Gyroscopic matrices of the right beams and the discs

Summary:

This document presents the formulation of the matrices of gyroscopic of the elements beams, voluminal and discrete damping and stiffness.

The beams are only right beams (Elements **POU_D_T** and **POU_D_E**). The section is constant over the length and of circular form. The material is homogeneous, isotropic.

The discs are cylinders of cross-section whose axis is confused with the axis of the beam. The disc is supposed to be indeformable.

The assumptions selected are the following ones:
- Assumption of Timoshenko: transverse shearing and all the terms of inertia are taken into account. This assumption is to be used for weak twinges (Elements **POU_D_T**).
- Assumption of Euler: transverse shearing is neglected. This assumption is checked for strong twinges (Elements **POU_D_E**).

The number of revolutions clean (along the axis of the beam) can be constant or variable.
In Code_Aster, adopted convention defines the positive direction following the axis of rotation as being the direction **trigonometrical** usual of rotation.

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1 The element beam of constant circular section

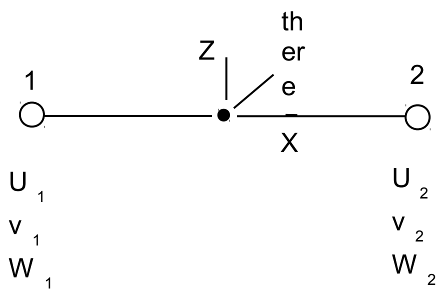
A beam is a solid generated by a surface of surface S, of which the geometrical centre of inertia G followed a curve C called average fibre or neutral fibre.

Within the framework of this modeling, only the right beams, with constant and circular section are taken into account.

For the study of the beams in general, one formulates the following assumptions:
•The cross-section of the beam is indeformable,
•Transverse displacement is uniform on the cross-section.

These assumptions make it possible to express displacements of an unspecified point of the section, according to an increase in displacement due to the rotation of the section around the transverse axes.

The discretization in "exact" elements of beam is carried out on a linear element with two nodes and six degrees of freedom by nodes. These degrees of freedom break up into three translations u, v, w (displacements according to the directions x, y and z) and three rotations θ_x, θ_y and θ_z. (around the axes x, y and z).



In the case of the right beams, the average line is along the axis x local base, displacement transverse being thus carried out in the plan (y, z).

For the storage of the sizes related to the degrees of freedom of an element in a vector or an elementary matrix (thus of dimension 12 or 12^2), one arranges initially the variables for node 1 then those of node 2. For each node, one stores initially the sizes related to the three translations, then those related to three rotations. For example, a vector displacement will be structured in the following way:

$$\underbrace{u_1,\ v_1,\ w_1,\theta_{x1},\theta_{y1},\theta_{z1}}_{sommet\ 1},\ \underbrace{u_2,\ v_2,\ w_2,\theta_{x2},\theta_{y2},\theta_{z2}}_{sommet\ 2}$$

1.1 Definition of the reference marks

One defines:
- x is the axis of neutral fibre of the beam,
- y and z are the main axes of inertia of the section,
- R_0 is the absolute reference mark related to a section in the initial configuration,
- R is the reference mark related to a section in the deformed configuration,

By not considering torsion, the passage of the reference mark R_0 with the reference mark R be carried out with the assistance 3 rotations, two following y and z, and a rotation around x, noted ϕ, such as:
$\dot{\phi}$: number of revolutions clean of the tree

1.2 Characteristics

Each element is an isoparametric element beam of circular and constant section. One takes into account transverse shearing in the formulation of this element (right beam of Timoshenko).

Notations:
- x is the axis of neutral fibre of the line of trees,
- density: ρ
- length of the element: L
- Young modulus: E
- Fish module: $G=\frac{E}{2(1+\nu)}$
- section:
  - interior ray: R_i
  - external ray: R_e
  - surface: $A=\pi\left(R_e^2-R_i^2\right)$
  - polar inertia: $I_x=\frac{\pi}{2}\left(R_e^4-R_i^4\right)$
  - inertia of section: $I_{yz}=I_y=I_z=\frac{\pi}{4}\left(R_e^4-R_i^4\right)$

1.3 Calculation of the kinetic energy of the beam of Timoshenko

One calculates the kinetic energy of the element beam of Timoshenko by considering the deformations of membrane and inflection. The expression of the kinetic energy is obtained while integrating over the length of the element beam:

$$T=\frac{1}{2}\rho.A\int_{x=0}^{L}\left[\dot{u}^2+\dot{v}^2+\dot{w}^2\right]dx+\frac{1}{2}\rho.\int_{x=0}^{L}\vec{\Omega}_{R/R0}.[J].\vec{\Omega}_{R/R0}\,dx$$

with: $[J]=\begin{bmatrix} I_x & 0 & 0\\ 0 & I_y & 0\\ 0 & 0 & I_z\end{bmatrix}$ with I (in m^4)

That is to say a right beam of axis $o\vec{x}$ for the not deformed configuration, it is necessary to define two intermediate bases to characterize the Flight Path Vector of rotation $\vec{\Omega}_{R/R0}$.

