Law of behavior `BETON_DOUBLE_DP` with double Drücker-Prager criterion for the cracking and the compression of the concrete

Summary

The model presented in this document, behavior `BETON_DOUBLE_DP`, is a nonlinear law of behavior for the concrete. It is based on the theory of plasticity, it is valid for the three-dimensional states of stress. The assumptions of modeling selected are the following ones:

- a field of reversibility of the constraints delimited by two criteria of the type Drücker Prager,
- a work hardening of each criterion,
- in compression, a positive work hardening to a peak, then a negative work hardening,
- in traction, a negative work hardening exclusively,
- a dependence of the shape of the curves post-peak in both cases (traction/compression) with the size of the finite element (the shape of this curve is related on negative work hardening and the energy of cracking),
- normal plastic rules of flow (associated plasticity) and an isotropic formulation of work hardening,
- the taking into account of the dependence of the thresholds of elasticity compared to the temperature,
- the taking into account of the dependence of the Young modulus compared to the temperature.

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1 Notations

$\boldsymbol{\sigma}$ indicate the tensor of constraint, arranged in the form of vector according to convention:

$$\begin{pmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{12}\\ \sigma_{13}\\ \sigma_{23}\end{pmatrix}$$

One notes:

$I_1 = Trace(\boldsymbol{\sigma})$

$\sigma_H = \frac{1}{3} tr(\boldsymbol{\sigma})$ the hydrostatic constraint

$\mathbf{s} = \boldsymbol{\sigma} - \frac{1}{3} tr(\boldsymbol{\sigma})\mathbf{I}$ the diverter of the constraints

$\varepsilon_H = \frac{1}{3} tr(\boldsymbol{\varepsilon})$ voluminal deformation

$\tilde{\boldsymbol{\varepsilon}} = \boldsymbol{\varepsilon} - \frac{1}{3} tr(\boldsymbol{\varepsilon})\mathbf{I}$ the diverter of the deformations

$\dot{\tilde{\varepsilon}}_{eq} = \sqrt{\frac{3}{2}\mathrm{trace}(\dot{\tilde{\boldsymbol{\varepsilon}}}.\dot{\tilde{\boldsymbol{\varepsilon}}})}$ the rate of deformation is equivalent

$J_2 = \frac{1}{2} trace(\mathbf{s}^2)$ the second invariant of the constraints

$\sigma^{eq} = \sqrt{3\mathrm{J}_2} = \sqrt{\frac{3}{2} trace(\mathbf{s}^2)}$ the equivalent constraint

$\tau_{oct} = \sqrt{\frac{2}{3} J_2} = \sqrt{\frac{trace(\mathbf{s}^2)}{3}}$

$\sigma_{oct} = \sigma_H = \frac{I_1}{3} = \frac{trace(\boldsymbol{\sigma})}{3}$

f'_c initial limit of rupture in simple compression

f'_{cc} initial limit of rupture out of Bi compression

$\varphi f'_c$ elastic limit in compression

f'_t initial limit of rupture in traction

$\alpha = \frac{f'_t}{f'_c}$ relationship between rupture limit in traction and compression

$\beta = \frac{f'_{cc}}{f'_c}$ relationship between rupture limit in bi--compression and simple compression

κ_t^p plastic deformation in traction

λ_t plastic multiplier in traction

κ_c^p	plastic deformation in compression
λ_c	plastic multiplier in compression
$f_c(\kappa_c^p)$	curve of work hardening in compression
$f_t(\kappa_t^p)$	curve of work hardening in traction
κ_t^u	ultimate plastic deformation in traction
k_c^u	ultimate plastic deformation in compression
G_c^f	energy of rupture in compression (characteristic of material)
G_t^f	energy of rupture in traction (characteristic of material)
θ	the maximum of temperature during the history of loading

2 Introduction

2.1 Principal characteristics of the model

The model presented in this document is a nonlinear law of behavior for the concrete. It is based on the theory of plasticity, it is valid for the three-dimensional states of stress. The assumptions of modeling selected partly take again the models developed per G. Heinfling [bib2] and J.F. Georgin [bib1] and are the following ones:

- there exists a field of reversibility of the constraints delimited by two criteria of the type Drücker Prager,
- each criterion is hammer-hardened, the field of rupture corresponds to the maximum of the field of reversibility,
- in compression, work hardening is positive to a peak, then it becomes negative,
- in traction, work hardening is negative exclusively,
- the curves post-peak in both cases (traction/compression) vary with size of the finite element (the shape of this curve is related on negative work hardening and the energy of cracking),
- the plastic flow is governed by a rule of normality (associated plasticity) the formulation of work hardenings is of isotropic type,
- the modulus of elasticity and the thresholds of reversibility vary with the temperature.

Notice :

The terminology of criterion of traction and criterion of compression is debatable. We will use it by practice, while being quite conscious that a state of tensile stresses can lead to the activation of the criterion known as of compression.

2.2 Why two criteria of Drücker Prager

The authors of the theses referred to [bib1] and [bib2] use a criterion of Drücker Prager in compression and a criterion of Rankine in traction. They justify these choices by physical considerations by showing that the field of reversibility thus obtained is close to experimental reality. On the other hand they limit their modelings in states of two-dimensional stresses. We preferred to also replace the criterion of traction by a surface of the type Drücker Prager. By this choice, one frees oneself from certain difficulties particularly in the three-dimensional formulations. Surface 3D defining the working states of stresses with respect to traction is not any more one pyramid (Rankine 3D) but a conical surface whose top is located on the hydrostatic axis. The trace of the criterion “known as of traction” on the plan deviatoric is not any more one triangle, but a circle. The formulation obtained is simpler. The difference between the two criteria is tiny for states of stress close to states of plane stress. On the other hand, for the strongly confined states of stress, the two approaches (of Rankine and Drücker Prager) are different, which is a limit of the model suggested.

3 Field of reversibility and functions thresholds

3.1 Pace of the field and the thresholds of reversibility

The field of reversibility is the field of the space of the constraints inside whose the ways of constraint are reversible. Within the space of principal constraints $(\sigma_{1,}\sigma_{2,}\sigma_3)$, they are two cones whose axis is the trisecting one of equation $\sigma_1=\sigma_2=\sigma_3$. [Figure 3.1-a] a chart gives some.

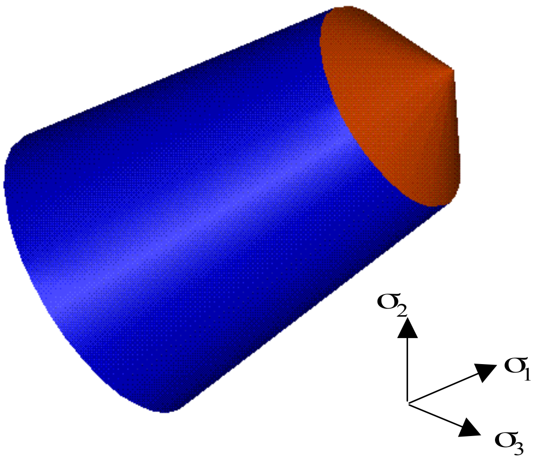


Figure 3.1 -has

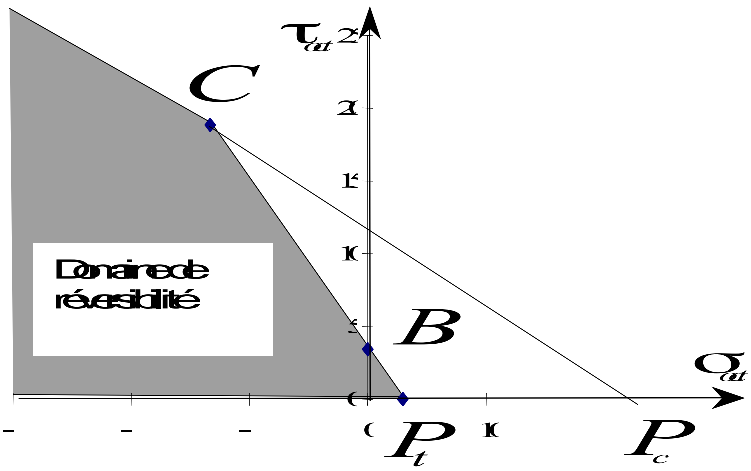


Figure 3.1 - B

In a plan $(\sigma_{oct}, \tau_{oct})$ the field of reversibility is determined by two straight lines as indicated on [Figure 3.1-b].
For a state of stress planes, the field of reversibility is the cut of the three-dimensional field by a plan of equation $\sigma_3 = \mathrm{cste}$, as indicated on [Figure 3.1-c], the result in a plan (σ_1, σ_2) being represented on the figure [Figure 3.1-d].

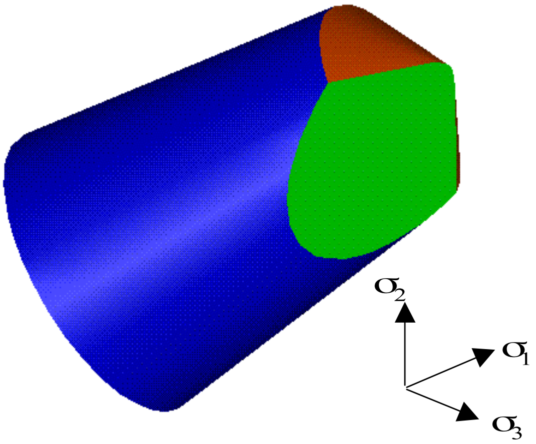


Figure 3.1 - C

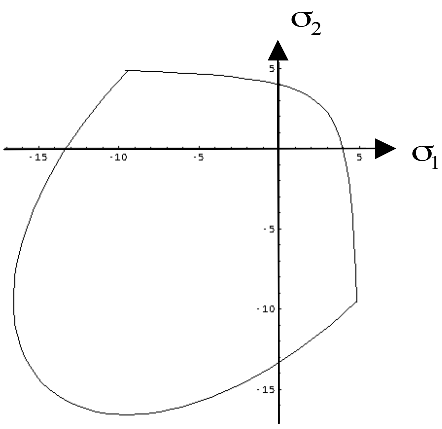


Figure 3.1 - D

3.2 Mathematical expression of the field of reversibility

It is defined by the inequation:

$$f(\boldsymbol{\sigma}, \boldsymbol{A}) \leq 0 \qquad \textbf{éq 3.2 -} \text{ 3.2-1}$$

in which $\boldsymbol{A}$ represent the thermodynamic forces associated with the internal variables (we note $\boldsymbol{\alpha}$ the whole of the internal variables).
For the model concrete that we present here, the equation [éq 3.2 - 3.2-1] takes the particular shape

$$f_{comp}(\boldsymbol{\sigma}, A_c) = \frac{\tau_{oct} + a.\sigma_{oct}}{b} - \varphi\ f'_c + A_c \leq 0 \qquad \textbf{éq 3.2 -} \text{ 3.2-2}$$

$$f_{trac}(\boldsymbol{\sigma}, A_t) = \frac{\tau_{oct} + c.\sigma_{oct}}{d} - f'_t + A_t \leq 0 \qquad \textbf{éq 3.2 -} \text{ 3.2-3}$$

$$f^H_{comp}(\boldsymbol{\sigma}, A_c) = \frac{a.\sigma_{oct}}{b} - \varphi\ f'_c + A_c \leq 0 \qquad \textbf{éq 3.2 -} \text{ 3.2-4}$$

$$f^H_{trac}(\boldsymbol{\sigma}, A_t) = \frac{c.\sigma_{oct}}{d} - f'_t + A_t \leq 0 \qquad \textbf{éq 3.2 -} \text{ 3.2-5}$$

Equations [éq 3.2 - 3.2-2] and [éq 3.2 - 3.2-3] correspond respectively to the thresholds of "compression" and "traction". Equations [éq 3.2 - 3.2-4] and [éq 3.2 - 3.2-5] limit the threshold of reversibility in the field of isotropic traction, they amount excluding the x-axis on [Figure 3.1-b] beyond the points P_t or P_c . It is clear that only one of these two last conditions is enough. For material not hammer-hardened, the choice of the coefficients is such as and $OP_t < OP_c$ the condition [éq 3.2 - 3.2-5] involves [éq 3.2 - 3.2-4].
We will see later that work hardening can reverse the order of the points P_t and P_c , returning the condition [éq 3.2 - 3.2-4] more constraining than [éq 3.2 - 3.2-5].

3.3 Criterion of rupture. choice of the coefficients has, B, C and D

When the state of stress reaches the edge of the field of reversibility, of the plastic deformations develop and the thresholds move: they are hammer-hardened. The threshold of compression "increases" initially, then decreases, whereas the threshold of traction can only decrease. The threshold of rupture corresponds to the maximum field being able to be reached, it is represented on [Figure 3.3-a] in a diagram of plane constraint:

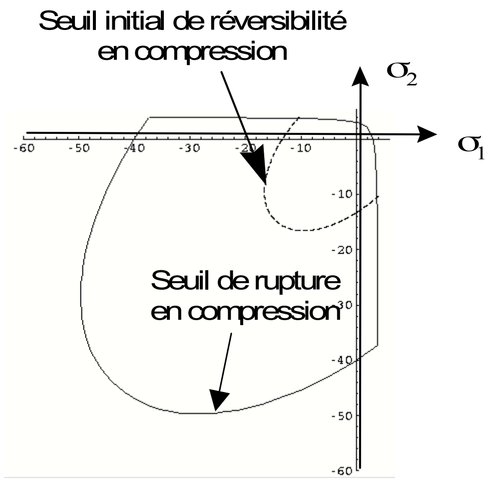


Figure 3.3 - has

The work hardening of the thresholds results mathematically in the evolution of the quantities A_c and A_t , thresholds of rupture corresponding to the maximum of the functions $f_c = \varphi f_c' - A_c$ and $f_t = f_t' - A_t$. In the models selected, these functions are such as: $Max\, f_c = f_c'$ and $Max\, f_t = f_t'$;

Coefficients *has, B, C* , and *D* are thus defined from:

- F $_T$ ‘: resistance in axial traction plain of the concrete,
- F $_C$ ‘: resistance in axial compression plain of the concrete,
- F $_{DC}$ ‘: resistance in axial compression Bi of the concrete,

One defines moreover coefficients: $\alpha = \frac{f_t'}{f_c'}$ and $\beta = \frac{f_{cc}'}{f_c'}$

To determine the coefficients has, B, C and D it is necessary to give oneself 4 equations which express in fact that the criteria are reached for states of stresses particular and judiciously selected.
A first possibility consists in writing that the two criteria are cut on the axes simple compression (points C of [Figure 3.3-b]).

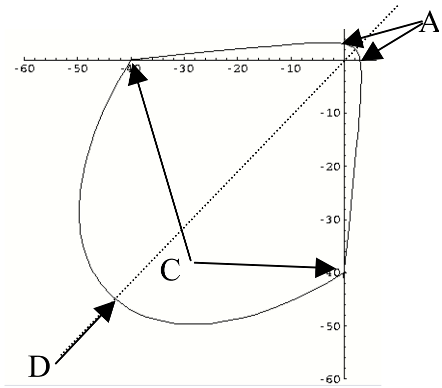


Figure 3.3 - B

By recalling that:

In simple compression: $\sigma < 0$; $\sigma_{oct} = \frac{\sigma}{3}$; $\tau_{oct} = -\frac{\sqrt{2}}{3}\sigma$

Out of Bi compression $\sigma < 0$; $\sigma_{oct} = 2\frac{\sigma}{3}$; $\tau_{oct} = -\frac{\sqrt{2}}{3}\sigma$

In simple traction $\sigma > 0$; $\sigma_{oct} = \frac{\sigma}{3}$; $\tau_{oct} = \frac{\sqrt{2}}{3}\sigma$

The following relations then are obtained:

Number of condition	State of stress	Criterion reached	relation obtained
1	Simple compression	Compression	$a + 3b = \sqrt{2}$

2	Bi compression	Compression	$2a+\frac{3}{\beta}b=\sqrt{2}$
3	Simple traction	Traction	$-c+3d=\sqrt{2}$
4	Simple compression	Traction	$c+3\alpha d=\sqrt{2}$

Table 3.3 - has

Who gives, while posing: $\alpha=\frac{f'_t}{f'_c}$ and $\beta=\frac{f'_{cc}}{f'_c}$

$$a=\sqrt{2}\frac{\beta-1}{2\beta-1} \quad b=\frac{\sqrt{2}}{3}\frac{\beta}{2\beta-1}$$ **éq 3.3 -** 3.3-1

$$c=\sqrt{2}\frac{1-\alpha}{1+\alpha} \quad d=\frac{2\sqrt{2}}{3}\frac{1}{1+\alpha}$$ **éq 3.3 -** 3.3-2

But this choice is problematic.

Indeed, after work hardening of the criterion of traction, and for a limit of traction become worthless the field of admissibility takes the shape indicated on [Figure 3.3-c], making nonacceptable of the Bi compressions states.

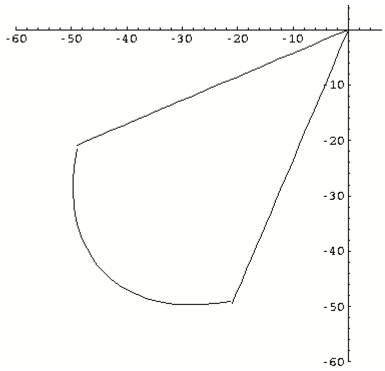


Figure 3.3 - C

Moreover, with this choice of the coefficients, certain ways of simple traction compression presented snap-back as indicated in appendix.

We then preferred to replace the condition number 4 of [Table 3.3-a] by a condition expressing that, after the limit of traction fell down to zero, the field of reversibility is that represented on [Figure 3.3-d].

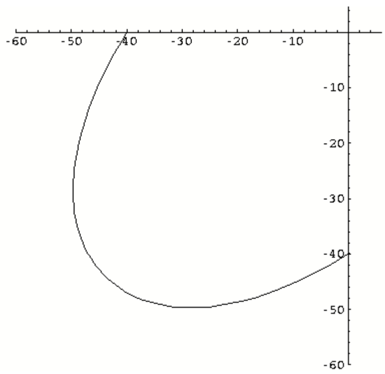


Figure 3.3 - D

This resulted in replacing the relation $c+3\alpha d=\sqrt{2}$ by $c=\sqrt{2}$
The choice of the coefficients has, B, C and D is finally:

$$a=\sqrt{2}\frac{\beta-1}{2\beta-1} \quad b=\frac{\sqrt{2}}{3}\frac{\beta}{2\beta-1}$$ **éq 3.3 -** 3.3-3

$$c=\sqrt{2} \quad d=\frac{2\sqrt{2}}{3}$$ **éq 3.3 -** 3.3-4

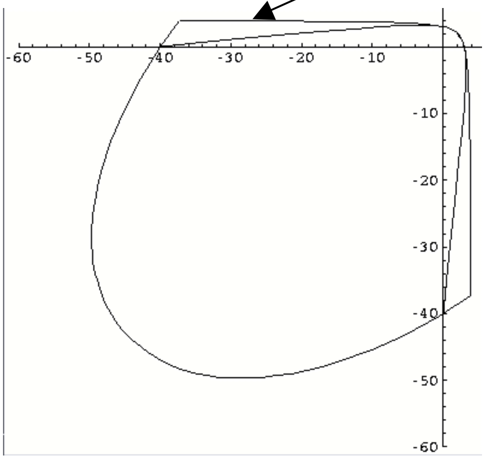


Figure 3.3 - E

[Figure 3.3-e] the watch the difference between the two models for a state in plane constraint.

