Dynamic nonlinear algorithm

Summary:

The operator DYNA_NON_LINE [U4.53.01] of *Code_Aster* get busy for the nonlinear dynamic analysis of the structures by a direct integration in time. Non-linearities can come from the behavior of material, the connections (contact-friction), or great geometrical transformations (great displacements and great rotations).
The organization of DYNA_NON_LINE are strongly connected with that of the nonlinear quasi-static operator STAT_NON_LINE [R5.03.01]. *A priori*, all relations of behavior developed within the framework of STAT_NON_LINE function in that of DYNA_NON_LINE.
One presents here the general formulation of the nonlinear dynamic problem in order to specify the articulations between the purely dynamic aspects and those already treated in other operators or formulations available in *Code_Aster* : management of the boundary conditions, the couplings fluid-structure, damping, calculation in relative reference mark, then properties of the diagram of digital integration temporal, which is worked out independently of any relation of behavior. It is exposed how this one is articulated with the algorithm of Newton to treat material and geometrical non-linearities.

One has three diagrams in complementary and effective implicit times:
- the family of Newmark,
- modified average acceleration (" HHT " in DYNA_NON_LINE, with the option MODI_EQUI=' NON'),
- complete diagram HHT (" HHT " in DYNA_NON_LINE, with the option MODI_EQUI=' OUI'),

Code_Aster propose also two explicit diagrams:
- centered differences,
- the dissipative diagram of Tchamwa-Wielgosz.

One gives some advices for a good use which U2.06.13 documentation comes to supplement.

Contents

1 Notations

u	field of continuous absolute displacement
$\mathbf{K}$, $\mathbf{K}_{\mathbf{i}}^{\mathbf{n}}$	matrix of rigidity, tangent matrix
$\mathbf{M}$	matrix of inertia
$\mathbf{R}$	vector of the interior forces
$\mathbf{L}$	vector second member of loadings (linear form)
$\mathbf{L}^{abso}$ $\mathbf{L}_{GR}^{iner}$, $\mathbf{L}^{anel}$	second members respectively due to an absorbing border, the nonlinear inertial terms in great rotations of beam, to unelastic chainings (coming from variables of order: temperature…)
$\mathbf{C}$	matrix of damping
$\mathbf{Q}$	matrix of assembled deformation
${}^{t}\mathbf{V}$ …	transposed of a vector $\mathbf{V}$: dual linear form…
t ; Δt	time; no time
α	parameter of the diagram of temporal integration of α method (and HHT)
β , γ	parameters of the diagram of temporal integration of NEWMARK
Δ	increment of various sizes during the step of time
δ	virtual variation of a field; increment of various sizes during iterations of correction
i ; n ; j	index of the step of time; index of the iteration of NEWTON ; index of component
$\boldsymbol{\lambda}$, $\boldsymbol{\mu}$	parameters of LAGRANGE: reactions of connection, reactions of contact
$\mathbf{U}$, $\dot{\mathbf{U}}$, $\ddot{\mathbf{U}}$	vector degrees of freedom of successive displacement and derivative compared to time
$\mathbf{P}$	vector degrees of freedom of disturbances of fluid pressure barotrope
$\boldsymbol{\phi}$	vector degrees of freedom of potential of disturbances of fluid displacement barotrope
$\boldsymbol{\Phi}$	configuration: vector position: $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ and possibly vector rotation, and other fields parameterizing the system
$\dot{\boldsymbol{\Phi}}$	temporal derivative of the configuration $\boldsymbol{\Phi}$ compared to time: speed of traverse and possibly angular velocity
$\ddot{\boldsymbol{\Phi}}$	temporal derivative of $\dot{\boldsymbol{\Phi}}$ compared to time: acceleration of translation and possibly angular acceleration
	Convention of the repeated indices: $U_d^k(t)\boldsymbol{\Psi}_{\mathbf{k}} = \sum_k U_d^k(t)\boldsymbol{\Psi}_{\mathbf{k}}$

2 Nonlinear dynamics: space discretization of the continuous problem

To solve a nonlinear problem of dynamics requires to describe the equations of the continuous problem first of all, then to present their space discretization, here in finite elements, and finally to describe the method of temporal integration, associated with the treatment of material and geometrical non-linearities.

2.1 Discretization of the linear problem of dynamics

One notes $\mathbf{u}$ the field of absolute displacements compared to the configuration of reference, and parameterized by the moment t, pertaining to space refines acceptable fields V_{adm}.

The direct method consists in solving the problem resulting from the discretization by finite elements of the formulation in displacement.

Discretization of the virtual work of the inertial forces, in a field $\delta \mathbf{v} \in V_{adm}^0$, directing vector space of V_{adm}, is written:

$$\int_\Omega \rho \ddot{\mathbf{u}}.\delta \mathbf{v}\, d\Omega = {}^t\delta \mathbf{V}.\mathbf{M}.\ddot{\mathbf{U}}$$

The discretization of the virtual variation of the work dissipated in viscosity (damping brought by a dependence of the constraints according to speeds of deformation) is:

$$\int_\Omega c\,\dot{\mathbf{u}}.\delta \mathbf{v}\, d\Omega = {}^t\delta \mathbf{V}.\mathbf{C}.\dot{\mathbf{U}}$$

One specifies with [§2.2.1] how the operator of damping $\mathbf{C}$ is built in DYNA_NON_LINE.

The discretization of the virtual work of the interior efforts in linear elasticity is written:

$$\int_\Omega \boldsymbol{\varepsilon}(\mathbf{u}).\mathbf{A}.\boldsymbol{\varepsilon}(\delta \mathbf{v})\, d\Omega = {}^t\delta \mathbf{V}.\mathbf{K}.\mathbf{U}$$

Lastly, $\mathbf{L}$ designate the second member resulting from the discretization of the virtual work of the external forces.

In linear elasticity, that thus leads to consider the hyperbolic différentio-algebraic system according to, for the degrees of freedom $\mathbf{U}$, with the initial conditions:

$$\begin{aligned}&\text{Trouver } \mathbf{U} \in \mathbb{R}^n :\\ &\mathbf{M}.\ddot{\mathbf{U}} + \mathbf{C}.\dot{\mathbf{U}} + \mathbf{K}.\mathbf{U} = \mathbf{L}\\ &\mathbf{U}(t_0) = \mathbf{U}_0\\ &\dot{\mathbf{U}}(t_0) = \dot{\mathbf{U}}_0\end{aligned}$$

accompanied by boundary conditions.

The initial conditions are provided to the algorithm by the keyword ETAT_INIT (operands DEPL and QUICKLY).

If the initial state results from a simulation in or not linear linear statics, one does not take into account initial speed, and displacement, as well as the variables of state (forced, internal variables), are extracted from the result of this simulation at the starting moment considered.

The dynamic system of balance becomes unstable if one can find a pulsation ω complex, the imaginary part is negative, for which one can cancel the determinant of: $-\omega^2\mathbf{M}+i\omega\mathbf{C}+\mathbf{K}$.

2.2 Discretization of the nonlinear problem of dynamics

One places oneself now within a nonlinear mechanical framework.

Virtual work is noted $\int_\Omega \boldsymbol{\sigma}(\mathbf{u},\dot{\mathbf{u}},t).\boldsymbol{\varepsilon}(\delta\mathbf{v})\,d\Omega$ of deformation (known as also of the internal forces) of the nonlinear problem of mechanics. However one frequently adds to the internal forces coming from the law of behavior, the internal forces of viscous damping built in a total way, *cf.* [§ 2.2.2]: $\boldsymbol{C}\cdot\dot{\mathbf{U}}$, where $\boldsymbol{C}$ is the matrix of damping. The virtual work of the internal forces is thus written after discretization:

$$^t\delta\mathbf{V}.\left(\mathbf{R}(\mathbf{U},\dot{\mathbf{U}},t)+\mathbf{C}.\dot{\mathbf{U}}\right)=^t\delta\mathbf{V}.\left(^t\mathbf{Q}(\mathbf{U}).\boldsymbol{\sigma}\left(\mathbf{U},\dot{\mathbf{U}},t\right)+\mathbf{C}.\dot{\mathbf{U}}\right)$$

where one voluntarily distinguished the linear viscous forces (operator $\mathbf{C}$) other internal forces. In the case of small displacements, the operator of deformation assembled $^t\mathbf{Q}$ is constant (and definite on the initial configuration confused with the deformation).

The mechanical assessment of energy is written:

$$\mathbf{L}.\delta\mathbf{v}=\int_\Omega \rho\,\ddot{\mathbf{u}}.\delta\mathbf{v}\,d\Omega+\int_\Omega \boldsymbol{\sigma}(\mathbf{u},\dot{\mathbf{u}},t).\boldsymbol{\varepsilon}(\delta\mathbf{v})\,d\Omega$$

The stress field $\boldsymbol{\sigma}$ at the moment t is written in a general way $\boldsymbol{\sigma}(\mathbf{u},\dot{\mathbf{u}},\mathbf{Z},t,\mathbf{H})$, if one notes $\mathbf{Z}$ the field of variables of order, such as for example T the field of temperatures, and $\mathbf{H}$ last history of the structure until the moment t . For the incrémentaux behaviors, the history is the whole of the states (fields of displacements, constraints and variables internal) at the previous moment.

In the linear case (*cf.* [§ 2.1]), the virtual work of deformation becomes:

$$\mathbf{R}\left(\mathbf{U},\dot{\mathbf{U}}\right)+\mathbf{C}.\dot{\mathbf{U}}=\mathbf{K}.\mathbf{U}+\mathbf{C}.\dot{\mathbf{U}}$$

where $\mathbf{K}$ is the elastic matrix of rigidity of the structure.

2.2.1 Term of inertia

One notes $^t\delta\mathbf{V}.\mathbf{M}(\mathbf{U},\dot{\mathbf{U}},\ddot{\mathbf{U}})$ after discretization, virtual work $\int_\Omega \rho\,\ddot{\mathbf{u}}.\delta\mathbf{v}\,d\Omega$ inertial forces of the system.

One notes $\mathbf{M}$ the matrix of mass of the system in small transformations. In great rotations of structure (such as for example the beams, *cf.* [R5.03.40]), this virtual work is a nonlinear function of $\mathbf{U}(t)$ and of its derivative temporal (specifically of the degrees of freedom of rotation); one thus reveals the usual term of acceleration with a nonlinear correction: $\mathbf{M}(\mathbf{U},\dot{\mathbf{U}},\ddot{\mathbf{U}})=\mathbf{M}(\mathbf{U}).\ddot{\mathbf{U}}+\mathbf{L}_{GR}^{iner}(\mathbf{U},\dot{\mathbf{U}},\ddot{\mathbf{U}})$. In the other cases, $\mathbf{M}(\mathbf{U},\dot{\mathbf{U}},\ddot{\mathbf{U}})=\mathbf{M}(\mathbf{U}).\ddot{\mathbf{U}}$, which can vary if the geometry is reactualized, or who is constant in small displacements.

Note: case of the geometrical reactualization (DEFORMATION=' PETIT_REAC'**) :**

For a dynamic problem with geometrical reactualization, while having admitted the conservation of the mass, one notes that the matrix of mass is not modified. Indeed: while noting respectively $det\,\mathbf{J}(\xi)$ *and* $det\,\mathbf{J}_{RG}(\xi)$ *the determinant of the matrix jacobienne of the transformation real element* Ω_e *towards element of reference* Ω_r *and that of the transformation reactualized real element* Ω_{RG} *towards element of reference – cf [R3.01.00], § 3.5 – the virtual work of the inertial forces is written:*

$$\int_{\Omega_{RG}} \rho_{RG}\,\ddot{\mathbf{u}}.\delta\mathbf{v}\;d\Omega_{RG}=\int_{\Omega_r}\rho_{RG}\,\ddot{\mathbf{u}}.\delta\mathbf{v}.det\,\mathbf{J}_{RG}(\xi)\;d\Omega_r=\int_{\Omega_r}\rho\,\ddot{\mathbf{u}}.\delta\mathbf{v}.det\,\mathbf{J}(\xi)\;d\Omega_r$$

because $\rho_{RG}=\rho.det\,\mathbf{J}_{RG}^{EL}=\rho.det\,\mathbf{J}/det\,\mathbf{J}_{RG}$ *following the conservation of the mass in the reactualization (jacobien* $\mathbf{J}_{RG}^{EL}$*).*

2.2.2 Damping

It is permissible to use discrete elements on which one makes carry a behaviour of damping via a matrix acting on the degrees of freedom, *cf.* [U4.42.01], but damping can also relate to the massive models or of structures. The operator of damping $\mathbf{C}$ of the latter can be in *Code_Aster* defined in two ways in DYNA_NON_LINE, *cf.* also [R5.05.04], [U2.06.03]:

1) **a total way on a basis of clean modes** $(\mathbf{\Phi_k})$ **, known as modal damping** (according to the assumption of BASILE) established as a preliminary on the elastic structure, expressed on the basis of "physical" modeling by finite elements. One thus defines a coefficient by selected mode. The keyword: AMOR_MODAL of the operator DYNA_NON_LINE allows to provide him the base of modes and the coefficients damping reduced (according to the assumption of BASILE). Indeed, depreciation is in experiments given by analysis modal on resonances.
Displacements $\mathbf{U}$ are thus projected on the modes to obtain their generalized coordinates: $\boldsymbol{\eta}^{\mathbf{k}}={}^t\Phi_k.\mathbf{U}$. The matrix of modal damping is:
$\mathbf{C}=(\mathbf{K}\mathbf{\Phi_k}).C_{modal}^k.{}^t(\mathbf{K}\mathbf{\Phi_k})$ with the scalar $C_{modal}^k=2\dfrac{\xi_k}{\omega_k.{}^t\mathbf{\Phi_k}.\mathbf{K}.\mathbf{\Phi_k}}$ where ξ_k is the ratio damping modal to the pulsation ω_k and $k_k={}^t\mathbf{\Phi_k}.\mathbf{K}.\mathbf{\Phi_k}$ is the generalized stiffness of the mode k. Unfortunately this matrix can have a very full profile and return expensive (matrices $\mathbf{C}$ and $\mathbf{K}$ not having the same profile), as one will see it with [§ 3.1], its integration in the first member: one will then choose to treat these forces of modal damping $-\mathbf{C}.\dot{\mathbf{U}}(t)$ with the second member by an explicit diagram.
The validity of this modeling of modal damping is acquired if one can neglect the couplings between clean modes. It thus should be made sure that the following criterion is checked:

$$\underset{k,\ell}{Min}\left(\frac{\xi_k\omega_k}{|\omega_k-\omega_\ell|}\right)\ll 1$$

2) **a total/local way known as viscous damping proportional** (according to the assumption of RAYLEIGH) starting from the matrices of elastic stiffness $\mathbf{K}$ and of mass $\mathbf{M}$. The parameters are given by material on the finite elements of the model (keywords AMOR_ALPHA/AMOR_BETA order DEFI_MATERIAU). The matrix of viscous damping is $\mathbf{C}=\alpha\mathbf{K}+\beta\mathbf{M}$. It is diagonalisable on the basis as of real modes clean, which makes possible to do a transitory calculation on modal basis by uncoupling the modes: to see [R5.06.04]. This formulation, into linear, led to a damping ratio related to the frequency f : $\xi=\alpha\pi f+\beta/(4\pi f)$. In the nonlinear case, this evaluation does not have any more course.
The coefficients in practice are adjusted α, β so that damping ξ that is to say almost uniform in the beach $[f_1,f_2]$ of frequency of interest for the studied structure. From where thus of reasonable

manner:

$$\alpha=\frac{\xi}{\pi(f_1+f_2)} \quad \text{and} \quad \beta=\frac{4\pi\cdot\xi\cdot f_1\cdot f_2}{f_1+f_2}$$

This modeling of damping is not very realistic if the structure answers on a broad waveband, because introduced damping varies then much on the studied range. If the law of behavior of material is nonlinear dissipative, the choice of the parameters of damping relates to the beach where the structure remains almost elastic.
During a non-linear transitory calculation, one will use for the resolution an assembled matrix of stiffness which can be the tangent matrix (reactualized with each step, for example) or the elastic matrix. Historically, for practical reasons, one built the damping of Rayleigh while making use of the matrix of stiffness such as it was specified by the user for the calculation of the internal efforts. Thus if the user chose the tangent matrix, one had:

$$\mathbf{C}=\alpha\,\mathbf{K}^{\mathbf{T}}+\beta\,\mathbf{M}$$

This makes delicate the interpretation of the effect of damping proportional. In particular in the event of appearance of negative eigenvalues of the matrix $\mathbf{K}^{\mathbf{T}}$ (for example in the event of damage of material), damping can become negative and reinforce instabilities!
Without going until that, on a simple elastoplastic example, one observes a fall of viscous damping because the tangent stiffness drops. This loss of damping could be partly compensated by dissipation material, but the total level of dissipation will be badly controlled.
Therefore one decided, in version 10 of the code, to give to the user the choice for the definition of the damping of Rayleigh. Either one adopts the matrix of stiffness already employed for the internal efforts (as in the previous models of the code), or one specifies the use of the elastic matrix. This second choice is particularly advised for the lenitive laws and the total laws of type GLRC, cf. [R7.01.32].

2.2.3 Connections

In practice, one can have bilateral or unilateral conditions of connection, or connections of type “impedance” or “absorbing”, *cf.* [R4.02.05].

Bilateral connections

The bilateral connections are written in the form of the following relation: $\mathbf{B}.\mathbf{u}=\mathbf{u}^{\mathbf{d}}(t)$.

They are perfect: they do not dissipate energy. They are connections “holonomists”, where speed $\dot{\mathbf{U}}$ does not intervene. These connections are in general dualisées by *Code_Aster*, *cf.* [R3.03.01].

Unilateral connections

System mechanical object of simulation by finite elements can to come into contact (unilateral connections) with “obstacle”, which is a solid which one knows a priori the movement, from where the definition of a game $\mathbf{d_0}(t)$. The unilateral connections (for example the unilateral contact) are written on the configuration at any moment t : $\mathbf{A}(\mathbf{u}).\mathbf{u}(t)\leq\mathbf{d_0}(t)$ (nonpenetration or checking that the effective game remains positive or null in any configuration). The operator $\mathbf{A}(\mathbf{u})$ can depend on the configuration in the presence of great displacements by reactualization during the step of time, *cf.* [R5.03.50].

One will consider only connections “holonomists”, type $\mathbf{A}(\mathbf{U},t)=0$, which utilizes only the values of the degrees of freedom $\mathbf{U}(t)$ and time explicitly if the obstacle is mobile. One will not consider connections “not holonomists”, for example of the standard bearing without slip, who utilize speed explicitly and are written $\mathbf{A_1}(\mathbf{U},t).\dot{\mathbf{U}}+\mathbf{A_2}(\mathbf{U},t)=0$, $\mathbf{A_2}$ and direct dependence in time being present only if it there a mobile obstacle.

If the obstacle is motionless, the connection as such will be explicitly independent of time (one also says “scleronomist”).

These “loads of contact” are defined by the operator DEFI_CONTACT. The presence of unilateral connections requires to define speeds of the solid in a particular functional space in order to ensure the existence of solutions of the system of dynamics. Indeed, at the time of the moments (countable) impact, speeds $\dot{\mathbf{U}}(t^-)$ and $\dot{\mathbf{U}}(t^+)$ can not coincide. It is necessary to guarantee this result that the data (loading, equations of connections) check a property of analycity, which is acquired in practice with the selected discretization by finite elements (see [bib6]).

One does not introduce *a priori* of relation constitutive of the impacts (dissipative behavior in simple rebound, expressed using a drainage efficiency normal $e\in[0,1]$), expressed on the differentials speeds of the two points in impact:

$$\dot{\mathbf{U}}_{2\_\mathrm{norm}}^{+}(t)=\dot{\mathbf{U}}_{1\_\mathrm{norm}}^{+}(t)-e\left(\dot{\mathbf{U}}_{2\_\mathrm{norm}}^{-}(t)-\dot{\mathbf{U}}_{1\_norm}^{-}(t)\right)$$

This kind of behavior is introduced usually to treat the contact-impact of rigid bodies, whereas digital modeling with deformable solids makes it possible to directly represent the vibratory behavior under the shock and material non-linearities. But it is possible to add in the modeling of the discrete elements of contact-shock placed on the interface in contact, provided with the law DIS_CHOC, which brings a dissipation of damping (on condition that supposing small movements…).

One can associate with the unilateral connections a behavior of friction (criterion of COULOMB), which dissipates energy in relative slip of surfaces in contact.

One refers sometimes to the fact that the dynamic coefficient of friction is lower than that measured into quasi-static (adherence), to see [bib27], [bib33]. That comes owing to the fact that in dynamics of the high frequency vibrations on the normal reaction appear and weaken the value of the threshold of friction of COULOMB. One would thus not need to provide two values of coefficients to *Code_Aster*, since one models the deformable solids in contact-friction (on condition that being able to simulate these vibrations high frequency, which are generally attenuated by the diagrams of temporal integration…).

It is known that dissipation is a requirement for the existence of theoretical solutions to the problem of dynamics with impact [bib32]. Use of the diagram HHT, to see [§ 5], which introduces digital dissipation can prove to be necessary.

Local dissipative connections

Specific relations (like DIS_CHOC) are conceived to treat certain types of dissipative specific connections, acting directly on the degrees of freedom of discrete elements of the system, to see [R5.03.17]. They constitute a law of behavior in generalized forces function of generalized displacements integrated like the whole of the internal forces $\mathbf{R}(\mathbf{U},\dot{\mathbf{U}},t)$ studied structures.

Absorbing connections

“Absorbing” connections of the type, *cf.* [R4.02.05], allow to simulate the “filtering” of part of the dynamic response, by preventing the exit of considered and diffracted waves, and prevent that the “incidental” field is not disturbed on a border of the model: to see it [§ 2.6]. They introduce damping terms of the type $\mathbf{A}_{abso}(\dot{\mathbf{u}})$ on a border of the solid considered.