- Passage de la bases B (O, $\vec{x}$, $\vec{y}$, $\vec{z}$) at the base B_1 (O, $\vec{x}_1$, $\vec{y}_1$, $\vec{z}_1$) by a rotation of axis $o\vec{y}$ of amplitude $\theta_y(x,t)$ such as:
  $\vec{y}_1=\vec{y}$
- Passage de la bases B (O , $\vec{x}_1$, $\vec{y}_1$, $\vec{z}_1$) at the base B_2 (O , $\vec{x}_2$, $\vec{y}_2$, $\vec{z}_2$) by a rotation of axis $o\vec{z}_1$ of amplitude $\theta_z(x,t)$ such as:
  $\vec{z}_2=\vec{z}_1$ and $\vec{y}_1=\cos\theta_z(x,t).\vec{y}_2+\sin\theta_z(x,t).\vec{x}_2$

- Rotation at the angular velocity $\dot{\phi}(t)$ place has along the axis $o\vec{x}_2$.
- Thus the vector of rotation is written: $\vec{\Omega}_{R/R0}=\dot{\phi}(t).\vec{x}_2+\dot{\theta}_y(x,t).\vec{y}_1+\dot{\theta}_z(x,t).\vec{z}_1$
- Since the operator $[J]$ element beam is written in the base B$_2$ who corresponds to the deformed position, it is imperative unless changing basic the operator of inertia, to write the Flight Path Vector of rotation $\vec{\Omega}_{R/R0}$ in the base B$_2$.
  $\vec{\Omega}_{R/R0}=\dot{\phi}(t).\vec{x}_2+\dot{\theta}_y(x,t).\left(\cos\theta_z(x,t).\vec{y}_2+\sin\theta_z(x,t).\vec{x}_2\right)+\dot{\theta}_z(x,t).\vec{z}_2$
- By considering that angles $\theta_y(x,t)$ and $\theta_z(x,t)$ are small, it is legitimate to carry out a development limited to order 1. The expression of the Flight Path Vector $\vec{\Omega}_{R/R0}$ becomes then:
  $\vec{\Omega}_{R/R0}=\left(\dot{\phi}(t)+\dot{\theta}_y(x,t).\theta_z(x,t)\right).\vec{x}_2+\dot{\theta}_y(x,t).\vec{y}_2+\dot{\theta}_z(x,t).\vec{z}_2$
- It remains to develop the following scalar product:

$$\frac{1}{2}\rho\int_{x=0}^{L}\vec{\Omega}_{R/R0}.[J].\vec{\Omega}_{R/R0}\,dx=\frac{1}{2}\rho I_{yz}\int_{x=0}^{L}\left[\dot{\theta}_y^2+\dot{\theta}_z^2\right]dx+\frac{1}{2}\rho I_x.L.\dot{\phi}^2+\rho\dot{\phi}I_x\int_{x=0}^{L}\dot{\theta}_y.\theta_z\,dx$$

For an element beam of constant section, the expression becomes:

$$T=\frac{1}{2}\rho A\int_{x=0}^{L}\left[\dot{u}^2+\dot{v}^2+\dot{w}^2\right]dx+\frac{1}{2}\rho I_{yz}\int_{x=0}^{L}\left[\dot{\theta}_y^2+\dot{\theta}_z^2\right]dx+\frac{1}{2}\rho I_x.L.\dot{\phi}^2+\rho\dot{\phi}I_x\int_{x=0}^{L}\dot{\theta}_y.\theta_z\,dx$$

With:

$$I_x=\frac{\pi}{2}.\left[R_e^4-R_e^4\right]$$

$$I_{yz}=I_y=I_z=\frac{\pi}{4}.\left[R_e^4-R_e^4\right]$$

The various terms of the kinetic energy represent:

- for the first term, the kinetic energy of translation,
- for the two following terms, the kinetic energy of rotation,
- for the fourth term, the gyroscopic term of effect.

1.4 Functions of interpolation

For the deformations of membrane (traction and compression), the field $u(x)$ is approached by a linear function of displacements of nodes 1 and 2 of the element beam:

$$u(x)=\langle N_1^L(x)\;N_2^L(x)\rangle\begin{Bmatrix}u_1\\u_2\end{Bmatrix}\quad\text{with}\quad\begin{cases}N_1^L(x)=1-\dfrac{x}{L}\\N_2^L(x)=\dfrac{x}{L}\end{cases}$$

For the deformations of inflection, one uses cubic functions of type modified Hermit. Degrees of freedom $v(x),\theta_y(x),w(x),\theta_z(x)$ are thus interpolated as follows:

$$v(x)=\langle \xi_1(x)\ \ -\xi_2(x)\ \ \xi_3(x)\ \ -\xi_4(x)\rangle \begin{Bmatrix} v_1 \\ \theta_{z1} \\ v_2 \\ \theta_{z2} \end{Bmatrix}$$

$$\theta_z(x)=\langle -\xi_5(x)\ \ \xi_6(x)\ \ -\xi_7(x)\ \ \xi_8(x)\rangle \begin{Bmatrix} v_1 \\ \theta_{z1} \\ v_2 \\ \theta_{z2} \end{Bmatrix}$$

$$w(x)=\langle \xi_1(x)\ \ \xi_2(x)\ \ \xi_3(x)\ \ \xi_4(x)\rangle \begin{Bmatrix} w_1 \\ \theta_{y1} \\ w_2 \\ \theta_{y2} \end{Bmatrix}$$

$$\theta_y(x)=\langle \xi_5(x)\ \ \xi_6(x)\ \ \xi_7(x)\ \ \xi_8(x)\rangle \begin{Bmatrix} w_1 \\ \theta_{y1} \\ w_2 \\ \theta_{y2} \end{Bmatrix}$$

One defines K_{yz} coefficient of shearing in the directions y and z.