3.4 Analysis of the field of reversibility retained

In this chapter, we give indications on the order of magnitude of working stresses within the meaning of the criterion selected. We endeavour to give indications on tensile stresses, in particular for three-dimensional states of stress.
[Figure 3.4-a] the watch initial fields (i.e. before work hardening) for the following values of the parameters materials:

f'_c	initial limit of rupture in simple compression:	f'_c = 40 Mpa
f'_{cc}	initial limit of rupture out of Bi compression	f'_{cc} =44 Mpa
$\beta=\frac{f'_{cc}}{f'_c}$	relationship between rupture limit in bi--compression and simple compression	β = 1.1
$\varphi f'_c$	elastic limit in compression;	$\phi=0,33$
f'_t	initial limit of rupture in traction	f'_t = 4 Mpa

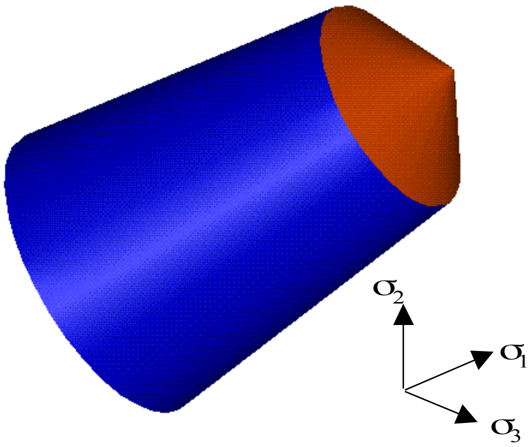


Figure 3.4 - has

Figures [Figure 3.4-b], [Figure 3.4-c] and [Figure 3.4-d] the cuts of the three-dimensional field by the plane ones show $\sigma_3=0$ and $\sigma_3=-25\,\text{Mpa}$

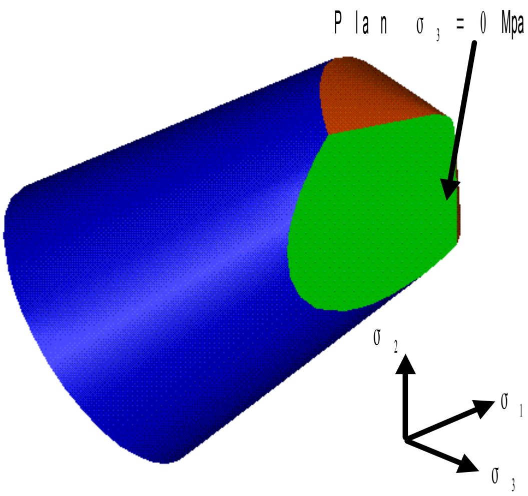


Figure 3.4 - B

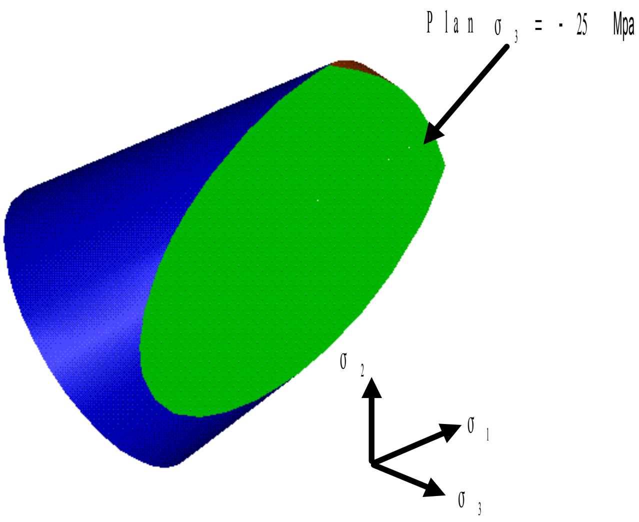


Figure 3.4 - C

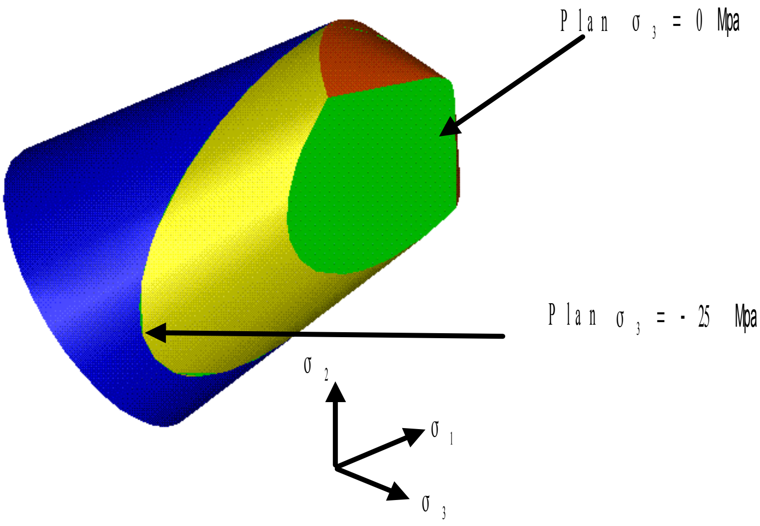


Figure 3.4 - D

[Figure 3.4-e] the watch fields of reversibility in a plan (σ_1, σ_2) for states of stress σ_3 constant, fields parameterized by the value of σ_3 . We represent the fields for $\sigma_3 = -25$ Mpa , $\sigma_3 = 0$ Mpa , $\sigma_3 = 4\,\text{Mpa}$, $\sigma_3 = 10\,\text{Mpa}$, $\sigma_3 = 15\,\text{Mpa}$. It is seen there that for a containment of 25 Mpa of compression, tensile stresses can reach $15\,Mpa$, and that, in parallel, the field of reversibility for $\sigma_3 = 15\,\text{Mpa}$ is not empty and corresponds to compressive stresses σ_1 and σ_2 order of $-25\,Mpa$. It is also seen, that, for a value given of σ_3 , the maximum value of traction

Obtained for σ_1 and σ_2 is reached with the intersection of the criteria of traction and compression.

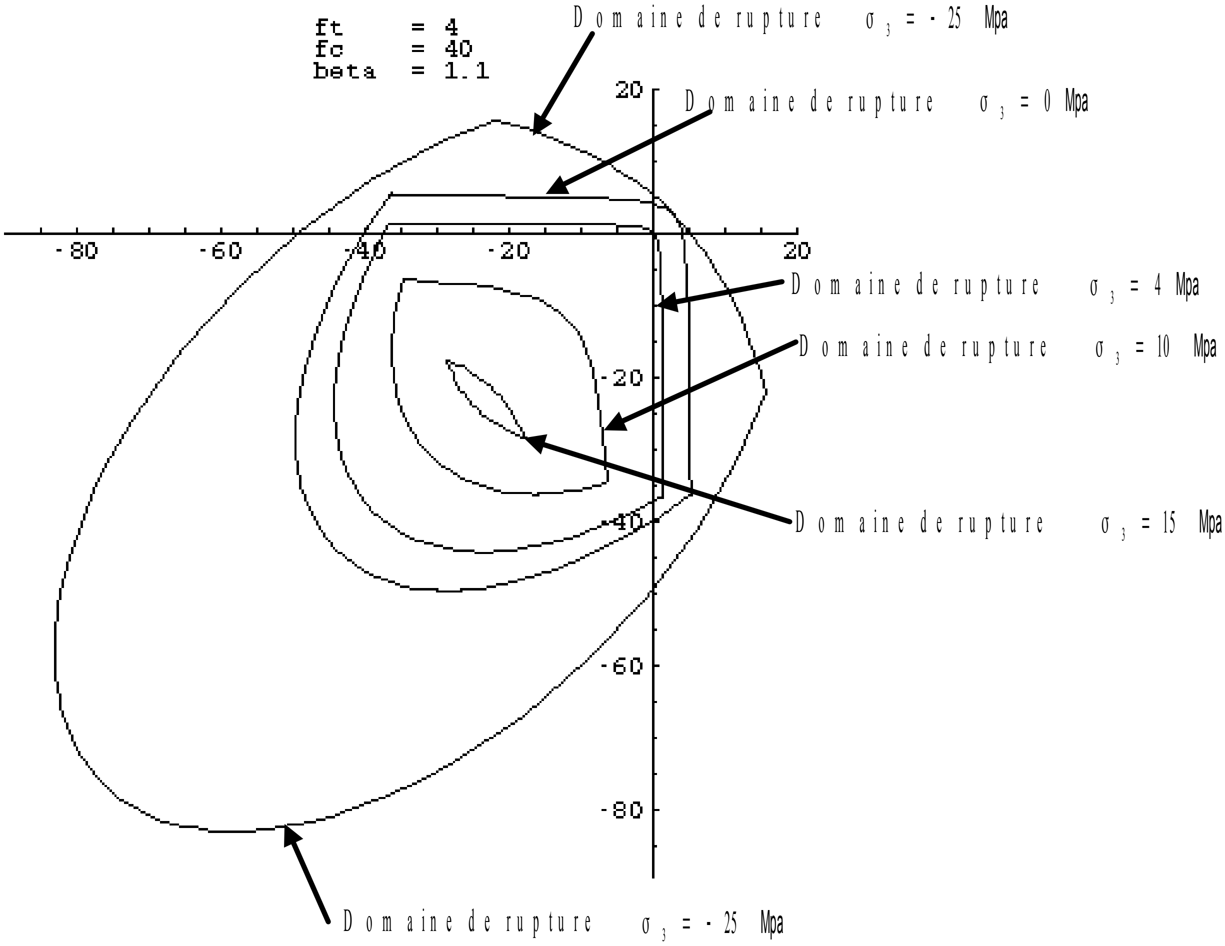


Figure 3.4 - E

We thus study the place of intersection of the criteria of traction and compression. We note $\left(\sigma_H^0, \sigma_0^{eq}\right)$ the point of intersection of the two criteria in the plan $\left(\sigma_H, \sigma^{eq}\right)$ (not C of [Figure 3.4-f]).

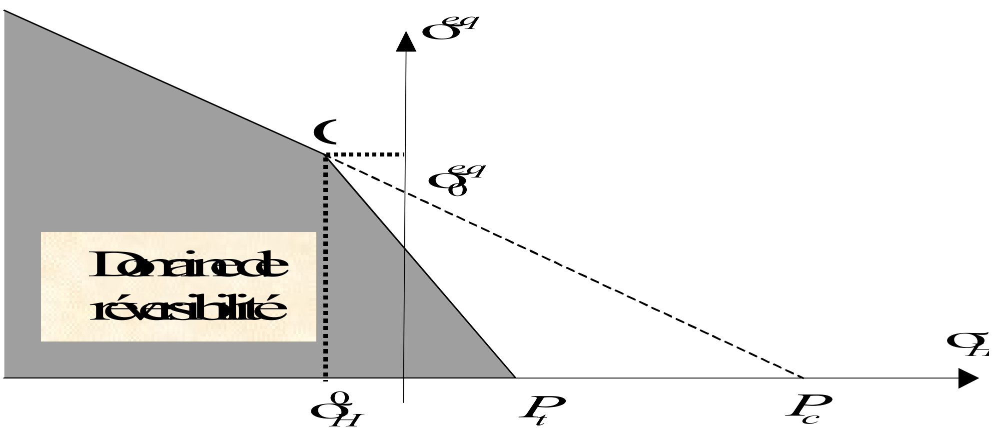


Figure 3.4 - F

The place of intersection of the two criteria within the space of constraints is given by:

$$\begin{cases} \sigma_1 = \frac{2}{3}\sigma_0^{eq}\sin\left(\theta+\frac{\pi}{6}\right)+\sigma_H^0 \\ \sigma_2 = \frac{2}{3}\sigma_0^{eq}\sin\left(-\theta+\frac{\pi}{6}\right)+\sigma_H^0 \\ \sigma_3 = 3\sigma_H^0-\sigma_1-\sigma_2 \end{cases}$$

Where θ is a parameter.

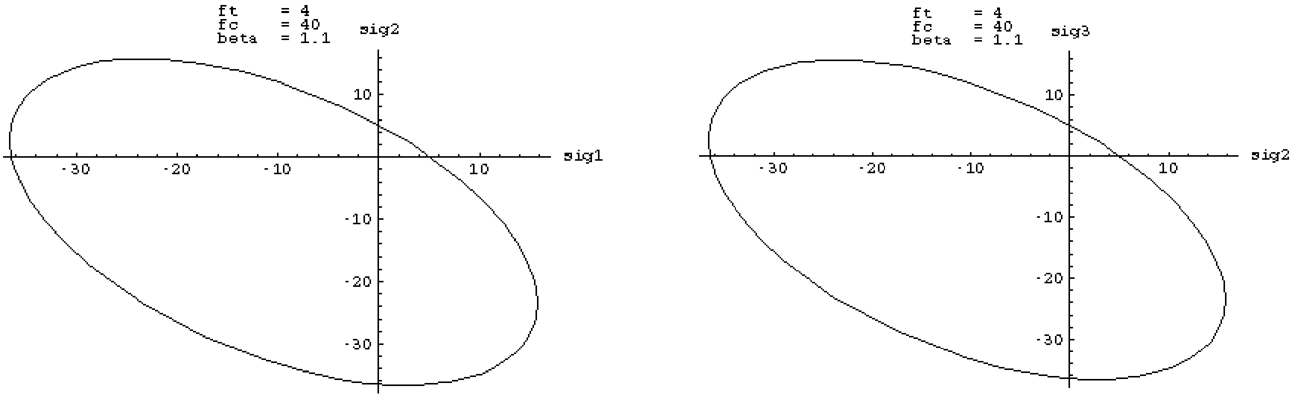


Figure 3.4 - G

[Figure 3.4-g] the watch projections of this place in the plans $(\sigma_{1,}\sigma_2)$ and $(\sigma_{2,}\sigma_3)$.
One can easily calculate the maximum value of the constraint along this curve:

$$\sigma_{max} = \frac{f_c'}{3}+\frac{2}{3\beta}f_t' \qquad \textbf{éq 3.4 -} \text{ 3.4-1}$$

This equation shows that, whatever the value chosen for the rupture limit in traction, the maximum constraint reached in traction is higher than the third of the rupture limit in compression.

[Figure 3.4-h] the watch three principal constraints according to the parameter θ .

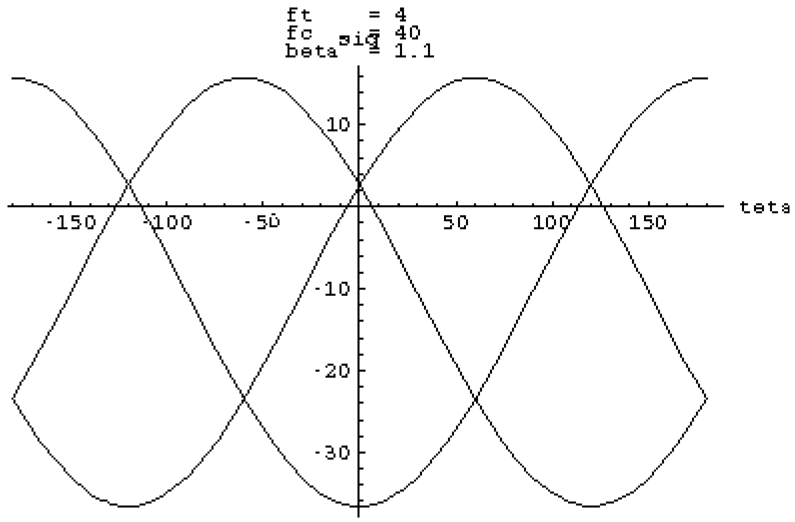


Figure 3.4 - H

It is seen that one can reach a level of traction of $15\,Mpa$, but for a containment of $\sigma_2=-25\,\text{Mpa}$ and $\sigma_3=-25\,\text{Mpa}$.

To try to avoid this disadvantage, which is important, one can try to exploit the values of resistance in compression and the parameter β .
As example, we chose the following game of parameters:

$$f'_c=20\,Mpa$$
$$f'_{cc}=40\,Mpa$$
$$\beta=2$$
$$f'_t=4\,Mpa$$

[Figure 3.4-i] the watch criteria with this choice of parameters. [Figure 3.4-j] the watch the value of the principal constraints to the intersection of the two criteria for this new choice of parameters. Maximum traction obtained is weaker ($8\,Mpa$), but it is reached for a level of containment also low (-7Mpa).

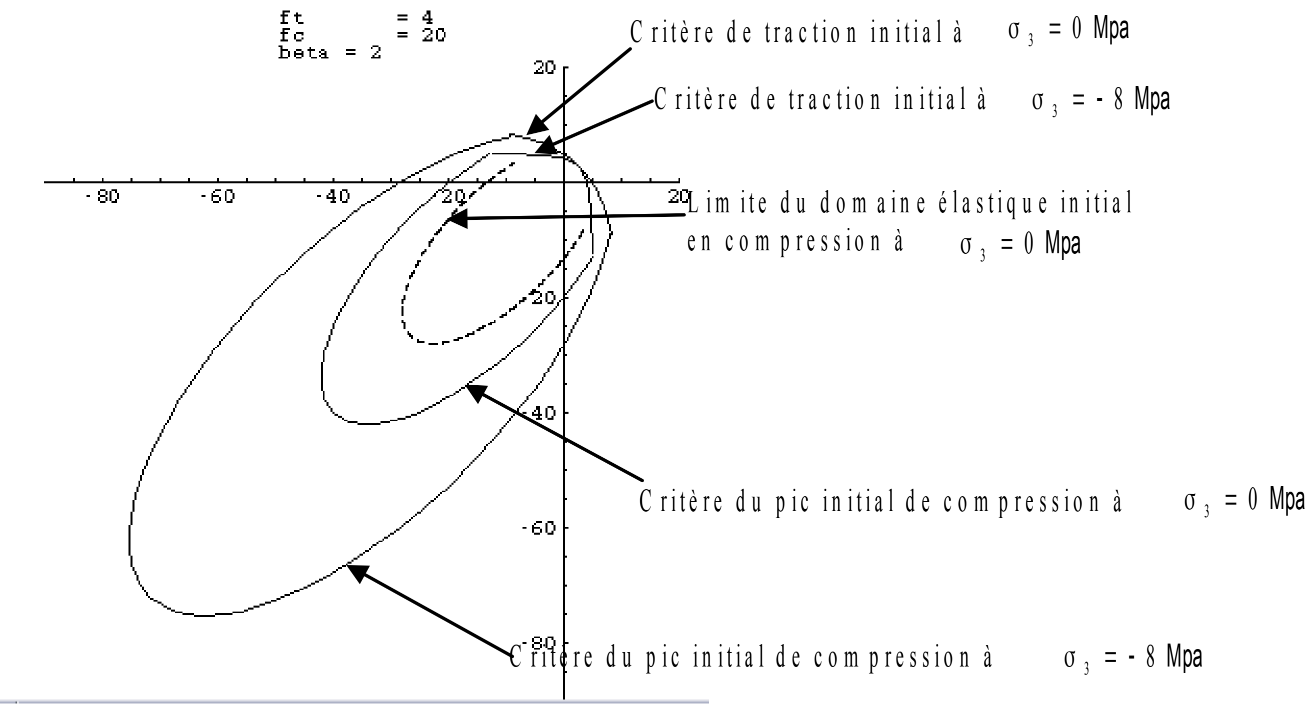


Figure 3.4 - I

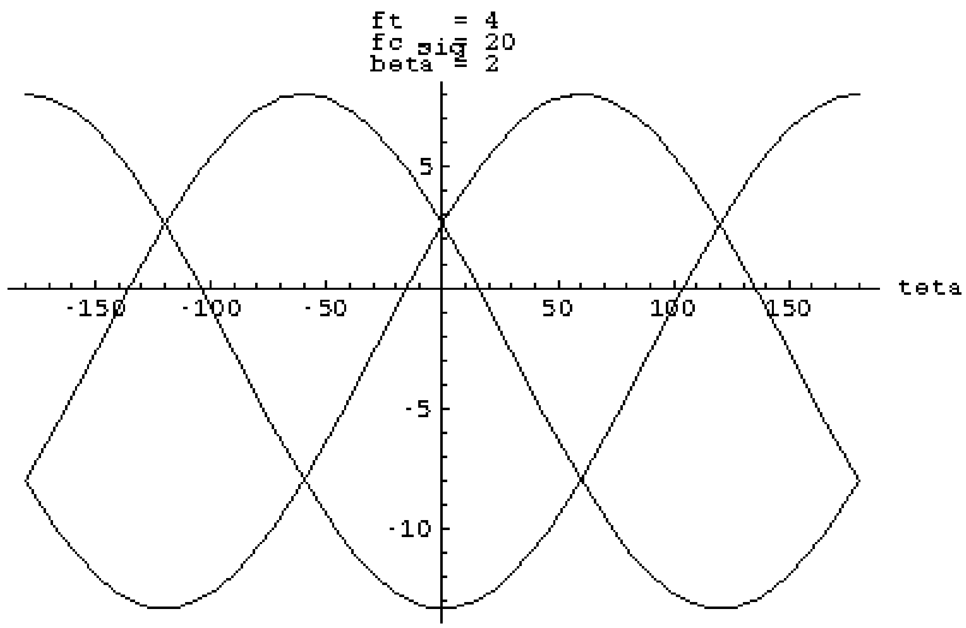


Figure 3.4 - J

3.5 Work hardening

As we already mentioned in the paragraph [§ 3.3], when the state of stress reach the edge of the field of reversibility, the plastic deformations and the variables internal develop, the thresholds move: they are hammer-hardened. For our model, the internal variables are two, they are noted κ_c^p for the internal variable "known as of compression" and κ_t^p for that "known as of traction". These variables determine the evolution of the thresholds of compression and of traction respectively, the thermodynamic forces theirs are connected by the relations:

$$A_c = \varphi\ f'_c - f_c\left(\kappa_c^p\right)$$ **éq 3.5 -** 3.5-1

and

$$A_t = f'_t - f_t\left(\kappa_t^p\right)$$ **éq 3.5 -** 3.5-2

where $f_c\left(\kappa_c^p\right)$ and $f_t\left(\kappa_t^p\right)$ the values of resistances in compression and traction represent respectively.