2.2.4 Discretized dynamic problem

The dualisation of the boundary conditions of DIRICHLET $\mathbf{B}.\mathbf{u}=\mathbf{u}^{\mathbf{d}}(t)$ and of the unilateral conditions results after discretization in defining the unknown factors at any moment t : $(\mathbf{U},\boldsymbol{\lambda},\boldsymbol{\mu})$,

where λ represent them “multiplying of LAGRANGE” of the boundary conditions of DIRICHLET [R3.03.01], and μ represent the “multipliers of Lagrange” of the unilateral conditions.

The nonlinear dynamic problem is written, with the initial conditions [R3.03.01], [R5.03.50]:

To find the trajectory $\mathbf{U}(t)$:

$$\begin{cases} \mathbf{M}(\mathbf{U},\dot{\mathbf{U}},\ddot{\mathbf{U}})+\mathbf{R}(\mathbf{U},\dot{\mathbf{U}},t)+\mathbf{C}.\dot{\mathbf{U}}+{}^{t}\mathbf{B}.\boldsymbol{\lambda}+{}^{t}\mathbf{A}.\boldsymbol{\mu}=\mathbf{L}(t) \\ \mathbf{B}.\mathbf{U}=\mathbf{U}_{d}(t) \\ \mathbf{A}.\mathbf{U}\leq\mathbf{d_0}(t) \\ \forall\, j,\ \boldsymbol{\mu}_{\mathbf{j}}\geq 0 \\ \forall\, j,\ (\mathbf{A}.\mathbf{U}-\mathbf{d_0})_{j}.\boldsymbol{\mu}_{\mathbf{j}}=0 \\ \mathbf{U}(t_0)=\mathbf{U}_0 \\ \dot{\mathbf{U}}(t_0)=\dot{\mathbf{U}}_0 \end{cases} \qquad \text{éq 2.2.4-1}$$

$\mathbf{L}$ represent the vector of the external forces (mechanical loadings). These forces can depend on time and space. It is supposed that, like the connections, they depend systematically on the parameters, which ensures the existence of the solution of the problem (theorem of CAUCHY). One can consider “following” forces $\mathbf{L}(\mathbf{U},t)$, for example pressure, if one takes into account the changes of geometry.

The vector ${}^{t}\mathbf{B}.\boldsymbol{\lambda}$ be interpreted like the opposite of the reactions of support to the corresponding nodes ($\mathbf{B}$ is the linear operator expressing the passage to the degrees of freedom of the supports). The vector ${}^{t}\mathbf{A}.\boldsymbol{\mu}$ be interpreted like the nodal forces due to the contact ($\mathbf{B}$ is the linear operator expressing the passage to the degrees of freedom of the zones in contact).

The analysis of stability of the dynamic system of balance [éq 2.2.4-1] is more complex than into linear, but a sufficient condition of loss of stability is the possibility of finding a pulsation ω with negative imaginary part for which one can cancel the determinant of: $-\omega^2\mathbf{M}+i\omega\mathbf{C}^{\mathbf{T}}+\mathbf{K}^{\mathbf{T}}$, definite on the tangent operators, at the moment considered.

2.2.5 Initial conditions

Initial conditions $\mathbf{U_0},\dot{\mathbf{U}}_0$ are provided to the code by the keyword ETAT_INIT (operands DEPL and QUICKLY).

If the initial state results from a simulation in or not linear linear statics, displacement, as well as the variables of state (forced, internal variables), are extracted from the result of this simulation, and initial speed is by default presumedly worthless.

2.3 Taking into account of a prestressed initial state

If the dynamic problem to solve “follows” an initial mechanical state, two principal situations are offered to us:

- one wishes to calculate displacements $\mathbf{u}$ and dynamic stresses starting from the “virgin” state initial, where all the fields are worthless;
- one wishes to calculate displacements $\mathbf{u}$ and dynamic stresses in “differential” starting from a preloaded state, within the elastic framework.

1) In the first situation, it is advisable to calculate as a preliminary static state, possibly nonlinear (material, great transformations), precondition to dynamics.

Thus, if the structure has a nonlinear behavior, one must proceed directly in nonlinear dynamics, after having evaluated the initial state by simulation in statics, possibly nonlinear (with the operator STAT_NON_LINE), and the field of displacement is evaluated since the beginning of the history. It can be necessary to take into account the variations of geometry. Necessary information to describe the initial state (results of a preceding simulation via the concept result EVOL_NOLI or mechanical fields necessary: DEPL, SIGM, VARI) are provided by the keyword ETAT_INIT, for example if one is within the framework of an incremental behavior, to see [U4.51.03].

The initial state can be also obtained by a simulation in “very slow” dynamics, by having care to put a “slow” slope of dependence at time on the static efforts applied, as well as a strong damping (physical, *cf.* [§ 2.2.1] ou/et digital *cf.* [§ 5]). This manner of proceeding has the advantage of injecting so much into the operator of the phase of prediction [éq 3.2.1-4] of the algorithm of NEWTON, that in that of the phase of correction [éq 3.3-1] of the terms $\hat{\mathbf{K}}$ coming from the matrix of mass and that of damping, to establish the static mechanical state. That is invaluable in situations of contact-friction, of damage… to improve convergence.

2) The second situation relates to the case of a structure which underwent an “ordinary” preloading thermo - mechanical, leading to a state of linear elastic balance. If one measures by $\mathbf{u}$ displacement starting from this preloaded state, which generated a state of stresses $\boldsymbol{\sigma}_1$, then the elastic deformation energy is supplemented by a geometrical term of stiffness:

$$\int_\Omega \boldsymbol{\varepsilon}(\mathbf{u}) . \mathbf{A} . \boldsymbol{\varepsilon}(\delta \mathbf{v}) \, d\Omega + \int_\Omega \boldsymbol{\sigma}_1 . \nabla \mathbf{u} \otimes \nabla(\delta \mathbf{v}) \, d\Omega = {}^t\delta \mathbf{V} . (\mathbf{K} + \mathbf{K}_\mathbf{g}) . \mathbf{U}$$

One then assembles simply the matrices $\mathbf{K}$ and $\mathbf{K}_\mathbf{g}$, and one carries out the resolution in linear dynamics, as with [§ 2.1]. CE can be for example the case of a seismic study on a stopping arches. With DYNA_NON_LINE, it is necessary to provide by the keyword ETAT_INIT, the stress field SIGM result of the preloaded state and to specify on the level of the keyword DEFORMATION under BEHAVIOR the taking into account of the nonlinear terms of deformation.

It is frequent that the contribution of $\mathbf{K}_\mathbf{g}$ that is to say negligible: one can then be satisfied with a dynamic analysis on the basis of completely virgin initial state. It will be also noted that the calculation of the matrix $\mathbf{K}_\mathbf{g}$ is not available for all the finite elements proposed by *Code_Aster*, to see Docs. [U3].

2.4 Coupled problems vibroacoustic fluid-structure

One will be able for more details to refer to the documents [R4.02.02], [R4.02.04], [R4.02.05].

One considers the small movements approaches eulérienne of a compressible true fluid it, possibly bathing a wall of a solid structure. The fluid is known as barotrope:

1. one considers small irrotational disturbances around the initial state (hydrostatic): $\vec{\mathbf{U}}_{fl} = \vec{\mathbf{u}}^0_{fl} + \vec{\mathbf{u}}_{fl}$, $P = P_0 + p$ and $\rho_f = \rho_0 + \rho$, index 0 indicating the permanent part of the fields,
2. the law of behavior of the fluid gives the “fluctuating” constraints: $\boldsymbol{\sigma} = -p\,\mathbf{Id} = -\rho\, c_0^2\,\mathbf{Id} \approx \rho_0 c_0^2 (\text{div}\ \vec{\mathbf{U}}_{fl})\ \mathbf{Id}$, because $\rho_0 = \rho_f (1 + \text{div}\ \vec{\mathbf{U}}_{fl})$,
3. fluid speeds derive from a potential $\vec{\mathbf{v}}_{fl} = \dot{\vec{\mathbf{u}}}_{fl} = \vec{\nabla}\dot{\varphi}$ and are modelled using the fields (p, φ) : pressure fluctuhandle and potential of displacement, which are not independent because: $\dot{p} = -\rho_0 c_0^2 \Delta \dot{\varphi}$ (by combining equation of continuity and law of behavior).

It is admitted that one does not consider fluctuating forces of volume $\vec{f}$ being exerted on the fluid. The dynamic balance of the fluid is written: $\overrightarrow{\nabla p}+\rho_f \ddot{\vec{\mathbf{u}}}_{fl}=\vec{0}$ (equation of linearized Euler), which is valid for a fluid not weighing compressible or weighing incompressible; on the other hand, for a heavy fluid compressible, the approaches eulérienne and Lagrangian do not coincide even in small movements: this case is not treated by *Code_Aster* .

The dynamic balance of the fluid will be written in variational form under the action of a fluctuating pressure p_{imp} imposed on part of the border. Into harmonic mode, the dynamic balance of the fluid results in the variational formulation of the equation of Helmholtz, [bib26].

Code_Aster have a symmetrized formulation, to see [R4.02.02], with elements $(\mathbf{P},\varphi)$ where $\mathbf{P}$ is the vector of the degrees of Liberté of pressure and φ is the potential to describe the disturbances (fluctuations) in the fluid, knowing that $\overrightarrow{\nabla p}+\rho_0\overrightarrow{\nabla \ddot{\varphi}}=\vec{0}$. An equation in $\mathbf{P}$ solves dynamic balance in the fluid field, that in φ translating the equation of derived continuity combined with the fluid law of behavior. Boundary conditions in $\mathbf{P}$ and φ supplement the system of equations to describe the evolutions of the fluid. Thus, because of the formulation $(\mathbf{P},\varphi)$, so on a border $\partial\Omega_{\text{f\_p}}$ fluid field, a fluctuating pressure is applied: $\mathbf{P}=\mathbf{P}_{imp}(t)$, one must also impose a condition on it on φ : $\varphi(t)=\varphi_{imp}(t)$, checking $\rho_0\ddot{\varphi}_{imp}(t)=-\mathbf{P}_{imp}(t)$.

As one is considered **border common fluid-structure** $\partial\Omega_{FS}$, where the normal is defined $\vec{\mathbf{n}}$ outgoing of the structure field towards the fluid, the loading of wall of the fluid is coupled with the displacement of the structure. Normal displacements are continuous: $(\vec{\mathbf{u}}_{\mathbf{fl}}+\vec{\mathbf{u}}^0_{fl})\cdot\vec{\mathbf{n}}=\vec{\mathbf{u}}_{st}\cdot\vec{\mathbf{n}}$ on $\partial\Omega_{FS}$, $\vec{\mathbf{u}}^0_{fl}$ indicating the permanent part of fluid displacement. By using the fluid potential, one a: $\overrightarrow{\nabla\dot{\varphi}}.\vec{\mathbf{n}}=\dot{\vec{\mathbf{u}}}_{st}.\vec{\mathbf{n}}$ on $\partial\Omega_{FS}$. In a dual way, the vectors forced are continuous: $-p\vec{\mathbf{n}}=\sigma.\vec{\mathbf{n}}$ on $\partial\Omega_{FS}$. The structure thus receives the loading fluctuating of the fluid: $\int_{\partial\Omega_{FS}} p\,\partial\vec{\mathbf{v}}.(-\vec{\mathbf{n}})\,dS$.

If one is considered **free surface** (subjected to a constant pressure), also treated of description eulérienne, to see [R4.02.04], one notes by z (and $\mathbf{Z}$ degrees of freedom associated after discretization) fluctuating altitude (small) with free surface $\partial\Omega_{SL}$ and the fluctuation in pressure eulérienne in the fluid checks: $\int_{\partial\Omega_{SL}}(p-\rho_0 gz)\,\delta z\,dS$, in any virtual altitude δz . It is the only consequence of gravity which one can take account in this formulation.

In addition, one can need to take into account an artificial border with an infinite medium (which must treat the condition of radiation ad infinitum): *Code_Aster* propose finite elements of absorbing border (paraxial or anechoic elements), to see [R4.02.05]. As they bring a priori a term in derived third of time, because of the introduction of the field φ , one prefers to treat it using a transformation into a nonsymmetrical term in $\dot{\mathbf{P}}$ who will be deferred to the second member, in an explicit way.

On the whole one obtains the différentio-algebraic semi-discrete equations of the coupled problem:

$$\begin{pmatrix} \mathbf{M} & \mathbf{0} & \mathbf{M}_{FS} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{M}_{fl} & \mathbf{0} \\ {}^t\mathbf{M}_{FS} & {}^t\mathbf{M}_{fl} & \mathbf{H}_{fl} & \mathbf{M}_z \\ \mathbf{0} & \mathbf{0} & {}^t\mathbf{M}_z & 0 \end{pmatrix}\begin{pmatrix} \ddot{\mathbf{U}} \\ \ddot{\mathbf{P}} \\ \dot{\varphi} \\ \ddot{\mathbf{Z}} \end{pmatrix}+\begin{pmatrix} \mathbf{C} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{fa} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}\begin{pmatrix} \dot{\mathbf{U}} \\ \dot{\mathbf{P}} \\ \dot{\varphi} \\ \dot{\mathbf{Z}} \end{pmatrix}+\begin{pmatrix} \mathbf{K} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_{fl} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}_z \end{pmatrix}\begin{pmatrix} \mathbf{U} \\ \mathbf{P} \\ \varphi \\ \mathbf{Z} \end{pmatrix}=\begin{pmatrix} \mathbf{L}_{st} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}\begin{matrix} \textit{structure} \\ \textit{fluide} \\ \textit{fluide} \\ \textit{surf. libre} \end{matrix}$$

éq 2.4-1

accompanied by the initial conditions:

$$\mathbf{U}(t_0)=\mathbf{U}_0\ ,\quad \dot{\mathbf{U}}(t_0)=\dot{\mathbf{U}}_0\ ,\quad \mathbf{P}(t_0)=\mathbf{P_0}\ ,\quad \varphi(t_0)=\varphi_0 \text{ and } \mathbf{Z}(t_0)=\mathbf{0}\ ,$$

and of the boundary conditions: $\mathbf{U}(t)=\mathbf{U}_{imp}$ on the edge $\partial\Omega_{S_u}$ structure, possible unilateral conditions, and $\mathbf{P}(t)=\mathbf{P}_{imp}(t)$ with $\varphi(t)=\varphi_{imp}(t)$, checking $\rho_0\ddot{\varphi}_{imp}(t)=\mathbf{P}_{imp}(t)$ on the edge $\partial\Omega_{\text{f\_p}}$ fluid.

Notice :

It is noted that in the nonlinear case, one replaces in [éq 2.4-1] the term $\mathbf{K}\mathbf{U}$ by the nonlinear internal forces $\mathbf{R}(\mathbf{U},\dot{\mathbf{U}},\mathbf{Z},t)$.

The various operators are:

1) matrices $\mathbf{K}$, $\mathbf{M}$, $\mathbf{C}$ defined higher for the solid structure,
2) $\mathbf{Q}_{fl}$ is the matrix built from $\int_{\Omega_f}\frac{1}{\rho_0 c_0^2}p.qd\Omega$, which has the physical direction of one elastic energy of the fluid,
3) $\mathbf{H}_{fl}$ is built from $\int_{\Omega_f}-\rho_0\vec{\nabla}\varphi.\vec{\nabla}\Psi\,d\Omega$, and described the fluid transport of mass,
4) $\mathbf{M}_{fl}$ is built from $\int_{\Omega_f}\frac{1}{c_0^2}\varphi.qd\Omega$, and described the inertia of the fluid,
5) $\mathbf{M}_{FS}$ is built from $\int_{\partial\Omega_{FS}}\rho_0\varphi.\delta\vec{\mathbf{v}}.(-\vec{\mathbf{n}})\,dS$ ($\vec{\mathbf{n}}$ is the normal of the structure field towards the fluid), and described the mass throughput eulérien with the interface fluid-structure,
6) $\mathbf{A}_{fa}$ is built from $\int_{\Omega_f}\rho_0 p.\Psi\,d\Omega$, and the operator absorbing border, acting indicates on $\dot{\mathbf{P}}$, modifying the equation in φ absorbing wall: the elements (model 3D_FLUI_ABSO) absorbents die-symmetrize the system resulting from the formulation $(\mathbf{P},\varphi)$ and one will defer this term to the second member, who will be noted $\mathbf{L}^{\mathbf{abso}}$, by temporal discretization clarifies with each iteration, to see the § 3,
7) $\mathbf{K}_{\mathbf{z}}$ is the "stiffness" of free surface, built from $\int_{\partial\Omega_{SL}}\rho_0 gz.\delta z\,dS$,
8) $\mathbf{M}_{\mathbf{z}}$ comes from the work of the fluctuating pressure in free surface, built from $\int_{\partial\Omega_{SL}}\rho_0 z.\Psi\,dS$,
9) the term $\mathbf{L}_{st}$ contains, inter alia loadings, the effect of the hydrostatic pressure exerted by the fluid on the structure.

In short, the taking into account of a fluid field in fluctuating evolution barotrope, interacting with the structure results in considering in the nonlinear dynamic system [éq 2.2.4-1] enriched on particular degrees of freedom $(\mathbf{P},\varphi)$ and Z :

1) an operator of inertial forces $\mathbf{M}(\mathbf{U}).\ddot{\mathbf{U}}$ nouveau riche by:

$$\mathbf{M}^{fs}\begin{pmatrix}\ddot{\mathbf{U}}\\ \ddot{\mathbf{P}}\\ \ddot{\varphi}\\ \ddot{\mathbf{Z}}\end{pmatrix}=\begin{pmatrix}\mathbf{0}&\mathbf{0}&\mathbf{M}_{FS}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{M}_{fl}&\mathbf{0}\\ {}^{t}\mathbf{M}_{FS}&{}^{t}\mathbf{M}_{fl}&\mathbf{H}_{fl}&\mathbf{M}_{\mathbf{z}}\\ \mathbf{0}&\mathbf{0}&{}^{t}\mathbf{M}_{\mathbf{z}}&\mathbf{0}\end{pmatrix}\begin{pmatrix}\ddot{\mathbf{U}}\\ \ddot{\mathbf{P}}\\ \ddot{\varphi}\\ \ddot{\mathbf{Z}}\end{pmatrix}\ ;$$

2) an operator of interior forces $\mathbf{R}(\mathbf{U},\dot{\mathbf{U}})$ nouveau riche by:

$$\mathbf{K}^{fs}\begin{pmatrix}\mathbf{U}\\ \mathbf{P}\\ \varphi\\ \mathbf{Z}\end{pmatrix}=\begin{pmatrix}\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{Q}_{fl} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}_z\end{pmatrix}\begin{pmatrix}\mathbf{U}\\ \mathbf{P}\\ \varphi\\ \mathbf{Z}\end{pmatrix};$$

3) a second member enriched by the carryforward with the second member by temporal discretization clarifies $-\mathbf{A}_{\mathbf{fa}}\dot{\mathbf{P}}$ on the degrees of freedom φ.

The fluid must remain in small movements (basic assumption of this modeling), but one can consider great movements of the structure, bathed by the fluid, via a reactualization of the geometry $\mathbf{X}\rightarrow\mathbf{X}+\Delta\mathbf{U}$ (via the keyword BEHAVIOR, operand DEFORMATION : 'PETIT_REAC', valid if one considers small rotations) borders $\partial\Omega_{FS}$, which makes recompute the term M_{FS} but also all others since the field Ω_f evolved; scalar fields $(\mathbf{P},\varphi)$ then are simply transported to identical on the reactualized geometry. See also CAS-test FDNV100 [V8.03.100].

2.5 Taking into account of laws of viscous behavior and damping

A law of viscous behavior, to see [R5.03.08], is translated like an elastoplastic law by an evolution of the work of the interior forces $\mathbf{R}(\mathbf{U},\dot{\mathbf{U}},t)$. Thus, while bringing a "physical" damping in dynamic balance, it does not produce a direct taxation at the end of damping $\mathbf{C}.\dot{\mathbf{U}}$ dynamic equilibrium equation, but however in an indirect way if one chose a damping of Rayleigh (*cf.* [§ 2.2.1]) via the matrix of tangent stiffness of the diagram of integration.

Indeed with a law of viscous behavior, the tensor of the deformations comprises an elastic part, a thermal part, an unelastic part (known) and a viscous, deviatoric part (diverter of the constraints noted $\tilde{\boldsymbol{\sigma}}$), for example checking:

$$\begin{aligned}&\boldsymbol{\varepsilon}_{tot}=\boldsymbol{\varepsilon}_{\mathrm{e}}+\boldsymbol{\varepsilon}_{th}+\boldsymbol{\varepsilon}_{\mathrm{a}}+\boldsymbol{\varepsilon}_{\mathrm{v}}\\ &\boldsymbol{\sigma}=\mathbf{A}(T).\boldsymbol{\varepsilon}_{\mathrm{e}}\\ &\dot{\boldsymbol{\varepsilon}}_{v}=g(\boldsymbol{\sigma}_{eq},\lambda,T)\frac{3}{2}\frac{\tilde{\boldsymbol{\sigma}}}{\sigma_{eq}}\end{aligned}$$

For the relation of viscous behavior LEMAITRE, the function g is explicit, but it is not always the case. After implicit discretization in time, the viscous flow is:

$$\frac{\Delta\boldsymbol{\varepsilon}_{\mathrm{v}}}{\Delta t}=\frac{3}{2}g\left(\boldsymbol{\sigma}_{eq},\lambda^{-}+(\Delta\varepsilon_{v})_{eq},T\right)\frac{\tilde{\boldsymbol{\sigma}}}{\sigma_{eq}}$$

One can also adopt an semi-implicit diagram, which seems to give better results:

$$\frac{\Delta\boldsymbol{\varepsilon}_{\mathrm{v}}}{\Delta t}=\frac{3}{2}g\left(\left(\boldsymbol{\sigma}^{-}+\frac{\Delta\sigma}{2}\right)_{eq},\lambda^{-}+\frac{(\Delta\varepsilon_{v})_{eq}}{2},T^{-}+\frac{\Delta T}{2}\right)\frac{\left(\tilde{\boldsymbol{\sigma}}^{-}+\frac{\Delta\tilde{\boldsymbol{\sigma}}}{2}\right)}{\left(\sigma^{-}+\frac{\Delta\sigma}{2}\right)_{eq}}$$

After solution of a local nonlinear equation per elimination of $\Delta\boldsymbol{\varepsilon}_{\mathrm{v}}$ to calculate $\sigma_{eq}=(\sigma^{-}+\Delta\sigma)_{eq}$, one thus obtains the constraint at the end of the step of current time $\boldsymbol{\sigma}=\boldsymbol{\sigma}^{-}+\Delta\boldsymbol{\sigma}$.