For an element beam of constant section:

$$K_{yz}=\frac{7+20.\alpha^2}{6} \quad \text{with} \quad \alpha=\frac{R_i}{R_e.\left(1+\frac{R_i^2}{R_e^2}\right)}$$

While noting $\phi_{yz}=\frac{12.E.I_{yz}}{K_{yz}.A.G.L^2}$, functions ξ_i are as follows defined:

$$\xi_1(x)=\frac{1}{1+\phi_{yz}}\left[2.\left(\frac{x}{L}\right)^3-3.\left(\frac{x}{L}\right)^2-\phi_{yz}.\left(\frac{x}{L}\right)+(1+\phi_{yz})\right]$$

$$\xi_5(x)=\frac{6}{L.(1+\varphi_{yz})}.\left(\frac{x}{L}\right).\left[1-\left(\frac{x}{L}\right)\right]$$

$$\xi_2(x)=\frac{L}{1+\phi_{yz}}\left[-\left(\frac{x}{L}\right)^3+\frac{4+\phi_{yz}}{2}\left(\frac{x}{L}\right)^2-\frac{2+\phi_{yz}}{2}.\left(\frac{x}{L}\right)\right]$$

$$\xi_6(x)=\frac{1}{1+\phi_{yz}}\left[3.\left(\frac{x}{L}\right)^2-(4+\phi_{yz}).\left(\frac{x}{L}\right)+(1+\phi_{yz})\right]$$

$$\xi_3(x)=\frac{1}{1+\phi_{yz}}\left[-2.\left(\frac{x}{L}\right)^3+3.\left(\frac{x}{L}\right)^2+\phi_{yz}.\left(\frac{x}{L}\right)\right]$$

$$\xi_7(x)=\frac{-6}{L.(1+\phi_{yz})}.\left(\frac{x}{L}\right).\left[1-\left(\frac{x}{L}\right)\right]$$

$$\xi_4(x)=\frac{L}{1+\phi_{yz}}\left[-\left(\frac{x}{L}\right)^3+\frac{2-\phi_{yz}}{2}\left(\frac{x}{L}\right)^2+\frac{\phi_{yz}}{2}.\left(\frac{x}{L}\right)\right]$$

$$\xi_8(x)=\frac{1}{1+\phi_{yz}}\left[3.\left(\frac{x}{L}\right)^2+(-2+\phi_{yz}).\left(\frac{x}{L}\right)\right]$$

Note:

In the case of elements beams of Euler (Elements **POU_D_E**) the term ϕ_{yz} is null.

The vector of the degrees of freedom of the element beam is defined by:

$$\langle q\rangle=\langle u_1\ v_1\ w_1\ \theta_{x1}\ \theta_{y1}\ \theta_{z1}\ u_2\ v_2\ w_2\ \theta_{x2}\ \theta_{y2}\ \theta_{z2}\rangle$$

One poses:

$$\langle\delta u\rangle=\langle u_1\ u_2\rangle$$
$$\langle\delta v\rangle=\langle v_1\ \theta_{z1}\ v_2\ \theta_{z2}\rangle$$
$$\langle\delta w\rangle=\langle w_1\ \theta_{y1}\ w_2\ \theta_{y2}\rangle$$

By replacing the preceding approximations in the expression of the kinetic energy, one obtains:

$$T=\frac{1}{2}\langle\delta\dot{u}\rangle[M_1]\{\delta\dot{u}\}+\frac{1}{2}\langle\delta\dot{w}\rangle([M_2]+[M_4])\{\delta\dot{w}\}+\frac{1}{2}\langle\delta\dot{v}\rangle([M_3]+[M_5])\{\delta\dot{v}\}$$
$$+\dot{\phi}.\langle\delta\dot{v}\rangle[M_6]\{\delta w\}+\frac{1}{2}.\rho.I_x.\dot{\phi}^2$$

With:

$$[M_1]=\int_{x=0}^{L}\rho.A.\begin{Bmatrix}N_1^L(x\\N_2^L(x)\end{Bmatrix}.\langle N_1^L(x)\ N_2^L(x)\rangle.dx$$

$$[M_2]=\int_{x=0}^{L}\rho.A.\begin{Bmatrix}\xi_1(x)\\\xi_2(x)\\\xi_3(x)\\\xi_4(x)\end{Bmatrix}.\langle\xi_1(x)\ \ \xi_2(x)\ \ \xi_3(x)\ \ \xi_4(x)\rangle.dx$$

$$[M_3]=\int_{x=0}^{L}\rho.A.\begin{Bmatrix}\xi_1(x)\\-\xi_2(x)\\\xi_3(x)\\-\xi_4(x)\end{Bmatrix}.\langle\xi_1(x)\ \ -\xi_2(x)\ \ \xi_3(x)\ \ -\xi_4(x)\rangle.dx$$

$$[M_4]=\int_{x=0}^{L}\rho.I_{yz}.\begin{Bmatrix}\xi_5(x)\\\xi_6(x)\\\xi_7(x)\\\xi_8(x)\end{Bmatrix}.\langle\xi_5(x)\ \ \xi_6(x)\ \ \xi_7(x)\ \ \xi_8(x)\rangle.dx$$

$$[M_5]=\int_{x=0}^{L}\rho.I_{yz}.\begin{Bmatrix}-\xi_5(x)\\ \xi_6(x)\\ -\xi_7(x)\\ \xi_8(x)\end{Bmatrix}.\langle -\xi_5(x)\ \ \xi_6(x)\ \ -\xi_7(x)\ \ \xi_8(x)\rangle.dx$$

$$[M_6]=\int_{x=0}^{L}\rho.I_{x}.\begin{Bmatrix}-\xi_5(x)\\ \xi_6(x)\\ -\xi_7(x)\\ \xi_8(x)\end{Bmatrix}.\langle \xi_5(x)\ \ \xi_6(x)\ \ \xi_7(x)\ \ \xi_8(x)\rangle.dx$$

1.5 Calculation of the equilibrium equations

The equations of Lagrange for the kinetic energy of the beam are written in the following form:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right)-\frac{\partial T}{\partial q_i}=0 \quad \text{with} \quad \langle q\rangle=\langle u\ v\ w\ \rangle$$