3.5.1 Functions of work hardening

The function $f_c\left(\kappa_c^p\right)$ is initially increasing then decreasing, the decreasing part being either linear [Figure 3.5.1-a], or quadratic [Figure 3.5.1-b],

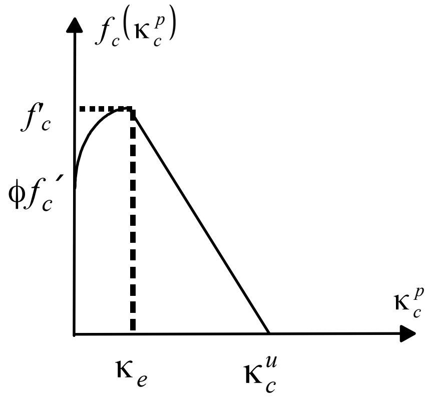


Figure 3.5.1-a

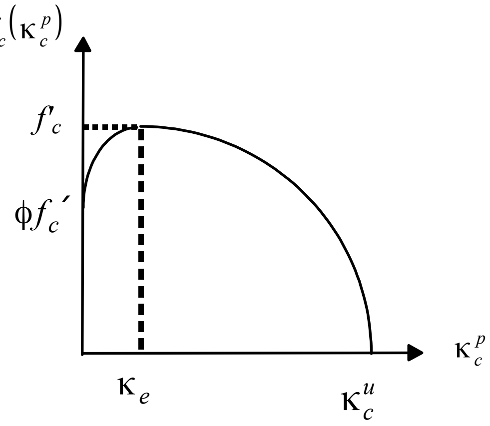


Figure 3.5.1-b

φ is a data of the model. The shape of the curve enters κ_c^e and κ_c^e (negative work hardening) depends on the element, and more precisely on its dimensions, according to a criterion similar to that chosen by G. Heinfling, [Error: Reference source not found] for the taking into account of the localization of the deformations.

In traction, shape of the curve giving the value of the elastic limit $f_t(\kappa_t^p)$ according to the cumulated plastic deformation κ_t^p do not comprise a part "pre-peak", the part "post-peak" being either linear [Figure 3.5.1-c], or exponential [Figure 3.5.1-d].

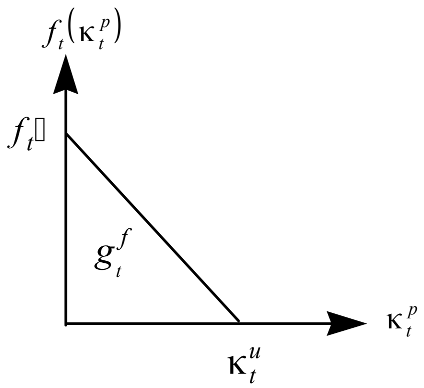


Figure 3.5.1-c

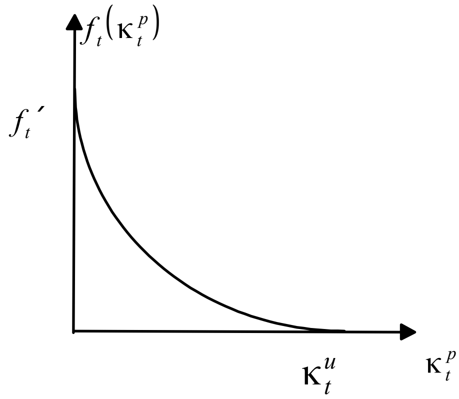


Figure 3.5.1-d

3.5.2 Curves of work hardening and modules post peak

3.5.2.1 Model of cracking distributed

The introduction of a behavior softening post-peak into the relations stress-strains poses major problems. Under stress statical, beyond a certain level of constraint, corresponding to the starter of the lenitive behavior, the equations governing the balance of the structure lose their elliptic nature. These equations of the mechanical problem then form a system of partial derivative equations evil posed of which the number of solutions is multiple. This problem results in an not-objectivity compared to the grid. It results from this a pathological sensitivity of the digital solution to the smoothness and the orientation of the grid.

In order to solve this problem, or at least, to limit the consequences of them on the reliability of the predicted solution, it is necessary to use techniques known as of regularization. The object of these techniques is to enrich the mechanical description of the medium, to be able to describe nonhomogeneous states of deformation, and to preserve the mathematical nature of the problem. One operates this regularization while introducing, in the law of behavior, a characteristic length or internal length, connected to the width of the zone of localization. Several techniques are possible to improve the mechanical description of the lenitive medium. They constitute limitings device of localization. The implementation of these techniques requires in general, of the delicate digital developments. An intermediate approach between the use of the classical models and the placement of these limitings device of localization consists in making depend the slope post-peak on the relation stress-strain, of the size of the element, so as to dissipate with the rupture a constant energy. This approach constitutes a step towards a nonlocal description of the continuous medium.

Let us consider initially a real crack of surface S whose measurement is A [3.5.2.1 Figure - has]. S is a surface of discontinuity of the field of displacement $\mathbf{u}$. It is supposed that to create this discontinuity, it is necessary to spend an energy W whose expression is: $W=\int_S G_f(\mathbf{x})dS$, G_f being a property of material.

Let us consider now that one wants to represent the same phenomenon, by not representing a discontinuity of displacement but a plastic deformation uniformly distributed in a volume V .

Dissipated energy will be: $W=\int_v dV\int_0^{t_r}\sigma^{ij}\frac{d\,\varepsilon_{ij}^p}{dt}dt$, where one noted t_r “time-to-failure”.

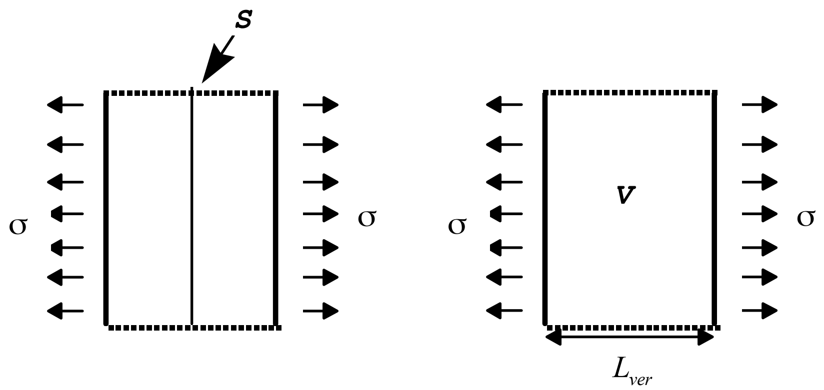


Figure 3.5.2.1 - has

By making the series of following assumptions:

- the crack is plane,
  - G_f is constant along the crack and thus $W = A.G_f$,
  - V is a basic cylinder S and a height L_{ver} ,
  - $g_{f} = \int_0^{t_t} \sigma^{ij} \frac{d\,\varepsilon_{ij}^p}{dt} dt$ is constant in V .

One leads finally to the relation:

$$W = V g_f = V \int_0^{t_t} \sigma^{ij} \frac{d\,\varepsilon_{ij}^p}{dt} dt = A.G_f \quad \text{éq 3.5.2.1 - 1}$$

Or:

$$g^f = \int_0^{t_r} \sigma^{ij} \frac{d\,\varepsilon_{ij}^p}{dt} dt = \frac{G_f}{L_{ver}} \quad \text{éq 3.5.2.1 - 2}$$

It is seen easily that: $g^f = \int_0^{\kappa_u} f(\kappa)\, d\kappa$, writing in which quantities $(g^f, f, \kappa_u, \kappa)$ represent respectively $(g_t^f, f_t, \kappa_t^u, \kappa_t^p)$ in traction and $(g_c^f, f_c, \kappa_c^u, \kappa_c^p)$ in compression. The data of g^f thus determine k_u , this in traction as in compression:

$$g_c^f = \int_0^{k_c^u} f(\kappa_c^p)\, d\,\kappa_c^p = \frac{G_c^f}{L_{ver}}$$

$$g_t^f = \int_0^{k_t^u} f(\kappa_t^p)\, d\,\kappa_t^p = \frac{G_t^f}{L_{ver}}$$

Quantity g^f is thus related to the slope of the curve post peak in a diagram constraint-variable of work hardening, which is related to the forced slope post peak in a diagram deformation.
Let us suppose for example that the forced relation deformation is linear in mode post peak. Let us call $E_T<0$ the slope post peak in the diagram $(\boldsymbol{\sigma},\boldsymbol{\varepsilon})$ and $h<0$ the corresponding slope in diagram (f,κ) [3.5.2.1 Figure - B]. There is the relation $h=\frac{EE_T}{E-E_T}\Leftrightarrow E_T=\frac{hE}{E+h}$ who show that one must have:
$-h<E$, or else the diagram $(\boldsymbol{\sigma},\boldsymbol{\varepsilon})$ present a snap back.

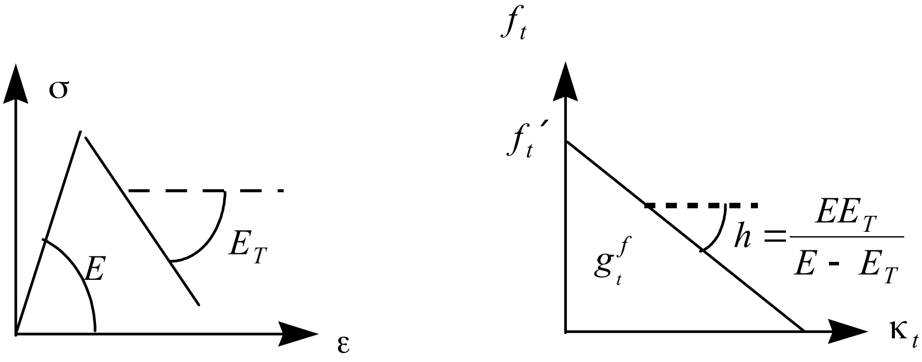


Figure 3.5.2.1 - B

The condition $-h<E$ is known as condition of applicability, it will result in an inequality on g^f and thus on L_{ver} .
Within the framework of a resolution by the finite element method, ground volume representative of the fissured medium can be comparable to an element of the grid. The characteristic length (noted thereafter l_c) introduced into the method of the energy of equivalent rupture corresponds to the length $Lver$. During a calculation corresponding to an unspecified structure, the determination this characteristic length is delicate. It depends on the position of the plan of crack, dimensions and the type of the elements…
A simple estimate for the two-dimensional cases can be expressed in the form:
$l_c=r\sqrt{A_e}$ where A_e is the surface of the element considered, and r , a correct factor, being worth 1 for the quadratic elements, and $\sqrt{2}$ for the linear elements.
One can extend this formulation to the case 3D: $l_c=r\sqrt[3]{V_e}$ where V_e indicate the volume of the element.
Concerning the evolution of work hardening with the temperature, we regard as in [bib2] that energies of rupture and resistances to rupture depend not on the current temperature T material point considered at time t , but of the maximum temperature reached in this point since the beginning of the loading until time t . When we need to show the dependence of the quantities compared to the temperature, we will note:

- θ the maximum of temperature since the beginning of loading,
- $f_c^{'}(\theta)$ resistance in compression,
- $f_t^{'}(\theta)$ indicate resistance in traction,
- $f_c(\theta,\kappa_c^p)$ the curve of work hardening in compression,
- $f_t(\theta,\kappa_t^p)$ the curve of work hardening in traction.

3.5.2.2 Behavior of the concrete in traction and linear curve post-peak

In this modeling, the concrete is supposed to be elastic until its resistance in traction f'_t. The curve $f_t(\kappa_t^p)$ in traction is represented on [Figure 3.5.1-c] and is entirely defined by resistance in traction of material, the energy of cracking G_t^f, and the characteristic length l_c.
The mathematical expression of this curve is:

$$f_t(\theta,\kappa_t^p)=f'_t(\theta)\left(1-\frac{\kappa_t^p}{\kappa_t^u(\theta)}\right) \quad \text{éq 3.5.2.2 - 1}$$

The equivalence of dissipated energy makes it possible to write:

$$G_t^f(\theta)=l_c\int_0^{\kappa_t^u} f_t(\theta,\kappa_t^p)=l_c f'_t(\theta)\int_0^{\kappa_t^u}\left(1-\frac{\kappa_t^p}{\kappa_t^u(\theta)}\right)d\kappa_t^p$$

from where

$$G_t^f(\theta)=\frac{l_c.f'_t(\theta).\kappa_t^u(\theta)}{2} \quad \text{éq 3.5.2.2 - 2}$$

and

$$\kappa_t^u(\theta)=\frac{2.G_t^f(\theta)}{l_c.f'_t(\theta)} \quad \text{éq 3.5.2.2 - 3}$$

The condition of applicability is written:

$$l_c \le \frac{2.E(\theta).G_t^f(\theta)}{f_t'^2(\theta)} \quad \text{éq 3.5.2.2 - 4}$$

3.5.2.3 Behavior of the concrete in traction and exponential curve post-peak

In this modeling, the concrete is supposed to be elastic until its resistance in traction f_t'. The curve $f_t(\kappa_t^p)$ in traction is represented on [Figure 3.5.1-d] and is entirely defined by resistance in traction of material, the energy of cracking G_t^f, and the characteristic length l_c.
The mathematical expression of this curve is:

$$f_t(\theta,\kappa_t^p)=f'_t(\theta).\exp\left(-a\frac{\kappa_t^p}{\kappa_t^u(\theta)}\right) \quad \text{éq 3.5.2.3 - 1}$$

The equivalence of dissipated energy makes it possible to write:

$$G_t^f(\theta)=l_c\int_0^{\infty} f_t(\theta,\kappa_t^p)=l_c f_{t'}(\theta)\int_0^{\infty}\exp\left(-a\frac{\kappa_t^p}{\kappa_t^u(\theta)}\right)d\kappa_t^p$$

From where:

$$G_t^f(\theta)=\frac{l_c.f'_t(\theta).\kappa_t^u(\theta)}{a} \quad \text{éq 3.5.2.3 - 2}$$

and

$$\frac{\kappa_t^u(\theta)}{a}=\frac{G_t^f(\theta)}{l_c.f'_t(\theta)} \quad \text{éq 3.5.2.3 - 3}$$

That is to say still: $f_t(\theta,\kappa_t^p)=f_t'(\theta).\exp\left(-l_c.f_t'(\theta)\dfrac{\kappa_t^p}{G_t^f(\theta)}\right)$

The maximum slope of the curve is then $h_{\max}(\theta)=\dfrac{-l_c.f_t'^2(\theta)}{G_t^f(\theta)}$

and the condition of applicability is written:

$$l_c\leq\frac{E(\theta).G_t^f(\theta)}{f_t'^2(\theta)}\qquad\forall\theta \qquad \textbf{éq 3.5.2.3 - 4}$$

3.5.2.4 Behavior of the concrete in compression and linear curve post-peak

In this modeling, the behavior of the concrete is supposed to be elastic until the elastic limit, given by a proportionality factor (noted φ. expressed as a percentage of resistance to the peak $f_c'(\theta)$. For the standard concretes φ is about 30%. The curve $f_c(\kappa_c^p)$ in compression is represented on [Figure 3.5.1-a] and is entirely defined by resistance in traction of material, the energy of cracking G_c^f, and the characteristic length l_c .
The mathematical expression of this curve is:

$$\begin{cases} f_c(\theta,\kappa_c^p)=f'_c(\theta)\left(\varphi+(2-2\phi)\dfrac{\kappa_c^u}{\kappa_e(\theta)}+(\varphi-1)\dfrac{\kappa_c^{p^2}}{\kappa_{e^2}(\theta)}\right) & \text{si} \quad \kappa_c^p\leq\kappa_e(\theta) \qquad \textit{éq 3.5.2.4−1}\\ f_c(\theta,\kappa_c)=f'_c(\theta)\left(\dfrac{(\kappa_c^p-\kappa_c^u(\theta))}{(\kappa_e(\theta)-\kappa_c^u(\theta))}\right) & \text{si} \quad \kappa_e(\theta)\leq\kappa_c^p\leq\kappa_c^u(\theta) \qquad \textit{éq 3.5.2.4−2}\end{cases}$$

Resistance in maximum compression is reached when: $\kappa_e(\theta)=(2-2\varphi)\dfrac{f_c'(\theta)}{E(\theta)}$

The equivalence of dissipated energy makes it possible to write:

$$G_c^f(\theta)=l_c\int_0^{\kappa_c^u}f_c(\theta,\kappa_c^p)\,d\kappa_c^p$$

from where

$$G_c^f(\theta)=l_c.f_c'(\theta)\left(\frac{2\varphi+1}{6}\kappa_e(\theta)+\frac{1}{2}\kappa_c^u(\theta)\right) \qquad \textbf{éq 3.5.2.4 - 3}$$

and

$$\kappa_c^u(\theta)=\frac{2.G_c^f(\theta)}{l_c.f_c'(\theta)}-\frac{2\varphi+1}{3}\kappa_e(\theta) \qquad \textbf{éq 3.5.2.4 - 4}$$

The slope of the curve is then $h(\theta)=-\dfrac{f_c'(\theta)}{\kappa_c^u(\theta)-\kappa_e(\theta)}$

and the condition of applicability is written:

$$l_c\leq\frac{E(\theta).G_c^f(\theta)}{f_c'^2(\theta)}\frac{6}{11-4\varphi-4\varphi^2}\qquad\forall\theta \qquad \textbf{éq 3.5.2.4 - 5}$$

3.5.2.5 Behavior of the concrete in compression and nonlinear curve post-peak

In this modeling, the behavior of the concrete is supposed to be elastic until the elastic limit, given by a proportionality factor (noted φ. expressed as a percentage of resistance to the peak . $f'_c(\theta)$. For the standard concretes φ is about 30%. The curve $f_c(\kappa_c^p)$ in compression is represented on [Figure 3.5.1-b] and is entirely defined by resistance in traction of material, the energy of cracking G_c^f , and the characteristic length l_c .
The mathematical expression of this curve is:

$$\begin{cases} f_c(\theta,\kappa_c^p)=f'_c(\theta)\left(\varphi+(2-2\varphi)\dfrac{\kappa_c^p}{\kappa_e(\theta)}+(\varphi-1)\dfrac{\kappa_c^{p^2}}{\kappa_{e^2}(\theta)}\right) & \text{si} \quad \kappa_c^p\le\kappa_e(\theta) \qquad \textit{éq 3.5.2.5-1} \\ f_c(\theta,\kappa_c)=f'_c(\theta)\left(1-\dfrac{\left(\kappa_c^p-\kappa_e(\theta)\right)^2}{\left(\kappa_c^u(\theta)-\kappa_e(\theta)\right)^2}\right) & \text{si} \quad \kappa_e(\theta)\le\kappa_c^p\le\kappa_c^u(\theta) \qquad \textit{éq 3.5.2.5-2} \end{cases}$$

Resistance in maximum compression is reached when: $\kappa_e(\theta)=(2-2\varphi)\dfrac{f'_c(\theta)}{E(\theta)}$

The equivalence of dissipated energy makes it possible to write:

$$G_c^f(\theta)=l_c\int_0^{\kappa_c^u} f_c(\theta,\kappa_c^p)\,d\kappa_c^p$$

from where:

$$G_c^f(\theta)=l_c.f'_c(\theta)\left(\frac{2}{3}\kappa_c^u(\theta)+\frac{\varphi}{3}\kappa_e(\theta)\right) \qquad \textbf{éq 3.5.2.5 - 3}$$

and:

$$\kappa_c^u(\theta)=\frac{3}{2}\frac{G_c^f(\theta)}{l_c.f'_c(\theta)}-\frac{\varphi}{2}\kappa_e(\theta) \qquad \textbf{éq 3.5.2.5 - 4}$$

The maximum slope of the curved post-peak is then $h_{\max}(\theta)=-\dfrac{2.f'_c(\theta)}{\kappa_c^u(\theta)-\kappa_e(\theta)}$

and the condition of applicability is written:

$$l_c\le\frac{3}{2}\frac{E(\theta).G_c^f(\theta)}{f_c'^2(\theta)}\frac{1}{4-\varphi-\varphi^2} \qquad \forall\theta \qquad \textbf{éq 3.5.2.5 - 5}$$

4 Plastic flow

In this paragraph, we give the expression speeds of plastic deformation, by distinguishing the case general says where the state of stress is located on a “regular” zone of the edge of the field of reversibility and the case where it is at the top of one of the cones.

4.1 General form of the rule of normality

In space $(\boldsymbol{\sigma}, A)$, inequalities [éq 3.2 - 3.2-2], [éq 3.2 - 3.2-3], [éq 3.2 - 3.2-4], [éq 3.2 - 3.2-5], define a convex field which we will note $C_{(\sigma, A)}$. We will note Ψ_c the indicating function of this convex:

$$\psi_c(\boldsymbol{\sigma}, A) = \begin{cases} 0 \text{ si } (\boldsymbol{\sigma}, A) \in C_{(\sigma, A)} \\ \infty \text{ sinon} \end{cases}$$ éq 4.1 - 4.1-1

When the border of the field of reversibility is reached, of the irreversible plastic deformations develop, according to the classical theory of plasticity.