2.6 Équations moving “relative” [R4.05.01]

2.6.1 General information

In many applications, in particular in earthquake, one wishes to directly calculate the field of displacement of the structure deduced from the movement of “training” coming from imposed displacements $\mathbf{U}_d(t)$ supports of the structure.

One notes then $\mathbf{u}_a$ the absolute displacement of the structure: $\mathbf{u}_a = \mathbf{u}_{ent} + \mathbf{u}$, $\mathbf{u}_{ent}$ being the displacement of training (for example, $\mathbf{u}_{ent}$ can be an incidental field: it is then called “pseudo-statics displacement”). In the case MONO_APPUI, it is a rigid movement of body. In the case MULTI_APPUI, $\mathbf{u}_{ent}$ is defined as being the linear static solution of the structure subjected to the displacements imposed on the supports, to see [§2.6.2].

And one calls $\mathbf{u}$ the “relative” displacement (called thus by abuse language if $\mathbf{u}_{ent}$ is not rigid body).

Indeed coding (RCC-M, ASME…) introduced the distinction between “primary constraints” due to the vibratory movement “relative” and “secondary constraints” due to the vibratory movement of “training”. The relevance of this distinction disappears *a priori* obviously as soon as one considers a nonlinear behavior of material.

If one deals with the dynamic problem of interaction with the ground (which is an infinite half space), in earthquake for example, the field of relative displacement $\mathbf{u}$ check: $\lim_{\mathbf{x}\to\infty} \mathbf{u}(\mathbf{x}) = 0$: only the incidental field $\mathbf{u}_{ent}$ is perceptible ad infinitum – it is the seismic data of loading. One uses to define this loading of displacement imposed the order AFFE_CHAR_MECA and the keyword factor ONDE_PLANE, on a given border of the grid considered.

In this case of problem of interaction with the ground, one does not know a priori the displacement of “training” directly applied to the structure, since it results from the coupled total answer: also the case MULTI_APPUI, where displacement is necessarily known with the interface, it is not applicable.

On the other hand, not being able to simulate in finite elements with *Code_Aster* the field in all the infinite half space (ground), one is led to place absorbing “elastic” borders, *cf.* [R4.02.05], at the edge of the grid of ground. The virtual work associated with these absorbing borders, of outgoing normal $\mathbf{n}$, is treated as a second member (for the finite elements paraxial absorbents of *Code_Aster*, who are of order 0), because it is integrated explicitly in the diagram of temporal integration (*cf.* [§3]) of DYNA_NON_LINE : the value indeed is taken $\dot{\mathbf{u}}^-$ speed at the previous moment. The associated linear form is worth:

$$\mathbf{L}_{abso}^{ordre\,0}\left(\mathbf{u}_{ent}, \dot{\mathbf{u}}_{ent}, \dot{\mathbf{u}}^-\right).\delta\mathbf{v} = \int_{\partial\Omega_{abso}} \left(\mathbf{A}_{abso}^{ordre\,0}\left(\dot{\mathbf{u}}^-\right) + \boldsymbol{\sigma}\left(\mathbf{u}_{ent}\right).\mathbf{n} - \mathbf{A}_{abso}^{ordre\,0}\left(\dot{\mathbf{u}}_{ent}\right)\right).\delta\mathbf{v}\, dS \qquad \textbf{éq 2.6-1}$$

After space discretization, one deduces the second member:

$${}^t\delta\mathbf{V}.\mathbf{L}_{abso}^{ordre\,0}\left(\mathbf{U}_{ent}, \dot{\mathbf{U}}_{ent}, \dot{\mathbf{U}}^-\right) = {}^t\delta\mathbf{V}.\mathbf{A}_{abso}^{ordre\,0}.\dot{\mathbf{U}}^- + {}^t\delta\mathbf{V}.\boldsymbol{\Sigma}\left(\dot{\mathbf{U}}_{ent}\right) - {}^t\delta\mathbf{V}.\mathbf{A}_{abso}^{ordre\,0}.\dot{\mathbf{U}}_{ent} \qquad \textbf{éq 2.6-2}$$

Note: case MONO_APPUI **:**

> *For with a dynamic problem dealt in relative displacements (case* MONO_APPUI*), only the first term of [éq 2.6.2] remains. One will be able to refer to [R4.02.05].*

Note: case of a problem with interaction fluid-structure:

> *For a structure undergoing of imposed displacements, in the presence of interaction fluid-structure, where the fluid field is not directly related to the "support" imposing the signal of training, it is possible to solve the dynamic system of balance in term of "relative" displacement* $\mathbf{U}$ *fluid structure and variables* $(\mathbf{P},\varphi)$ *"absolute" and of dimension of free surface of the fluid* $\mathbf{Z}$ *"absolute", cf [§ 2.4]. Indeed, one can show that* $(\mathbf{P}_{ent},\ \varphi_{ent})=0$ *and* $\mathbf{Z}_{ent}=0$ *, on the basis of [eq 2.4-1]. One will be able to refer to [R4.02.05].*

One will not be able to consider such a type of decomposition field of "training" – "relative" field in the event of loadings of type fluctuating pressure imposed on a wall of the fluid field.

One seeks to exploit this distinction in the discrete nonlinear dynamic system [éq 2.2.4-1] to simplify the taking into account of imposed displacements $\mathbf{U}_d(t)$.

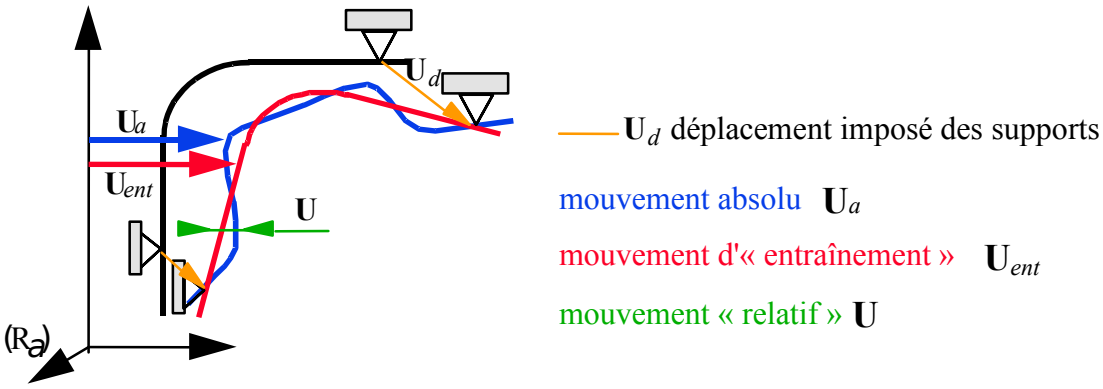

Figure 2.6.1-a: Decomposition of the absolute movement.

One separates the conditions from connection, to see [éq 2.2.4-1], in a group of conditions of "training" by the supports: $\mathbf{B}_s.\mathbf{U}_a=\mathbf{U}_d(t)$ and a group of conditions of connections noted by $\mathbf{B}_L.\mathbf{U}_a=0$ that one wishes to impose directly on absolute displacement $\mathbf{U}_a$ structure (for example of the internal connections like " 3D_POU "...) which their parameters are associated $\boldsymbol{\lambda}_L$ LAGRANGE. One thus considers thereafter these two families: $\mathbf{B}_s$ for the movements of "training" by the supports Γ and $\mathbf{B}_L$ for the bilateral connections "ordinary".

If the supports are of finished number (what will be the case in any case after discretization), one notes V_e vector space of the fields of displacements of the "trained" structure $\mathbf{u}_{ent}$, of finished size N_s, that one will define hereafter. The conditions of DIRICHLET are broken up $\mathbf{U}_d(t)$, on a basis $(\mathbf{X}_{ks})$ displacements of the supports: $\mathbf{U}_d(t)=\mathbf{U}_d^k(t)\,\mathbf{X}_{ks}$, $k=1\rightarrow N_s$ traversing all the "degrees of freedom trained" by the supports.

2.6.2 Elastostatic decomposition by raising

One builds a "raising elastostatic", i.e. a base of V_e starting from the linear elastic static solutions of the structure under only basic imposed displacements $\left(\mathbf{X}_{ks}\right)$ supports (no loading in imposed force). After discretization by finite elements, that amounts solving $k=1\rightarrow N_s$ problems of elastostatic (matrix of stiffness $\mathbf{K}$):

To find $\left(\mathbf{\Psi}_k, \boldsymbol{\lambda}_{s\Psi_k}, \boldsymbol{\lambda}_{L\Psi_k}\right)$ such as

$$\begin{cases} \mathbf{K}.\mathbf{\Psi}_\mathbf{k} + {}^t\mathbf{B}_\mathbf{s}.\boldsymbol{\lambda}_{s\Psi_k} + {}^t\mathbf{B}_\mathbf{L}.\boldsymbol{\lambda}_{\mathbf{L}\Psi_\mathbf{k}} = 0 \ \ dans\ le\ domaine \\ \mathbf{B}_\mathbf{s}.\mathbf{\Psi}_\mathbf{k} = \mathbf{X}_{ks} \ \ sur\ les\ supports\ \ \Gamma_s \\ \mathbf{B}_\mathbf{L}.\mathbf{\Psi}_\mathbf{k} = 0 \end{cases} \qquad \textbf{éq 2.6-3}$$

One calls "linear static modes" these N_s solutions $\mathbf{\Psi}_\mathbf{k}$ (well informed via the operand "MODE_STAT" of DYNA_NON_LINE). They are calculated as a preliminary by the operator MODE_STATIQUE [U4.52.14] with the option MODE_STAT.

One necessarily has with [éq 2.6-3]: ${}^t\mathbf{\Psi}_\boldsymbol{\ell}.\mathbf{K}.\mathbf{\Psi}_\mathbf{k} + {}^t\mathbf{X}_{\boldsymbol{\ell s}}.\boldsymbol{\lambda}_{\mathbf{s}\Psi_\mathbf{k}} = 0$, $\forall\, \ell, k=1\rightarrow N_s$.

Note:

It is essential that the same connections $\mathbf{B}_\mathbf{L}$ are represented in the resolution of the problem of search for static modes [éq. 2.6.3] that in the complete dynamic problem.

The field of displacements of the "trained" structure $\mathbf{u}_{ent}$, after discretization by finite elements, is thus described by $\mathbf{U}_{ent} = \mathbf{U}_{ent}^k(t)\mathbf{\Psi}_\mathbf{k}$, checking in particular $\mathbf{B}_\mathbf{s}.\mathbf{U}_{ent} = \mathbf{U}_d^k(t)\mathbf{X}_{ks}$ on the supports, traversing the discrete subspace V_e , "degrees of freedom trained" by the supports.

The discrete subspace V_e is thus generated by the base $\left(\mathbf{\Psi}_\mathbf{k}\right)$. One necessarily has by construction:

$$\mathbf{U}_{ent}^k(t) = \mathbf{U}_d^k(t), \forall\, k$$

The degrees of freedom of displacements of the "trained" structure thus are directly given by the value of $U^d(t)$.

Having characterized the discrete subspace V_e starting from the linear static modes, after discretization, let us study a field of displacement $\mathbf{W}$ structure, kinematically acceptable unspecified, but no one on the supports: $\mathbf{B}_\mathbf{s}.\mathbf{W}=0$ and checking the "ordinary" connections $\mathbf{B}_\mathbf{L}.\mathbf{W}=0$. Under the terms of [éq 2.6-3], one necessarily has:

$$\begin{cases} {}^t\mathbf{W}.\mathbf{K}.\mathbf{\Psi}_\mathbf{k} + {}^t\mathbf{W}.{}^t\mathbf{B}_\mathbf{s}.\lambda_{s\Psi_k} + {}^t\mathbf{W}.{}^t\mathbf{B}_\mathbf{L}.\lambda_{L\Psi_k} = 0 \\ {}^t\mathbf{W}.\mathbf{B}_\mathbf{s}.\mathbf{\Psi}_\mathbf{k} = {}^t\mathbf{W}.X_{ks} \\ {}^t\mathbf{W}.\mathbf{B}_\mathbf{L}.\mathbf{\Psi}_\mathbf{k} = 0 \end{cases} \qquad \forall W,\ \text{tel que}\ \ \mathbf{B}_\mathbf{s}.\mathbf{W}=0$$

From where simply the linear static problem:

$${}^t\mathbf{W}.\mathbf{K}.\mathbf{\Psi}_\mathbf{k} = 0 \quad \forall\, \mathbf{W},\ \text{tel que}\ \ \mathbf{B}_\mathbf{s}.\mathbf{W}=0\ \ \text{et}\ \ \mathbf{B}_\mathbf{L}.\mathbf{W}=0 \qquad \textbf{éq 2.6-4}$$

The elastic operator of stiffness $\mathbf{K}$ being definite positive (having eliminated the rigid modes of body), it is noted that any field of absolute displacement $\mathbf{W_a}$ kinematically acceptable of the structure, after discretization, can be written by single decomposition:

$$\mathbf{W_a}=\mathbf{W}+\mathbf{W_e} \text{ on the sum of the additional subspaces } V_a=V\oplus V_e \qquad \textbf{éq 2.6-5}$$

with V_e generated starting from the static modes $\left(\Psi_k\right)$, and V containing the fields known as "active degrees of freedom" such as $\mathbf{B_s}.\mathbf{W}=\mathbf{0}$ (worthless on the supports). One calls V the discrete subspace of the "active degrees of freedom".

2.6.3 Case MONO_APPUI and MULTI_APPUI

For a loading of the type MONO_APPUI, the static modes are simply the rigid modes of body of the structure: $\mathbf{K}.\Psi_k=0$, checking the "ordinary" connections $\mathbf{B_L}\cdot\Psi_k=0$.
If the loading is MULTI_APPUI, static modes $\left(\Psi_k\right)$ are unspecified.

This decomposition on two additional subspaces $V_a=V\oplus V_e$ of any field kinematically acceptable built using the operator of elasticity $\mathbf{K}$ is applicable in any nonlinear evolution of the structure, including with shocks…, provided that the bilateral connections remain the same ones during the history.

Let us exploit this decomposition and now project the nonlinear dynamic problem [éq 2.2.4 - 1] separately on the subspace V then on the subspace V_e, while exploiting [éq 2.6-3]. The result is simplified (a little only) because $\mathbf{B_s}.\mathbf{W}=0$ (i.e. the "active degrees of freedom" do not work in the reactions of the supports supports $\mathbf{B_s}$), $\mathbf{B_L}.\Psi_\mathbf{k}=0$ (i.e. the static modes do not work in the reactions of connection $\mathbf{B_L}$), $\mathbf{B_s}.\Psi_\mathbf{k}=\mathbf{X}_{ks}$:

To find $\mathbf{U_a}=\mathbf{U_e}+\mathbf{U}$, $\lambda_\mathbf{s}$, $\lambda_\mathbf{L},\mu$ such as:

$$\left\{\begin{array}{l}
\mathbf{U_e}=U_d^\ell(t)\Psi_\ell \\
{}^t\mathbf{W}.\mathbf{M}.\left(\mathbf{U}+\mathbf{U}_e,\dot{\mathbf{U}}+\dot{\mathbf{U}}_e,\ddot{\mathbf{U}}+\ddot{\mathbf{U}}_e\right)+{}^t\mathbf{W}.C.\left(\dot{\mathbf{U}}+\dot{\mathbf{U}}_e\right)+{}^t\mathbf{W}.\mathbf{R}\left(\mathbf{U},\dot{\mathbf{U}}+\dot{\mathbf{U}}_e,t\right)= \\
\qquad {}^t\mathbf{W}.\left(\mathbf{L}(t)+\mathbf{L}^{abso}\left(\mathbf{U}_e,\dot{\mathbf{U}}_e,\dot{\mathbf{U}}\right)\right)-{}^t\mathbf{W}.{}^t\mathbf{B_L}.\lambda_L-{}^t\mathbf{W}.{}^t\mathbf{A}.\mu \qquad \forall\,\mathbf{W}\in V \\
{}^t\Psi_\mathbf{k}.\mathbf{M}\left(\mathbf{U}+\mathbf{U}_e,\dot{\mathbf{U}}+\dot{\mathbf{U}}_e,\ddot{\mathbf{U}}+\ddot{\mathbf{U}}_e\right)+{}^t\Psi_\mathbf{k}.C.\left(\dot{\mathbf{U}}+\dot{\mathbf{U}}_e\right)+{}^t\Psi_\mathbf{k}.\mathbf{R}\left(\mathbf{U},\dot{\mathbf{U}}+\dot{\mathbf{U}}_e,t\right)= \\
\qquad {}^t\Psi_\mathbf{k}.\left(\mathbf{L}(t)+\mathbf{L}^{abso}\left(\mathbf{U}_e,\dot{\mathbf{U}}_e,\dot{\mathbf{U}}\right)\right)-{}^t\mathbf{X}_{ks}.\lambda_\mathbf{s}-{}^t\Psi_\mathbf{k}.{}^t\mathbf{A}.\mu \qquad \forall\,k \\
\mathbf{B_s}.\mathbf{U}=0 \\
\mathbf{B_L}.\mathbf{U}=0 \\
\mathbf{A}.\left(\mathbf{U}+\mathbf{U}_e\right)\le d_0(t) \\
\mu\ge 0 \\
\forall\,j,\ \left(\mathbf{A}.\left(\mathbf{U}+\mathbf{U}_e\right)-d_0\right)_j.\mu_j=0 \\
\left(\mathbf{U}+\mathbf{U}_e\right)(t_0)=\mathbf{U_0} \\
\left(\dot{\mathbf{U}}+\dot{\mathbf{U}}_e\right)(t_0)=\dot{\mathbf{U}}_0
\end{array}\right.$$

éq 2.6-6

It is noted that is rather complicated.

We restrict initially with the dynamic problem where operators of inertia $\mathbf{M}$ and of interior forces $\mathbf{R}$ are linear, and in absence of absorbing borders: $\mathbf{L}^{abso}\left(\mathbf{U}_e, \dot{\mathbf{U}}_e, \dot{\mathbf{U}}\right)=0$.

One makes the assumption in *Code_Aster* that: $\mathbf{C}.\dot{\mathbf{U}}_e=0$, including in multi-supports (whereas that is not exact that in mono-support, where the static modes are the rigid modes $\mathbf{K}.\mathbf{\Psi}_\mathbf{k}=0$ without deformation). That amounts neglecting the contribution of displacements of training of the structure to the forces of viscous damping.

Static modes $\mathbf{\Psi}_\mathbf{k}$ checking [éq 2.6-3], the system [éq 2.6-6] is restricted with:

$$\left\{\begin{array}{ll}
{}^t\mathbf{W}.\mathbf{M}.\ddot{\mathbf{U}}+{}^t\mathbf{W}.\mathbf{C}.\dot{\mathbf{U}}+{}^t\mathbf{W}.\mathbf{K}.\mathbf{U}={}^t\mathbf{W}.\mathbf{L}(t)-{}^t\mathbf{W}.{}^t\mathbf{B}_\mathbf{L}.\lambda_\mathbf{L}-{}^t\mathbf{W}.{}^t\mathbf{A}.\boldsymbol{\mu}-{}^t\mathbf{W}.\mathbf{M}.\ddot{\mathbf{U}}_e & \forall W\in V \\
{}^t\mathbf{\Psi}_\mathbf{k}.\mathbf{M}.\left(\ddot{\mathbf{U}}+\ddot{\mathbf{U}}_d^\ell(t)\mathbf{\Psi}_\mathbf{\ell}\right)+{}^t\mathbf{\Psi}_\mathbf{k}.\mathbf{C}.\dot{\mathbf{U}}+\mathbf{U}_\mathbf{d}^\ell(t){}^t\mathbf{\Psi}_\mathbf{k}.\mathbf{K}.\mathbf{\Psi}_\mathbf{\ell}={}^t\mathbf{\Psi}_\mathbf{k}.\mathbf{L}(t)-{}^t\mathbf{X}_{ks}.\Lambda_\mathbf{s}-{}^t\mathbf{\Psi}_\mathbf{k}.{}^t\mathbf{A}.\boldsymbol{\mu} & \forall k \\
\mathbf{U}_e=\mathbf{U}_d^k(t)\mathbf{\Psi}_\mathbf{k} & \\
\mathbf{B}_\mathbf{s}.\mathbf{U}=0 & \\
\mathbf{B}_\mathbf{L}.\mathbf{U}=0 & \\
\mathbf{A}.\left(\mathbf{U}+\mathbf{U}_d^k(t)\mathbf{\Psi}_\mathbf{k}\right)\le d_0(t) & \\
\mu\ge 0 & \\
\forall j,\ \left(\mathbf{A}.\left(\mathbf{U}+\mathbf{U}_d^k(t)\mathbf{\Psi}_\mathbf{k}\right)-d_0\right)_j.\mu_j=0 & \\
\left(\mathbf{U}+\mathbf{U}_e\right)(t_0)=\mathbf{U}_0 & \\
\left(\dot{\mathbf{U}}+\dot{\mathbf{U}}_e\right)(t_0)=\dot{\mathbf{U}}_0 &
\end{array}\right.$$

éq 2.6-7

One notes on the first of these equations [éq 2.6-7], that thanks to the made assumptions, one can restrict oneself to solve a dynamic problem on the field of "relative" displacement $\mathbf{U}$, having blocked the degrees of freedom on the supports ($\mathbf{B}_\mathbf{s}.\mathbf{U}=0$), on condition that providing the term as a preliminary ${}^t\mathbf{W}.\mathbf{M}.\ddot{\mathbf{U}}_e$, as well as the static modes $\left(\mathbf{\Psi}_\mathbf{k}\right)$.