That is to say:

$$\begin{cases}\frac{d}{dt}\left(\frac{\partial T}{\partial(\delta\dot{u})}\right)-\frac{\partial T}{\partial(\delta u)}=[M_1]\{(\delta\ddot{u})\}\\ \frac{d}{dt}\left(\frac{\partial T}{\partial(\delta\dot{v})}\right)-\frac{\partial T}{\partial(\delta v)}=([M_3]+[M_5])\{(\delta\ddot{v})\}-\dot{\varphi}.[M_6]\{(\delta\dot{w})\}-\ddot{\varphi}.[M_6]\{(\delta w)\}\\ \frac{d}{dt}\left(\frac{\partial T}{\partial(\delta\dot{w})}\right)-\frac{\partial T}{\partial(\delta w)}=([M_2]+[M_4])\{(\delta\ddot{w})\}+\dot{\varphi}.[M_6]^T\{(\delta\dot{v})\}\end{cases}$$

These equations can be put in the form:

$$[M]\langle\ddot{q}\rangle+[C_{gyro}]\langle\dot{q}\rangle+([K]+[K_{gyro}])\langle q\rangle=\langle 0\rangle$$

The gyroscopic matrix of damping $[C_{gyro}]$ system is made up starting from the matrix $[M_6]$ and of its transposed. It is antisymmetric, and its contribution must be multiplied by the angular velocity $\dot{\phi}$.

While noting: $\phi=\phi_{yz}$

$$[M_6]=\frac{\rho.I_x}{30L(1+\phi)^2}\begin{bmatrix}-36 & 3L(1-5\phi) & 36 & 3L(1-5\phi)\\ -3L(1-5\phi) & L^2(4+5\phi+10\phi^2) & 3L(1-5\phi) & -L^2(1+5\phi-5\phi^2)\\ 36 & -3L(1-5\phi) & -36 & 3L(-1+5\phi)\\ -3L(1-5\phi) & -L^2(1+5\phi-5\phi^2) & -3L(-1+5\phi) & L^2(4+5\phi+10\phi^2)\end{bmatrix}$$

$$[C_{gyro}]=\frac{\rho . I_x}{30\,L(1+\phi)^2}\times$$

$$\begin{bmatrix}
- & - & - & - & - & - & - & - & - & - & - & - \\
 & 0 & 36 & - & -3L(1-5\phi) & 0 & - & 0 & -36 & - & -3L(1-5\phi) & 0 \\
 & & 0 & - & 0 & -3L(1-5\phi) & - & 36 & & - & 0 & -3L(1-5\phi) \\
 & & & - & - & - & - & - & & - & - & - \\
 & & & & 0 & L^2(4+5\phi+10\phi^2) & - & -3L(1-5\phi) & & - & 0 & -L^2(1+5\phi-5\phi^2) \\
 & & & & & 0 & - & 0 & -3L(1-5\phi) & - & L^2(1+5\phi-5\phi^2) & 0 \\
 & & & & & & - & - & & - & - & - \\
 & & & & & & & 0 & 36 & - & 3L(1-5\phi) & 0 \\
 & & & & & & & & 0 & - & 0 & 3L(1-5\phi) \\
 & & & & & & & & & - & - & - \\
 & & & & & & & & & & 0 & L^2(4+5\phi+10\phi^2) \\
 & & & & & & & & & & & 0
\end{bmatrix}$$

Like the matrix $[C_{gyro}]$ is antisymmetric, only the higher triangle is represented.
(–) mean that the degree of freedom is not concerned with the gyroscopic matrices.

The gyroscopic matrix of stiffness $[K_{gyro}]$ system is made up starting from the matrix $[M_6]$. Its contribution must be multiplied by the angular acceleration $\ddot{\phi}$.

$$[K_{gyro}]=\frac{\rho . I_x}{30\,L(1+\phi)^2}\times$$

$$\begin{bmatrix}
- & - & - & - & - & - & - & - & - & - & - & - \\
- & 0 & 36 & - & -3L(1-5\phi) & 0 & - & 0 & -36 & - & -3L(1-5\phi) & 0 \\
- & 0 & 0 & - & 0 & 0 & - & 0 & & - & 0 & 0 \\
- & - & - & - & - & - & - & - & & - & - & - \\
- & 0 & 0 & - & 0 & 0 & - & 0 & & - & 0 & 0 \\
- & 0 & 3L(1-5\phi) & - & -L^2(4+5\phi+10\phi^2) & 0 & - & 0 & -3L(1-5\phi) & - & L^2(1+5\phi-5\phi^2) & 0 \\
- & - & - & - & - & - & - & - & & - & - & - \\
- & 0 & -36 & - & 3L(1-5\phi) & 0 & - & 0 & 36 & - & 3L(1-5\phi) & 0 \\
- & 0 & 0 & - & 0 & 0 & - & 0 & 0 & - & 0 & 0 \\
- & - & - & - & - & - & - & - & - & - & - & - \\
- & 0 & 0 & - & 0 & 0 & - & 0 & 0 & - & 0 & 0 \\
- & 0 & 3L(1-5\phi) & - & L^2(4+5\phi+10\phi^2) & 0 & - & 0 & -3L(1-5\phi) & - & -L^2(4+5\phi+10\phi^2) & 0
\end{bmatrix}$$

The full matrix $[K_{gyro}]$ is filled in entirety (triangles higher and inferior).

Recall:

- with $\langle q\rangle=\langle u_1\ v_1\ w_1\ \theta_{x1}\ \theta_{y1}\ \theta_{z1}\ u_2\ v_2\ w_2\ \theta_{x2}\ \theta_{y2}\ \theta_{z2}\rangle$
- in the case of elements beams of Euler (Elements **POU_D_E**) the term ϕ_{yz} is null.