For a standard material, [bib4] the law of flow checks the principle of maximum work plastic, which results in the equation:

$$(\dot{\boldsymbol{\varepsilon}}^p, \dot{\boldsymbol{\alpha}}) \in \partial \Psi_c$$ **éq 4.1 -** 4.1-2

where $\partial \Psi_c$ note under differential of the function Ψ_c . We point out [bib3] that under differential of a convex function in a point x is the whole of the vectors z such as:

$$f(x^*) \geq f(x) + \langle z, x^* - x \rangle \quad \forall x^*$$

It is then seen easily that [éq 4.1 - 4.1-2] involves:

$$\Psi_c(\boldsymbol{\sigma}^*, \mathbf{A}^*) \geq \Psi_c(\boldsymbol{\sigma}, \mathbf{A}) + \dot{\boldsymbol{\varepsilon}}^p(\boldsymbol{\sigma}^* - \boldsymbol{\sigma}) + \dot{\boldsymbol{\alpha}}(\mathbf{A}^* - \mathbf{A}) \qquad \forall \boldsymbol{\sigma}^* et\, \mathbf{A}^*$$ **éq 4.1 -** 4.1-3

Taking into account the definition of the characteristic function, one to see easily that [éq 4.1 - 4.1-3] is equivalent to:

$$\dot{\boldsymbol{\varepsilon}}^p \boldsymbol{\sigma} + \dot{\boldsymbol{\alpha}} \mathbf{A} \geq \dot{\boldsymbol{\varepsilon}}^p \boldsymbol{\sigma}^* + \dot{\boldsymbol{\alpha}} \mathbf{A}^* \qquad \forall \boldsymbol{\sigma}^* et\, \mathbf{A}^* \in C_{(\sigma, A)}$$ **éq 4.1 -** 4.1-4

In other words the plastic flow is such as the couple $(\boldsymbol{\sigma}, \mathbf{A})$ carry out the maximum of plastic dissipation among the acceptable thermodynamic forces.

4.2 Expression of the plastic flow partly current

When the function f is differentiable at the point considered $(\boldsymbol{\sigma}, \mathbf{A})$ the rule of normality is written simply

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda} \frac{\partial f}{\partial \boldsymbol{\sigma}}$$ **éq 4.2 -** 4.2-1

$$\dot{\boldsymbol{\alpha}} = \dot{\lambda} \frac{\partial f}{\partial \mathbf{A}}$$ **éq 4.2 -** 4.2-2

$\dot{\lambda}$ and f checking the conditions of Kuhn-Tucker:

$$\left. \begin{array}{r} \dot{\lambda} \geq 0 \\ f \leq 0 \\ \dot{\lambda}.\, f = 0 \end{array} \right\}$$ **éq 4.2 -** 4.2-3

The variable of work hardening is related to the plastic multiplier by the law of work hardening. By using plastic work, one can write: $\dot{\kappa} f = \boldsymbol{\sigma} \dot{\boldsymbol{\varepsilon}}^p$.

If f is a homogeneous function of order 1 compared to the tensorial variable σ , one has $\boldsymbol{\sigma} \dfrac{\partial f}{\partial \boldsymbol{\sigma}} = f$, which leads to the equality: $\dot{\lambda} = \dot{\kappa}$ and thus finally with the equations:

$$\dot{\boldsymbol{\varepsilon}}_c^p = \dot{\kappa}_c^p \frac{\partial f_{comp}}{\partial \boldsymbol{\sigma}}$$ **éq 4.2 -** 4.2-4

$$\dot{\boldsymbol{\varepsilon}}_t^p = \dot{\kappa}_t^p \frac{\partial f_{trac}}{\partial \boldsymbol{\sigma}}$$ **éq 4.2 -** 4.2-5

4.3 Expression of the plastic flow at the top of a cone

We give two presentations of the same result. The first presentation uses the theory of standard materials generalized and under differentials, the second share of an equality posed a priori on plastic work.

4.3.1 Demonstration by the general theory of standard materials

The field $C_{(\sigma, A)}$ consists of two cones. The function Ψ_c is not differentiable either with the intersection of these two cones, or at the top of each one of these cones. When the point $(\boldsymbol{\sigma}, \mathbf{A})$ belongs to the intersection of the two cones, the preceding equations remain valid, with the precision that the deformations figure of compression and traction develop at the same time. This case known as "multi criterion" moreover is treated in [bib4]. We will be satisfied here to treat the case where $(\boldsymbol{\sigma}, \mathbf{A})$ is at the top of a cone, and we will choose the most frequent case of the top of the cone of traction, knowing that the case of the top of the cone of compression is treated exactly in the same way.
The criteria are rewritten by using the variables σ^{eq} and σ_H, more practical in the development analytical.

$$f_{trac}(\boldsymbol{\sigma}, A_t) = \frac{\sqrt{2}}{3\mathrm{d}} \sigma^{eq} + \frac{c}{d} \sigma_H - f_t^{'} + A_t \le 0$$ 4.3.1-1

$$f_{trac}^H(\boldsymbol{\sigma}, A_t) = \frac{c}{d} \sigma_H - f_t^{'} + A_t \le 0$$ **4.3.1-2**

We thus consider a case where:

$$\left.\begin{array}{l} \sigma^{eq} = 0 \\ \frac{c}{d} \sigma_H - f_t^{'} + A_t = 0 \end{array}\right\}$$ **4.3.1-3**

On the basis of [éq 4.1 - 4.1-4], we will calculate plastic dissipation as being the maximum of $\dot{\boldsymbol{\varepsilon}}^p \boldsymbol{\sigma}^* + \dot{\boldsymbol{\alpha}} \mathbf{A}^*$ for all the couples $\boldsymbol{\sigma}^*, \mathbf{A}^* \in C_{(\sigma, A)}$

$$D^p = \underset{\boldsymbol{\sigma}^*, A^* \in C_{(\sigma, A)}}{Max} (\dot{\boldsymbol{\varepsilon}}^p \boldsymbol{\sigma}^* + \dot{\boldsymbol{\alpha}} \mathbf{A}^*)$$ **4.3.1-4**

By writing whereas this maximum is finished and reached when $\boldsymbol{\sigma}^* = \boldsymbol{\sigma}$ and $A^* = A$, we will find conditions on $\dot{\boldsymbol{\varepsilon}}^p$ and $\dot{\boldsymbol{\alpha}}$. In fact, the finished character will be enough.
By using the decomposition of the tensors partly isotropic and déviatoire, and the particular form of the variables of work hardening, one finds easily:

$$\dot{\boldsymbol{\varepsilon}}^p \boldsymbol{\sigma}^* + \dot{\boldsymbol{\alpha}} A^* = \dot{\tilde{\boldsymbol{\varepsilon}}}^p \boldsymbol{s}^* + 3 \sigma_H^* \dot{\varepsilon}_H^p + \dot{\kappa}_t^p A_t^*$$ 4.3.1-5

Let us consider the unit then Σ_1 vectors forced of worthless trace and of which equivalent constraint of Von Mises is worth 1: $\Sigma_1 = \{\boldsymbol{\sigma}, \sigma^{eq} = 1, \mathrm{trace}(\boldsymbol{\sigma}) = 0\}$

$$(\boldsymbol{\sigma}, \mathbf{A}) \in C_{(\sigma, A)} \Leftrightarrow \begin{cases} \boldsymbol{\sigma} = \sigma^{eq} \mathbf{s_1} + \sigma_H \boldsymbol{I} \\ \mathbf{s_1} \in \Sigma_1 \\ \frac{\sqrt{2}}{3\mathrm{d}} \sigma^{eq} + \frac{c}{d} \sigma_H - f_t' + A_t \le 0 \\ \frac{c}{d} \sigma_H - f_t' + A_t \le 0 \end{cases} \quad \mathbf{4.3.1\text{-}6}$$

In other words, the "direction" of the diverter of the constraints is unspecified for a couple $(\boldsymbol{\sigma}, \boldsymbol{A}) \in C_{(\sigma, A)}$.
One can thus write:

$$D^p = \underset{\substack{\sigma^{eq^*}, \sigma_H^*, A_t^*, s_1^* \in \Sigma_1 \\ \left\{ \begin{array}{l} \frac{\sqrt{2}}{3\mathrm{d}} \sigma^{eq^*} + \frac{c}{d} \sigma_H^* - f_t' + A_t^* \le 0 \\ \frac{c}{d} \sigma_H^* - f_t' + A_t^* \le 0 \end{array} \right.}}{Max} \left(\sigma^{eq^*} \underset{s_1 \in \Sigma_1}{Max} \dot{\boldsymbol{\varepsilon}}^p \mathbf{s_1^*} + 3 \sigma_H^* \dot{\varepsilon}_H^p + \dot{\kappa}_t^p A_t^* \right) \quad \mathbf{4.3.1\text{-}7}$$

It is clear that the maximum of $\dot{\boldsymbol{\varepsilon}}^p \boldsymbol{s_1^*}$ is reached when $\boldsymbol{s_1^*}$ is "parallel" with $\dot{\boldsymbol{\varepsilon}}^p$ and that one has then: $\underset{s_1 \in \Sigma_1}{Max} \dot{\boldsymbol{\varepsilon}}^p \boldsymbol{s_1^*} = \frac{2}{3} \dot{\varepsilon}_{eq}^p$.

[éq 4.3.1-7] can thus be written:

$$D^p = \left(\underset{\substack{\sigma^{eq^*}, \sigma_H^*, A_t^* \\ \left\{ \begin{array}{l} \frac{\sqrt{2}}{3\mathrm{d}} \sigma^{eq^*} + \frac{c}{d} \sigma_H^* - f_t' + A_t^* \le 0 \\ \frac{c}{d} \sigma_H^* - f_t' + A_t^* \le 0 \end{array} \right.}}{Max} \left(\frac{2}{3} \dot{\varepsilon}_{eq}^p \sigma^{eq^*} \right) + \underset{\substack{\sigma^{eq^*}, \sigma_H^*, A_t^* \\ \left\{ \begin{array}{l} \frac{\sqrt{2}}{3\mathrm{d}} \sigma^{eq^*} + \frac{c}{d} \sigma_H^* - f_t' + A_t^* \le 0 \\ \frac{c}{d} \sigma_H^* - f_t' + A_t^* \le 0 \end{array} \right.}}{Max} \left(3 \boldsymbol{\sigma_H^*} \dot{\boldsymbol{\varepsilon}}_{\boldsymbol{H}}^p \right) + \underset{\substack{\sigma^{eq^*}, \sigma_H^*, A_t^* \\ \left\{ \begin{array}{l} \frac{\sqrt{2}}{3\mathrm{d}} \sigma^{eq^*} + \frac{c}{d} \sigma_H^* - f_t' + A_t^* \le 0 \\ \frac{c}{d} \sigma_H^* - f_t' + A_t^* \le 0 \end{array} \right.}}{Max} \left(\dot{\kappa}_t^p A_t^* \right) \right) \quad \mathbf{4.3.1\text{-}8}$$

Like $\dot{\varepsilon}_{eq}^p \ge 0$, one has for the first term:

$$\underset{\sigma^{eq^*},\sigma^*_H,A^*_t}{Max}\left(\frac{2}{3}\dot{\tilde{\varepsilon}}_{eq}^{\,p}\sigma^{eq^*}\right)=\frac{2}{3}\dot{\tilde{\varepsilon}}_{eq}^{\,p}\left(\frac{-3\mathrm{d}}{\sqrt{2}}A^*_t+\frac{3\mathrm{d}}{\sqrt{2}}f'_t\frac{3\mathrm{c}}{\sqrt{2}}\sigma^*_H\right)$$

$$\begin{cases}\dfrac{\sqrt{2}}{3\mathrm{d}}\sigma^{eq^*}+\dfrac{c}{d}\sigma^*_H-f'_t+A^*_t\le 0\\ \dfrac{c}{d}\sigma^*_H-f'_t+A^*_t\le 0\end{cases} \quad \textbf{4.3.1-9}$$

[éq 4.3.1-9] deferred in [éq 4.3.1-8] gives:

$$D^p=\sqrt{2}\,d\,\dot{\tilde{\varepsilon}}_{eq}^{\,p}f'_t+\underset{\substack{\sigma^*_H,A^*_t\\ \frac{c}{d}\sigma^*_H-f'_t+A^*_t\le 0}}{Max}\left(\sigma^*_H\left(3\dot{\varepsilon}_H^{\,p}-\sqrt{2}\,c\,\dot{\tilde{\varepsilon}}_{eq}^{\,p}\right)\right)+\underset{\substack{\sigma^*_H,A^*_t\\ \frac{c}{d}\sigma^*_H-f'_t+A^*_t\le 0}}{Max}\left(A^*_t\left(\dot{\kappa}_t^{\,p}-\sqrt{2}\,d\,\dot{\tilde{\varepsilon}}_{eq}^{\,p}\right)\right) \quad \textbf{4.3.1-10}$$

Let us pose then:

$m=3\dot{\varepsilon}_H^{\,p}-\sqrt{2}\,c\,\dot{\tilde{\varepsilon}}_{eq}^{\,p}$, $n=\dot{\kappa}_t^{\,p}-\sqrt{2}\,d\,\dot{\tilde{\varepsilon}}_{eq}^{\,p}$ and $q=\sqrt{2}\,d\,\dot{\tilde{\varepsilon}}_{eq}^{\,p}f'$

With these notations, [éq 4.3.1-10] becomes:

$$D^p=q+\underset{\substack{\sigma^*_H,A^*_t\\ \frac{c}{d}\sigma^*_H-f'_t+A^*_t\le 0}}{Max}\left(m\,\sigma^*_H+n\,A^*_t\right) \quad \textbf{4.3.1-11}$$

It is about a problem of type “simplex”. The field of σ^*_H, A^*_t is represented on [Figure 4.3.1 - has].

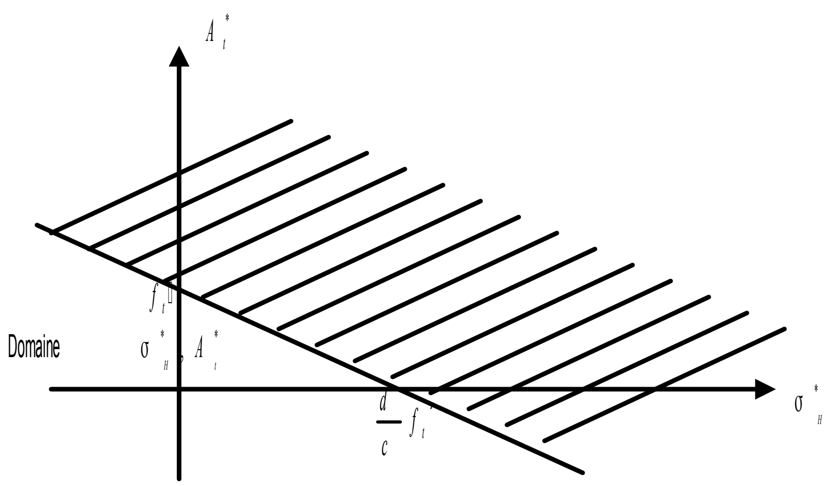


Figure 4.3.1-a

Like the field σ^*_H, A^*_t extends towards $-\infty$ at the same time for σ^*_H and A^*_t , so that D^p that is to say finished, it is necessary that m and n are positive. The maximum of $m\,\sigma^*_H+n\,A^*_t$ is reached for a couple σ^*_H, A^*_t located on the edge of the field of σ^*_H, A^*_t .

One has then: $D^p=q+n\,f'_t+\underset{\sigma^*_H}{Max}\left(\sigma^*_H\left(m-n\frac{c}{d}\right)\right)$

So that D^p that is to say finished, it is necessary that:

$$m=n\frac{c}{d}$$

Taking again the definitions of m and n , this relation gives:

$$3\dot{\varepsilon}_H^p=\frac{c}{d}\dot{\kappa}_t^p \quad \textbf{éq 4.3.1-12}$$

In addition, constraints $m\geq 0$ and $n\geq 0$ give:

$$3\dot{\varepsilon}_H^p\geq\sqrt{2}c\ \dot{\tilde{\varepsilon}}_{eq}^p \quad \textbf{éq 4.3.1-13}$$

and

$$\dot{\kappa}_t^p\geq\sqrt{2}d\ \dot{\tilde{\varepsilon}}_{eq}^p \quad \textbf{éq 4.3.1-14}$$

these two last inequalities being equivalent because of [éq 4.3.1-11].
The equations [éq 4.3.1-11] and [éq 4.3.1-12] define the plastic flow in the top of one of the cones of the field of reversibility.

4.3.2 Demonstration by plastic work

The starting point is to consider that compared to the developments made into cubes regular points, they are primarily the relations [éq 4.2-1] and [éq 4.2-2], known as rules of normality, which cannot to be written more. However the relation [éq 4.2-1] implies the equality $\dot{\kappa}_t^p f_t(\kappa_t^p)=\boldsymbol{\sigma}\,\dot{\boldsymbol{\varepsilon}}^{\boldsymbol{p}}$, which can, it, being maintained.
We will thus leave the equation:

$$\dot{\kappa}_t^p f_t(\kappa_t^p)=\boldsymbol{\sigma}\,\dot{\boldsymbol{\varepsilon}}^{\boldsymbol{p}} \quad \textbf{éq 4.3.2-1}$$

We use the partly isotropic decomposition and déviatoire tensors and find:

$$\dot{\kappa}_t^p f_t(\kappa_t^p)=\dot{\tilde{\boldsymbol{\varepsilon}}}^{\boldsymbol{p}}\boldsymbol{s}+3\sigma_H\dot{\varepsilon}_H \quad \textbf{éq 4.3.2-2}$$

At the top of the cone of traction, there are the relations [éq 4.3.1-3], which, carried in [éq 4.3.2-2] give, while also using [éq 3.5 - 3.5-2]:

$$\dot{\kappa}_t^p f_t(\kappa_t^p)=3\frac{d}{c}f_t(\kappa_t^p)\dot{\varepsilon}_H \quad \textbf{éq 4.3.2-3}$$

And one thus finds the relation [éq 4.3.1-12]: . $3\dot{\varepsilon}_H=\frac{c}{d}\dot{\kappa}_t^p$

4.4 Together equations of behavior (summarized)

One notes H the matrix of elasticity:

$$H=\begin{bmatrix}\lambda+2\mu & \lambda & \lambda & 0 & 0 & 0\\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0\\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0\\ 0 & 0 & 0 & 2\mu & 0 & 0\\ 0 & 0 & 0 & 0 & 2\mu & 0\\ 0 & 0 & 0 & 0 & 0 & 2\mu\end{bmatrix}$$

With:

$$\lambda=\nu\frac{E}{(1+\nu)(1-2\nu)} \text{ and } \mu=\frac{E}{2(1+\nu)} \text{ , and } K=\frac{3\lambda+2\mu}{3}$$

The forced relations deformations are written finally:

$$\boldsymbol{\sigma}=\boldsymbol{H}\left(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}_c^p-\boldsymbol{\varepsilon}_t^p\right)$$ **éq 4.4 -** 4.4-1

For a regular point of the cone of compression:

$$f_{comp}(\boldsymbol{\sigma}, A_c) = \frac{\sqrt{2}}{3\mathrm{b}}\sigma^{eq}+\frac{a}{b}\sigma_H-\varphi f_c^{'}+A_c \leq 0$$ **éq 4.4 -** 4.4-2

$$\dot{\kappa}_c^p f_{comp}=0 \quad ; \quad \dot{\varepsilon}_c^p=\dot{\kappa}_c^p\frac{\partial f_{comp}}{\partial\boldsymbol{\sigma}}$$ **éq 4.4 -** 4.4-3

For a regular point of the cone of traction:

$$f_{trac}(\boldsymbol{\sigma}, A_t) = \frac{\sqrt{2}}{3\mathrm{d}}\sigma^{eq}+\frac{c}{d}\sigma_H-\varphi f_t^{'}+A_t \leq 0$$ **éq 4.4 -** 4.4-4

$$\dot{\kappa}_t^p f_{trac}=0 \quad ; \quad \dot{\varepsilon}_t^p=\dot{\kappa}_t^p\frac{\partial f_{trac}}{\partial\boldsymbol{\sigma}}$$ **éq 4.4 - 5**

For a point at the top of the cone of compression:

$$\mathbf{s}=0 \quad \text{éq 4.4 - 6}$$

$$f^{H}_{comp}(\boldsymbol{\sigma}, A_c)=\frac{a}{b}\sigma_H-\varphi\ f'_c+A_c=0 \quad \text{éq 4.4 - 7}$$

$$3\dot{\varepsilon}^{p}_{c\ H}=\frac{a}{b}\dot{\kappa}^{p}_{c} \quad \text{éq 4.4 - 8}$$

$$3\dot{\varepsilon}^{p}_{c\ H}\geq\sqrt{2}\,a\,\dot{\bar{\varepsilon}}^{p}_{c\ eq} \quad \text{éq 4.4 - 9}$$