The object of the operator CALC_CHAR_SEISME [U4.63.01] is precisely to calculate the term $-{}^t\mathbf{W}.\mathbf{M}.\ddot{\mathbf{U}}_e$, transformed into a concept of the type "load", using the operator AFFE_CHAR_MECA [U4.25.01]. One can also simply introduce a load of unit "gravity" into the direction wanted, and amplified by the temporal signal of acceleration. The advantage is to exploit the data in accélérogramme directly (for example produced starting from a spectrum), without having to twice integrate it in time with uncertainties which this operation generates. One can directly produce the constraints known as "primary" induced by dynamics in "relative movement".

This advantage is lost so unilateral connections are present, since the condition $\mathbf{A}.\left(\mathbf{U}+\mathbf{U}_d^k(t)\mathbf{\Psi}_\mathbf{k}\right)\le d_0(t)$ request the value of $U_d^k(t)$ to be expressed correctly, except if there is the chance that the movement of training does not have a component on the interface where the game of the unilateral connection is calculated, i.e. if the thrusts of these unilateral supports are fixed in the absolute reference mark!

The second equation of [éq 2.6-7] provides the reactions on the supports $\lambda_\mathbf{s}$ (the rest being determined by the resolution of the problem in "relative" displacement $\mathbf{U}$); but it is noted there too that it is necessary to know $U_d^k(t)=U_e^k(t)$ on the supports.

It is the same thing if one wants to reconstitute the total solution for postprocessing in constraints with their contribution known as “secondary”, related to $\mathbf{U}_e=\mathbf{U}_d^k(t)\mathbf{\Psi_k}$. This contribution is worthless in mono-support, since it acts a rigid mode then.

Now one considers the dynamic problem of a structure interacting with a ground (“infinite” medium), source of an incidental seismic wave, *cf.* [R4.05.01] and [R4.02.05]. One is necessarily within a framework “ MONO_APPUI ”, where the static modes are the rigid modes of the structure, from where: $\mathbf{K}.\mathbf{\Psi_k}=0$ and it is usually admitted that ${}^t\mathbf{\Psi}_\mathbf{k}.\mathbf{C}.\dot{\mathbf{U}}=0$ (indeed, physically, damping comes from the deformations in the solid). One thus uses elements of absorbing border, and the term $\mathbf{L}_{abso}\left(\mathbf{U}_e,\dot{\mathbf{U}}_e,\dot{\mathbf{U}}\right)$ is present in [éq 2.6-7]. The incidental wave is provided directly by the signal $\mathbf{U_d}\left(\vec{x},t\right)$. One thus builds the terms of [éq 2.6-2], as well as the term ${}^t\mathbf{W}.\mathbf{M}.\ddot{\mathbf{U}}_e$. The system [éq 2.6-7] becomes in the case mono-support:

$$\left\{\begin{array}{l}
{}^t\mathbf{W}.\mathbf{M}.\ddot{\mathbf{U}}+{}^t\mathbf{W}.\mathbf{C}.\dot{\mathbf{U}}+{}^t\mathbf{W}.\mathbf{K}.\mathbf{U}= \\
{}^t\mathbf{W}.\mathbf{L}(t)+{}^t\mathbf{W}.A_0^{abso}.\left(\dot{\mathbf{U}}-\dot{\mathbf{U}}_e\right)+{}^t\mathbf{W}.\mathbf{\Sigma}(\dot{\mathbf{U}}_\mathbf{e})-{}^t\mathbf{W}.{}^t\mathbf{B}_\mathbf{L}.\lambda_\mathbf{L}-{}^t\mathbf{W}.{}^t\mathbf{A}.\mu-{}^t\mathbf{W}.\mathbf{M}.\ddot{\mathbf{U}}_e \quad \forall\,\mathbf{W}\in V \\
{}^t\mathbf{\Psi_k}.\mathbf{M}.\left(\ddot{\mathbf{U}}+\ddot{\mathbf{U}}_d^l(t)\mathbf{\Psi_l}\right)={}^t\mathbf{\Psi_k}.\mathbf{L}(t)+{}^t\mathbf{\Psi_k}.\mathbf{A_0^{abso}}.\left(\dot{\mathbf{U}}-\dot{\mathbf{U}}_e\right)+{}^t\mathbf{\Psi_k}.\mathbf{\Sigma}\left(\dot{\mathbf{U}}_e\right)-{}^t\mathbf{X}_{ks}.\lambda_\mathbf{s}-{}^t\mathbf{\Psi_k}.{}^t\mathbf{A}.\mu \quad \forall\,k \\
\mathbf{U}_e=\mathbf{U}_d^k(t)\mathbf{\Psi_k} \\
\mathbf{U}_s.\mathbf{U}=0 \\
\mathbf{U}_L.\mathbf{U}=0 \\
\mathbf{A}.\left(\mathbf{U}+\mathbf{U}_d^k(t)\mathbf{\Psi_k}\right)\leq d_0(t) \\
\mu\geq 0 \\
\forall\,j,\ \left(\mathbf{A}.\left(\mathbf{U}+\mathbf{U}_d^k(t)\mathbf{\Psi_k}\right)-d_0\right)_j.\mu_j=0 \\
\left(\mathbf{U}+\mathbf{U}_e\right)(t_0)=\mathbf{U_0} \\
\left(\dot{\mathbf{U}}+\dot{\mathbf{U}}_e\right)(t_0)=\dot{U}_0
\end{array}\right.$$

éq 2.6-8

Let us return now to the nonlinear problem general [éq 2.6-6]. It is noted that the first equation on the “active” degrees of freedom comprises necessarily a coupling with the field $\mathbf{U}_e=\mathbf{U}_d^k(t)\mathbf{\Psi_k}$ in the operator of inertia as in that of internal forces. One thus must, in accordance with the diagram of temporal integration and nonlinear resolution, developed with [§ 3], at every moment to reconstitute calculation the value of absolute displacement $\mathbf{U_a}=\mathbf{U_e}+\mathbf{U}$, constraints…
In conclusion, one can to treat problem dynamic moving relative (by admitting that the forces of damping depend only on him), according to the following conditions:

situation	Possibility of MONO_APPUI	Possibility of MULTI_APPUI
linear behavior in small transformations, without absorbing border, with unilateral conditions such as the games are not modified by the movement of training, that the fluid field is not directly charged, and with a loading of imposed acceleration $\ddot{\mathbf{U}}_e$, *cf.* [éq 2.6-7]	yes	yes
unspecified behavior and with absorbing border, with unilateral conditions such as the games are not modified by the movement of training, and with a loading of imposed acceleration $\ddot{\mathbf{U}}_e$	yes	NOT

In the case “NOT”, one can only deal with the dynamic problem in the absolute reference mark.

3 Diagram of temporal integration: diagram of NEWMARK and method of NEWTON

The mechanical problem to analyze being modelled in finite elements, according to the approach described with [§ 2], one calculates the fields of displacements, speeds and accelerations with the nodes in a discrete succession of moments of calculation $t_1, t_2 \ldots t_{i-1}, t_i \ldots t_N$: $\left\{\{t_i\}\right\}_{1\leq i\leq N}$.

The user of DYNA_NON_LINE can currently choose enters five temporal diagrams with a step, of which three implicit: that of NEWMARK (1959), its alternative known as “modified average acceleration”, or that of HILBER-HUGUES-TAYLOR (HHT, 1977): to see the paragraph [§5] and two explicit: that of the centered differences (which is a typical case of the family of Newmark) and the diagram of Tchamwa-Wielgosz (which introduces a digital dissipation high frequency)): to see the paragraph [§8]. The state of the structure being known at the moment t_{i-1} , one seeks to calculate his state at the moment t_i by a method of prediction-correction.

Notice :

One must note that Code_Aster does not propose method multi-field in time and space, which would make it possible to define a diagram by zone in the studied solid.

3.1 Diagram of NEWMARK

One presents here this diagram in his classical form ([bib1] and [bib2]) relative to a rotation or translatory movement small. For great rotations of elements of structure [bib3], the formulas are more complicated, but they in the same way make it possible to bring up to date speed and the angular acceleration according to the increase in déplacemenT, which is in this case vector-rotation.

One notes hereafter by Φ configuration, i.e. the parameter setting of the system by the degrees of freedom of the finite elements: displacements and rotations $\mathbf{U}$, pressure P , potential φ …

The diagram of NEWMARK rests on the following developments of the vector-configuration, function of time, when β and γ are two parameters:

$$\Phi\left(t+\Delta t\right)\approx\Phi\left(t\right)+\Delta t\,\dot{\Phi}\left(t\right)+\frac{\Delta t^2}{2}\left[\left(1-2\beta\right)\ddot{\Phi}\left(t\right)+2\beta\,\ddot{\Phi}\left(t+\Delta t\right)\right] \qquad \textbf{éq 3.1-1}$$

$$\dot{\Phi}\left(t+\Delta t\right)\approx\dot{\Phi}\left(t\right)+\Delta t\left[\left(1-\gamma\right)\ddot{\Phi}\left(t\right)+\gamma\,\ddot{\Phi}\left(t+\Delta t\right)\right] \qquad \textbf{éq 3.1-2}$$

It is parameterization in acceleration, known as “A-form”, where one expresses displacements and speeds with accelerations. The equation [éq 3.1-1] is also written with [éq 3.1-2], if $\gamma\neq 0$:

$$\Phi\left(t+\Delta t\right)\approx\Phi\left(t\right)+\frac{\Delta t}{\gamma}\left(\left(\gamma-\beta\right)\dot{\Phi}\left(t\right)+\beta\,\dot{\Phi}\left(t+\Delta t\right)\right)+\frac{\Delta t^2}{2\gamma}\left(\gamma-2\beta\right)\ddot{\Phi}\left(t\right)$$

These parameters β and γ are provided respectively via the operands BETA and GAMMA keyword SCHEMA_TEMPS (SCHEMA=' NEWMARK') commandE DYNA_NON_LINE.

See it [§ 4] for the characteristics of the diagram according to the values of these parameters.

The hooks with the second members of the equations [éq 3.1-1] and [éq 3.1-2] are thus weighted averages of $\ddot{\Phi}\left(t\right)$ and of $\ddot{\Phi}\left(t+\Delta t\right)$.

It is noticed that one cannot have $\beta=0$. With the first step of calculation one exploits the initial conditions directly: $\Phi(0)$ and $\dot{\Phi}(0)$. It is necessary too $\ddot{\Phi}(0)$, except if $\gamma=1$: to see the remark passed with [§ 3.2].

In practice these expressions are not usable because one will have to express the values in displacements at the moment $t+\Delta t$ from those at the moment t . Between these two moments, the increment and the average of the fields are noted:

$$\Delta\Phi=\Phi(t+\Delta t)-\Phi(t) \quad ; \quad \bar{\Phi}=\frac{1}{2}\left(\Phi(t)+\Phi(t+\Delta t)\right)$$

These results will be used with [§ 4.2]. The equation [éq 3.1-1] gives:

$$\begin{aligned}&\ddot{\Phi}(t+\Delta t)=\frac{1}{\beta\,\Delta t^2}\left[\Phi(t+\Delta t)-\Phi(t)\right]-\frac{1}{\beta\,\Delta t}\,\dot{\Phi}(t)+\frac{2\beta-1}{2\beta}\,\ddot{\Phi}(t)\\ \Leftrightarrow&\Delta\ddot{\Phi}=\frac{1}{\beta\,\Delta t^2}\Delta\Phi-\frac{1}{\beta\,\Delta t}\,\dot{\Phi}(t)-\frac{1}{2\beta}\,\ddot{\Phi}(t)\end{aligned} \qquad \textbf{éq 3.1-3}$$

And, according to [éq 3.1-2]:

$$\begin{aligned}&\dot{\Phi}(t+\Delta t)=\frac{\gamma}{\beta\,\Delta t}\left[\Phi(t+\Delta t)-\Phi(t)\right]+\frac{\beta-\gamma}{\beta}\dot{\Phi}(t)+\frac{(2\beta-\gamma)\,\Delta t}{2\beta}\ddot{\Phi}(t)\\ \Leftrightarrow&\Delta\dot{\Phi}=\frac{\gamma}{\beta\,\Delta t}\Delta\Phi-\frac{\gamma}{\beta}\dot{\Phi}(t)+\frac{(2\beta-\gamma)\,\Delta t}{2\beta}\ddot{\Phi}(t)\\ \Leftrightarrow&\Delta\dot{\Phi}=\Delta t\left(\bar{\ddot{\Phi}}-\frac{1-2\gamma}{2}\Delta\ddot{\Phi}\right)\end{aligned} \qquad \textbf{éq 3.1-4}$$

Expressions [éq 3.1-3) and [éq 3.1-4] constitute parameterization in displacement, known as “D-form”, where one expresses accelerations and speeds with displacements, which will be advantageous into nonlinear to express the laws of behavior. Lastly, one can write:

$$\Delta\Phi=\Delta t.\,\dot{\Phi}(t)+\frac{\Delta t^2}{4}(4\beta-1)\,.\Delta\ddot{\Phi}+\frac{\Delta t^2}{2}\bar{\ddot{\Phi}}=\Delta t.\bar{\dot{\Phi}}+\frac{\Delta t^2}{2}(2\beta-\gamma)\,.\Delta\ddot{\Phi} \qquad \textbf{éq 3.1-5}$$

In the case of great rotations of elements of structure the homologous expressions with [éq 3.1-3] and [éq 3.1-4] are more complex [bib5], but the relations which follow are rather easily transposable.

During a step of time (of t with $t+\Delta t$), where values at the moment t are known, the equations [éq 3.1-4] and [éq 3.1-3] the increases speed define $\delta\dot{\Phi}$ and of acceleration $\delta\ddot{\Phi}$ correspondent with a accroissemearbitrary NT of displacement $\delta\Phi$ starting from the position at the moment t , which one will need at the time of the iterations of correction of NEWTON (within the step of time, to see it [§ 3.3]):

$$\delta \dot{\Phi} = \frac{\gamma}{\beta \Delta t} \delta \Phi \qquad \text{éq 3.1-6}$$

$$\delta \ddot{\Phi} = \frac{1}{\beta \Delta t^2} \delta \Phi \qquad \text{éq 3.1-7}$$

We place at the moment $t = t_{i-1}$, and one writes balance [éq 2.2.4-1] after space discretization at the moment $t_i = t_{i-1} + \Delta t$, possibly with the complementary elements brought in [éq 2.2-6]. One notes $\mathbf{U}_i = \mathbf{U}(t_i)$ degrees of freedom at the new moment t_i and by exploiting the terms of the temporal diagram [éq 3.1-4] and [éq 3.1-3], one leads to the nonlinear system of dynamic balance:

$$\begin{cases} \hat{\mathbf{K}}\,\mathbf{U}_i + \mathbf{R}(\mathbf{U}_i, \dot{\mathbf{U}}_i, t) + {}^t\mathbf{B}.\lambda_i + {}^t\mathbf{A}.\mu_i = \hat{\mathbf{L}}(t_i) - \mathbf{L}_{GR}^{iner}(\mathbf{U}_i, \dot{\mathbf{U}}_i, \ddot{\mathbf{U}}_i) \\ \mathbf{B}.\mathbf{U}_i = \mathbf{U}^d(t_i) \\ \mathbf{A}.\mathbf{U}_i \leq d_0(t_i) \\ \forall\, j,\ \mu_j \geq 0 \\ \forall\, j,\ (\mathbf{A}.\mathbf{U}_i - d_{0_i})_j . \mu_j = 0 \end{cases} \qquad \text{éq 3.1-8}$$

with:

$$\hat{\mathbf{K}} = \mathbf{K}^{fs} + \frac{1}{\beta \Delta t^2}(\mathbf{M} + \mathbf{M}^{fs}) + \frac{\gamma}{\beta \Delta t}\mathbf{C} \qquad \text{éq 3.1-9}$$

$$\hat{\mathbf{L}}(t_i) = \mathbf{L}(t_i) + \mathbf{L}^{abso}(t_i) + \frac{1}{\beta \Delta t^2}(\mathbf{M} + \mathbf{M}^{fs}) . \left(\mathbf{U}_{i-1} + \Delta t\, \dot{\mathbf{U}}_{i-1} + \Delta t^2 \left(\frac{1-2\beta}{2} \right) \ddot{\mathbf{U}}_{i-1} \right) + \frac{1}{\beta \Delta t}\mathbf{C} . \left(\gamma\, \mathbf{U}_{i-1} + \Delta t (\gamma - \beta)\, \dot{\mathbf{U}}_{i-1} + \Delta t^2 \left(\frac{\gamma - 2\beta}{2} \right) \ddot{\mathbf{U}}_{i-1} \right) \qquad \text{éq 3.1-10}$$

Notice :

It is noted that the matrix $\hat{\mathbf{K}}$ contributing to the stiffness generalized of the system to solve is enriched by terms coming from the matrix from mass $\mathbf{M}$ system (provided that one has affected a density on all the finite elements of the model) and of the matrix of damping $\mathbf{C}$. One will further see than the linearization from the internal forces $\mathbf{R}(\mathbf{U}_i, \dot{\mathbf{U}}_i, t)$ also contribute to $\hat{\mathbf{K}}$. Thus, even if the mechanical system considered comprises rigid modes, for example in a study where a solid is in freefall, the matrix of mass comes "to prevent" that the matrix of "stiffness" $\hat{\mathbf{K}}$ is not factorisable. To some extent, they are the inertial forces of the solid body (in its rigid modes) which ensures balance with the external forces, which could not be done into quasi-static.
However, it is observed that the step of time Δt appears too. If it is too large, the terms of mass will not be important enough vis-a-vis those of stiffness, and stamps it risk to be quasi not factorisable ("quasi-worthless pivots").

Notice :

Terms with the exhibitor ^{fs} appear if the system contains fluid fields (see [§ 2.4] and terms coming from the fluid field in [éq 2.4-1]); it is pointed out that it is not envisaged of fluid loading $\mathbf{L}^{fl}$.

Notice :

Contrary to the case of damping of Rayleigh (see [§ 2.2.1]), the matrix of damping $\mathbf{C}$ *does not appear in the matrix* $\hat{K}$ *in the presence of modal damping (keyword* AMOR_MODAL *), because in this case, the force* $\mathbf{C}.\dot{\mathbf{U}}_{i-1}$ *is built directly without storage of the matrix* C *, and is deferred to the second member* $\hat{\mathbf{L}}(t_i)$ *.*
The term $\mathbf{L}^{abso}$ *in the second member of [éq 3.1-10] appears in the presence of absorbing borders of a fluid or a solid (elastic formulation), to see it [§ 2.6]. It is treated according to an explicit diagram according to the fields solutions obtained with* t_{i-1} *.*

Notice :

Contrary to θ *– diagram also used in transitory thermics, cf [R5.02.01], where the derivative are written in an explicit way, while the conservation equation is written at one fictitious moment resulting from a combination with the parameter* θ *values at the moments* t_{i-1} *and* t_i*, cf [§ 6], dynamic balances are checked over the moments* t_{i-1} *and* t_i *, while the derivative are combinations determined by the diagram of Newmark [éq 3.1-1] and [éq 3.1-2]. On the other hand, complete diagram HHT, to see it [§ 5.2] exploits a shift, in the manner of* θ *– diagram.*

It still remains to treat the nonlinear terms: internal forces $\mathbf{R}(\mathbf{U}_i, \dot{\mathbf{U}}_i, t)$ and the nonlinear contribution of the inertial forces in great rotations of structures $\mathbf{L}_{GR}^{iner}(\mathbf{U}_i, \dot{\mathbf{U}}_i, \ddot{\mathbf{U}}_i)$ (*cf.* [§ 2.2.2]).

At the first moment t_1 , one sees in the second member [éq 3.1-10] whom one needs for $\mathbf{U}_0, \dot{\mathbf{U}}_0$, provided by the initial conditions, but also of $\ddot{\mathbf{U}}_0$ because of diagram of NEWMARK (except if $\gamma = 2\beta = 1$): to see the remark passed with [§ 3.2].

One thus uses a method of Newton to solve this nonlinear problem [éq 3.1.7].

3.2 Phase of prediction

The system [éq 3.1-8] is nonlinear and is integrated, after a prediction of EULER by linearization, using an iterative method of NEWTON, as in nonlinear statics [R5.03.01]. The calculation of this prediction can be slightly erroneous, as long as the phase of correction by iterations of NEWTON [§ 3.3] is able to correct with convergence…

3.2.1 Pas de time general

With the phase of prediction, one exploits the solution with the preceding step or the values of the initial state, and one notes the matrix of initial tangent stiffness of the step (*cf.* definition of $\mathbf{Q}$ with [§ 2.2]):

$$\mathbf{K}_{i-1} = \frac{\partial \mathbf{R}}{\partial \mathbf{U}}\Big|_{(\mathbf{U}_{i-1}, \dot{\mathbf{U}}_{i-1}, t_{i-1})} = \frac{d\,{}^t\mathbf{Q}}{d\,\mathbf{U}}\Big|_{(\mathbf{U}_{i-1})} \cdot \boldsymbol{\sigma}_{(\mathbf{U}_{i-1}, \dot{\mathbf{U}}_{i-1}, \mathbf{t}_{i-1})} + {}^t\mathbf{Q}_{(\mathbf{U}_{i-1})} \cdot \frac{\partial\, \boldsymbol{\sigma}}{\partial \mathbf{U}}\Big|_{(\mathbf{U}_{i-1}, \dot{\mathbf{U}}_{i-1}, t_{i-1})} \qquad \textbf{éq 3.2.1-1}$$

These terms (*cf.* [§ 2.2]) are evaluated on the preceding step, the first term of the second member of [éq 3.2.1-1] appearing only in great displacements ($\mathbf{Q}$ not being then constant).