2 The circular disc

The objective of this chapter is to characterize the gyroscopic matrices of an infinitely rigid circular disc, subjected at a number of constant or variable revolutions.

The characteristics of the disc are the following ones:

- axis of the disc confused with the axis of neutral fibre of the beam (axis $\vec{x}$)
- centre of gravity of the disc: C

- interior ray: R_i
- external ray: R_e
- thickness: h
- presumedly uniform density: ρ

Deduced values:

- mass of the disc: $M=\pi\,\rho\,h\,(R_e^2-R_i^2)$
- moment of inertia mass/axes y or z calculated in the centre of gravity C :

$$I_{yz}=\frac{M}{12}\,(3.R_e^2+3.R_i^2+h^2)$$

- mass moment of inertia compared to the axis x calculated in the centre of gravity C :

$$I_x=\frac{M}{2}\,(R_e^2+R_i^2)$$

Note:

- Axes $C\vec{x}$, $C\vec{y}$ and $C\vec{z}$ being main axes of inertia of the disc, products of inertia I_{xy} , I_{yz} and I_{xz} are worthless.
- The symmetry of the disc compared to the axes $C\vec{y}$ and $C\vec{z}$ impose: $I_{yz}=I_y=I_z$

The displacement of the center of the disc is given by: $u.\vec{x}+v.\vec{y}+w.\vec{z}$

One notes:

- $\vec{\Omega}_{R/R0}$: the Flight Path Vector of rotation of the disc
- $\vec{x}.\vec{\Omega}_{R/R0}=\dot{\phi}(t)$: number of revolutions clean

2.1 Calculation of the kinetic energy of the disc

One calculates the kinetic energy of the disc by applying the formula of Huygens:

$$T=\frac{1}{2}M\ .\left(\vec{V}_{C,D/R0}\right)^2+\frac{1}{2}\vec{\Omega}_{R/R0}.[J].\vec{\Omega}_{R/R0}$$

$$T=\frac{1}{2}M\ .\left(\dot{u}^2+\dot{v}^2+\dot{w}^2\right)^2+\frac{1}{2}\vec{\Omega}_{R/R0}.[J].\vec{\Omega}_{R/R0}$$

with: $[J]=\begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix}$ with $I_{yz}=I_y=I_z$

By developing the preceding expression, one obtains:

$$T=\frac{1}{2}\rho.\left(\dot{u}^2+\dot{v}^2+\dot{w}^2\right)^2+\frac{1}{2}I_{yz}.(\dot{\theta}_y^2+\dot{\theta}_z^2)+\frac{1}{2}\ I_x.(\dot{\phi}^2+2\,\dot{\theta}_y.\dot{\phi}.\theta_z)$$

The various terms of the kinetic energy represent:

- for the first term, the kinetic energy of translation,
- for the second term, the kinetic energy of rotation,
- for the term $\frac{1}{2}\,I_x.\dot{\phi}^2$, the “clean” energy of rotation,
- and for the term $I_x.(\dot{\theta}_y.\dot{\phi}.\theta_z)$, the gyroscopic effect.

2.2 Calculation of the equilibrium equations

The equations of Lagrange are used to formulate the dynamic balance of the disc. In this case, typical case the deformation energy is worthless (infinitely rigid disc) and no external effort is considered, one thus has:

$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right)-\frac{\partial T}{\partial q_i}=0$ with $\langle q\rangle=\langle u\ v\ w\ \theta_y\ \theta_z\rangle$: vector of the degrees of freedom of the element disc.

One does not take account of the degree of freedom ϕ because it is considered that the number of revolutions clean is imposed and thus known. The following equations then are obtained:

$$\begin{cases}\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right)-\frac{\partial T}{\delta u}=M.\ddot{u}\\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{v}}\right)-\frac{\partial T}{\delta v}=M.\ddot{v}\\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{w}}\right)-\frac{\partial T}{\delta w}=M.\ddot{w}\\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_y}\right)-\frac{\partial T}{\partial \theta_y}=I_{yz}.\ddot{\theta}_y+I_x.\dot{\phi}.\dot{\theta}_z+I_x.\ddot{\phi}.\theta_z\\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_z}\right)-\frac{\partial T}{\partial \theta_z}=I_{yz}.\ddot{\theta}_z-I_x.\dot{\phi}.\dot{\theta}_y\end{cases}$$

These equations can be put in the form:

$$[M]\langle\ddot{q}\rangle+[C_{gyro}]\langle\dot{q}\rangle+([K]+[K_{gyro}])\langle q\rangle=\langle 0\rangle$$

The gyroscopic matrix of damping of the disc is obtained as from the moment of inertia I_x. It is antisymmetric, and its contribution must be multiplied by the clean angular velocity $\dot{\phi}$.

$$[C_{gyro}]=\dot{\phi}.\begin{bmatrix}0&0&0&-&0&0\\0&0&0&-&0&0\\0&0&0&-&0&0\\-&-&-&-&-&-\\0&0&0&-&0&I_x\\0&0&0&-&-I_x&0\end{bmatrix}$$