For a point at the top of the cone of traction:

$$\mathbf{s}=0 \quad \text{éq 4.4 - 10}$$

$$f^{H}_{trac}(\sigma, A_t)=\frac{c}{d}\sigma_H-f'_t+A_t=0 \quad \text{éq 4.4 - 11}$$

$$3\dot{\varepsilon}^{p}_{t\ H}=\frac{c}{d}\dot{\kappa}^{p}_{t} \quad \text{éq 4.4 - 12}$$

$$3\dot{\varepsilon}^{p}_{t\ H}\geq\sqrt{2}\,c\,\dot{\bar{\varepsilon}}^{p}_{t\ eq} \quad \text{éq 4.4 - 13}$$

5 Digital integration of the law of behavior

5.1 The total problem and the local problem: recalls

For a given structure (geometry and material), and for a given loading, the fields of displacement, constraint and variables internal are by solving a set of partial derivative equations nonlinear formed starting from and the law equilibrium equations of behavior. The document [bib5] presents the algorithm of which we give a summary here:

$\mathbf{u_0}$ and $\boldsymbol{\sigma_0}$ known

Buckle urgent t_i : loading $\mathbf{L}_i=\mathbf{L}(t_i)$

$\mathbf{u}_{i-1}$ known; calculation of the prediction $\Delta\mathbf{u}^{0}_{i}$

Iterations of balance of Newton N

$\mathbf{u}^{n}_{i}$ known; $\Delta\mathbf{u}^{n}_{i}=\mathbf{u}^{n}_{i}-\mathbf{u}_{i-1}$

Buckle elements el

Buckle points of gauss G

calculation $\Delta\boldsymbol{\varepsilon}^{el^n}_{g_i}=\boldsymbol{\varepsilon}^{el}_{g}(\Delta\mathbf{u}^{n}_{i})$

law of behavior :

calculation of: $\boldsymbol{\sigma}^{el^n}_{g_i}$ and $\boldsymbol{\alpha}^{el^n}_{g_i}$ from $\boldsymbol{\sigma}^{el}_{g_{i-1}}$, $\boldsymbol{\alpha}^{el}_{g_{i-1}}$ and $\Delta\boldsymbol{\varepsilon}^{el^n}_{g_i}$

calculation of $\dfrac{\partial\boldsymbol{\sigma}^{el^n}_{g_i}}{\partial\Delta\boldsymbol{\varepsilon}^{el^n}_{g_i}}$ (according to option)

Accumulation in vectors and matrices assembled :

Accumulation of $\boldsymbol{Q}^{T^{el}}_{g}.\boldsymbol{\sigma}^{el^n}_{g_i}$ in $\boldsymbol{Q}^{T}.\boldsymbol{\sigma}^{n}_{i}$

Accumulation of calculation of $\boldsymbol{Q}^{T^{el}}_{g}\dfrac{\partial\boldsymbol{\sigma}^{el^n}_{g_i}}{\partial\Delta\boldsymbol{\varepsilon}^{el^n}_{g_i}}\boldsymbol{Q}^{el}_{g}$ in $\boldsymbol{K}^{n}_{i}$ (according to option)

Calculation of $\delta\mathbf{u}^{n}_{i}$ by:

$\boldsymbol{K}^{n}_{i}.\delta\mathbf{u}^{n}_{i}=-\boldsymbol{Q}^{T}.\boldsymbol{\sigma}^{n}_{i}+\mathbf{L_i}$

linear iteration of research to determine ρ
Actualization :
$\Delta \mathbf{u}_i^{n+1} = \Delta \mathbf{u}_i^n + \rho\, \delta \mathbf{u}_i^n$
IF test convergence OK
| fine Newton: no time following I = i+1
If not
| N = n+1

The calculation of the constraints and internal variables $\boldsymbol{\alpha}_{g_i}^{el^n}$, $\boldsymbol{\sigma}_{g_i}^{el^n}$ with the iteration of Newton n and at time t_i starting from the internal constraints and variables $\boldsymbol{\sigma}_{g_{i-1}}^{el}$, $\boldsymbol{\alpha}_{g_{i-1}}^{el}$ at time t_{i-1} and of the value and $\Delta \boldsymbol{\varepsilon}_{g_i}^{el^n}$ increase in deformation in the time interval estimated with the iteration of Newton n consist in integrating the equations [éq 4.4-1], [éq 4.4-2] with [éq 4.4-5] or [éq 4.4-6] with [éq 4.4-9] or [éq 4.4-10] with [éq 4.4-13] according to the cases with the initial conditions:

$$\boldsymbol{\alpha}(t_{i-1}) = \boldsymbol{\alpha}_{g_{i-1}}^{el}$$ **éq 5.1 -** 5.1-1

$$\boldsymbol{\sigma}(t_{i-1}) = \boldsymbol{\sigma}_{g_{i-1}}^{el}$$ **éq 5.1 -** 5.1-2

$$\boldsymbol{\varepsilon}^p(t_{i-1}) = \boldsymbol{\varepsilon}_{g_{i-1}}^{p^{el}}$$ **éq 5.1 -** 5.1-3

With the condition of loading in imposed deformation:

$$\boldsymbol{\varepsilon}(t_i) = \boldsymbol{\varepsilon}_{g_i}^{el^n}$$ **éq 5.1 -** 5.1-4

The result of this integration will provide:

$$\boldsymbol{\sigma}_{g_i}^{el^n} = \boldsymbol{\sigma}(t_i)$$
$$\boldsymbol{\varepsilon}_{g_i}^{p^{el^n}} = \boldsymbol{\varepsilon}^p(t_i)$$
$$\boldsymbol{\alpha}_{g_i}^{el^n} = \boldsymbol{\alpha}(t_i)$$

The object of this chapter is to present the digital integration of these equations. It is about a system of nonlinear differential equations which we solve by a method of implicit Euler. From now, the quantity at the beginning of the step of time (known) will be noted with an index $\tilde{}$, whereas unknown factors at the end of the step of time (all unknown factors except $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_{g_i}^{el^n}$) will be noted without index. For an unspecified quantity a one notes $\Delta a = a - a^-$.
One always starts by calculating an elastic solution $\boldsymbol{\sigma}^e$, by supposing that there are no evolution of the plastic deformations and variables internal. So at least one of the criteria is violated by this elastic solution, it has reasons there to calculate plastic flows. It is then necessary to distinguish the regular cases for which the solution $\boldsymbol{\sigma}$ is on the regular part of one of the cones or with their intersection of the cases known as singular where the solutio $\boldsymbol{\sigma}$ is not at the top of one of the two cones. Logic allowing to examine and choose these various cases, and the algorithm which results from this are relatively complex. We thus present in first the treatment of each case and explain their sequence subsequently, in the chapter [§ 5.7].

5.2 Digital processing of the regular case.

One presents in detail only the case where at the same time plastic deformations in traction and compression develop and where thus the solution $\boldsymbol{\sigma}$ belongs to the intersection of the two cones. Let us note however that, even if $\boldsymbol{\sigma}^e$ violate at the same time the two criteria, for as much the final solution $\boldsymbol{\sigma}$ can belong very well finally only to one of the cones hammer-hardened. One is thus brought to search balanced which one applies that they belong to one of the two cones or both. The case or it belongs to only one of the two cones results easily from the case plus general introduced here. The equations which we have to solve are finally:

$$\mathbf{s}^{\mathbf{e}}=\frac{\mu}{\mu^-}\mathbf{s}^-+2\mu\Delta\tilde{\boldsymbol{\varepsilon}}$$ **éq 5.2 -** 5.2-1

$$\sigma_{H^e}=\frac{K}{K^-}\sigma_{H^-}+3\mathrm{K}\Delta\varepsilon_H$$ **éq 5.2 -** 5.2-2

$$\mathbf{s}=\mathbf{s}^{\mathbf{e}}-2\mu\left(\Delta\tilde{\boldsymbol{\varepsilon}}_c^p+\Delta\tilde{\boldsymbol{\varepsilon}}_t^p\right)$$ **éq 5.2 -** 5.2-3

$$\sigma_H=\sigma_H^e-3\mathrm{K}\left(\Delta\varepsilon_{H_c}^p+\Delta\varepsilon_{H_t}^p\right)$$ **éq 5.2 -** 5.2-4

$$f_{comp}(\boldsymbol{\sigma},\kappa_c^p)\;=\;\frac{\sqrt{2}}{3\mathrm{b}}\sigma^{eq}+\frac{a}{b}\sigma_H-f_c(\kappa_c^p)\;=\;0$$ **éq 5.2 -** 5.2-5

$$\Delta\varepsilon_c^p=\Delta\kappa_c^p\frac{\partial f_{comp}}{\partial\boldsymbol{\sigma}}$$ **éq 5.2 -** 5.2-6

$$f_{trac}(\boldsymbol{\sigma},\kappa_t^p)\;=\;\frac{\sqrt{2}}{3\mathrm{d}}\sigma^{eq}+\frac{c}{d}\sigma_H-f_t(\kappa_t^p)\;=\;0$$ **éq 5.2 -** 5.2-7

$$\Delta\varepsilon_t^p=\Delta\kappa_t^p\frac{\partial f_{trac}}{\partial\boldsymbol{\sigma}}$$ **éq 5.2 -** 5.2-8

By taking the isotropic and deviatoric parts plastic deformations, the equations [éq 5.2-6] and [éq 5.2 - 5.2-8] give:

$$\Delta\tilde{\boldsymbol{\varepsilon}}_c^p=\frac{\Delta\kappa_c^p}{\sqrt{2}b}\frac{\boldsymbol{s}}{\sigma^{eq}}$$ **éq 5.2 -** 5.2-9

$$\Delta\boldsymbol{\varepsilon}_{H_c}^p=\Delta\kappa_c^p\frac{a}{3\mathrm{b}}$$ **éq 5.2 -** 5.2-10

$$\Delta\tilde{\boldsymbol{\varepsilon}}_t^p=\frac{\Delta\kappa_t^p}{\sqrt{2}d}\frac{\boldsymbol{s}}{\sigma^{eq}}$$ **éq 5.2 -** 5.2-11

$$\Delta\varepsilon_{H_t}^p=\Delta\kappa_t^p\frac{c}{3\mathrm{d}}$$ **éq 5.2 -** 5.2-12

While deferring [éq 5.2-9] and [éq 5.2-11] in [éq 5.2 - 5.2-3], one finds:

$$\boldsymbol{s}=\boldsymbol{s}^e-2\mu\left(\frac{\Delta\kappa_c^p}{\sqrt{2}b}+\frac{\Delta\kappa_t^p}{\sqrt{2}d}\right)\frac{\boldsymbol{s}}{\sigma^{eq}}$$ **éq 5.2 -** 5.2-13

who shows that $\mathbf{s}$ is parallel to $\mathbf{s}^e$ from where one deduces:

$$\frac{\boldsymbol{s}}{\sigma^{\mathrm{eq}}}=\frac{\mathbf{s}^e}{\sigma^{e^{eq}}}$$ **éq 5.2 -** 5.2-14

While deferring [éq 5.2-14] in [éq 5.2-9] and [éq 5.2-11] one finds:

$$\Delta\tilde{\boldsymbol{\varepsilon}}_c^p=\frac{\Delta\kappa_c^p}{\sqrt{2}b}\frac{\boldsymbol{s}^e}{\sigma^{e^{eq}}}$$ éq 5.2 - 5.2-15

$$\Delta\tilde{\boldsymbol{\varepsilon}}_t^p=\frac{\Delta\kappa_t^p}{\sqrt{2}d}\frac{\boldsymbol{s}^e}{\sigma^{e^{eq}}}$$ éq 5.2 - 5.2-16

One defers then [éq 5.2 - 5.2-15] and [éq 5.2 - 5.2-16] in [éq 5.2 - 5.2-3] and [éq 5.2 - 5.2-4], and one expresses the criteria [éq 5.2-5] and [éq 5.2 - 5.2-7] with these new results. That led to two equations having like unknown factors $\Delta\kappa_c^p$ and $\Delta\kappa_t^p$:

$$\frac{\sqrt{2}}{3\mathrm{b}}\sigma^{e^{eq}}+\frac{a}{b}\sigma_H^e-\Delta\kappa_c^p\left(\frac{2\mu}{3\mathrm{b}^2}+\frac{Ka^2}{b^2}\right)-\Delta\kappa_t^p\left(\frac{2\mu}{3\,bd}+\frac{Kac}{bd}\right)-f_c\left(\kappa_c^{p^-}+\Delta\kappa_c^p\right)=0$$ éq 5.2 - 5.2-17

$$\frac{\sqrt{2}}{3\mathrm{d}}\sigma^{e^{eq}}+\frac{c}{d}\sigma_H^e-\Delta\kappa_c^p\left(\frac{2\mu}{3\,bd}+\frac{Kac}{bd}\right)-\Delta\kappa_t^p\left(\frac{2\mu}{3\mathrm{d}^2}+\frac{Kc^2}{d^2}\right)-f_t\left(\kappa_t^{p^-}+\Delta\kappa_t^p\right)=0$$ éq 5.2 - 5.2-18

It is this system of two equations to two unknown factors which should finally be solved. If functions f_c and f_t are linear, i.e. if one is in linear mode post-peak in compression as in traction, it is about a linear system which will thus be solved in an iteration. In the case, is mode pre-peak in compression (which is always nonlinear), that is to say modelings with nonlinear modes post peak, the system [éq 5.2 - 5.2-17] and [éq 5.2 - 5.2-18] is solved by a method of Newton:

One notes $f^*_{comp}\left(\Delta\kappa_c^p,\Delta\kappa_t^p\right)$ the criterion of compression regarded as function of the only variables $\Delta\kappa_c^p$ and $\Delta\kappa_t^p$, in the same way for traction:

$$f^*_{comp}\left(\Delta\kappa_c^p,\Delta\kappa_t^p\right)=\frac{\sqrt{2}}{3b}\sigma^{e^{eq}}+\frac{a}{b}\sigma_H^e-\Delta\kappa_c^p\left(\frac{2\mu}{3b^2}+\frac{Ka^2}{b^2}\right)-\Delta\kappa_t^p\left(\frac{2\mu}{3bd}+\frac{Kac}{bd}\right)-f_c\left(\kappa_c^{p^-}+\Delta\kappa_c^p\right)$$

$$f^*_{trac}\left(\Delta\kappa_c^p,\Delta\kappa_t^p\right)=\frac{\sqrt{2}}{3d}\sigma^{e^{eq}}+\frac{c}{d}\sigma_H^e-\Delta\kappa_c^p\left(\frac{2\mu}{3bd}+\frac{Kac}{bd}\right)-\Delta\kappa_t^p\left(\frac{2\mu}{3d^2}+\frac{Kc^2}{d^2}\right)-f_t\left(\kappa_t^{p^-}+\Delta\kappa_t^p\right)$$

The ième iteration of Newton for system [éq 5.2 - 5.2-17] - [éq 5.2 - 5.2-18] is:

$$\begin{Bmatrix}\Delta\kappa_c^p\\ \Delta\kappa_t^p\end{Bmatrix}^{i+1}=\begin{Bmatrix}\Delta\kappa_c^p\\ \Delta\kappa_t^p\end{Bmatrix}^{i}-\boldsymbol{J}_i^{-1}\begin{Bmatrix}f^*_{comp}\left(\Delta\kappa_c^p,\Delta\kappa_t^p\right)\\ f^*_{tracp}\left(\Delta\kappa_c^p,\Delta\kappa_t^p\right)\end{Bmatrix}^{i}$$

The jacobien $\boldsymbol{J}_i$ is worth:

$$\boldsymbol{J}_i=\begin{bmatrix}\frac{\partial f^*_{comp}}{\Delta\kappa_c^p} & \frac{\partial f^*_{comp}}{\Delta\kappa_t^p}\\ \frac{\partial f^*_{trac}}{\Delta\kappa_c^p} & \frac{\partial f^*_{trac}}{\Delta\kappa_t^p}\end{bmatrix}$$

With:

$$\frac{\partial f^*_{comp}}{\Delta\kappa_c^p}=-\left(\frac{2\mu}{3b^2}+\frac{Ka^2}{b^2}\right)-\frac{\partial f_c\left(\kappa_c^{p^-}+\Delta\kappa_c^p\right)}{\partial\Delta\kappa_c^p}$$

$$\frac{\partial f^*_{comp}}{\partial\Delta\kappa_t^p}=-\left(\frac{2\mu}{3bd}+\frac{Kac}{bd}\right)$$

$$\frac{\partial f^*_{trac}}{\partial\Delta\kappa_c^p}=-\left(\frac{2\mu}{3bd}+\frac{Kac}{bd}\right)$$

$$\frac{\partial f^*_{trac}}{\partial\Delta\kappa_t^p}=-\left(\frac{2\mu}{3d^2}+\frac{Kc^2}{d^2}\right)-\frac{\partial f_t\left(\kappa_t^{p^-}+\Delta\kappa_t^p\right)}{\partial\Delta\kappa_t^p}$$

Initial Jacobien of the system results from the values of the derivative in $\Delta\kappa_c^p=0$ and $\Delta\kappa_t^p=0$, which amounts solving the nonlinear system on the basis of the worthless solution. Nonthe linearities are introduced by the curves of softening. In the post-peak part, when they are linear, convergence is done in an iteration. When they are nonlinear, convergence requires only some iterations. To leave the worthless solution thus does not pose a problem of convergence. That returns starting from the linearization of the criteria in the vicinity of the elastic prediction.

5.3 Existence of a solution and condition of applicability

We point out that the solution of the problem [éq 5.2 - 5.2-17] and [éq 5.2 - 5.2-18] must check the conditions [éq 4.2-3] and thus inter alia the positivity of the increases in the plastic multipliers.

$$\Delta \kappa_c^p \geq 0 \quad \textbf{éq 5.3 -} \text{ 5.3-1}$$

$$\Delta \kappa_t^p \geq 0 \quad \textbf{éq 5.3 -} \text{ 5.3-2}$$

Let us suppose that we are in a case of linear behavior post peak in traction as in compression and call respectively h_c and h_t slopes of the parts post peak. The increases in the plastic multipliers are obtained by solving the linear system:

$$\begin{bmatrix} \frac{2\mu}{3b^2}+\frac{Ka^2}{b^2}+h_c & \frac{2\mu}{3bd}+\frac{Kac}{bd} \\ \frac{2\mu}{3bd}+\frac{Kac}{bd} & \frac{2\mu}{3d^2}+\frac{Kc^2}{d^2}+h_t \end{bmatrix} \begin{pmatrix} \Delta\kappa_c^p \\ \Delta\kappa_t^p \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{3b}\sigma^{e^{eq}}+\frac{a}{b}\sigma_H^e - f_c\left(\kappa_c^{p^-}\right) \\ \frac{\sqrt{2}}{3d}\sigma^{e^{eq}}+\frac{c}{d}\sigma_H^e - f_t\left(\kappa_t^{p^-}\right) \end{pmatrix} \quad \textbf{éq 5.3 -} \text{ 5.3-3}$$

Since the criteria of traction and compression were activated in traction as in compression, the second member of this system is positive. But nothing ensures in so far as the solution of [éq 5.3 - 3] will be positive.
If one poses:

$$\boldsymbol{HTC} = \begin{bmatrix} \frac{2\mu}{3b^2}+\frac{Ka^2}{b^2}+h_c & \frac{2\mu}{3bd}+\frac{Kac}{bd} \\ \frac{2\mu}{3bd}+\frac{Kac}{bd} & \frac{2\mu}{3d^2}+\frac{Kc^2}{d^2}+h_t \end{bmatrix} \quad \text{é}\textbf{Q 5.3 -} \text{ 5.3-4}$$

One a:

$$\begin{Bmatrix} \Delta\kappa_c^p \\ \Delta\kappa_t^p \end{Bmatrix} = \boldsymbol{HTC}^{-1} \begin{pmatrix} \frac{\sqrt{2}}{3b}\sigma^{e^{eq}}+\frac{a}{b}\sigma_H^e - f_c\left(\kappa_c^{p^-}\right) \\ \frac{\sqrt{2}}{3d}\sigma^{e^{eq}}+\frac{c}{d}\sigma_H^e - f_t\left(\kappa_t^{p^-}\right) \end{pmatrix} \quad \textbf{éq 5.3 -} \text{ 5.3-5}$$

With:

$$|\mathbf{HTC}| = 3\frac{h_t}{\beta^2}(3K+\mu) - 6\frac{h_t}{\beta} + h_c h_t + \frac{9}{4}h_c K + 9 h_t K + \frac{3\mu}{4}(h_c + 16 h_t + 9K) \quad \textbf{éq 5.3 -} \text{ 5.3-6}$$

$$\mathbf{HTC}^{-1} = \frac{1}{|\mathbf{HTC}|} \begin{bmatrix} h_t + \frac{3}{4}(3K+\mu) & \frac{9K}{2\beta} - \frac{9K}{2} + \frac{3\mu}{2\beta} - 3\mu \\ \frac{9K}{2\beta} - \frac{9K}{2} + \frac{3\mu}{2\beta} - 3\mu & h_c + 9K - 18\frac{K}{\beta} + 12\mu\frac{(\beta-1)}{\beta} \end{bmatrix} \quad \textbf{éq 5.3 -} \text{ 5.3-7}$$

It is seen that conditions of positivity [éq 5.3 - 5.3-1] and [éq 5.3 - 5.3-2] lead to relatively complicated relations. If the solution of the problem [éq 5.2 - 5.2-17] and [éq 5.2 - 5.2-18] does not check the conditions of positivity [éq 5.3 - 5.3-1] and [éq 5.3 - 5.3-2], that can correspond is with the fact that the coefficients h_c and h_t are such as there is no positive solution (that would correspond to a snap-back in a diagram (σ, K)), that is to say with the fact that the solution activates finally only one of the two criteria.
Let us examine the simpler case of only one activated criterion. Let us suppose to fix the notations that the only activated criterion is the criterion of traction.
One must have:

$$\Delta \kappa_t^p \left(\frac{2\mu}{3d^2} + \frac{Kc^2}{d^2} + h_t \right) = \frac{\sqrt{2}}{3d} \sigma^{e^{eq}} + \frac{c}{d} \sigma_H^e - f_t \left(\kappa_t^{p^-} \right)$$ **éq 5.3 -** 5.3-8

One sees reappearing the condition known as of applicability:

$$\left(\frac{2\mu}{3d^2} + \frac{Kc^2}{d^2} + h_t \right) > 0$$ **éq 5.3 -** 5.3-9

This condition is the generalization of the condition $-h < E$ presented to the paragraph [§ 3.5.2.1] in a typical case of axial request plain.