If the behavior is linear, the matrix $\mathbf{K}_{i-1}$ is simply the elastic matrix of rigidity $\mathbf{K}$ structure.

One can also decide to save time calculation not to reactualize this matrix (only if the step of time does not vary), and to take the elastic matrix of rigidity, to see [U4.51.03], keyword NEWTON, operand PREDICTION, value 'RUBBER BAND', rather than 'TANGENT'. If nothing is specified, the by default choice made by *Code_Aster* is coherent with that done on the iterations of correction of NEWTON described hereafter with [§3.3].

In addition, in the presence of great rotations (elements of structure: beams…), one must also derive the nonlinear term from inertia $\mathbf{L}_{GR}^{iner}\left(\mathbf{U}_i, \dot{\mathbf{U}}_i, \ddot{\mathbf{U}}_i\right)$:

$$\mathbf{K}_{\mathbf{i-1}}^{\mathbf{M1}} = \frac{\partial \mathbf{L}_{GR}^{iner}}{\partial \mathbf{U}}\Big|_{\left(\mathbf{U}_{i-1}, \dot{\mathbf{U}}_{i-1}, \ddot{\mathbf{U}}_{i-1}\right)} \qquad \textbf{éq 3.2.1-2}$$

The matrix $\hat{\mathbf{K}}$ having been established by [éq 3.1-9], this new matrix $\mathbf{K}_{\mathbf{i-1}}^{\mathbf{M1}}$ is combined with the matrix $\hat{\mathbf{K}} + \mathbf{K}_{i-1}$, to see below [éq 3.2.1-4].

One notes the increase in loading $\Delta \hat{\mathbf{L}}\left(t_i\right)$, $\hat{\mathbf{L}}$ being defined with [éq 3.1-10]:

$$\Delta \hat{\mathbf{L}}\left(t_i\right) = \hat{\mathbf{L}}\left(t_i\right) - \hat{\mathbf{L}}\left(t_{i-1}\right) \qquad \textbf{éq 3.2.1-3}$$

One defines also the increases in imposed conditions $\Delta \mathbf{U}_d\left(t_i\right)$, and one gathers in $\Delta \mathbf{L}^{anél}\left(t_i\right)$ dependences of the constraints according to the various parameters or “variables of order” $\mathbf{Z}$ law of behavior of constitutive material: such as the temperature…:

$$\Delta \mathbf{L}^{anél}\left(t_i\right) = -{}^t\mathbf{Q}\left(\mathbf{U}_{i-1}\right).\frac{d\,\boldsymbol{\sigma}}{d\,\mathbf{Z}}\Big|_{\left(\mathbf{U}_{i-1}, \dot{\mathbf{U}}_{i-1}, \dot{\mathbf{U}}_{i-1}, \mathbf{Z}\right)}.\Delta \mathbf{Z}$$

In the presence of modal damping the forces of damping are deferred to the second member. One adds then to $\Delta \hat{\mathbf{L}}\left(t_i\right)$ the term corresponding: $-\mathbf{C}.\dot{\mathbf{U}}_{i-1}$.

The term $\Delta \mathbf{L}^{abso}\left(t_i\right) = -\mathbf{A}_{fa}\Delta \dot{\mathbf{P}}\left(t_{i-1}\right) + \Delta \mathbf{L}_{abso}^{ordre\,0}\left(\mathbf{U}_{ent}\left(t_i\right), \dot{\mathbf{U}}_{ent}\left(t_i\right), \dot{\mathbf{U}}\left(t_{i-1}\right)\right)$ indicate the contribution integrated into explicit starting from the solutions to t_{i-1} (to avoid having to treat a nonsymmetrical matrix, *cf.* [R4.02.05]) of an absorbing fluid border and an absorbing elastic border, *cf.* [éq 2.4-1] and [éq 2.6-2].

One then calculates predictive values for the step of time in progress $\left(\Delta \mathbf{U}_i^0, \Delta \boldsymbol{\lambda}_i^0, \Delta \boldsymbol{\mu}_i^0\right)$:

Prediction

$$\begin{cases} \left(\hat{\mathbf{K}} + \mathbf{K}_{i-1} + \mathbf{K}_{i-1}^{MI}\right)\Delta \mathbf{U}_i^0 + {}^t\mathbf{B}.\Delta \boldsymbol{\lambda}_i^0 + {}^t\mathbf{A}.\Delta \boldsymbol{\mu}_i^0 = \Delta \hat{\mathbf{L}}\left(t_i\right) + \Delta L^{anél}\left(t_i\right) \\ \mathbf{B}.\Delta \mathbf{U}_i^0 = \Delta \mathbf{U}_d\left(t_i\right) \\ \mathbf{A}.\left(\mathbf{U}_i^0\right) \le d_0(t_i) \\ \forall\, j,\ \boldsymbol{\mu}_i^{0_j} \ge 0 \\ \forall\, j,\ \left(\mathbf{A}.\mathbf{U}_i^0 - d_{0_i}\right)_j.\boldsymbol{\mu}_i^{0_j} = 0 \end{cases} \qquad \textbf{éq 3.2.1-4}$$

where the prediction is defined $\mathbf{U}_i^0 = \mathbf{U}_{i-1} + \Delta \mathbf{U}_i^0$ for the new moment $t_i = t_{i-1} + \Delta t$, under the conditions of contact established with the step of previous time.

If one chose MATRIX = ‘ELASTIQUE' in the keyword NEWTON, one does not revalue with each step of time $\hat{\mathbf{K}} + \mathbf{K}_{i-1} + \mathbf{K}_{i-1}^{MI} = \hat{\mathbf{K}} + \mathbf{K}_0 + \mathbf{K}_0^{MI}$, which avoids the cost of re-assembly and inversion, but the iteration count of correction increases.

After the establishment of a candidate solution of [éq 3.2.1-4], on the basis of connection already in contact with the preceding step, without ensuring of new checking of the criterion of contact, one launches the algorithm of active constraints to satisfy the conditions with contact: one corrects thus $\mathbf{U}_i^0, \boldsymbol{\lambda}_i^0, \boldsymbol{\mu}_i^0$ [R5.03.50].

Note: singular tangent matrix:

One checks on [éq 3.2.1-4] whom if the tangent matrix $\mathbf{K}_{i-1}$ is singular (case of a rigid mode, a damaged material, or ductile plate…), dynamics is controlled by the inertial forces, and that, the matrix $\hat{\mathbf{K}}$ being in general regular (cf notices made with [§ 3.1]), one finds despite everything good a predictor $\mathbf{U}_i^0$, provided that precision in the matrix $\hat{\mathbf{K}}$ is not lost (pivot quasi-no one) because of a choice of step of too large time which plays in $1/\beta\Delta t^2$ in [éq 3.1 - 9]. It is for example the case in dynamics of freefall (cf the remark of equation 3.1-10).

3.2.2 First step of time

If one is with the first step of time, a study or a recovery (continuation), the predictor is calculated differently to take account of the initial state ($\mathbf{U}_0$, $\dot{\mathbf{U}}_0$, $\boldsymbol{\sigma}_0$):

$$\begin{cases} \left(\hat{\mathbf{K}}+\mathbf{K}_0+\mathbf{K}_0^{\mathbf{M1}}\right).\mathbf{U}_1^0+{}^tB.\boldsymbol{\lambda}_1^0+{}^t\mathbf{A}.\boldsymbol{\mu}_1^0=\hat{\mathbf{L}}(t_1)+\Delta\mathbf{L}^{anél}(t_1)-{}^t\mathbf{Q}.\boldsymbol{\sigma}_0-\mathbf{C}\dot{\mathbf{U}}_0 \\ \mathbf{B}.\mathbf{U}_1^0=\mathbf{U}_d(t_1) \\ \mathbf{A}\left(\mathbf{U}_1^0\right)\le\mathbf{d}_0(t_1) \\ \forall j,\ \mu_1^{0,j}\ge 0 \\ \forall j,\ \left(\mathbf{A}.\mathbf{U}_1^0-\mathbf{d}_{0_1}\right)_j.\mu_1^{0,j}=0 \end{cases} \quad \textbf{éq 3.2.2-1}$$

with:

$$\hat{\mathbf{L}}(t_1)=\mathbf{L}(t_1)+\frac{1}{\beta\Delta t^2}\left(\mathbf{M}+\mathbf{M}^{fs}\right)\left(\mathbf{U}_0+\Delta t\,\dot{\mathbf{U}}_0+\Delta t^2\left(\frac{1-2\beta}{2}\right)\ddot{\mathbf{U}}_0^0\right) +\frac{1}{\beta\Delta t}C\left(\gamma\mathbf{U}_0+\Delta t(\gamma-\beta)\dot{\mathbf{U}}_0+\Delta t^2\left(\frac{\gamma-2\beta}{2}\right)\ddot{\mathbf{U}}_0^0\right) \quad \textbf{éq 3.2.2-2}$$

and acceleration $\ddot{\mathbf{U}}_0^0$ evaluated by the preliminary resolution of the system (one simplifies by supposing the blocked connections of contact, the iterations of NEWTON undertaking to correct):

$$\begin{cases} \mathbf{M}.\ddot{\mathbf{U}}_0^0+{}^t\mathbf{B}.\boldsymbol{\lambda}_0+{}^t\mathbf{A}.\boldsymbol{\mu}_0=\mathbf{L}(t_0)+\mathbf{L}^{anél}(t_0)-{}^t\mathbf{Q}.\boldsymbol{\sigma}_0-\mathbf{C}\dot{\mathbf{U}}_0 \\ \mathbf{B}.\mathbf{U}_0^0=0 \\ \mathbf{A}.\mathbf{U}_0^0=0 \end{cases} \quad \textbf{éq 3.2.2-3}$$

Note:

It would be more exact to calculate: $\mathbf{M}\ddot{\mathbf{U}}_0^0+{}^tB.\boldsymbol{\lambda}_0+{}^t\mathbf{A}.\boldsymbol{\mu}_0=\mathbf{L}(t_0)+\mathbf{L}^{anél}(t_0)-{}^t\mathbf{Q}.\boldsymbol{\sigma}_0-\mathbf{C}\dot{\mathbf{U}}_0$, but to simplify knowing that will have only little influence on the continuation of the solutions, one neglects $\mathbf{C}\dot{\mathbf{U}}_0$ in the second member of [éq 3.2.2-3].

One must note that:

1. the matrix $\mathbf{M}$ must be invertible: one will have to affect a density on all the finite elements of the model,
2. it can be necessary to establish by a static calculation (possibly nonlinear) the state of balance under the initial loading, therefore the constraints σ_0, before the imposition of the initial dynamic conditions $\mathbf{U}_0, \dot{\mathbf{U}}_0$. Indeed, if not, acceleration $\ddot{\mathbf{U}}_0^0$ could be "excessive" and lead on a nondesired branch of balance,
3. in the presence of "loads kinematics", *cf.* [U4.44.03], these last are put at zero with this stage of calculation of $\ddot{\mathbf{U}}_0^0$.

However, certain finite elements do not have mass on all the degrees of freedom, for example the beams with warping POU_D_TG precisely on the degrees of freedom of warping, or in fluid coupling/structure. For the vibroacoustic elements of coupling [§ 2.4] indeed, the matrix of mass of the problem [éq 2.4-1] is not invertible. In these cases, the matrix $\mathbf{M}$ is not invertible, and one is satisfied to take a worthless initial acceleration on the whole of the model and the continuation of the iterations will have to be given the responsibility to correct this "good" prediction less. One will then choose a small step of time to ensure convergence at least the beginning of the transient.
The term $\mathbf{L}^{abso}(t_0)$ is evaluated starting from initial speed $\dot{\mathbf{U}}_0$ (*cf.* [R4.02.05]).

3.3 Phase of correction by the method of NEWTON

The values are sought $(\Delta \mathbf{U}_i, \Delta \boldsymbol{\lambda}_i, \Delta \boldsymbol{\mu}_i)$ increments of displacements and parameters of LAGRANGE since the values $(\mathbf{U}_{i-1}, \boldsymbol{\lambda}_{i-1}, \boldsymbol{\mu}_{i-1})$ obtained with preceding balance (urgent t_{i-1}). One takes as initial values $(\Delta \mathbf{U}_i^0, \Delta \boldsymbol{\lambda}_i^0, \Delta \boldsymbol{\mu}_i^0)$ obtained at the conclusion of the phase of prediction, before beginning the iterations of the method of NEWTON.

With each iteration n of NEWTON, one notes by δ evolutions leading to the estimate of the increments Δ (step of time i with the step of time $i+1$) with convergence of the iterations: $\Delta \mathbf{U}_i^{n+1} = \mathbf{U}_i^n + \delta \mathbf{U}_i^{n+1} - \mathbf{U}_{i-1}$ and $\Delta \boldsymbol{\lambda}_i^{n+1} = \boldsymbol{\lambda}_i^n + \delta \boldsymbol{\lambda}_i^{n+1} - \boldsymbol{\lambda}_{i-1}$ (in the same way for μ).
One must then solve a system allowing to determine $(\delta \mathbf{U}_i^{n+1}, \delta \boldsymbol{\lambda}_i^{n+1}, \delta \boldsymbol{\mu}_i^{n+1})$, increments of displacements and the parameters of LAGRANGE since the result $(\mathbf{U}_i^n, \boldsymbol{\lambda}_i^n, \boldsymbol{\mu}_i^n)$ preceding iteration:

Correction (iteration ° n)

$$\begin{cases} \left(\hat{\mathbf{K}} + \mathbf{K}_i^n + \mathbf{K}_i^{\mathbf{M}n}\right).\delta \mathbf{U}_i^{n+1} + {}^t\mathbf{B}.\delta \boldsymbol{\lambda}_i^{n+1} = \hat{\mathbf{L}}(t_i) - \mathbf{F}_i^n \\ \mathbf{B}.\delta \mathbf{U}_i^{n+1} = \mathbf{U}^d(t_i) - \mathbf{U}_i^n \\ \mathbf{A}.\mathbf{U}_i^{n+1} \leq \mathbf{d}_0(t_i) \\ \forall j,\ \left(\boldsymbol{\mu}_i^{n+1}\right)_j \geq 0 \\ \forall j,\ \left(\mathbf{A}.\mathbf{U}_i^{n+1} - d_{0_i}\right)_j.\left(\mu_i^{n+1}\right)_j = 0 \end{cases} \quad \textbf{éq 3.3-1}$$

with a second member called "residue", because it tends towards zero to convergence. One notes (*cf.* balance [éq 2.2.4-1]):

$$\mathbf{F}_i^n = {}^t\mathbf{Q}\left(\mathbf{U}_i^n\right).\boldsymbol{\sigma}_{\left(\mathbf{U}_i^n, \dot{\mathbf{U}}_i^n, \mathbf{t}_i\right)} + \mathbf{C}.\dot{\mathbf{U}}_i^n + \mathbf{M}\left(\mathbf{U}_i^n\right).\ddot{\mathbf{U}}_i^n - \mathbf{L}_{\boldsymbol{GR}}^{\boldsymbol{iner}}{}_{\left(\mathbf{U}_i^n, \dot{\mathbf{U}}_i^n, \ddot{\mathbf{U}}^n\right)} + {}^t\mathbf{B}.\boldsymbol{\lambda}_i^n + {}^t\mathbf{A}.\boldsymbol{\mu}_i^n \quad \textbf{éq 3.3-2}$$

The term $\hat{\mathbf{L}}(t_i)$ is defined as in [éq 3.1-10]; the matrix $\hat{\mathbf{K}}$ is given by [éq 3.1-9].

In the presence of modal damping, the forces of damping are deferred to the second member. According to the value given to the keyword AMOR_MODAL, REAC_VITE, one adds to $\hat{\mathbf{L}}(t_i)$ the reactualized term: $-\mathbf{C}.\dot{\mathbf{U}}_i^n$, or not reactualized $\mathbf{C}.\dot{\mathbf{U}}_{i-1}$.

The matrix $\mathbf{K}_i^n$ is the matrix of the tangent linear application of the part "forces internal" of the system of nonlinear equations [éq 3.1-8]; it is thus worth:

$$\mathbf{K}_i^n = \frac{\partial \mathbf{F}}{\partial \mathbf{U}}\Big|_{(\mathbf{U}_i^n, \dot{\mathbf{U}}_i^n, t_i)} = \frac{\partial \mathbf{R}}{\partial \mathbf{U}}\Big|_{(\mathbf{U}_i^n, \dot{\mathbf{U}}_i^n, t_i)} - \frac{\partial \mathbf{L}_i^{méca}}{\partial \mathbf{U}}\Big|_{(\mathbf{U}_i^n, t_i)} \quad \textbf{éq 3.3-3}$$

In the absence of following forces, the last term is null. The following forces can be: pressure exerted on the edges of solid elements, the loading of gravity for the elements of cable, the centrifugal force in great displacements, the loading of gravity for all modelings THM unsaturated porous environments [R7.01.10].

If one considers the reactualization of the geometry (in great displacements), one has more precisely:

$$\mathbf{K}_i^n = {}^t\mathbf{Q}(\mathbf{U}).\frac{\partial \boldsymbol{\sigma}}{\partial \mathbf{U}}\Big|_{(\mathbf{U}_i^n, \dot{\mathbf{U}}_i^n, t_i)} + \frac{\partial^t \mathbf{Q}(\mathbf{U})}{\partial \mathbf{U}}\Big|_{(\mathbf{U}_i^n, \dot{\mathbf{U}}_i^n, t_i)}.\sigma - \frac{\partial \mathbf{L}_i^{méca}}{\partial \mathbf{U}}\Big|_{(\mathbf{U}_i^n, t_i)} \qquad \textbf{éq 3.3-4}$$

The first term is the contribution of the behavior as in small transformations, with the difference which this contribution is evaluated here in current configuration. The second term is the contribution of the geometry which is not present in small transformations. Within the framework of the reactualization PETIT_REAC, this term is not present in the calculation of the tangent matrix. He is taken into account for the other options of nongeometrical linearity (GREEN and SIMO_MIEHE)

The matrix $\mathbf{K}_i^{Mn}$ is the matrix of the tangent linear application of the part “inertial forces” of the system of nonlinear equations [éq 3.1-8] which is thus worth:

$$\mathbf{K}_i^{Mn} = \frac{\partial \mathbf{L}_{GR}^{iner}}{\partial \mathbf{U}}\Big|_{(\mathbf{U}_i^n, \dot{\mathbf{U}}_i^n, \ddot{\mathbf{U}}_i^n)} \qquad \textbf{éq 3.3-5}$$

In practice one can use “the true” tangent matrix $\mathbf{K}_i^n$, but that presents a cost unquestionable calculation (calculation and factorization), or to be satisfied with a reactualization from time to time: to see in [U4.51.03] the keyword factor NEWTON, keyword MATRIX.

Note: singular tangent matrix:

One checks on [éq 3.3-1] whom if the tangent matrix $\mathbf{K}_i^n$ is singular (case of a damaged material, or ductile plate…), dynamics is controlled by the inertial forces, and that, the matrix $\hat{\mathbf{K}}$ being regular, one finds despite everything good a corrector $\delta \mathbf{U}_i^{n+1}$ (as if one were for example in situation of freefall). However, if the step of time is “too high”, the matrix $\left(\hat{\mathbf{K}} + \mathbf{K}_i^n + \mathbf{K}_i^{Mn}\right)$ can be badly conditioned and the solvor finds a pivot to him quasi-no one.

After each iteration of NEWTON having established a candidate solution of [éq 3.3-1] without checking the criterion of contact, one launches the algorithm of active constraints to satisfy the conditions with contact: one corrects thus $\delta \mathbf{U}_i^{n+1}$. For the other algorithms of contact-friction of *Code_Aster*, such as that used by the discrete elements “with shock” (DIS_CHOC), *cf.* [bib17], the iteration of NEWTON [éq 3.3-1] milked all non-linearities at the same time.

3.4 Update

In the case of small rotations, for usual modelings (solid elements, discrete beams, plates, hulls, elements…), the update at the end of the iteration $n+1$ be based on the formulas [éq 3.1-6] and [éq 3.1-7].:

$$\begin{cases} \mathbf{U}_i^{n+1} = \mathbf{U}_i^n + \delta \mathbf{U}_i^{n+1} \\ \Delta \mathbf{U}_i^{n+1} = \Delta \mathbf{U}_i^n + \delta \mathbf{U}_i^{n+1} \\ \dot{\mathbf{U}}_i^{n+1} = \dot{\mathbf{U}}_i^n + \frac{\gamma}{\beta \Delta t} \delta \mathbf{U}_i^{n+1} \\ \ddot{\mathbf{U}}_i^{n+1} = \ddot{\mathbf{U}}_i^n + \frac{1}{\beta \Delta t^2} \delta \mathbf{U}_i^{n+1} \end{cases} \quad \textbf{éq 3.4-1}$$

In the case of great rotations of the elements of structure (beams…) the update, definitely more complex, is indicated in [R5.03.40].

3.5 Criterion of stop

The convergence criteria total of the algorithm of NEWTON are identical to that practised in STAT_NON_LINE, *cf.* [bib19]. It represents the checking of dynamic balance.