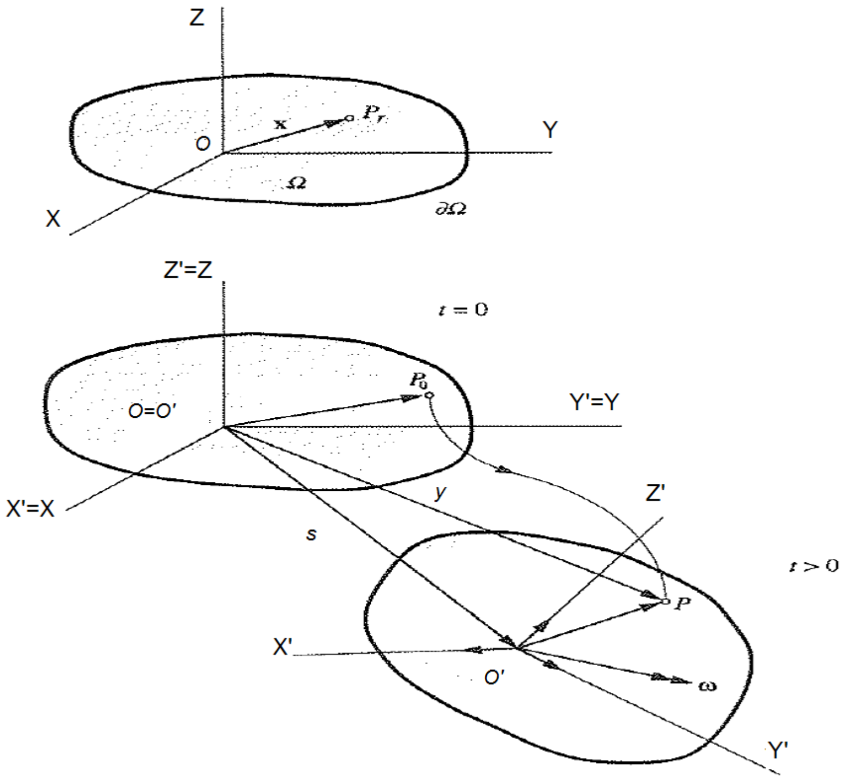
with $\langle u\ v\ w\ \theta_x\ \theta_y\ \theta_z\rangle$ vector of the degrees of freedom of the element disc and such as: $\dot{\theta}_x=\dot{\phi}$
The indent corresponds to the degree of freedom of rotation along the axis of the beam and leads obviously to worthless terms.

The gyroscopic matrix of stiffness of the disc is also obtained as from the moment of inertia I_x. Its contribution must be multiplied by the clean angular acceleration $\ddot{\phi}$.

$$[K_{gyro}]=\ddot{\phi}.\begin{bmatrix} 0 & 0 & 0 & - & 0 & 0 \\ 0 & 0 & 0 & - & 0 & 0 \\ 0 & 0 & 0 & - & 0 & 0 \\ - & - & - & - & - & - \\ 0 & 0 & 0 & - & 0 & I_x \\ 0 & 0 & 0 & - & 0 & 0 \end{bmatrix}$$

3 The voluminal element 3D

The setting in equation of a mobile structure can be done, either in the fixed Galilean reference mark (OXYZ), or in the reference mark of inertia (O' X' Y' Z') attache with the structure.



For a dynamic analysis, the choice of the reference mark of inertia as frame of reference makes it possible to simplify the formulation of the differential equations governing their dynamic behavior.

This choice makes it possible to characterize the effect Coriolis (gyroscopic effect seen of the rotor) which is added to damping. Also, two other effects of type apparent stiffness are added to the matrix of stiffness: it is the centrifugal effect and the gyroscopic effect associated with the variation speed.

3.1 Position moving in the revolving reference mark

In any general information, it will be considered that mobile reference frame, whose origin is characterized by a translation $\mathbf{s}(t)={}^t(s_x, s_y, s_y)$ compared to the reference mark of inertia, turns at the angular velocity $\boldsymbol{\omega}={}^t(\omega_x, \omega_y, \omega_z)$ around an unspecified axis passing by the origin. It is important to note that the components number of revolutions are measured in reference mark corotationnel. One will note $\boldsymbol{\omega}'$ the projection this number of revolutions in the fixed reference mark, IE $\boldsymbol{\omega}={}^t\mathbf{R}\,\boldsymbol{\omega}'$ where $\mathbf{R}(t)$ is the matrix of basic change, formed by the cosine directors of the basic vectors of the reference mark of inertia, expressed in the fixed reference frame.

The position of a point P in the reference frame of inertia (OXYZ), noted $\mathbf{y}$, has then as an expression:

$$\mathbf{y}=\mathbf{s}(t)+\mathbf{R}(t)[\mathbf{x}+\mathbf{u}(\mathbf{x},t)]$$

where $\mathbf{x}$ is the initial position of the point P in the inertial system and $\mathbf{u}$ is the vector displacement resulting from the dynamic deformation of the structure at a moment given T.

Absolute velocity $\dot{\mathbf{y}}$ point P is defined as being the derivative first compared to the time of the vector $\mathbf{y}$. She is written:

$$\dot{\mathbf{y}}=\dot{\mathbf{s}}+\dot{\mathbf{R}}(\mathbf{x}+\mathbf{u})+\mathbf{R}\,\dot{\mathbf{u}}$$

where $\dot{\mathbf{s}}$ is the speed of traverse of the O' origin of the turning reference mark, written in the fixed reference mark.

It is shown that the derivative $\dot{\mathbf{R}}$ matrix of basic change can be also written as being the vector product between the vector of rotation in the fixed reference mark $\boldsymbol{\omega}'$ and D stamps itE basic change $\mathbf{R}$ or then as being the product of the matrix $\mathbf{R}$ and of the antisymmetric matrix $\boldsymbol{\Omega}$ who includes the three components of the tensor of rotation in the reference mark of inertia $\boldsymbol{\omega}$:

$$\dot{\mathbf{R}}=\boldsymbol{\omega}'\wedge\mathbf{R}=\mathbf{R}\,\boldsymbol{\Omega} \text{ with } \boldsymbol{\Omega}=\begin{bmatrix}0 & -\omega_z & \omega_y\\ \omega_z & 0 & -\omega_x\\ -\omega_y & \omega_x & 0\end{bmatrix}$$