The following strategy will thus be retained:

If $\frac{\sqrt{2}}{3b} \sigma^{e^{eq}} + \frac{a}{b} \sigma_H^e - f_c \left(\kappa_c^{p^-} \right) > 0$ and $\frac{\sqrt{2}}{3b} \sigma^{e^{eq}} + \frac{a}{b} \sigma_H^e - f_c \left(\kappa_c^{p^-} \right) > 0$

- Activation a priori of the two criteria: problem resolution [éq 5.2 - 5.2-17] and [éq 5.2 - 5.2-18]
  So not convergence or so not checking conditions of positivity [éq 5.3 - 5.3-1] and [éq 5.3 - 5.3-2]
  - Research solution with only one activated criterion
    So not convergence or not checking condition positivity
    - Stop on diagnostic of nonchecking of condition of applicability of the type [éq 5.3 - 5.3-9]

5.4 Treatment of the nonregular cases

In this paragraph, we describe the discrete treatment of the equations corresponding to projection at the top of the cone of traction, [éq 4.4-10] with [éq 4.4-13], knowing that projection at the top of the cone of compression is done in the same way. The equations [éq 4.4-10] and [éq 4.4-12] define the plastic flow in this case, whereas the equation [éq 4.4-13] is a condition of acceptability of projection at the top of the cone.

5.4.1 Calculation of the constraints and plastic deformations

Discrete forms of [éq 4.4-10] with [éq 4.4-13], are:

$$\mathbf{s} = 0$$ **éq 5.4.1-1**

$$\sigma_H = \frac{d}{c} f_t \left(\kappa_t^{p^-} + \Delta \kappa_t^p \right)$$ **éq 5.4.1-2**

$$3 \Delta \varepsilon_{t_H}^p = \frac{c}{d} \Delta \kappa_t^p$$ **éq 5.4.1-3**

The relation [éq 5.2 - 5.2-4] established in the regular case is always valid, one jointly uses it with [éq 5.4.1-3] in [éq 5.4.1-1] and one obtains:

$$\sigma_H^e - \frac{c}{d} K \Delta \kappa_t^p = \frac{d}{c} f_t \left(\kappa_t^{p^-} + \Delta \kappa_t^p \right)$$ **éq 5.4.1-4**

The relation [éq 5.4.1-4] is a nonlinear equation compared to the variable $\Delta \kappa_t^p$ that one solves by an algorithm of Newton, which makes it possible to calculate $\Delta \varepsilon_{t_H}^p$ by [éq 5.4.1-3] and σ_H by [éq 5.4.1-2]. Taking into account [5.4.1-1], the constraints are thus completely known. [éq 5.4.1-1] still gives:

$$\mathbf{s} = \mathbf{s^e} - 2\mu \Delta \tilde{\boldsymbol{\varepsilon}}_t^p = 0$$ **éq 5.4.1-5**

This last equation makes it possible to calculate $\Delta \tilde{\boldsymbol{\varepsilon}}_t^p$ and the plastic deformations are completely known.

5.4.2 Acceptability

The discrete form of the relation [éq 4.4-13] is:

$$3\,\Delta\,\varepsilon^{p}_{t\;H} \geq \sqrt{2}\,c\,\Delta\,\tilde{\varepsilon}^{p}_{t\;eq} \qquad \text{éq 5.4.2-1}$$

[éq 5.4.1-5] gives:

$$\Delta\,\tilde{\varepsilon}^{p}_{t\;eq} = \frac{\sigma^{e^{eq}}}{2\mu} \qquad \text{éq 5.4.2-2}$$

While using [éq 5.2 - 5.2-4] and [éq 5.4.2-2], [éq 5.4.2-1] is written:

$$\sigma^{e^{eq}}\frac{cK}{\sqrt{2}\mu} \leq \sigma^{e}_{H} - \sigma_{H} \qquad \text{éq 5.4.2-3}$$

5.4.2.1 Acceptability a priori and a posteriori

For the criterion of traction and the part post peak of the criterion of compression, $\sigma_H = \frac{d}{c} f_t\left(\kappa^{p-}_{t} + \Delta\,\kappa^{p}_{t}\right)$ is a decreasing function of the variable of work hardening $\Delta\,\kappa^{p}_{t}$. One from of deduced that $\sigma^{e}_{H} - \sigma^{-}_{H} \leq \sigma^{e}_{H} - \sigma_{H}$ and thus that:

$$\sigma^{e^{eq}}\frac{cK}{\sqrt{2}\mu} \leq \sigma^{e}_{H} - \sigma^{-}_{H} \quad \Rightarrow \quad \sigma^{e^{eq}}\frac{cK}{\sqrt{2}\mu} \leq \sigma^{e}_{H} - \sigma_{H}$$

The condition $\sigma^{e^{eq}}\frac{cK}{\sqrt{2}\mu} \leq \sigma^{e}_{H} - \sigma^{-}_{H}$ is known as condition of acceptability a priori because it can be calculated as of the elastic prediction. The condition $\sigma^{e^{eq}}\frac{cK}{\sqrt{2}\mu} \leq \sigma^{e}_{H} - \sigma_{H}$ is known as condition of acceptability a posteriori.

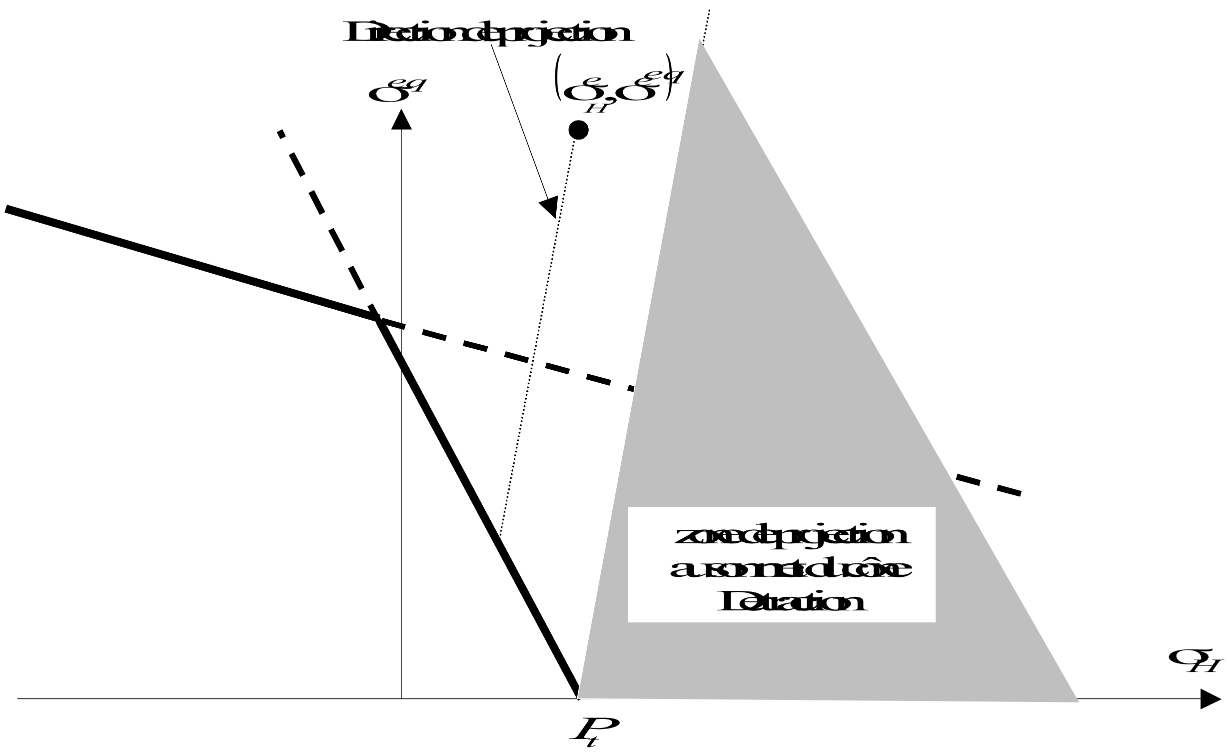


Figure 5 - 5.4.2.1-a

These conditions have a simple graphic interpretation. One can see easily that, in the case of a regular solution, one a:

$$\frac{\sigma^{e^{eq}} - \sigma^{eq}}{\sigma^{e}_{H} - \sigma_{H}} = \frac{\sqrt{2}\mu}{cK}$$

That shows that the solution in constraint is obtained by projecting the point $\left(\sigma_H^e, \sigma^{e^{eq}}\right)$ parallel to a direction $\left(cK, \sqrt{2}\mu\right)$ in a diagram $\left(\sigma_H, \sigma^{eq}\right)$, as indicated on [Figure 5.4.2-a]. zones of acceptability of projection at the top are cones of which the top and that of the cone of reversibility and delimited on the one hand by the axis $\sigma_H > OP_t$ and by a half-line resulting from the same point and from direction $\left(cK, \sqrt{2}\mu\right)$.

5.4.3 Existence of a regular solution and a singular solution.

If projection at the top of the cone is acceptable a posteriori, it may be which exists also a regular solution as one can see it on [Figure 5.4.2-b].

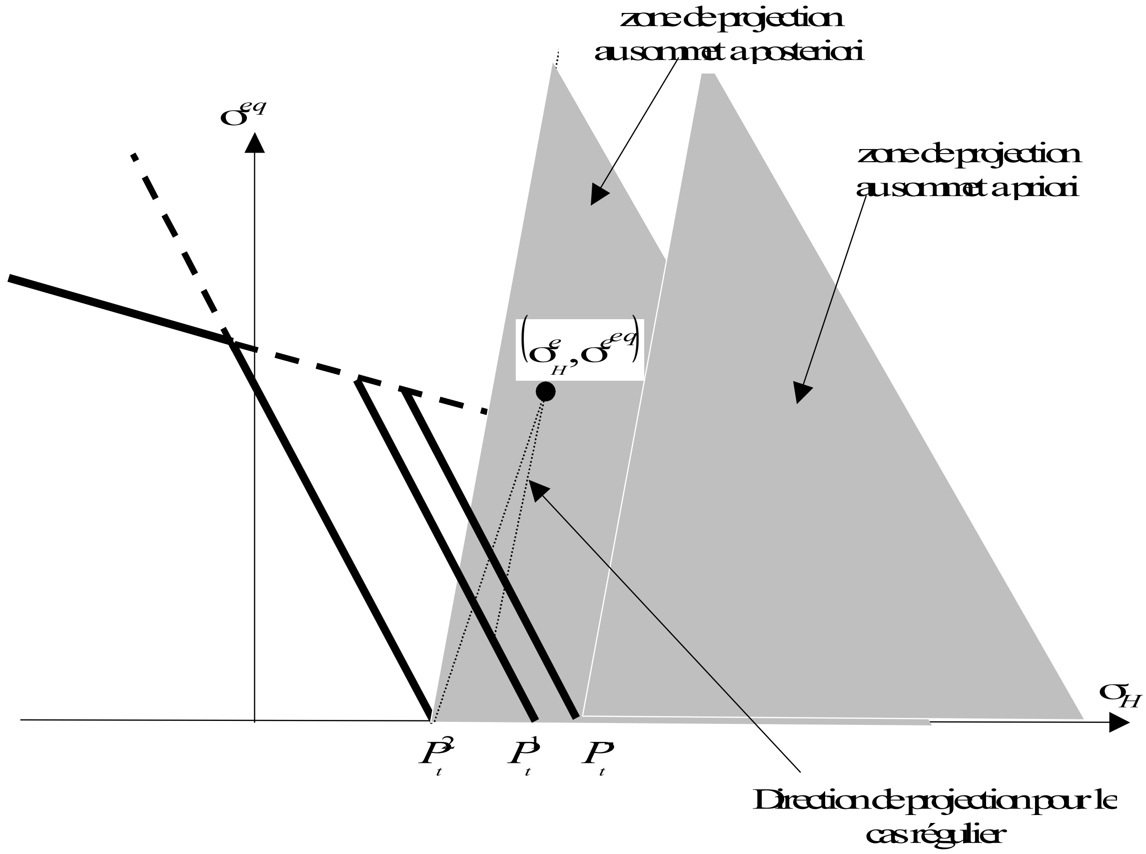


Figure 5.4.2-b

The top of the cone of traction before work hardening is noted P_t^- , that of the cone hammer-hardened with an increase in variable of work hardening $\Delta\kappa_t^{p^1}$ is noted P_t^1 , that of the cone hammer-hardened with an increase in variable of work hardening $\Delta\kappa_t^{p^2} > \Delta\kappa_t^{p^1}$ is noted P_t^2 . It is seen that there exists a regular solution with $\Delta\kappa_t^{p^1}$ and a solution with projection at the top of the cone for $\Delta\kappa_t^{p^2}$. Since the regular solution corresponds to a less work hardening, in the process of evolution, it will be met before the solution with projection at the top: it is thus the regular solution which it is necessary to adopt.

For this reason the sequence of regular research of solution and with projection at the top is the following:

<u>I projection at the acceptable top a priori</u> : $\sigma^{e^{eq}}\frac{cK}{\sqrt{2}\mu}\leq\sigma_H^e-\sigma_H^{-}$

Calculation of the solution with projection at the top: $\Delta\kappa_t^p$ by [éq 5.4-4]

<u>So not</u>

Research regular solution

<u>So not convergence or not checking condition positivity</u>

Calculation of the solution with projection at the top: $\Delta\kappa_t^p$ by [éq 5.4-4]

Checking of acceptability a posteriori: $\sigma^{e^{eq}}\frac{cK}{\sqrt{2}\mu}\leq\sigma_H^e-\sigma_H$

<u>If not acceptable</u> : $\sigma^{e^{eq}}\frac{cK}{\sqrt{2}\mu}>\sigma_H^e-\sigma_H$

Stop on diagnostic of nonchecking of condition of applicability

5.4.4 Inversion of the tops of the cones of traction and compression

A priori, the top of the cone of compression corresponds to a hydrostatic pressure of traction much larger than that of the top of the cone of traction. But, as one can see it on [Figure 5.4.4-a], one can find a history of loading which never activates the criterion of traction, which activates and strongly hammer-hardens the criterion of compression until it to return strictly included in the field of reversibility of the criterion of traction.

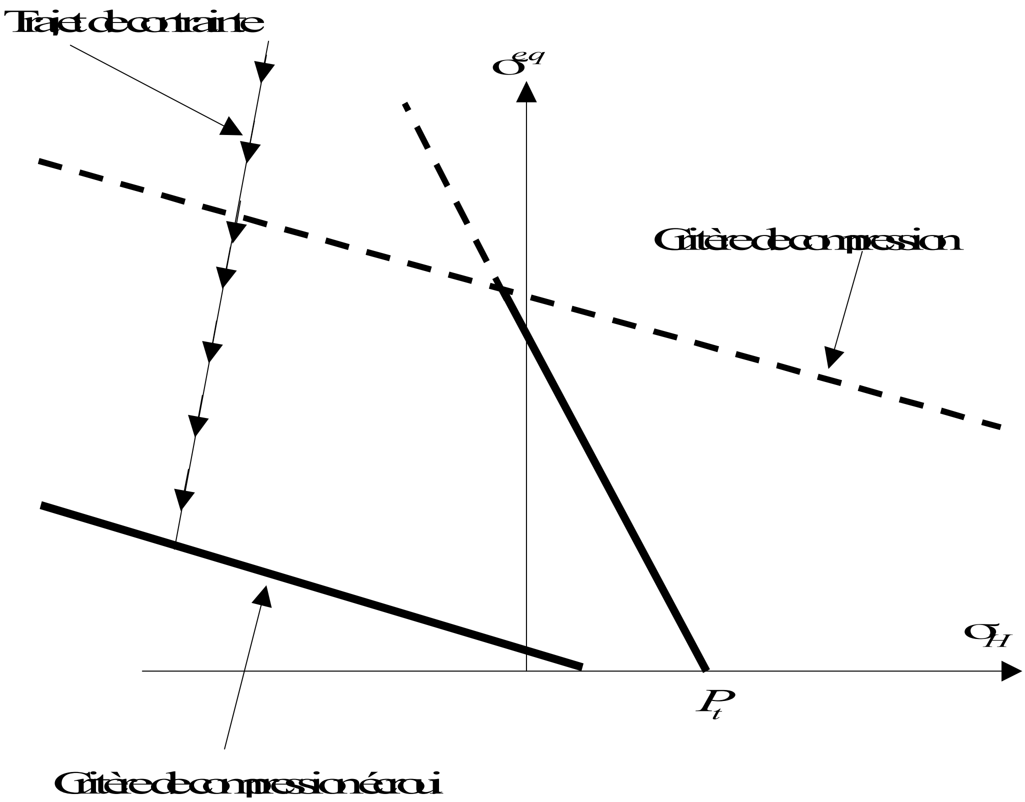


Figure 5.4.4-a

When the two criteria were thus reversed, but the criterion of compression should not intervene a priori any more. It may be whereas the solution is a projection at the top of the cone of compression, which is treated exactly like projection with the top of the cone of traction.

5.4.5 Projection at the top of the two cones

If the two cones were inverted and if the elastic prediction violates the two criteria, it may be which are finally acceptable at the same time the solution of projection at the top of the cone of compression and the top of the cone of traction. In these situations, no criterion makes it possible to select a solution rather than the other and one will thus search a simultaneous projection with the top of the two cones, which will have to thus share the same top, as indicated on [Figure 5.4.5-a].

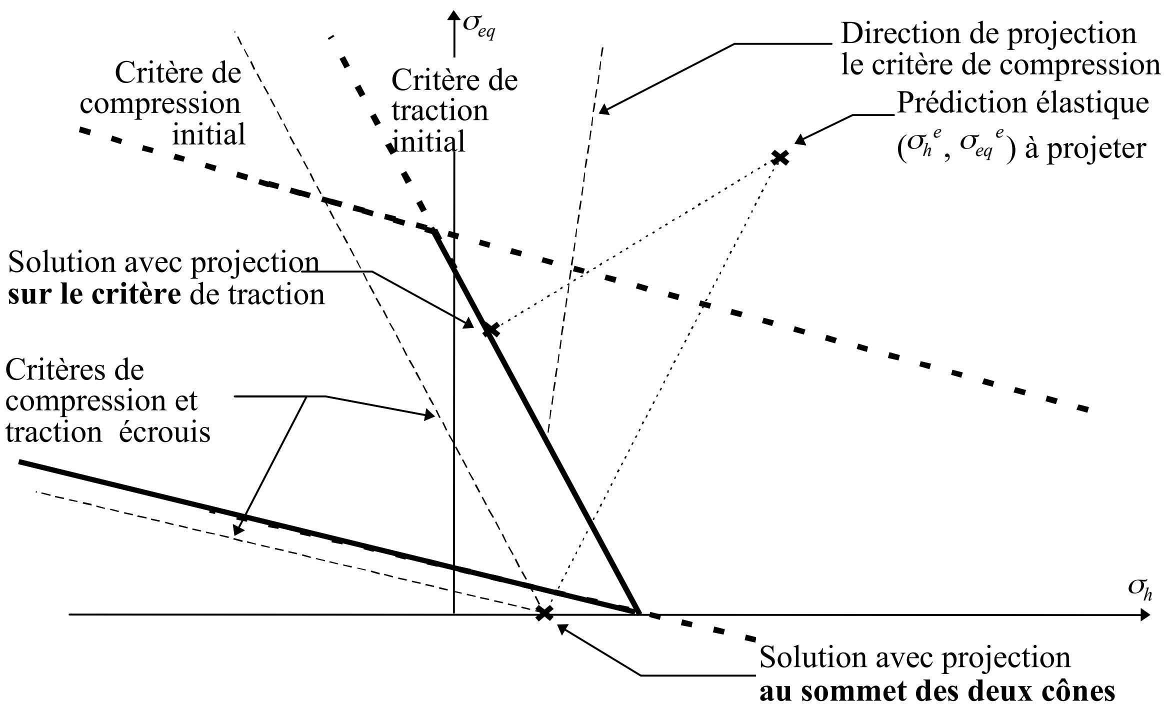


Figure 5.4.5-a

The solution with projection at the two tops is obtained by solving the system:

$$\sigma_H^e - \frac{a}{b} K \Delta \kappa_c^p - \frac{c}{d} K \Delta \kappa_t^p = \frac{b}{a} f_c\left(\kappa_c^{p-} + \Delta \kappa_c^p\right) \quad \textbf{éq 5.4.5-1}$$

$$\sigma_H^e - \frac{a}{b} K \Delta \kappa_c^p - \frac{c}{d} K \Delta \kappa_t^p = \frac{d}{c} f_t\left(\kappa_t^{p-} + \Delta \kappa_t^p\right) \quad \textbf{éq 5.4.5-2}$$

The state of stress is given by:

$$\boldsymbol{s} = 0$$
$$\sigma_H = \frac{d}{c} f_t\left(\kappa_t^{p-} + \Delta \kappa_t^p\right) = \frac{b}{a} f_c\left(\kappa_c^{p-} + \Delta \kappa_c^p\right) \quad \textbf{éq 5.4.5-3}$$

5.5 Determination of the tangent operator

During iterations of the algorithm of Newton-Raphson, it is necessary to calculate the matrix of tangent stiffness. The construction of this one plays an important role in stability, the speed and the precision of the method of resolution. To preserve these properties, the matrix of tangent stiffness must be built starting from an operator binding the increment of constraint to the increment of deformation in a

precise way at the end of the process of return on surfaces of load. The matrix of Hooke, as well as the thermal deformations intervene like constants at the time of the determination of the coherent tangent operator, built at the end of the iteration in the increment concerned.
The calculation of the operator of coherent tangent behavior takes into account the plastic deformations. For reasons of simplicity, we chose to calculate the operator of tangent behavior of speed.