At the moment t_i , one stops the iterations with the row n , as soon as the following inequality is satisfied:

$$\frac{\|\mathbf{F}_i^n - \hat{\mathbf{L}}(t_i)\|_\infty}{\|\hat{\mathbf{L}}(t_i) + \mathbf{L}^{anél}(t_i) - {}^t\mathbf{B}.\boldsymbol{\lambda}_i^n - {}^t\mathbf{A}.\boldsymbol{\mu}_i^n\|_\infty} = \frac{\|{}^t\mathbf{Q}(\mathbf{U}_i^n).\boldsymbol{\sigma}_\mathbf{i}^\mathbf{n} + \mathbf{C}.\dot{\mathbf{U}}_i^n + \mathbf{M}(\mathbf{U}_i^n).\ddot{\mathbf{U}}_i^n + {}^t\mathbf{B}.\boldsymbol{\lambda}_i^n + {}^t\mathbf{A}.\boldsymbol{\mu}_i^n - \mathbf{L}_{GR}^{iner}(\mathbf{U}_i^n, \dot{\mathbf{U}}_i^n, \ddot{\mathbf{U}}_i^n) - \hat{\mathbf{L}}(t_i)\|_\infty}{\|\hat{\mathbf{L}}(t_i) + \mathbf{L}^{anél}(t_i) - {}^t\mathbf{B}.\boldsymbol{\lambda}_i^n - {}^t\mathbf{A}.\boldsymbol{\mu}_i^n\|_\infty} \leq \eta \quad \textbf{éq 3.5-1}$$

η is a tolerance, introduced in data by the user (keyword CONVERGENCE, operand RESI_GLOB_RELA), being worth by default 10^{-6}, and $\|.\|_\infty$ is the standard of the maximum on the degrees of freedom.

The denominator of [éq 3.5-1] is a standard of the loading at the moment t_i , to which one reports the numerator, which is a standard of the forces not (still) balanced.

Notice :

It comprises a contribution of the unelastic, useful loads to standardize the error on balance in absence of external load. It must contain the internal forces due to the variables of order. One adds to it moreover inertial forces, so that in the rare cases where no other force acts, the criterion remains calculable.

Just like in statics, one must attach significance to a correct convergence because if not, the estimates of interior forces and reactions of contact, checking the relations of behavior and connection of contact-friction move away from balance, and it is not an inertial force $\mathbf{M}(\mathbf{U}_i^n).\ddot{\mathbf{U}}_i^n$ too much far away from the “exact” value which will be enough to produce with the step of time following a transitory dynamic good answer.

One will be able to also employ other convergence criteria as proposed in STAT_NON_LINE [U4.51.03].

4 Qualities and defects of the diagram of NEWMARK

4.1 Properties of the diagram of NEWMARK

This paragraph and the following takes again partially certain parts of [bib2]. One will be able to also consult [bib22], [bib25] and [bib30].

4.1.1 General information

One defines using the methods of analysis digital several types of properties for a diagram of temporal integration. Here their significance:

1. Convergence: the solution tends towards a limit when the step of time Δt tends towards 0;
2. Precision: rate of convergence when the step of time Δt tends towards 0;
3. Consistency and order of the diagram: the residue is limited by $(\Delta t)^{k+1}$ (example $k=2$: for the rule of the trapezoid): a polynomial of order $k+1$ is integrated exactly;
4. Stability: a finished disturbance of the initial state does not involve a disturbance exaggeratedly amplified ("digital explosion") in a later state; it is necessary that amplification (spectral ray) is lower than 1: for an Schéma of general form $\mathbf{U}_i=\mathbf{A}\,\mathbf{U}_{i-1}+\mathbf{L}_i$, one must have $\rho(\mathbf{A})\leq 1$. It is a requirement not to diverge! If amplification $\rho(\mathbf{A})<1$ there is digital attenuation.

The error in time e induced by the diagram of NEWMARK [éq 3.1-1], [éq 3.1-2], which approaches a development of Taylor of the solution, is given by, *cf.* [bib26]:

$$e\approx\frac{\Delta t^3}{6}\|\Delta\ddot{\mathbf{U}}_i\|\approx\frac{\Delta t^2}{6}\|\Delta\ddot{\mathbf{U}}_i\| \qquad \text{éq 4.1-1}$$

that one can standardize by the vector position $\|\mathbf{X}\|$ or the amplitude $\|\mathbf{U}\|$. One can also replace in [éq 4.1-1] the standard L_2 ($\|\Delta\ddot{\mathbf{U}}_i\|$) by the standard L_∞ : $|\Delta\ddot{\mathbf{U}}_i|_\infty=\underset{noeuds\,k}{max}\left(|\Delta\ddot{\mathbf{U}}_i^k|\right)$. *Code_Aster* does not provide, currently, the information of this estimator.

The digital analysis of the diagram of NEWMARK is done on the treatment of the equation of the oscillator **linear** with a degree of freedom, without dissipative phenomenon:

$$m\ddot{x}+kx=0 \qquad \textbf{éq 4.1-2}$$

The temporal diagrams do not give **pas la exact solution** of [éq 4.1-2]:

$$x=C_1\cos\,\omega t+C_2\sin\,\omega t$$

with $\omega=\sqrt{k/m}$, C_1 and C_2 depending on the initial conditions, but one **solution approached** :

$$x_i=\left[\exp\left(-\xi\,\tilde{\omega}\,t_i\right)\right]\left(C_1'\cos\,\tilde{\omega}\,t_i+C_2'\sin\,\tilde{\omega}\,t_i\right)$$

Two errors are thus brought:

1) on the one hand, it is introduced a definite artificial damping by the reduced rate of depreciation ξ , which is the decrement logarithmic curve divided by 2π ;
2) in addition, the pulsation ω corresponding to the exact period T is replaced by the pulsation $\tilde{\omega}$ correspondent at one period $\tilde{T}$.

These errors depend on the report $\frac{\Delta t}{T}$ and of the diagram itself.

By writing the balance of the system with 1 degree of freedom [éq 4.1-2], effortlessly imposed, and the equations of the diagram of NEWMARK [éq 3.1-1], [éq 3.1-2] one obtains:

$$\begin{pmatrix} x_{i+1} \\ \Delta t\,\dot{x}_{i+1} \\ \Delta t^2\,\ddot{x}_{i+1} \end{pmatrix} = \frac{1}{1+\beta\,\omega^2\Delta t^2} \begin{pmatrix} 1 & 1 & \frac{1}{2}-\beta \\ -\gamma\,\omega^2\Delta t^2 & 1+\omega^2\Delta t^2(\beta-\gamma) & 1-\gamma-\frac{1}{2}\omega^2\Delta t^2(\gamma-2\beta) \\ -\omega^2\Delta t^2 & -\omega^2\Delta t^2 & -\omega^2\Delta t^2\left(\frac{1}{2}-\beta\right) \end{pmatrix} \begin{pmatrix} x_i \\ \Delta t\,\dot{x}_i \\ \Delta t^2\,\ddot{x}_i \end{pmatrix} \quad \textbf{éq 4.1-3}$$

from where in term of increases:

$$\begin{pmatrix} \Delta x \\ \Delta t\,\Delta\dot{x} \\ \Delta t^2\,\Delta\ddot{x} \end{pmatrix} = \frac{1}{1+\beta\,\omega^2\Delta t^2} \begin{pmatrix} -\beta\,\omega^2\Delta t^2 & 1 & \frac{1}{2}-\beta \\ -\gamma\,\omega\,\Delta t & -\gamma\,\omega^2\Delta t^2 & 1-\gamma-\frac{1}{2}\omega^2\Delta t^2(\gamma-2\beta) \\ -\omega^2\Delta t^2 & -\omega^2\Delta t^2 & -\frac{1}{2}\omega^2\Delta t^2-1 \end{pmatrix} \begin{pmatrix} x_i \\ \Delta t\,\dot{x}_i \\ \Delta t^2\,\ddot{x}_i \end{pmatrix} \quad \textbf{éq 4.1-4}$$

Properties of this matrix of amplification – which must preserve the harmonic character of the answer, more precisely those of its characteristic polynomial, - are used to characterize the properties of the diagram, in linear mode, [bib25].

The properties of the diagram of NEWMARK are summarized hereafter.

Because of the selected manner to express the diagram (*cf.* [éq 3.1-1] and [éq 3.1-2]), one cannot take $\beta=0$.

Pour problems linear, the diagram is **unconditionally stable**, even in the presence of physical damping – - a disturbance is not amplified by the diagram – i.e. stable whatever the size of the step of time, if the parameters satisfy the inequalities:

$$\begin{cases} 2\gamma \geq 1 \\ 4\beta \geq \left(\gamma+\frac{1}{2}\right)^2 \end{cases} \quad \textbf{éq 4.1-5}$$

If $2\gamma > 1$ and $\beta < \frac{1}{4}\left(\gamma+\frac{1}{2}\right)^2$ the diagram is **conditionally stable** : the step of time must be selected lower than: $\Delta t \leq \frac{T_{\min}}{\pi}\left(\left(\gamma+\frac{1}{2}\right)^2-4\beta\right)^{-1/2}$ - this relation being valid in physical absence of damping $\xi=0$ but also in the presence of physical damping – - according to the smallest period T_{min} of vibration of the studied system, to see [bib22], [bib25].

In the presence of physical damping $\xi>0$, one can allow oneself Δt slightly higher.

In the presence of contact with impact, the stability of the diagram is assured if $2\beta \geq \gamma \geq ½$, [bib31].

The following table shows some typical cases:

β	γ	method	type	properties
$\frac{1}{12}$	$\frac{1}{2}$	"Fox terrier-Goodwin"	implicit	Diagram of order 4 in time; no digital dissipation in absence of damping material: thus not of attenuation of amplitude due to the diagram. Conditionally stable: $\Delta t \leq \sqrt{3/2}/\left(\pi f_{\max}\right)^{-1}$, $f_{\max}$ being the maximum vibratory frequency "aimed" in simulation, not of digital dissipation. Error in time: $1+\left(\omega.\Delta t\right)^4/480+\ldots$
$\frac{1}{6}$	$\frac{1}{2}$	"linear" acceler ation	implicit identical to θ-method WILSON, with $\theta=1$	Diagram of order 2 in time, not of digital dissipation in absence of damping material: thus not of attenuation of amplitude due to the diagram. Conditionally stable: $\Delta t \leq \sqrt{3}/\left(\pi f_{\max}\right)^{-1}$, $f_{\max}$ being the maximum vibratory frequency "aimed" in simulation, not of digital dissipation Error in time: $1-\left(\omega.\Delta t\right)^2/24+\ldots$
$\frac{1}{4}$	$\frac{1}{2}$	"rule of the trapezoid" or average acceleration	implicit	Diagram of order 2 in time, not of digital dissipation in absence of damping material: thus not of attenuation of amplitude due to the diagram. Unconditional stability in Δt. Error in time: $1-\left(\omega.\Delta t\right)^2/12+\ldots$: the frequencies are shifted to the bottom, but a matrix of consistent mass limits this defect (since the adverse effect causes)
β	γ	α - method "modified average acceleration"	Implicit to see [§ 5]	Diagram of order 2 in time Error in time: $1-\left(\omega.\Delta t\right)^2\left(3\alpha^2+1\right)/12+\ldots$ Digital dissipation: $\left\|\alpha\right\|\omega.\Delta t+\ldots$ to see [§ 5].

The diagram of NEWMARK is of the second order in time (in the worst case) if and only if $\gamma=1/2$. As soon as $\gamma>1/2$, the diagram of NEWMARK is of a nature 1, and introduced a digital dissipation proportional to $\left(\gamma-\frac{1}{2}\right)\Delta t$. In order to have a digital damping growing with the frequency, it is advisable to choose: $\beta\geq\frac{\left(\gamma+1/2\right)^2}{4}$, equality being the best alternative, *cf.* diagram HHT, to see it [§ 5].

If one chose $\gamma<1/2$, the diagram of NEWMARK would bring a negative digital damping which would bring an instability.

By using the increases [éq 4.1-4], one deduces the error in time [éq 4.1-1] from the diagram, realised over one period $2\pi/\omega$, and standardized compared to the amplitude, for the oscillator with 1 d.d.l., *cf.* [bib26]:

$$e_{moy}^{norm}\approx\frac{\omega^3\Delta t^3}{2\pi\left(1+\beta\omega^2\Delta t^2\right)}\sqrt{4+\omega^2\Delta t^2} \qquad \textbf{éq 4.1-6}$$

This expression can be used as error of reference to the diagram applied to the dynamic response of an unspecified structure.

Notice :

The properties of the diagram of Newmark obtained on the case of the oscillator with 1 degree of freedom are generalizable with the case of the linear dynamic system to several degrees of freedom (by using a projection on modal basis) provided that the operator of damping is proportional to the operators of reduced mass and stiffness.

Notice complementary :

> *The diagram of Newmark is known to introduce a notable error as soon as a noninfinitesimal rotation intervenes in the dynamic response of the solid. In addition, the diagram of Newmark can also generate an error if one intervenes too much "brutally" on variations of parameters (not of time, β or γ) in the succession of the temporal discretization over the duration of the transient.*

4.1.2 Diagram of average acceleration or rule of the trapezoid

The "rule of the trapezoid" ($\beta=1/4$, $\gamma=1/2$), **unconditionally stable**, is most commonly adopted, associated with a consistent mass (option by default `MASS_DIAG = 'NOT'`). It is shown, for example in [bib30], that the distortion in relative frequency brought by the diagram of integration, for a system deadened (reduced damping is noted $\xi=\frac{c}{2\sqrt{km}}=\frac{c}{2\mathrm{m}\,\omega}$) to 1 degree of freedom in free vibration, deadened own pulsation $\omega_d=\omega\sqrt{1-\xi^2}$:

$$m\ddot{x}+c\dot{x}+kx=0 \qquad \textbf{éq 4.1-7}$$

is:

$$\frac{\omega_d-\omega_d^{Newmark}}{\omega_d}=\left(\frac{1}{12}-\frac{\xi^2}{3}\right)\omega^2\Delta t^2+o\left(\omega^4\Delta t^4\right) \qquad \textbf{éq 4.1-8}$$

where $\omega_d^{\mathrm{Newmark}}$ indicate the approximate value of ω_d by the diagram (via a spectral analysis of the answer).
In forced vibrations, dependence of [éq 4.1-8] in $\omega^2\Delta t^2$ remain valid. In a general way, the frequencies are shifted downwards, and this shift grows if the step of time increases and decrease for an increasing damping.

If one wishes to use a diagram where the frequencies are shifted upwards, it is advisable to choose the option `MASS_DIAG = 'YES'`.

One also shows in [bib30], that the relative distortion in damping brought by the diagram of integration of the "rule of the trapezoid" is not worthless for a system comprising a damping material. For small steps of time:

$$\frac{\xi-\xi^{Newmark}}{\xi}=\frac{\omega^2\Delta t^2}{6}+o\left(\omega^4\Delta t^4\right) \qquad \textbf{éq 4.1-9}$$

where $\xi^{Newmark}$ indicate the value of ξ resulting from the diagram.

Total damping decreases, and decrease when the step of time increases. In forced vibrations, the dependence of distortion in damping [éq 4.1-9] becomes in $\omega\Delta t$, *cf.* [bib30].

For steps of time more important close relations of *digital pulsation of cut* had with the diagram, such as $\hat{\omega}_d^{Newmark}\Delta t=\pi$, *cf.* [bib30], one obtains the estimate (free cases of vibrations):

$$\xi^{Newmark}=\xi\frac{4\sqrt{1-\xi^2}}{\pi\,\omega\Delta t}+o\left(\omega^{-2}\Delta t^{-2}\right) \qquad \text{éq 4.1-10}$$

One can thus summarize by the fact that the precision is good if one chooses $\Delta t=\frac{1}{10\,\omega}$ (*cf.* [éq 4.1-8]), for the pulsation ω system highest that one seeks to describe, *cf.* [§ 4.4].

4.2 Energy point of view

In the case of a linear elastic conservative system in small transformations without physical damping, one can easily evaluate the variations of various energies during the step of time Δt enter t_i and t_{i+1} . Between these two moments, the increment and the average of the fields are noted:

$$\Delta\mathbf{\Phi}=\mathbf{\Phi}(t_{i+1})-\mathbf{\Phi}(t_i) \qquad \bar{\mathbf{\Phi}}=\frac{1}{2}\left(\mathbf{\Phi}(t_{i+1})+\mathbf{\Phi}(t_i)\right)$$

Thus one has respectively for the kinetic energy, the deformation energy; $\mathbf{M}$ and $\mathbf{K}$ being symmetrical, and potential of the external efforts independent of time it system being conservative:

$$\Delta E_{cin}=\frac{1}{2}\Delta\left({}^t\dot{\mathbf{\Phi}}.\mathbf{M}.\dot{\mathbf{\Phi}}\right)=\Delta{}^t\dot{\mathbf{\Phi}}\ .\mathbf{M}.\bar{\dot{\mathbf{\Phi}}}$$
$$\Delta E_{déf}=\frac{1}{2}\Delta\left({}^t\mathbf{\Phi}.\mathbf{K}.\mathbf{\Phi}\right)=\Delta{}^t\mathbf{\Phi}\ .\mathbf{K}.\bar{\mathbf{\Phi}}$$
$$\Delta V_{ext}=-\Delta\left({}^t\mathbf{\Phi}.\mathbf{F}\right)=-\Delta{}^t\mathbf{\Phi}\ .\bar{\mathbf{F}}$$

By using balance at the moments t_i and t_{i+1} . under the action of the external efforts, one obtains:

$$\mathbf{M}.\Delta\ddot{\mathbf{\Phi}}+\mathbf{K}.\Delta\mathbf{\Phi}=0 \quad ; \quad \mathbf{M}.\bar{\ddot{\mathbf{\Phi}}}+\mathbf{K}.\bar{\mathbf{\Phi}}=\bar{\mathbf{F}} \qquad \text{éq 4.2-1}$$

Variations of various energies, of which the total energy of the system – or Hamiltonian: $\Delta E_{tot}=\Delta E_{cin}+\Delta E_{déf}+\Delta V_{ext}$, express themselves, while exploiting [éq 4.2-1] to eliminate the loading $\mathbf{F}$ and terms of the diagram [éq 3.1-4] and [éq 3.1-5]:

$$\Delta E_{cin}=\Delta t.\left({}^t\bar{\ddot{\mathbf{\Phi}}}-\frac{1-2\gamma}{2}\Delta{}^t\ddot{\mathbf{\Phi}}\right).\mathbf{M}.\bar{\dot{\mathbf{\Phi}}}$$
$$\Delta E_{déf}=\Delta t.\left(\bar{\dot{\mathbf{\Phi}}}+\frac{\Delta t}{2}(2\beta-\gamma).\Delta\ddot{\mathbf{\Phi}}\right).\mathbf{K}.\bar{\mathbf{\Phi}} \qquad \text{éq 4.2-2}$$
$$\Delta V_{ext}=-\Delta t.\left(\bar{\dot{\mathbf{\Phi}}}+\frac{\Delta t}{2}(2\beta-\gamma).\Delta\ddot{\mathbf{\Phi}}\right).\bar{\mathbf{F}}$$
$$\Delta E_{tot}={}^t\bar{\dot{\mathbf{\Phi}}}.\mathbf{M}.\Delta\dot{\mathbf{\Phi}}-{}^t\bar{\ddot{\mathbf{\Phi}}}.\mathbf{M}.\Delta\mathbf{\Phi}$$

And it is obtained that the variation of total energy of the system is expressed:

$$\Delta E_{tot}=-\frac{\Delta t}{2}.\left((1-2\gamma).\Delta{}^t\ddot{\mathbf{\Phi}}.\mathbf{M}.\bar{\dot{\mathbf{\Phi}}}+\Delta t.(2\beta-\gamma)\,\Delta{}^t\ddot{\mathbf{\Phi}}.\mathbf{M}.\bar{\ddot{\mathbf{\Phi}}}\right) \qquad \text{éq 4.2-3}$$

It is checked that when $\gamma=1/2$ and $\beta=1/4$ (rule of the trapezoid), there is not digital dissipation of energy brought by the diagram. It is noticed that a choice different from γ and/or $2\beta\neq\gamma$ bring a dissipation proportional to the step of time.

It is also checked that if $\gamma > 1/2$ and $2\beta - \gamma > 0$, digital damping is positive, *cf.* [bib25], which is in conformity also with what was stated with [§ 4.1.1].

4.3 Properties of the diagram of NEWMARK for the nonlinear problems

Pour problems not linear (great deformations, non-linearities material), the diagram is **unconditionally stable**, if $\beta \geq \frac{1}{4}$ and with $\gamma = 1/2$, *cf.* proof in [bib23].

In **presence of unilateral contact**, one finds in the literature, *cf.* [bib24], the council to take $\beta = \gamma$ (for example with $\beta = \gamma = 1/2$) what ensures the compatibility speeds during the phase where the contact is maintained between two solids.

That is obtained directly starting from the equations [éq 3.1-1] and [éq 3.1-2]; the jump $[\,.\,]$ of speed enters the two points staying in touch on the step Δt ($[\Phi(t)] = 0$ and $[\Phi(t + \Delta t)] = 0$) at the moment $t + \Delta t$ is indeed:

$$[\dot{\Phi}(t + \Delta t)] = (1 - \gamma/\beta) . [\dot{\Phi}(t)] + (1 - \gamma/(2\beta)) . \Delta t . [\ddot{\Phi}(t)]$$

If one chooses $\beta = \gamma$, then $[\dot{\Phi}(t + \Delta t)] = o(\Delta t)$

. However this choice is not compatible with that of an optimum in digital term of dissipation, as defined by α - method (*cf.* [§ 5]): it would be necessary to take $\alpha = -1 - \sqrt{2}$ who gives an enormous digital damping! One thus does not incite the user to follow this recommendation.

If the unilateral contact is solved by a method of penalization, one does not escape the risk from instability even by introducing a digital damping.

4.4 Choice of the steps of time

The step of time to choose must observe a certain number of conditions.

The first, obvious, is that it must be **adapted to temporal sampling loadings** applied to the studied system. Incidentally, it can be convenient to reconsider a modeling" with a temporal dependence too much "violent of the loadings applied, by softening changes of incline for example.

One advises, for **reasons of precision** (criterion of the type "Shannon" on the cut-off frequency), a step of time to choose such as:

$$\Delta t \leq 0,10\, T_{\min} \qquad \textbf{éq 4.4-1}$$

where $T_{\min}$ indicate the smallest period of vibration of the system which one wishes to study.

The step of selected space of the meshs finite elements also intervenes: the step of time Δt maximum to choose is about h/c , where c is the celerity of the elastic compression waves of material and h a size characteristic of the meshs, if one seeks to describe partially of the phenomena in high frequency, for which however this digital formulation of elastodynamic is not truly adapted (there exist other digital methods with this intention).