While inserting this definition in the expression absolute velocity, it results from it that:

$$\dot{\mathbf{y}}=\dot{\mathbf{s}}+\boldsymbol{\omega}'\wedge\mathbf{R}(\mathbf{x}+\mathbf{u})+\mathbf{R}\,\dot{\mathbf{u}}$$

Absolute acceleration $\ddot{\mathbf{y}}$ point P is defined as being the derivative second compared to the time of the Flight Path Vector $\mathbf{y}$. She is written:

$$\ddot{\mathbf{y}}=\ddot{\mathbf{s}}+\dot{\boldsymbol{\omega}}'\wedge\mathbf{R}(\mathbf{x}+\mathbf{u})+\boldsymbol{\omega}'\wedge[\,'\boldsymbol{\omega}'\wedge\mathbf{R}(\mathbf{x}+\mathbf{u})]+2\,\boldsymbol{\omega}'\wedge\mathbf{R}\,\dot{\mathbf{u}}+\mathbf{R}\,\ddot{\mathbf{u}}$$

with $\ddot{\mathbf{s}}$ the acceleration of translation of the O' origin of the turning reference mark, written in the fixed reference mark and $\dot{\boldsymbol{\omega}}'$ is the instantaneous acceleration of rotation defined by the relation $\dot{\boldsymbol{\omega}}={}^t\mathbf{R}\,\dot{\boldsymbol{\omega}}'$.

By concern of clearness and without losing the general information, one will suppose in the continuation of the document that the O' origin of the turning reference mark is fixed, IE. $\dot{\mathbf{s}}=\ddot{\mathbf{s}}=0$.

After pre-multiplication by the reverse transformation ${}^t\mathbf{R}$, one then obtains the absolute speed and acceleration expressed in the mobile reference frame:

$${}^t\mathbf{R}\,\dot{\mathbf{y}}=\dot{\mathbf{u}}+\boldsymbol{\omega}\wedge\mathbf{R}(\mathbf{x}+\mathbf{u})$$

$$^{t}\mathbf{R}\,\ddot{\mathbf{y}}=\ddot{\mathbf{u}}+\dot{\boldsymbol{\omega}}\wedge(\mathbf{x}+\mathbf{u})+\boldsymbol{\omega}\wedge[\boldsymbol{\omega}\wedge(\mathbf{x}+\mathbf{u})]+2\,\boldsymbol{\omega}\wedge\dot{\mathbf{u}}$$

In this expression, which represents the theorem of the composition of accelerations of a material point, the terms are recognized:

- of relative acceleration $\ddot{\mathbf{u}}$, which contributes to the matrix of mass;
- of acceleration of training $\dot{\boldsymbol{\omega}}\wedge(\mathbf{x}+\mathbf{u})+\boldsymbol{\omega}\wedge[\boldsymbol{\omega}\wedge(\mathbf{x}+\mathbf{u})]$ (nap of the effect Euler due to acceleration of rotation and of the effect of softening centrifugal, which contributes to the matrix of stiffness);
- of complementary acceleration or Coriolis $2\boldsymbol{\omega}\wedge\dot{\mathbf{u}}$, who contributes to the matrix of damping.

3.2 Expression of the energies kinetic and potential

In its general form, the kinetic energy is obtained starting from the absolute velocity $\mathbf{y}$ as follows:

$$\mathbf{T}=\frac{1}{2}\int_{\Omega}\rho\,\dot{\mathbf{y}}^{t}\dot{\mathbf{y}}\;d\,\Omega$$

The development of the terms gives the following expression according to displacement $\mathbf{u}$:

$$\mathbf{T}=\frac{1}{2}\int_{\Omega}\rho\,\dot{\mathbf{u}}^{t}\dot{\mathbf{u}}\;d\,\Omega+\int_{\Omega}\rho\,\dot{\mathbf{u}}^{t}\boldsymbol{\Omega}\,\mathbf{u}\;d\,\Omega-\frac{1}{2}\int_{\Omega}\rho\,\mathbf{u}^{t}\boldsymbol{\Omega}^{2}\mathbf{u}\;d\,\Omega-\int_{\Omega}\rho\,\mathbf{u}^{t}\boldsymbol{\Omega}^{2}\mathbf{x}\;d\,\Omega+\int_{\Omega}\rho\,\dot{\mathbf{u}}^{t}\boldsymbol{\Omega}\,\mathbf{x}\;d\,\Omega$$
$$-\frac{1}{2}\int_{\Omega}\rho\,\mathbf{x}^{t}\boldsymbol{\Omega}^{2}\mathbf{x}\;d\,\Omega$$

The potential energy of the system (internal deformation energy and work of the external forces) has as a classical expression:

$$\mathbf{U}=\frac{1}{2}\int_{\Omega}\boldsymbol{\varepsilon}^{t}\boldsymbol{\Lambda}\,\boldsymbol{\varepsilon}\;d\,\Omega-\int_{\Omega}\mathbf{u}^{t}\mathbf{f}\;d\,\Omega-\int_{\partial\Omega_{\sigma}}\mathbf{u}^{t}\mathbf{t}\;d(\partial\Omega)$$

where $\boldsymbol{\varepsilon}$ is the vector associated with the tensor with the deformations, $\boldsymbol{\Lambda}$ is the matrix of behavior and where $\mathbf{f}$ and $\mathbf{t}$ are, respectively, the vectors of the voluminal and surface forces external.

One approaches the vector displacement by the finite element method. With this intention, one will use the classical functions of form described in the document `[R3.01.01]`. Displacement is written then in the shape of the product of a matrix of interpolation of displacements, noted $\mathbf{B}$, and of a vector of the generalized coordinates $\mathbf{q}$.

