-\frac{\xi}{\sqrt{1-\xi^2}}\sin\left(\omega_d\Delta t\right)\right)-\right.$$
$$\left.\left(\frac{2\xi}{\omega^3\Delta t}\right).\left(\omega_d\sin\left(\omega_d\Delta t\right)+\xi\omega\cos\left(\omega_d\Delta t\right)\right)\right]-\frac{1}{\omega^2\Delta t}$$

with $\omega_d=\omega\sqrt{1-\xi^2}$

# 5 Calcul of acceleration $\ddot{q}(\tau)$

Knowing $q(\tau)$ and $\dot{q}(\tau)$, it is consequently possible to give the analytical expression of acceleration $\ddot{q}(\tau)$.

$$\begin{aligned}
\dot{q}(\tau) &= -(\xi\omega)e^{-\xi\omega}\left[C_1\cos(\omega_d\tau)+C_2\sin(\omega_d\tau)\right] \\
&\quad + e^{-\xi\omega}\left(-C_1\omega_d\sin(\omega_d\tau)+C_2\omega_d\cos(\omega_d\tau)\right)-\frac{1}{\omega^2} \\
\ddot{q}(\tau) &= +(\xi\omega)^2 e^{-\xi\omega}\left[C_1\cos(\omega_d\tau)+C_2\sin(\omega_d\tau)\right] \\
&\quad +(\xi\omega)e^{-\xi\omega}\left(-C_1\omega_d\sin(\omega_d\tau)+C_2\omega_d\cos(\omega_d\tau)\right) \\
&\quad -(\xi\omega)e^{-\xi\omega}\left(-C_1\omega_d\sin(\omega_d\tau)+C_2\omega_d\cos(\omega_d\tau)\right) \\
&\quad + e^{-\xi\omega}\left[-C_1\omega_d^2\cos(\omega_d\tau)-C_2\omega_d^2\sin(\omega_d\tau)\right] \\
\ddot{q}(\tau) &= \left[(\xi\omega)^2-\omega_d^2\right]e^{-\xi\omega}.\left[C_1\cos(\omega_d\tau)+C_2\sin(\omega_d\tau)\right] \\
&\text{or} \\
\omega_d^2 &= \omega^2(1-\xi^2)\text{, d'où} \\
\ddot{q}(\tau) &= \omega^2 e^{-\xi\omega}\left[C_1\cos(\omega_d\tau)+C_2\sin(\omega_d\tau)\right]
\end{aligned}$$

however

$$\omega_d^2=\omega^2(1-\xi^2)$$

from where:

$$\ddot{q}(\tau)=\omega^2 e^{-\xi\omega}\left[C_1\cos(\omega_d\tau)+C_2\sin(\omega_d\tau)\right]$$

# 6 Bibliography

1) R.J. GIBERT: Vibrations of the structures, Collection of the Management of the Studies and Searchs for Électricité de France, n°69, Eyrolles 1988.

2) N.C. NIGAM & PC JENNINGS: Calculation of Réponse will spectra from motion earthquake Bull. of the Seismological society of America, Vol.59 n°2 pp 909 - 922 April 1969.

3) D. SELIGMANN, L. VIVAN: Seismic answer by spectral method [R4.05.03].

# 7 Description of the versions of the document

| Version Aster | Author (S) Organization (S) | Description of the modifications |
|---|---|---|
| 6 | D.Selligmann, EDF/DER/MMN<br>O.Boiteau, EDF-R&D/SINETICS | Initial text |