Finally in the case of solid presenting rigid modes (freefall for example), in accordance with the remark passed with [§ 3.1], it is advisable to choose a step of time Δt sufficient small so that the terms of mass are of the same order as those of stiffness (in the matrix $\hat{\mathbf{K}}$, *cf.* [éq 3.1-9]). Thus, one will be able to choose:

$$\Delta t \approx \sqrt{\frac{L}{50\,g}} \qquad \textbf{éq 4.4-2}$$

where L is the diameter of the solid considered in "freefall" under the acceleration of gravity g. Thus the increment of displacement $g\Delta t^2/2$ at the time of this step of time is weak in front of dimensions of the solid, or, in a another way, similar to an elastic displacement under of a the same sphere of activity width.

It is possible to make follow one another several dynamic analyses, on successive time intervals, communicating by recoveries by choosing like initial state the result of the last step studied before, by using steps of very distinct times according to the idea a priori which one has of the answer of the studied system. One does not have a result general with this kind of choice in term of convergence…

One also knows that the treatment of the collisions is sensitive to the clocking of the steps of time, compared to the "real" moments of shock: one will have to study the sensitivity of the answer obtained. However, a step of "too small" time can exacerbate the oscillations induced by discontinuity.
One will be able to also direct oneself towards the choice of an explicit diagram, of less low nature (order 1), *cf.* for example [bib31] even if such a choice led to take steps of very weak times.

It is however advised to maintain constant the step of time during a phase of linear dynamic response stationary, to keep the properties stated previously.

Remarks :

One must note that Code_Aster does not propose today method multi-field in time and space, which would make it possible to define a step of time per zone in the studied solid, nor of criterion of error in dynamics.
In the operator `DYNA_NON_LINE`*, one can choose to solve balance in displacement or of speed for the implicit schemes (for the diagrams clarifies it is in acceleration). The following keyword allows this choice:* `FORMULATION = 'DISPLACEMENT'` or `'SPEED'`.

5 An alternative of the diagram of NEWMARK: diagrams HHT and of modified average acceleration

5.1 Motivation

In mechanical analysis, one wishes that the low frequencies be reproduced most accurately possible.

It is wished on the other hand that the high frequencies be attenuated by calculation because they can generate digital instabilities and that the associated mechanical constraints are in general low.

Curves giving the damping ratio ξ according to $\frac{\Delta t}{T}$ (period $T=2\pi/\omega$) thus must:

- to leave the origin with a horizontal tangent to give a very weak damping to the low frequencies,
- to be increasing functions to attenuate the high frequencies and this more especially as they are higher.

To try to achieve these goals, Hilber, Hughes and Taylor (HHT) proposed in [bib4] defining the parameters β and γ NEWMARK according to a third parameter α negative by the following relations, which are copied on the stability conditions [éq 3.1-2]:

$$\gamma=\frac{1}{2}-\alpha \quad ; \quad \beta=\frac{(1-\alpha)^2}{4} \qquad \textbf{éq 5.1-1}$$

This choice offers the best compromise on the precision and damping in high frequencies.

This parameter α, negative, is provided via the operand ALPHA keyword SCHEMA_TEMPS (SCHEMA='HHT') of DYNA_NON_LINE.

5.2 Diagram HHT and method of modified average acceleration

One obtains thus [bib4], [bib25]:

$$\ddot{\Phi}(t+\Delta t)=\frac{4}{(1-\alpha)^2\,\Delta t^2}\left[\Phi(t+\Delta t)-\Phi(t)\right]-\frac{4}{(1-\alpha)^2\,\Delta t}\dot{\Phi}(t)+\frac{(1-\alpha)^2-2}{(1-\alpha)^2}\ddot{\Phi}(t) \qquad \text{éq 5.2-1}$$

$$\dot{\Phi}(t+\Delta t)=\frac{2-4\alpha}{(1-\alpha)^2\,\Delta t}\left[\Phi(t+\Delta t)-\Phi(t)\right]+\frac{\left(\alpha^2+2\alpha-1\right)}{(1-\alpha)^2}\dot{\Phi}(t)+\frac{\alpha^2\,\Delta t}{(1-\alpha)^2}\ddot{\Phi}(t) \qquad \text{éq 5.2-2}$$

In addition, the dynamic balance [éq 2.2.4-1], discretized in time at the moment $t_i=t_{i-1}+\Delta t$ is modified by introducing a "shift", also controlled by the coefficient $\alpha\leq 0$, on the interior forces:

$$\begin{aligned}\mathbf{M}(\mathbf{U}_i)\,\ddot{\mathbf{U}}_i+(1+\alpha)\,\mathbf{C}\dot{\mathbf{U}}_i+(1+\alpha)\,\mathbf{R}(\mathbf{U}_i,\dot{\mathbf{U}}_i,t_i)+{}^t\mathbf{B}.\boldsymbol{\lambda}_\mathbf{i}+{}^t\mathbf{A}.\boldsymbol{\mu}_\mathbf{i}=\\ \mathbf{L}(t_{i+\alpha})+\alpha\,\mathbf{R}(\mathbf{U}_{i-1},\dot{\mathbf{U}}_{i-1},t_{i-1})+\alpha\,\mathbf{C}\dot{\mathbf{U}}_{i-1}-\mathbf{L}_{\boldsymbol{GR}}^{\boldsymbol{iner}}(\mathbf{U}_i,\dot{\mathbf{U}}_i,\ddot{\mathbf{U}}_i)\end{aligned} \qquad \text{éq 5.2-3}$$

where the external forces are evaluated at the moment $t_{i+\alpha}=(1+\alpha)\,t_i-\alpha\,t_{i-1}=t_i+\alpha\Delta t$:

$$\mathbf{L}(t_{i+\alpha})=(1+\alpha)\mathbf{L}(t_i)-\alpha\,\mathbf{L}(t_{i-1})$$

The system of nonlinear equations [éq 3.1-8], récrit thus:

$$\begin{cases}\hat{\mathbf{K}}\,\mathbf{U}_i+(1+\alpha)\,\mathbf{R}(\mathbf{U}_i,\dot{\mathbf{U}}_i,t_i)+{}^t\mathbf{B}.\boldsymbol{\lambda}_\mathbf{i}+{}^t\mathbf{A}.\boldsymbol{\mu}_\mathbf{i}=\hat{\mathbf{L}}(t_i)-\mathbf{L}_{\boldsymbol{GR}}^{\boldsymbol{iner}}(\mathbf{U}_i,\dot{\mathbf{U}}_i,\ddot{\mathbf{U}}_i)\\ \mathbf{B}.\mathbf{U}_i=\mathbf{U}^d(t_i)\\ \mathbf{A}.\mathbf{U}_i\leq\mathbf{d_0}(t_i)\\ \mu\geq 0\\ \forall\, j,\ (\mathbf{A}.\mathbf{U}_i-\mathbf{d}_{\mathbf{0}_i})_j.\mu_j=0\end{cases} \qquad \text{éq 5.2-4}$$

with, while following [éq 3.1-9] and [éq 3.1-10] and β and γ function of the parameter α, *cf.* [éq 4.1-1]:

$$\hat{\mathbf{K}}=(1+\alpha)\,\mathbf{K}^{\boldsymbol{fs}}+\frac{1}{\beta\,\Delta t^2}(\mathbf{M}+\mathbf{M}^{\boldsymbol{fs}})+\frac{\gamma}{\beta\,\Delta t}(1+\alpha)\,\mathbf{C} \qquad \text{éq 5.2-5}$$

$$\begin{aligned}\hat{\mathbf{L}}(t_i)=&\mathbf{L}(t_i)+\mathbf{L}_{\boldsymbol{abso}}(t_{i-1})+\alpha\,\mathbf{K}^{\boldsymbol{fs}}.\mathbf{U}_{i-1}\\ &+\frac{1}{\beta\,\Delta t^2}(\mathbf{M}+\mathbf{M}^{\boldsymbol{fs}})\left(\mathbf{U}_{i-1}+\Delta t\,\dot{\mathbf{U}}_{i-1}+\Delta t^2\left(\frac{1-2\beta}{2}\right)\ddot{\mathbf{U}}_{i-1}\right)+\alpha\,\mathbf{R}(\mathbf{U}_{i-1},\dot{\mathbf{U}}_{i-1},t_{i-1})\\ &+\frac{1}{\beta\,\Delta t}\mathbf{C}.\left(\gamma\,\mathbf{U}_{i-1}+\Delta t\,(\gamma-\beta)\dot{\mathbf{U}}_{i-1}+\Delta t^2\left(\frac{\gamma-2\beta}{2}\right)\ddot{\mathbf{U}}_{i-1}\right)+\alpha\,C\,\dot{\mathbf{U}}_{i-1}\end{aligned} \qquad \text{éq 5.2-6}$$

The modifications that this diagram brings to the phases of prediction and correction of the method of NEWTON [§ 3.2] and [§3.3] are the following ones:

1. phase of prediction [éq 3.2.1-4]: to put the new expression of $\hat{\mathbf{K}}$ [éq 5.2-5], like $(1+\alpha)\,\mathbf{K}_{i-1}$ instead of $\mathbf{K}_{i-1}$ in [éq 3.2.1-4] and $(\alpha-1)\,\mathbf{R}(\mathbf{U}_{i-1},\dot{\mathbf{U}}_{i-1},t_{i-1})$ instead of $-\mathbf{R}(\mathbf{U}_{i-1},\dot{\mathbf{U}}_{i-1},t_{i-1})$ in the expression of $\Delta\hat{\mathbf{L}}(t_i)$ in [éq 3.2.1-3];

2. phase of correction [éq 3.3-1]: to put $(1+\alpha)\,\mathbf{K}_i^n$ and $\alpha\,\mathbf{F}_i^n$ instead of $\mathbf{K}_i^n$ and $\mathbf{F}_i^n$.

Notice :

The total algorithmic organization is thus unchanged compared to the diagram with modified average acceleration: only the matrix of the system and the second member are modified, and it is advisable to store the vectors obtained with the preceding steps.

When $\alpha=-1$ (from where $\beta=1$, $\gamma=3/2$), diagram HHT becomes explicit then (*cf.* [éq 5.2-3]):

$$\mathbf{M}\,\ddot{\mathbf{U}}_i+{}^t\mathbf{B}\,.\,\boldsymbol{\lambda}_i+{}^t\mathbf{A}\,.\,\boldsymbol{\mu}_i=\mathbf{L}\left(t_{i-1}\right)-\mathbf{C}\,\dot{\mathbf{U}}_{i-1}-\mathbf{R}\left(\mathbf{U}_{i-1},\dot{\mathbf{U}}_{i-1},t\right)-\mathbf{L}_{GR}^{iner}\left(\mathbf{U}_i,\dot{\mathbf{U}}_i,\ddot{\mathbf{U}}_i\right) \qquad \textbf{éq 5.2-7}$$

with: $\ddot{\mathbf{U}}(t+\Delta t)=\dfrac{1}{\Delta t^2}\left[\mathbf{U}(t+\Delta t)-\mathbf{U}(t)\right]-\dfrac{1}{\Delta t}\dot{\mathbf{U}}(t)$

and $\dot{\mathbf{U}}(t+\Delta t)=\dfrac{3}{2\,\Delta t}\left[\mathbf{U}(t+\Delta t)-\mathbf{U}(t)\right]-\dfrac{1}{2}\dot{\mathbf{U}}(t)+\dfrac{\Delta t}{4}\ddot{\mathbf{U}}(t)$

ensuring at the same time greatest possible dissipation in high frequencies.

Notice :

In Code_Aster, one can use the two following alternatives with the keyword factor `SCHEMA_TEMPS (SCHEMA=' HHT')` *:*

- On the one hand, while choosing for the simple keyword `MODI_EQUI=' NON'`, one uses the classical equilibrium equation where nor terms of [éq 5.2-3] with [éq 5.2-6] "shifted" by $(1+\alpha)$ or α on damping and the stiffnesses, nor the evaluation with $t_{i+\alpha}$ second members are not treated, which makes lose an order on the diagram (from 2 to 1). It is thus actually about what one notes in the literature the "method of modified average acceleration", which is thus limited to define the optimal relation between the parameters of the diagram of Newmark according to [éq 4.1-1] with [éq 4.1-3];
- In addition, while choosing for the simple keyword `MODI_EQUI=' OUI'` (value by default of the code), that amounts adopting complete diagram HHT presented above with shift of the terms of interior and external efforts.

Remarks :

> *With complete diagram HHT (* `HHT` *with* `MODI_EQUI=' OUI'` *), one treats the terms coming from displacements and the following forces imposed like in the case of the diagram of average acceleration (* `HHT` *with* `MODI_EQUI=' NON'` *).*

5.3 Properties of the diagram of modified average acceleration

It is necessary:

$$\boxed{\alpha\leq 0} \qquad \textbf{éq 5.3-1}$$

so that the stability conditions of the diagram are met, *cf.* [§4.3]: the diagram is **unconditionally stable**.

The diagram "of modified average acceleration" (called too α - method), thus brings $\beta=\left(\gamma+\dfrac{1}{2}\right)^2/4$,

which is the best alternative to bring damping growing on the high frequencies.

When $\alpha=0$, the diagram "of modified average acceleration" becomes again the "rule of the trapezoid" and damping is null [Figure 5.3-a].

The value $\alpha=-1$, that is to say $\gamma=3/2$ and $\beta=1$ product strongest dissipation in high frequency, but destroyed much the precision on the low frequency modes.

In practice, in the diagram “of modified average acceleration” original, one limits to $\alpha\in\left[-1/3,\ 0\right]$, which ensures the monotony of the increase in damping according to the frequency. The choice $\alpha=-0,10$ seem to be effective, even if it is possible to take values of α higher in absolute value.

The figure [Figure 5.3-a], extracted from [bib5], gives the variations of digital damping ξ according to $\frac{\Delta t}{T}$ for some values of α , where T of the system with 1 d.d.l is the period. This figure calls for the following observations:

1. the “rule of the trapezoid” ($\alpha=0$) is tempting because it does not bring digital damping for a system not deadened physically, *cf.* [§ 4.1.2], but it can be unstable into nonlinear,
2. when $\alpha\neq0$, the curves are not with horizontal tangent in the beginning. Like the diagram “of modified average acceleration” (α - method) brings an important artificial damping, certain users [bib5] choose a step of time, then determine the value of the parameter α so that, in the beach of frequencies which interests them, the digital rate of depreciation is of the same order as the mechanical rate of depreciation real which, then, is not taken into account,
3. the error in time does not depend on α , *cf.* [bib30].

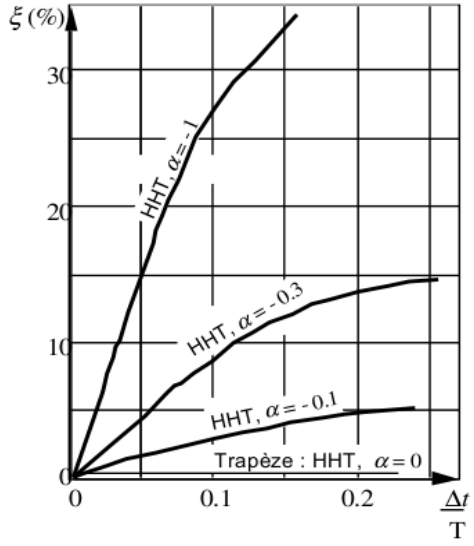

Figure 5.3-a: Digital rate of depreciation according to the step of time of the diagram of modified average acceleration

It is noted that one has for the small steps of time $\frac{\Delta t}{T}$ following digital damping:

$$\xi=|\alpha|\frac{\pi\Delta t}{T}+o\left(\left(\frac{\Delta t}{T}\right)^2\right) \qquad \textbf{éq 5.3-2}$$

what makes it possible to estimate in the frequency band concerned the average digital damping brought by the diagram of modified average acceleration (α - method), which comes to be added to physical damping possibly already present.

The figure [Figure 5.3-b], also extracted from [bib4], gives the variations of the relative error in period according to $\frac{\Delta t}{T}$. The error is an increasing function of $|\alpha|$.

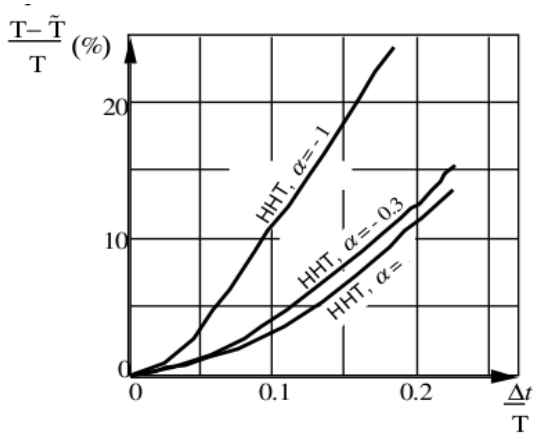

Figure 5.3-b: Relative error over the period according to the step of time of the diagram of modified average acceleration

5.4 Properties of diagram HHT

It is necessary:

$$\alpha \leq 0 \qquad \textbf{éq 5.4-1}$$

so that the stability conditions of the diagram are met, *cf.* [§4.3]: the diagram is **unconditionally stable**.

When $\alpha = 0$, diagram HHT becomes again the “rule of the trapezoid” and digital damping is null [Figure 5.3-a]. The value $\alpha = -1$ resulted in writing balance with the preceding step and is thus disadvised.

In practice, in diagram HHT, one limits oneself to $\alpha \in [-1/3,\ 0]$, which ensures the monotony of the increase in damping according to the frequency. The choice $\alpha = -0,10$ seem to be effective. The error in time grows less quickly than digital damping according to α [bib5].

The figure [Figure 5.4-a], extracted from [bib25], gives the variations of digital damping ξ according to $\frac{\Delta t}{T}$ for some values of α , by comparing them with the diagram of modified average acceleration.
It is noted that:

- method HHT deadens less the low frequencies than the diagram of modified average acceleration;
- moreover, for the same value of the parameter α , digital damping is definitely more important, in “average” and “high” frequency $(\omega \Delta t > 1)$ in the case of diagrams of type modified average acceleration, that for diagram HHT. For example, for $\omega \Delta t = 2$, and for $\alpha = -0,05$, there is a report 2 on the values of digital damping;
- one thus advises with diagram HHT to consider the taking into account of a physical damping [§2.2.1], that one could not have taken into account while having chosen the diagram of modified average acceleration.

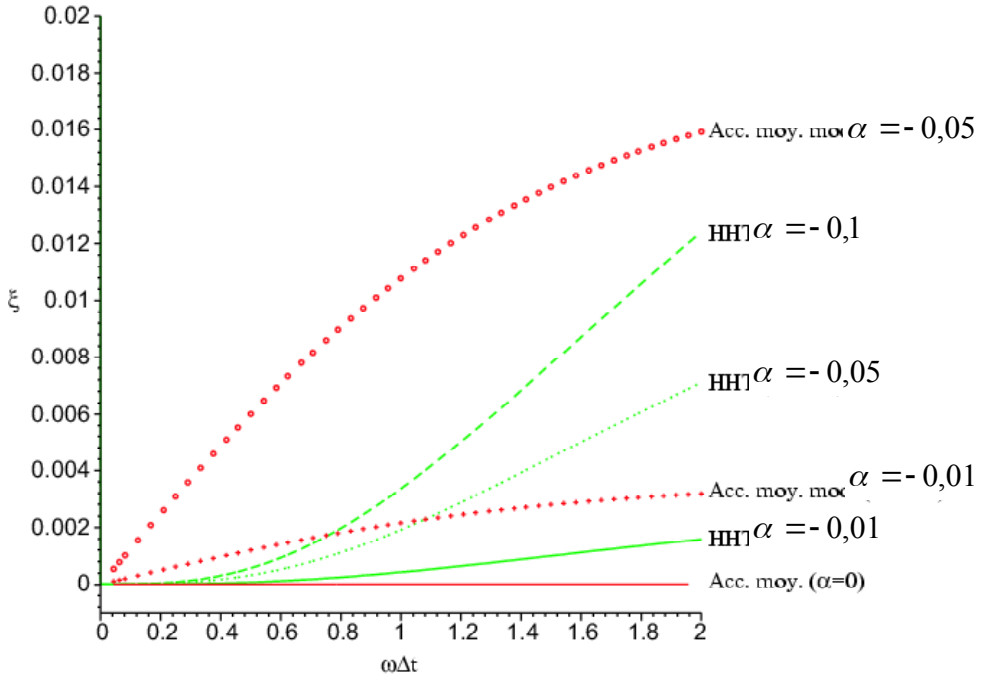

Figure 5.4-a: Digital rate of depreciation according to the step of time of the diagram HHT, comparison with that produced by the diagram of modified average acceleration, *cf.* [bib29].

6 Explicit diagrams

Code_Aster the use of two different explicit diagrams allows: that of the centered differences and that of Tchamwa-Wielgosz [bib34], which, is dissipative for him. These diagrams, contrary to the implicit schemes, are conditionally stable: there exists a step of time limits which one cannot exceed under penalty of divergence.

These diagrams are very well adapted to the problems of fast dynamics, where one seeks to analyze the phenomena of wave propagation. Their cost becomes prohibitory, at least on physical basis, when one must make digital simulations over long times, as it is the case when one seeks to obtain a vibratory answer or when the imposed loading is of long life: as from several seconds, like, for example, in earthquake.

It should however be announced that, compared with a resolution with implicit scheme, the explicit approaches are much less expensive on a step of time. Indeed, whereas into implicit one must truly solve the assembled total system of great dimension to each step (and even with each iteration into nonlinear), into explicit one frees oneself from this very expensive inversion by using a diagonal operator (based on the only matrix of mass, therefore which does not require a reactualization: with the keyword MASS_DIAG = 'YES'). With an explicit diagram, one can solve only in acceleration (keyword FORMULATION = 'ACCELERATION').