The energies kinetic and potential of the deformable body are written then according to the structural matrices as follows:

$$\mathbf{T}=\frac{1}{2}\dot{\mathbf{q}}^{t}[\mathbf{M}]\dot{\mathbf{q}}+\frac{1}{2}\dot{\mathbf{q}}^{t}[\mathbf{G}]\mathbf{q}-\frac{1}{2}\mathbf{q}^{t}[\mathbf{N}]\mathbf{q}-\mathbf{q}^{t}\int_{\Omega}\rho\,\mathbf{B}^{t}\boldsymbol{\Omega}^{2}\mathbf{x}\;d\,\Omega+\dot{\mathbf{q}}^{t}\int_{\Omega}\rho\,\mathbf{B}^{t}\boldsymbol{\Omega}\,\mathbf{x}\;d\,\Omega$$
$$-\frac{1}{2}\int_{\Omega}\rho\,\mathbf{x}^{t}\boldsymbol{\Omega}^{2}\mathbf{x}\;d\,\Omega$$

$$\mathbf{U}=\frac{1}{2}\mathbf{q}^{t}[\mathbf{K}]\mathbf{q}-\mathbf{q}^{\mathbf{t}}\int_{\Omega}\mathbf{B}^{t}\mathbf{f}\;d\,\Omega-\mathbf{q}^{\mathbf{t}}\int_{\partial\Omega_{\sigma}}\mathbf{B}^{t}\mathbf{t}\;d(\partial\Omega)$$

The first three terms of the kinetic energy highlight the matrices:

- of mass $[\mathbf{M}]=\int_{\Omega}\rho\,\mathbf{B}^t\,\mathbf{B}\;d\Omega$;
- of Coriolis $[\mathbf{G}]=2\int_{\Omega}\rho\,\mathbf{B}^t\boldsymbol{\Omega}\,\mathbf{B}\;d\Omega$;
- of centrifugal acceleration $[\mathbf{N}]=\int_{\Omega}\rho\;\mathbf{B}^t\boldsymbol{\Omega}^2\mathbf{B}\;d\Omega$.

The first term of the potential energy highlights the matrix of stiffness $[\mathbf{K}]=\int_{\Omega}\nabla\,\mathbf{B}^t\,\boldsymbol{\Lambda}\nabla\,\mathbf{B}\;d\Omega$.

3.3 Calculation of the equilibrium equations of the system in rotation

By disregarding a possible classical function of dissipation, characterized by the matrix of damping, the equations of Lagrange for the energies kinetic and potential of the solid are written as follows:

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial\mathbf{T}}{\partial\dot{\mathbf{q}}}\right)-\frac{\partial\mathbf{T}}{\partial\mathbf{q}}+\frac{\partial\mathbf{U}}{\partial\mathbf{q}}=0$$

While inserting the expressions of energies in the equations of Lagrange, one finds:

$$\frac{\mathrm{d}}{\mathrm{dt}}\left[[\mathbf{M}]\dot{\mathbf{q}}+\frac{1}{2}[\mathbf{G}]\mathbf{q}+\frac{1}{2}[\mathbf{G}]\dot{\mathbf{q}}+[\mathbf{N}]\,\mathbf{q}+\int_{\Omega}\rho\,\mathbf{B}^t\boldsymbol{\Omega}\,\mathbf{x}\;d\Omega+\int_{\Omega}\rho\,\mathbf{B}^t\boldsymbol{\Omega}^2\mathbf{x}\;d\Omega+[\mathbf{K}]\,\mathbf{q}-\int_{\Omega}\mathbf{B}^t\mathbf{f}\;d\Omega\right]$$
$$-\frac{\mathrm{d}}{\mathrm{dt}}\left[\int_{\partial\Omega_\sigma}\mathbf{B}^{\mathbf{t}}\,\mathbf{t}\;d(\partial\Omega)\right]=0$$

By clarifying the temporal derivative and by taking account of the derivative of the matrix of transformation in the equations of Lagrange, one obtains, after simplification and rearrangement of the terms, the following matric form:

$$[\mathbf{M}]\langle\ddot{\mathbf{q}}\rangle+[\mathbf{G}]\langle\dot{\mathbf{q}}\rangle+([\mathbf{K}]+[\mathbf{P}]+[\mathbf{N}])\langle\mathbf{q}\rangle=\langle\mathbf{r}\rangle$$

$[\mathbf{P}]$ is the matrix of angular acceleration, defined by $[\mathbf{P}]=\frac{1}{2}[\dot{\mathbf{G}}]=\int_{\Omega}\rho\,\mathbf{B}^t\dot{\boldsymbol{\Omega}}\,\mathbf{B}\;d\Omega$, with:

$$\dot{\boldsymbol{\Omega}}=\begin{bmatrix}0 & -\dot{\omega}_z & \dot{\omega}_y\\ \dot{\omega}_z & 0 & -\dot{\omega}_x\\ -\dot{\omega}_y & \dot{\omega}_x & 0\end{bmatrix}$$

$\langle\mathbf{r}\rangle$ is the vector joining together the terms of the member of right-hand side. He gathers the external excitations $\int_{\Omega}\mathbf{B}^{\mathbf{t}}\mathbf{f}\;d\Omega+\int_{\partial\Omega_\sigma}\mathbf{B}^t\,\mathbf{t}\;d(\partial\Omega)$ and centrifugal prestressing $-\int_{\Omega}\rho\,\mathbf{B}^t(\dot{\boldsymbol{\Omega}}\,\mathbf{x}+\boldsymbol{\Omega}^2\,\mathbf{x})\;d\Omega$).

4 Description of the versions

Version Aster	Author (S) Organization (S)	Description of the modifications
9.4	E. BOYERE, X. RAUD EDF/R & D AMA	Initial text
9,8	Mr. Torkhani	Correction of shells

	EDF/R & D AMA	