5.5.1 Tangent operator of speed with only one active criterion

In the case of an only active criterion, for example, the criterion in compression, the calculation of the operator of tangent behavior speed is the following:
One thus uses the equations of speed, in elastoplastic load:

$$\dot{\boldsymbol{\sigma}}-\boldsymbol{H}\left(\dot{\boldsymbol{\varepsilon}}-\dot{\kappa}_c^p\frac{\partial f_{comp}}{\partial\boldsymbol{\sigma}}\right)=\boldsymbol{0}$$ **éq 5.5.1-1**

$$\frac{\partial f_{comp}}{\partial\boldsymbol{\sigma}}^T\dot{\boldsymbol{\sigma}}+\frac{\partial f_{comp}}{\partial\kappa_c^p}\dot{\kappa}_c^p=0$$ **éq 5.5.1-2**

The tangent operator of speed is defined by:

$$\dot{\boldsymbol{\sigma}}=\boldsymbol{D}\,\dot{\varepsilon}$$ **éq 5.5.1-3**

While identifying [éq 5.5.1-3] with [éq 5.5.1-1] and [éq 5.5.1-2], one finds classically:

$$\boldsymbol{D}=\boldsymbol{H}-\frac{1}{\delta}\boldsymbol{H}\frac{\partial f_{comp}}{\partial\boldsymbol{\sigma}}^T\frac{\partial f_{comp}}{\partial\boldsymbol{\sigma}}\boldsymbol{H}$$ **éq 5.5.1-4**

with:

$$\delta=\frac{\partial f_{comp}}{\partial\boldsymbol{\sigma}}^T\boldsymbol{H}\frac{\partial f_{comp}}{\partial\boldsymbol{\sigma}}-\frac{\partial f_{comp}}{\partial\kappa_c^p}$$ **éq 5.5.1-5**

5.5.2 Tangent operator of speed with two active criteria

If the two criteria are activated, the criterion in compression and the criterion in traction, the calculation of the operator of tangent behavior speed is the following:
One leaves:

$$\dot{\boldsymbol{\sigma}}-\boldsymbol{H}\left(\dot{\boldsymbol{\varepsilon}}-\dot{\kappa}_c^p\frac{\partial f_{comp}}{\partial\boldsymbol{\sigma}}-\dot{\kappa}_t^p\frac{\partial f_{trac}}{\partial\boldsymbol{\sigma}}\right)=\boldsymbol{0}$$ **éq 5.5.2-1**

$$\frac{\partial f_{comp}}{\partial\boldsymbol{\sigma}}^T\dot{\boldsymbol{\sigma}}+\frac{\partial f_{comp}}{\partial\kappa_c^p}\dot{\kappa}_c^p=0$$ **éq 5.5.2-2**

$$\frac{\partial f_{trac}}{\partial\boldsymbol{\sigma}}^T\dot{\boldsymbol{\sigma}}+\frac{\partial f_{trac}}{\partial\kappa_t^p}\dot{\kappa}_t^p=0$$ **éq 5.5.2-3**

One leads to:

$$\boldsymbol{D}=\boldsymbol{H}-\boldsymbol{H}\left[\frac{\partial f_{comp}}{\partial\boldsymbol{\sigma}}\left(\delta_{cc}\frac{\partial f_{comp}}{\partial\boldsymbol{\sigma}}^T+\delta_{ct}\frac{\partial f_{trac}}{\partial\boldsymbol{\sigma}}^T\right)+\frac{\partial f_{trac}}{\partial\boldsymbol{\sigma}}\left(\delta_{tc}\frac{\partial f_{comp}}{\partial\boldsymbol{\sigma}}^T+\delta_{tt}\frac{\partial f_{trac}}{\partial\boldsymbol{\sigma}}^T\right)\right]$$ éq 5.5.2-4

with:

$$\delta_{cc}=\frac{\left(\frac{\partial f_{trac}}{\partial\boldsymbol{\sigma}}^T\boldsymbol{H}\frac{\partial f_{trac}}{\partial\boldsymbol{\sigma}}-\frac{\partial f_{trac}}{\partial\kappa_t^p}\right)}{\left(\frac{\partial f_{comp}}{\partial\boldsymbol{\sigma}}^T\boldsymbol{H}\frac{\partial f_{comp}}{\partial\boldsymbol{\sigma}}-\frac{\partial f_{comp}}{\partial\kappa_c^p}\right)\left(\frac{\partial f_{trac}}{\partial\boldsymbol{\sigma}}^T\boldsymbol{H}\frac{\partial f_{trac}}{\partial\boldsymbol{\sigma}}-\frac{\partial f_{trac}}{\partial\kappa_t^p}\right)-\left(\frac{\partial f_{comp}}{\partial\boldsymbol{\sigma}}^T\boldsymbol{H}\frac{\partial f_{trac}}{\partial\boldsymbol{\sigma}}\right)\left(\frac{\partial f_{trac}}{\partial\boldsymbol{\sigma}}^T\boldsymbol{H}\frac{\partial f_{comp}}{\partial\boldsymbol{\sigma}}\right)}$$

éq 5.5.2-5

$$\delta_{ct}=\frac{-\left(\frac{\partial f_{comp}}{\partial \boldsymbol{\sigma}}^{T}\boldsymbol{H}\frac{\partial f_{trac}}{\partial \boldsymbol{\sigma}}\right)}{\left(\frac{\partial f_{comp}}{\partial \boldsymbol{\sigma}}^{T}\boldsymbol{H}\frac{\partial f_{comp}}{\partial \boldsymbol{\sigma}}-\frac{\partial f_{comp}}{\partial \kappa_c^p}\right)\left(\frac{\partial f_{trac}}{\partial \boldsymbol{\sigma}}^{T}\boldsymbol{H}\frac{\partial f_{trac}}{\partial \boldsymbol{\sigma}}-\frac{\partial f_{trac}}{\partial \kappa_t^p}\right)-\left(\frac{\partial f_{comp}}{\partial \boldsymbol{\sigma}}^{T}\boldsymbol{H}\frac{\partial f_{trac}}{\partial \boldsymbol{\sigma}}\right)\left(\frac{\partial f_{trac}}{\partial \boldsymbol{\sigma}}^{T}\boldsymbol{H}\frac{\partial f_{comp}}{\partial \boldsymbol{\sigma}}\right)}$$

éq 5.5.2-6

$$\delta_{ct}=\frac{-\left(\frac{\partial f_{trac}}{\partial \boldsymbol{\sigma}}^{T}\boldsymbol{H}\frac{\partial f_{comp}}{\partial \boldsymbol{\sigma}}\right)}{\left(\frac{\partial f_{comp}}{\partial \boldsymbol{\sigma}}^{T}\boldsymbol{H}\frac{\partial f_{comp}}{\partial \boldsymbol{\sigma}}-\frac{\partial f_{comp}}{\partial \kappa_c^p}\right)\left(\frac{\partial f_{trac}}{\partial \boldsymbol{\sigma}}^{T}\boldsymbol{H}\frac{\partial f_{trac}}{\partial \boldsymbol{\sigma}}-\frac{\partial f_{trac}}{\partial \kappa_t^p}\right)-\left(\frac{\partial f_{comp}}{\partial \boldsymbol{\sigma}}^{T}\boldsymbol{H}\frac{\partial f_{trac}}{\partial \boldsymbol{\sigma}}\right)\left(\frac{\partial f_{trac}}{\partial \boldsymbol{\sigma}}^{T}\boldsymbol{H}\frac{\partial f_{comp}}{\partial \boldsymbol{\sigma}}\right)}$$

é **Q 5.5.2-7**

$$\delta_{cc}=\frac{\left(\frac{\partial f_{comp}}{\partial \boldsymbol{\sigma}}^T \boldsymbol{H} \frac{\partial f_{comp}}{\partial \boldsymbol{\sigma}}-\frac{\partial f_{comp}}{\partial \kappa_c^p}\right)}{\left(\frac{\partial f_{comp}}{\partial \boldsymbol{\sigma}}^T \boldsymbol{H} \frac{\partial f_{comp}}{\partial \boldsymbol{\sigma}}-\frac{\partial f_{comp}}{\partial \kappa_c^p}\right)\left(\frac{\partial f_{trac}}{\partial \boldsymbol{\sigma}}^T \boldsymbol{H} \frac{\partial f_{trac}}{\partial \boldsymbol{\sigma}}-\frac{\partial f_{trac}}{\partial \kappa_t^p}\right)-\left(\frac{\partial f_{comp}}{\partial \boldsymbol{\sigma}}^T \boldsymbol{H} \frac{\partial f_{trac}}{\partial \boldsymbol{\sigma}}\right)\left(\frac{\partial f_{trac}}{\partial \boldsymbol{\sigma}}^T \boldsymbol{H} \frac{\partial f_{comp}}{\partial \boldsymbol{\sigma}}\right)}$$

éq 5.5.2-8

The expression seems expensive to express in term of products of matrix and calculation. But, when the operations are made in the order which is appropriate, it is enough to calculate terms little. Moreover, in fact the same terms intervene on several occasions. It is necessary to calculate the derivative of the criteria compared to the constraint, and compared to the plastic multipliers, then the sums and the products with the actual values, to finish by the constitution of the matrices and theirs sums.
Lastly, the resulting matrix with the advantage of being symmetrical, which is appropriate for the standard resolution with *Code_Aster* .

5.5.3 Derivative successive of the criteria in traction and compression

5.5.3.1 Successive drifts of the criteria compared to the constraint

The derivative of the isotropic and deviatoric components of the constraints compared to the tensor of constraints are expressed in the following way:

By defining the vector $\boldsymbol{\pi}_0=\begin{bmatrix}1\\1\\1\\0\\0\\0\end{bmatrix}$

the derivative of the criteria compared to the tensor of constraints are expressed in the following way:

$$\frac{\partial f_{comp}}{\partial \boldsymbol{\sigma}}=\frac{\boldsymbol{s}}{\sqrt{2} b \sigma^{eq}}+\frac{a}{3\mathrm{b}}\boldsymbol{\pi}_0$$

$$\frac{\partial f_{trac}}{\partial \boldsymbol{\sigma}}=\frac{\boldsymbol{s}}{\sqrt{2} d \sigma^{eq}}+\frac{c}{3\mathrm{d}}\boldsymbol{\pi}_0$$

5.5.3.2 Successive drifts of the criteria compared to the plastic multipliers

Derived from the criterion of compression in the case of a curved linear post-peak:

$$\frac{\partial f_{comp}}{\partial \kappa_c^p}=\frac{-4}{3}.f'_c.\left(\frac{1}{\kappa_e}-\frac{\kappa_c^p}{\kappa_e{}^2}\right) \quad si \quad \kappa_c^p \le \kappa_e$$

$$\frac{\partial f_{comp}}{\partial \kappa_c^p}=f'_c.\left(\frac{1}{(\kappa_t^u-\kappa_e)}\right) \quad si \quad \kappa_e \le \kappa_c^p \le \kappa_t^u$$

Derived from the criterion of compression in the case of a curved nonlinear post-peak:

$$\frac{\partial f_{comp}}{\partial \kappa_c^p}=\frac{-4}{3}.f'_c.\left(\frac{1}{\kappa_e}-\frac{\kappa_c^p}{\kappa_e{}^2}\right) \quad si \quad \kappa_c^p \le \kappa_e$$

$$\frac{\partial f_{comp}}{\partial \kappa_c^p}=2.f'_c.\left(\frac{(\kappa_c^p-\kappa_e)}{(\kappa_c^u-\kappa_e)^2}\right) \quad si \quad \kappa_e \le \kappa_c^p \le \kappa_c^u$$

Derived from the criterion of traction in the case of a curved linear post-peak:

$$\frac{\partial f_{trac}}{\partial \kappa_t^p}=\frac{f'_t}{\kappa_t^u} \quad si \quad \kappa_t^p \leq \kappa_t^u$$

Derived from the criterion of traction in the case of a curved post-peak exponential:

$$\frac{\partial f_{trac}}{\partial \kappa_t^p}=f'_t\left(\frac{a}{\kappa_t^u}\right)e^{-\frac{a.\kappa_t^p}{\kappa_t^u}}$$

5.6 Internal variables of the model

We assemble here the internal variables stored in each point of Gauss in the implementation of the model

Internal number of variable	Physical direction
1	κ_c^p : plastic deformation cumulated in compression
2	κ_t^p : plastic deformation cumulated in traction
3	θ : maximum temperature attack at the point of gauss
4	Indicator of plasticity

5.7 Top-level flowchart of resolution

The flow chart understands the various stages of the resolution, with the treatment of projections at the tops of the cones of compression and traction in the following way:

at the beginning of algorithm,

one carries out a projection at the top of the cone of traction:

- when the elastic prediction checks the condition of projection **a priori in traction** ,
- when the elastic prediction checks the condition of projection **a priori in compression** and that the tops of the cones of traction and compression were inverted on the hydrostatic axis,

one carries out a simultaneous projection with the tops of the cones of traction and compression:

- when the elastic prediction checks the condition of projection **a priori in compression** and that the tops of the cones of traction and compression were inverted on the hydrostatic axis, and that projection at the top of the cone of traction did not give a valid solution,

one carries out a projection at the top of the cone of compression:

- when the elastic prediction checks the condition of projection **a priori in compression** and that the tops of the cones of traction and compression were inverted on the hydrostatic axis, and that projection at the top of the cone of traction did not give a valid solution, and that simultaneous projection with the tops of the two cones did not give a valid solution,

in medium of algorithm,

one carries out one, two or three standards resolutions with projection on the criterion of compression or the criterion of traction or the two criteria at the same time,

and **at the end of the algorithm** ,

when that the standards resolutions with activation of a criterion (traction or compression) or of the two criteria at the same time did not give a solution,

one carries out a projection at the top of the cone of traction:

- when the elastic prediction checks the condition of projection **a posteriori in traction** ,
- when the elastic prediction checks the condition of projection **a posteriori in compression** (and that the tops of the cones of traction and compression were inverted on the hydrostatic axis,

one carries out a simultaneous projection with the tops of the cones of traction and compression:

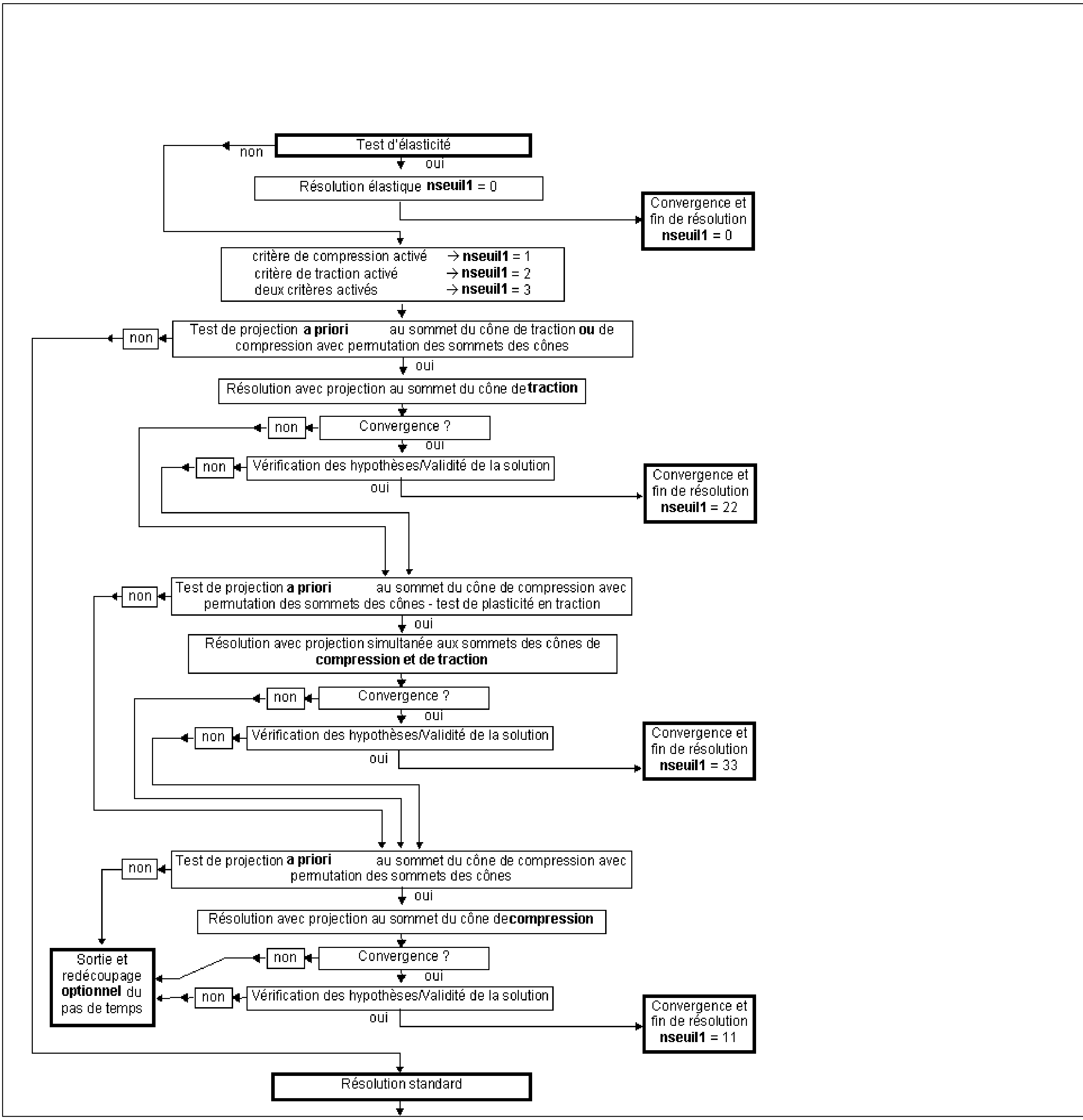
- when the elastic prediction checks the condition of projection **a posteriori in compression** and that the tops of the cones of traction and compression were inverted on the hydrostatic axis, and that projection at the top of the cone of traction did not give a valid solution,