6.1 The diagram of the centered differences

This diagram (keyword SCHEMA_TEMPS, SCHEMA='DIFF_CENT'), more running in the codes of fast dynamics, is a typical case of the family of Newmark. It is built while taking $\beta=0$ and $\gamma=1/2$. This method of integration in time, which does not introduce digital dissipation, is of order 2.
Its step of time of stability (condition CFL: Running Friedrichs and Levy [bib25]) is of $2/\omega$, with ω who is the highest own pulsation of the structure considered.

Remarks :

On physical basis, one can show that the critical step of time is equivalent to the minimal time which a wave spends to cross the smallest finite element of the grid. In practice, one approximates this value by calculating the minimum, for all the finite elements of: $\Delta t=D/C$ with D which is the diameter of the largest registered sphere and c who is the celerity of the elastic waves of compression in the element considered.

This method of calculating of condition CFL, directly applicable on the solid elements, requires to be corrected for the elements of structure.

For the classical industrial applications, condition CFL obliges to take steps of very small times generally about ($10^{-5}s$. for the metal structures).
On the other hand, on modal basis, from the strong truncation generally made for the industrial cases, the critical step of time, directly given by $2/\omega$ with ω who is the pulsation of cut, will be much taller than on physical basis.

The resolution clarifies is always done in acceleration (the operator to be reversed is reduced then to the matrix of mass alone which one recommends of “lumper”, therefore to make it diagonal).

6.2 The diagram of Tchamwa-Wielgosz [bib34]

This recent diagram can be seen like during diagram HHT for the explicit one. The required objective is the same one: to introduce a digital dissipation high frequency, which does not disturb “too much” the solution low frequency. This diagram of Tchamwa-Wielgosz [bib34] (keyword SCHEMA_TEMPS, SCHEMA='TCHAMWA') does not result from the family of Newmark.
Just like HHT, this diagram are parameterized by a reality, are named ϕ. If one takes ϕ =1, the diagram does not dissipate. In practice it is recommended to choose a value of ϕ ranging between 1 and 1.1.

The critical step of time is worth that of the diagram of the centered differences divided by ϕ.

6.3 Performance of the explicit diagrams in *Code_Aster*

Code_Aster is not a code dedicated to fast dynamics, it is a code general practitioner directed towards implicit approaches. He is thus not optimized for the resolution explicit, in particular as regards the calculation of the interior efforts and the structuring of the generated objects, like the structure of data solution.

It is thus not recommended to use an explicit diagram for problems of big size, more especially as one wants to calculate on one a large number of steps very.

6.4 Explicit calculations on the basis of Ritz with under-fields in *Code_Aster*

The comments of the preceding paragraph are however to moderate insofar as it is possible to carry out explicit calculations on the basis of Ritz with under-fields (with “generalized” degrees of freedom) in *Code_Aster* . I.e. one can by means of the keyword factor PROJ_MODAL to carry out calculation with a diagram of integration in explicit time on a modal basis (or of Ritz) of beforehand calculated projection.

This option intervenes if one wants to dynamically condense part of the model to the linear behavior, while not calculating strictly by DYNA_NON_LINE that under-fields with the nonlinear behavior. This, in order to reduce the size of the model of calculation. In this case, it is necessary to calculate a modal base of Ritz on the whole of the model: the nonlinear under-field modelled for calculation which calls on DYNA_NON_LINE and other under-fields presumedly linear digests dynamically. This base must be orthogonalized compared to the matrices of mass and linear rigidity of the whole of the model. It must simply be representative of the movements activating the whole of the model.

The field of modelled calculation thus understands a nonlinear under-field of calculation I and a series of linear under-fields E “external” with calculation which will be condensed dynamically.
Matrices $\mathbf{M}$, $\mathbf{C}$, $\mathbf{K}$, forces $\mathbf{F}$ can break up into a part associated with the field I and a part associated with (X) the under-field (S) E. The balance of the structure solved by DYNA_NON_LINE can be written then in term of degrees of freedom of total finite elements $\mathbf{U}$:

$$\left(\mathbf{M}_I+\mathbf{M}_E\right).\ddot{\mathbf{U}}=\mathbf{F}_I^{\text{ext}}+\mathbf{F}_E^{\text{ext}}-\mathbf{F}_I^{\text{int}}\left(\mathbf{U},\dot{\mathbf{U}}\right)-\mathbf{K}_E\mathbf{U}-\left(\mathbf{C}_I+\mathbf{C}_E\right).\dot{\mathbf{U}} \qquad \textbf{éq 8.4-1}$$

That is to say $\{\Phi\}$ a modal base of Ritz orthogonalized by elementary under-structuring on the whole of the under-fields I and E. One uses the transform of Ritz then: $\mathbf{U}=\Phi.\mathbf{q}$ and thus while projecting **éq 7.4-1** by ${}^t\Phi$ on the left one obtains:

$$\left(\tilde{\mathbf{M}}_I+\tilde{\mathbf{M}}_E\right).\ddot{\mathbf{q}} = {}^t\Phi.\mathbf{F}_I^{\text{ext}}+\tilde{\mathbf{F}}_E^{\text{ext}}-{}^t\Phi.\left(\mathbf{F}_I^{\text{int}}+\mathbf{C}_I.\dot{\mathbf{U}}\right)-\tilde{\mathbf{K}}_E.\mathbf{q}-\tilde{\mathbf{C}}_E.\dot{\mathbf{q}} \qquad \textbf{éq 8.4-2}$$

Modes Φ reduced to I are noted ϕ and degrees of freedom $\mathbf{U}$ reduced to I are noted $\mathbf{u}$ thus **éq 7.4-2** becomes for each step of time:

$$\tilde{\mathbf{M}}.\ddot{\mathbf{q}}=\phi^t.\mathbf{F}_I^{\text{ext}}-\phi^t.\left(\mathbf{F}_I^{\text{int}}(\mathbf{u},\dot{\mathbf{u}})+\mathbf{C}_I.\dot{\mathbf{u}}\right)+\tilde{\mathbf{F}}_E^{\text{ext}}-\tilde{\mathbf{K}}_E.\mathbf{q}-\tilde{\mathbf{C}}_E\dot{\mathbf{q}} \qquad \textbf{éq 8.4-3}$$

One thus needs the generalized matrices of mass $\tilde{\mathbf{M}}=\left(\tilde{\mathbf{M}}_I+\tilde{\mathbf{M}}_E\right)$ diagonal as well as matrices of the under-fields E : $\tilde{\mathbf{K}}_E$ and $\tilde{\mathbf{C}}_E$ full and with the generalized vector $\tilde{\mathbf{F}}_E^{\text{ext}}$ who take part under the terms complementary to the dynamic problem standard tiny room to the under-field I who would be solved without dynamic condensation in absence of under-fields E by DYNA_NON_LINE :

$$\tilde{\mathbf{M}}.\ddot{\mathbf{q}}=\phi^t.\mathbf{F}_I^{\text{ext}}-\phi^t.\left(\mathbf{F}_I^{\text{int}}(\mathbf{u},\dot{\mathbf{u}})+\mathbf{C}_I.\dot{\mathbf{u}}\right)$$

With each step of time, one evaluates and adds the complementary terms resulting from dynamic condensation, cf. **éq 7.4-3**, with those of the standard dynamic problem without dynamic condensation and one solves with the diagram of explicit temporal integration, cf [§7.1] and [§7.2], in generalized coordinates $\mathbf{q}$, having expressed with the diagram clarifies the terms in $\mathbf{u}$, $\dot{\mathbf{u}}$ on the under-field I.

The knowledge of the modal base of Ritz orthogonalized $\{\phi\}$ then allows to obtain at the same time the degrees of freedom $\mathbf{u}$ on the under-field of resolution I and $\mathbf{U}$ on the whole of the model made up of its under-fields.

One will not inform behind the keyword MODE_MECA that modes $\{\phi\}$ obtained by reduction of the base of Ritz to the nonlinear field of calculation I treaty by DYNA_NON_LINE, behind the keyword MASSE_GENE the matrix of generalized mass $\tilde{\mathbf{M}}$, like behind the keywords RIGI_GENE AMOR_GENE generalized operators of the under-fields E : $\tilde{\mathbf{K}}_E$, $\tilde{\mathbf{C}}_E$ and finally with the keyword factor EXCIT_GENE $\tilde{\mathbf{F}}_E$. An ExemplE of calculation is provided by the case test SDNV107A [V5.03.107].

The principal nuance compared to the restriction of use of an explicit diagram for problems of big size consists at the same time by the possibility of reducing the size of the model of calculation and by the use of a condition CFL on the step of time related to the maximum frequency of the base of projection, which makes it possible to strongly reduce the number of steps of computing time.

7 Shift of mass

It is possible to enter a coefficient $coef$ behind the operand COEF_MASS_SHIFT in the keyword SCHEMA_TEMPS so of to carry out a "shift" of the matrix of mass $\mathbf{M}$ who becomes: $\mathbf{M}'=\mathbf{M}+coef.\mathbf{K}$.

The introduction of this coefficient makes it possible to reverse in dynamics with diagram clarifies on physical basis the matrix of mass when this one has worthless terms for certain specific degrees of freedom, for example the pressure for the elements of modeling HM. It should be noted that this shift is not necessary in dynamics with explicit diagram on modal basis (presence of the keyword PROJ_MODAL) because in this case the matrix to be reversed is the matrix of generalized mass which is always invertible.

The entry of this coefficient also makes it possible to strongly improve convergence in dynamics with implicit scheme or clarifies whatever the type of modeling by imposing a cut-off frequency inversely proportional to the value of $coef$. This practice, being connected with the method of "selective farmhouse scaling" proposed by Lars Olovsson in multiple publications, makes it possible to however satisfy the convergence criteria with steps with more important times at the cost of a distortion of the whole of the Eigen frequencies of the system, light for the low frequencies.

An own pulsation of origin ω becomes then ω' it that: $\omega'^2=\frac{\omega^2}{1+coef.\omega^2}$

One notes according to this expression that ω' tends towards a maximum value $coef^{-0.5}$.
Thus, for example, for a maximum value in practice for $coef=10^{-6}$, the corrected maximum own pulsation system ω' is worth $1000\,rd.s^{-1}$, and a value for an Eigen frequency of origin of $30\,Hz$ is corrected by this method with $29{,}481\,Hz$.

Remarks :

It is advisable to announce that even if one activates the option of "shift" of mass, for the calculation of initial acceleration, it is always the matrix of not shiftée mass which is used. The fact that it can be not-invertible does not pose nevertheless a problem, because in this case, one imposes initial acceleration on 0 in any point.
The matrix of rigidity used for this "shift" is in practice that which is calculated at the time of the phase of prediction of the algorithm of Newton.

8 Example

It is [Figure 6-a] the movement plan of pendulaison of an extensible bar AB of length unit, rotulée in A with a fixed, free support in B and abandoned without speed in the terrestrial field of gravity starting from a position defined by the angle θ_0. One neglects all the phenomena of mechanical dissipation.

Like the amplitude θ_0 can be large – we will take it $90°$ – the point B sudden of great displacements and the problem is nonlinear.

The theoretical period is:

$$T=1,6744\,s$$

The calculation of the movement of the pendulum by diagram HHT (α - method) with $\alpha=0$ (“rule of the trapezoid”) CAS-test SDNL100 constitutes.

The figure [Figure 6-b] represents the evolution for one period of the dimension of the point B calculated by the diagram “of average acceleration modified” with three values of α. The period is divided into 40 pas de equal times.

The curve in full feature is relative to $\alpha=0$. One observes practically no error.

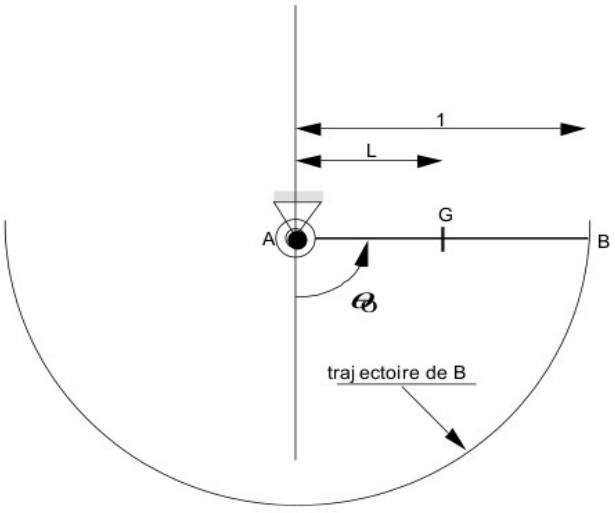

Figure 6-a: Pendulum of great amplitude

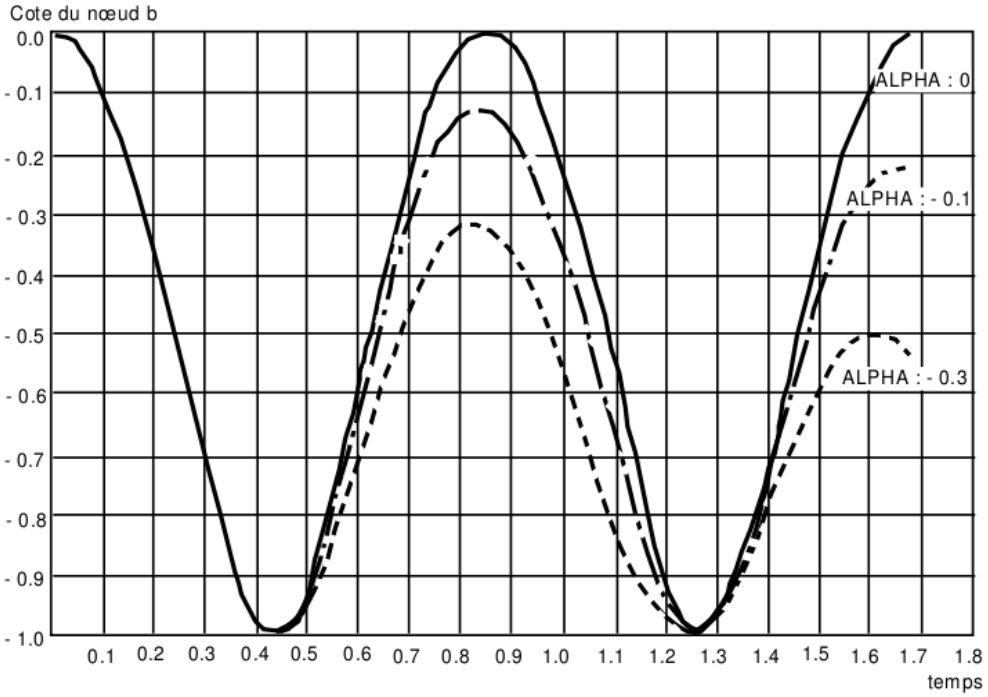

Figure 6-b: Diagram "of modified average acceleration", 40 pas de 0.0419 S

The curve in feature of axis is relative to $\alpha=-0,1$. One observes a rate of depreciation of approximately 2 % whereas the figure [Figure 6-a], for $\frac{\Delta t}{T}=0,025$, 0.8 % envisage. It is that this curve was established into linear, whereas the movement of our pendulum is nonlinear.

The curve in dotted lines is relative to $\alpha=-0,3$. The rate of depreciation is approximately 5.8 %, whereas that envisaged by the figure [Figure 5.3-a] is approximately 2.2 %. The variation is still due to the non-linearity of the problem.

Lastly, the curves in feature of axis and especially in dotted lines reveal a shortening of the period calculated compared to the theoretical period.

9 Conclusion

The diagram of temporal integration of Newmark and its alternatives known as “modified average acceleration” and HHT, accompanied by the method of Newton, make it possible to treat many types of problems of nonlinear dynamics, for material or geometrical non-linearities.

The treatment of the highly nonlinear and nonregular dynamic problems, such as the analysis of the structures in great displacements or contact-friction, is prone to digital instability, even with unconditionally stable methods of temporal integration in the linear field. One then manages to integrate the equations of the movement compared to time only by introducing artificial dissipation. All art is to proportion this dissipation so that, in the range of the frequencies which are of mechanical interest, it is about equivalent to natural damping, without shifting the vibratory spectrum of the structure too much.

Dthem diagrams explicit are also proposed. One recommends to the users of the operator DYNA_NON_LINE of reading the documentation [U2.06.13] whose role is to help with using these methods of transient resolutions well.

10 Bibliography

1) K.J. BATHE: Finite element procedures in engineering analysis. Prentice-hall (1982).

2) MR. AUFAURE: Direct methods of dynamic analysis of the structures into nonlinear. Note HI-70/93/124 (January 27th, 1994).

3) M.AUFAURE: Static and dynamic modeling of the beams in great rotations. [R5.03.40].

4) H.M. HILBER, T.J.R. HUGHES and R.L. TAYLOR: Improved numerical dissipation for time integration algorithms in structural dynamics. Earthq. Engng Struct. Dyn. 5,283-292 (1977).

5) J. - L. LILIEN: Constraints and electromechanical consequences dependent on the way of an intensity of current in the structures in cables. Thesis, University of Liege (1983).

6) P. BALLARD: Dynamics of the discrete mechanical systems with perfect unilateral connections. C.R.Acad.Sci. Series IIb, Paris, pp.953-958, 1999.

7) [R5.02.01] Algorithm of linear thermics transitory.

8) [R5.03.01] quasi-static nonlinear Algorithm (operator STAT_NON_LINE).

9) [R5.06.01] Methods of RITZ in nonlinear dynamics.

10) [R3.03.01] Dualisation of the boundary conditions.

11) [R5.03.08] Integration of the viscoelastic relations of behavior in the operator STAT_NON_LINE.

12) [R5.03.50] unilateral Contact by conditions kinematics.

13) [R4.02.02] Éléments vibroacoustic.

14) [R4.02.04] Fluid Coupling - Structure with Free surface.

15) [R4.05.01] seismic Answer by transitory analysis.

16) [R5.03.40] static and dynamic Modeling of the beams in great rotations.

17) [R5.03.17] Relations of behavior of the discrete elements.

18) [R4.02.05] elements of absorbing border.

19) [U4.51.03] Operator STAT_NON_LINE.

20) [U4.53.01] Operator DYNA_NON_LINE.

21) [R5.05.04] Modeling of damping in linear dynamics.

22) P. WITHOUT: Physical damping and digital damping in the digital diagrams of integration of the structural studies. Note HT-61/01/028/B, October 2002.

23) T.J.R. HUGHES: With note one the stability of Newmark' S algorithm in nonlinear structural dynamics, pp. 383-386 (8/19/1975).

24) A. MILLARD: Contact, friction, adhesion: bases and recent projections in modeling and digital simulation. Course IPSI 10/2004, ch.3-8.

25) NR. GREFFET: Worms of new digital methods for temporal integration in *Code_Aster*. Note EDF/AMA HT-62/04/016/A, 12/2004.

26) L. NOELS, and al.: Automatic time stepping algorithms for implicit numerical simulations of not linear dynamics, Adv. Eng. Software, vol. 33, pp. 589-603, 2002.

27) M.O OTHMAN, A.H. ELKHOLY: Measurement of friction coefficient under reciprocating sliding conditions. Eur. J. Mech. A/Solids, n°2, 215-225, 1994.

28) M.OHAYON: Interactions fluid-structures. Éd. Masson, Paris, 1992.

29) M.GERADIN, D.RIXEN: Theory of the vibrations; application to the dynamics of the structures, Éd. Masson, Paris, 1996.

30) P.PEGON: Alternative characterization of time integration designs. Comput. Meth. Appl. Mech. Engrg. 190, pp. 2707-2727, 2001.

31) A.MILLARD: Digital methods in dynamics with contact. Note CEA/DM2S, 1998.

32) J.A.C.MARTINS, J.T.ODEN: Existence and uniqueness results for dynamic normal contact problems with nonlinear and friction interfaces laws. Nonlinear Analysis, Theor. Meth. Appl., vol. 11, n°3, pp. 407-428, 1987. See too : J.T.ODEN, J.A.C.MARTINS: Models and computational methods for dynamic friction phenomena. Comp. Meth. Appl. Mech. Eng., vol. 52, pp. 527-634, 1985.

33) A. IBRAHIMBEGOVIC, E.L.WILSON: Unified computational model for static and dynamic frictional contact analysis. Int. J. for Num. Meth. In Eng., vol. 34, pp. 233-247, 1992.

34) B. TCHAMWA, C. WIELGOSZ: A new method clarifies precise direct integration and with controllable digital dissipation, Acts of the 13th French Congress of Mechanics, FLIGHT 1, Poitiers, pp 251-254, September 1997.

11 Description of the versions of the document

Version Aster	Author (S) or contributor (S), organization	Description of the modifications
4.4	M.Aufaure EDF/R & D /MMN	Initial text
7.4	F.Voldoire, G.Devesa EDF/R & D /AMA	Update supplements document: description of the diagram of Newmark with the integration of the method of Newton.
8.4	N.Greffet, F.Voldoire; G.Devesa EDF/R & D /AMA	New complete algorithm HHT, diagram of integration in dissipative time
8.5	N.Greffet, F.Voldoire; G.Devesa EDF/R & D /AMA	Small corrections
9.4	N.Greffet EDF/R & D /AMA	Addition of the theta-diagram and description of the choice of resolution in displacement, speed or acceleration.
10.1	N.Greffet EDF/R & D /AMA	Addition of the choice of the matrix of stiffness for the damping of Rayleigh
10.1	F.Voldoire EDF/R & D /AMA	Complements on Θ- diagram: description of its properties. Small corrections here and there.
10.2	N.Greffet EDF/R & D /AMA	Addition of the comment on the choice of variable of resolution according to the diagram.
10.2	F.Voldoire EDF/R & D /AMA	Addition of a remark on the case of the geometrical reactualization, with the § 2.2.1.
10.4	G.Devesa EDF/R & D /AMA	Addition of the § 7.4: Explicit calculations on the basis of Ritz with under-fields.
10.5	N.Greffet EDF/R & D /AMA	Addition diagram of Krenk