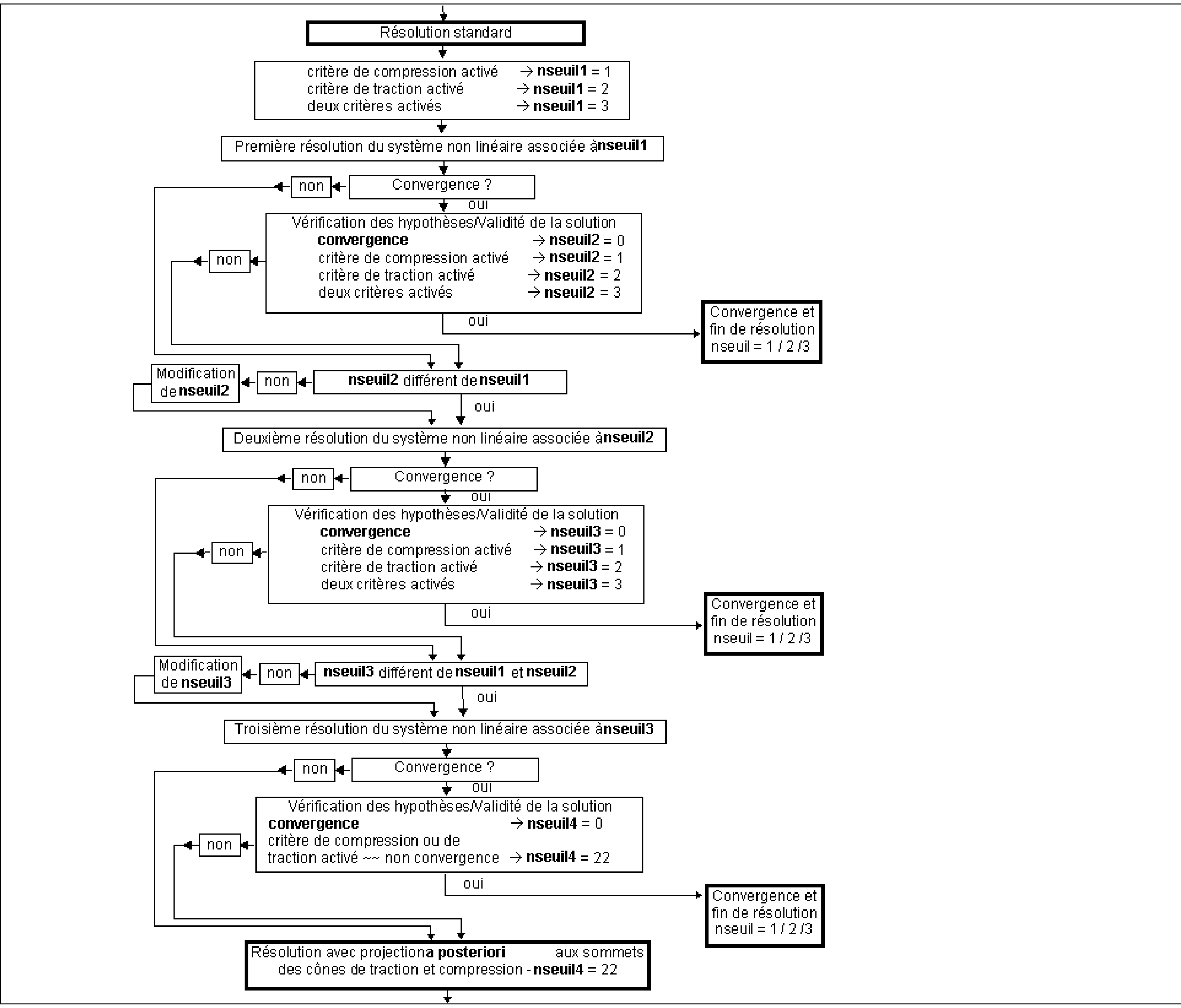
one carries out a projection at the top of the cone of compression:

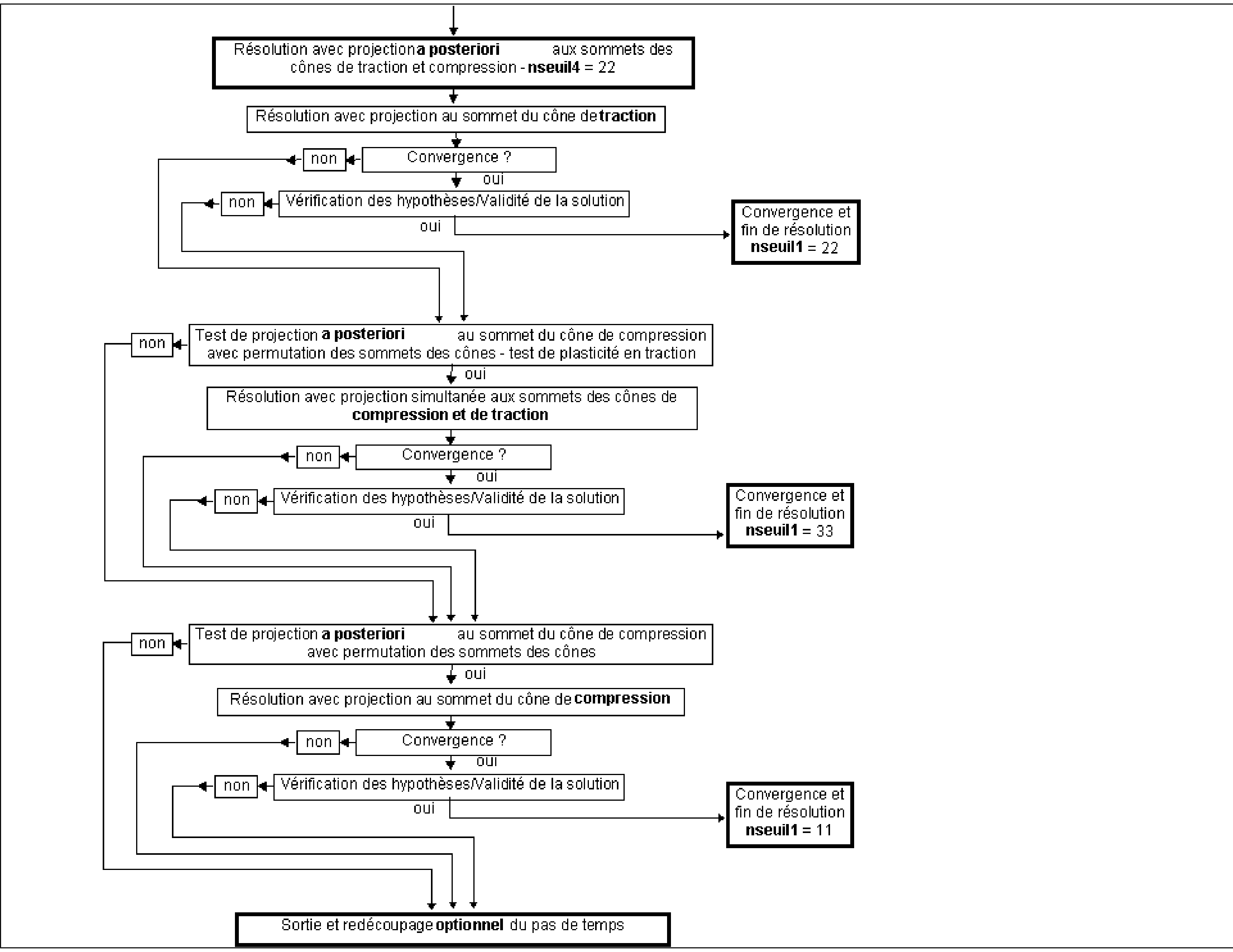
- when the elastic prediction checks the condition of projection **a posteriori in compression** and that the tops of the cones of traction and compression were inverted on the hydrostatic axis, and that projection at the top of the cone of traction did not give a valid solution, and that simultaneous projection with the tops of the two cones did not give a valid solution.

At the conclusion of each resolution having converged, one carries out the checks of conformity of the following solution:

- validity of the solution compared to the second criterion, when the resolution was made with only one criterion. In all the cases, it is enough to check that the two computed criterions with the final constraint, are negative or worthless,
- conformity of the solution: one calculates in the course of resolution the final equivalent constraint. It happens sometimes that the solution is beyond the top of the cone which is hammer-hardened, which leads to an equivalent constraint “negative”. Numerically, that results in a strictly positive final criterion. To check the conformity of the solution, it is enough to check that the two computed criterions with the final constraint, are negative or worthless,
- validity of projections at the tops of the cones. It should be checked that after resolution, when the work hardening of the criterion is known, the slope of the right-hand side connecting the elastic prediction to projection is lower than the slope of the direction of projection. (condition of projection a posteriori at the top of the cones). In the contrary case, that means that there exists a solution with standard resolution.


Test d'élasticité
non
oui
Résolution élastique nseuil1 = 0
Convergence et fin de résolution nseuil1 = 0
critère de compression activé → nseuil1 = 1
critère de traction activé → nseuil1 = 2
deux critères activés → nseuil1 = 3
Test de projection a priori au sommet du cône de traction ou de compression avec permutation des sommets des cônes
non
oui
Résolution avec projection au sommet du cône de traction
Convergence ?
non
oui
Vérification des hypothèses/Validité de la solution
non
oui
Convergence et fin de résolution nseuil1 = 22
Test de projection a priori au sommet du cône de compression avec permutation des sommets des cônes - test de plasticité en traction
non
oui
Résolution avec projection simultanée aux sommets des cônes de compression et de traction
Convergence ?
non
oui
Vérification des hypothèses/Validité de la solution
non
oui
Convergence et fin de résolution nseuil1 = 33
Test de projection a priori au sommet du cône de compression avec permutation des sommets des cônes
non
oui
Résolution avec projection au sommet du cône de compression
Convergence ?
non
oui
Sortie et redécoupage optionnel du pas de temps
Vérification des hypothèses/Validité de la solution
non
oui
Convergence et fin de résolution nseuil1 = 11
Résolution standard


Résolution standard
critère de compression activé → nseuil1 = 1
critère de traction activé → nseuil1 = 2
deux critères activés → nseuil1 = 3
Première résolution du système non linéaire associée à nseuil1
Convergence ?
non
oui
Vérification des hypothèses/Validité de la solution
convergence → nseuil2 = 0
critère de compression activé → nseuil2 = 1
critère de traction activé → nseuil2 = 2
deux critères activés → nseuil2 = 3
non
oui
Convergence et fin de résolution nseuil = 1 / 2 /3
Modification de nseuil2
non
nseuil2 différent de nseuil1
oui
Deuxième résolution du système non linéaire associée à nseuil2
Convergence ?
non
oui
Vérification des hypothèses/Validité de la solution
convergence → nseuil3 = 0
critère de compression activé → nseuil3 = 1
critère de traction activé → nseuil3 = 2
deux critères activés → nseuil3 = 3
non
oui
Convergence et fin de résolution nseuil = 1 / 2 /3
Modification de nseuil3
non
nseuil3 différent de nseuil1 et nseuil2
oui
Troisième résolution du système non linéaire associée à nseuil3
Convergence ?
non
oui
Vérification des hypothèses/Validité de la solution
convergence → nseuil4 = 0
critère de compression ou de traction activé ~~ non convergence → nseuil4 = 22
non
oui
Convergence et fin de résolution nseuil = 1 / 2 /3
Résolution avec projection a posteriori aux sommets des cônes de traction et compression - nseuil4 = 22



Note:

In the case of projections at the tops of the cones, one starts systematically with projection at the top of the cone of traction. If this solution is valid, this one is preserved. In the contrary case, if the criterion of traction is activated, one carries out a resolution with projection at the tops of the two cones. If the new solution is valid, that one is preserved. If not, one carries out a resolution with projection at the top of the cone of compression alone.

Note:

If the conditions of projection at the tops of the cones are activated, one of the three solutions must be valid, but for particularly important elastic jumps, it may be that the resolution does not succeed. The solution is then to carry out a recutting of the step of time.

Note:

Projection at the tops of the two cones simultaneously supposes that the criterion of traction is activated. It may be very well that in the event of permutation of the tops, only the criterion of compression is found activated. It is necessary to then make a projection at the top of the cone of compression alone.

Annexe 1snap-back with the initial values of the coefficients C and D

We show in this appendix the problem of snap-back met in the simulation of a simple tensile test followed compression, if the choice of the coefficients c and d criterion of traction corresponds to a situation where the criterion of traction cuts the axes in a diagram of constraint plane, i.e. the choice of the coefficients [éq 3.3-1] and [éq 3.3-2] leading to a field of reversibility represented on [Figure 3.3-b]:
The assumptions are the following ones:

- one takes into account only the criterion of traction,
- the marrow of work hardening is of the type: $f_t(\kappa_t^p)=f_t^{'}+h\kappa_t^p$,
- one notes simply: $\lambda=\kappa_t^p$ so that the curve of work hardening is written: $f_t=f_t^{'}+h\lambda$,
- the null Poisson's ratio,
- work hardening is negative,
- the condition of applicability is met: $-E<h<0$.

One considers an axial plain test controlled in deformation according to X, as indicated on Figure 5 -

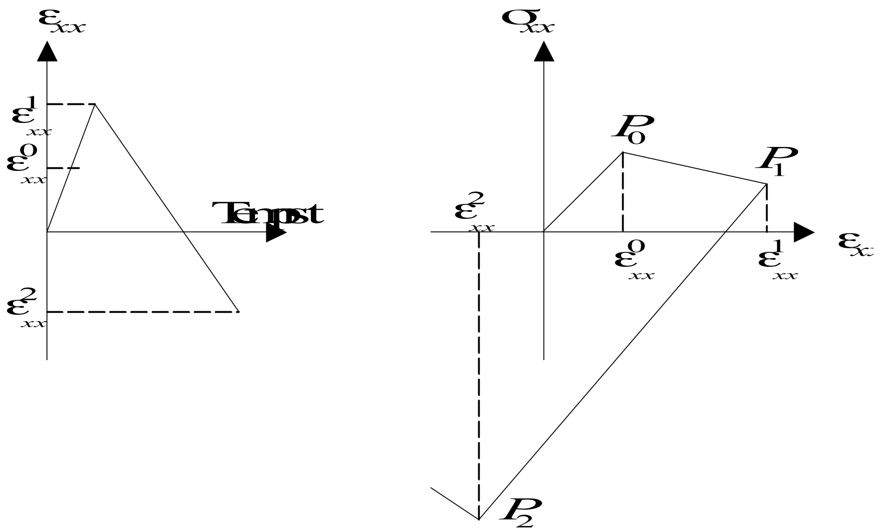


Figure 5 - has

In the other directions y and z , the imposed conditions are conditions of worthless constraints: $\sigma_{yy}=\sigma_{zz}=0$.

One starts by imposing a deformation of traction ε_{xx}^1 such that there is a plasticization in traction, but without the limit of traction falling down to 0. It is the point P_1 in the diagram stress-strain. One notes ε_{xx}^0 the deformation for which the rupture limit in traction appears for the first time. Beyond the point P_1 , one imposes a decrease of the deformation, which involves an elastic unloading of material, up to the value $\varepsilon_{xx}^2<0$ deformation for which one has a plasticization again, but this time under compressive stress. The object of this appendix is primarily to study the behavior of the model not retained (that corresponding to the formulas [éq 3.3-1] and [éq 3.3-2]) beyond the point P_2 .

A1.1 Calculation of the constraints and the deformations during the loading

Taking into account the assumptions pointed out higher, the invariants of constraint are worth:

$$I_1=\sigma_{xx} \quad \text{éq A1.1-1}$$

$$\sqrt{J_2}=\frac{|\sigma_{xx}|}{\sqrt{3}} \quad \text{éq A1.1-2}$$

$$\sigma^{eq}=|\sigma_{xx}| \quad \text{éq A1.1-3}$$

The plastic flow is calculated by:

$$\dot{\varepsilon}_{xx}^{p}=\frac{c+\sqrt{2}}{3d}\dot{\lambda} \quad \text{éq A1.1-4}$$

A1.1.1 Ways 0P $_0$ and P $_0$ P $_1$

Taking into account the relations [éq A1.1-1] with [éq A1.1-4], and the fact that along this way constraints σ_{xx} are positive, one finds easily

$$\sigma_{xx_1}=E\left(\frac{h\,\varepsilon_{xx_1}+\frac{c+\sqrt{2}}{3\mathrm{d}}f_t^{'}}{h+\frac{(c+\sqrt{2})^2 E}{9\mathrm{d}^2}}\right) \quad \text{éq A1.1.1-1}$$

A1.1.2 Chemin P1P2

By definition, $P1P2$ is an elastic way of discharge, the point $P2$ being such as the criterion is again reached there

$$\sigma_{xx_2}=\frac{3\mathrm{d}}{c-\sqrt{2}}\sigma_{xx_1} \quad \text{éq A1.1.2-1}$$

$$\varepsilon_{xx_2}=\left(\frac{3\mathrm{d}}{c-\sqrt{2}}\right)\frac{\sigma_{xx_1}}{E}+\varepsilon_{xx_1} \quad \text{éq A1.1.2-2}$$

A1.1.3 Beyond the P2 point

One is interested now in the slope of the curve at the point $P2$ in the reference mark $(\sigma_{xx},\varepsilon_{xx})$. More precisely one is interested in the slope of this curve for a dissipative solution.
By writing that the material remains plastic beyond the point $P2$, i.e. that the state of stress remains on criterion-which is hammer-hardened, one finds:

$$\dot{\sigma}_{xx}=\frac{3\mathrm{d}}{c-\sqrt{2}}h\dot{\lambda} \quad \text{éq A1.1.3-1}$$

In addition, by calculating the increase in flow plastic, and by deferring it in the calculation of the increase in constraint, one finds:

$$\dot{\varepsilon}_{xx}=\frac{\dot{\sigma}_{xx}}{E}+\frac{\dot{\lambda}}{3d}\left(c-\sqrt{2}\right) \qquad \textbf{éq A1.1.3-2}$$

One can then eliminate $\dot{\lambda}$ between [éq A1.1.3-1] and [éq A1.1.3-2] and one obtains:

$$\frac{\dot{\sigma}_{xx}}{\dot{\varepsilon}_{xx}}=\frac{h\,E}{h+E\,\frac{\left(c-\sqrt{2}\right)^2}{9d^2}}=E_{T_2} \qquad \textbf{éq A1.1.3-3}$$

This formula gives the slope of the answer in the plan $\left(\sigma_{xx},\varepsilon_{xx}\right)$
The numerator is always negative since h is negative.

The sign of $\left.\frac{\dot{\sigma}_{xx}}{\dot{\varepsilon}_{xx}}\right|_2=E_{T_2}$ depends on the sign of the denominator, thus two cases are posed:

If $h<-E\frac{\left(c-\sqrt{2}\right)^2}{9d^2}$ then $\left.\frac{\dot{\sigma}_{xx}}{\dot{\varepsilon}_{xx}}\right|_2=E_{T_2}$ one is positive has a configuration of snap-back.

If $h>-E\frac{\left(c-\sqrt{2}\right)^2}{9d^2}$ then $\left.\frac{\dot{\sigma}_{xx}}{\dot{\varepsilon}_{xx}}\right|_2=E_{T_2}$ is negative and there is no snap-back.

A new condition appears to avoid the snap back, condition which we compare with that already evoked, but which related to in fact possible the snap back at the point $P1$.

Slope at the point $P1$ in the reference mark $\left(\varepsilon_{xx},\sigma_{xx}\right)$: $E_{T_1}=\frac{h\,E}{h+E}$,

Slope at the point $P2$ in the reference mark $\left(\varepsilon_{xx},\sigma_{xx}\right)$: $E_{T_2}=\frac{h\,E}{h+E\frac{\left(c-\sqrt{2}\right)^2}{9d^2}}$.

However $\frac{\left(c-\sqrt{2}\right)^2}{9d^2}=3\,\alpha^2=3\left(\frac{f'_t}{f'_c}\right)^2$.

One can for example express E_{T_2} according to E_{T_1} while eliminating h :

$$E_{T_2}=\frac{E_{T_1}E}{3\,\alpha^2E+E_{T_1}\left(1-3\,\alpha^2\right)} \quad .$$

For example,

$E_{T_2}=\infty$ for $E_{T_1}=-\frac{3\alpha^2 E}{(1-3\alpha^2)}$.

As example, for $E=32000\,Mpa$, $f'_t=3\,Mpa$ and $f'_c=38{,}3\,Mpa$ one finds $E_{T_1}\approx-601$. Thus E_{T_1} is very weak compared to E . as illustrated on [A1.1.3-a Figure].

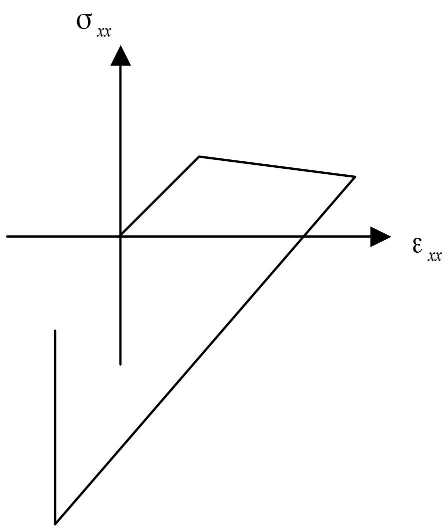


Figure A1.1.3-a

Thus, a condition implying that there is no snap-back at the point $P2$ would be too restrictive and would lead to practically choose a material not fragile in traction.
For this reason we preferred to modify the expression of the coefficients c and d as indicated in the paragraph [§ 3.3].

This said, and even if the adopted solution, consisting in modifying the coefficients c and d seem reasonable, the example treated in this appendix shows that a very simple problem can finally be a problem of structure: there is in this example of the equilibrium conditions, they are the conditions: $\sigma_{yy}=\sigma_{zz}=0$.

6 Bibliography

- J.F. GEORGIN "Contribution to the modeling of the concrete under fast request of dynamics. The taking into account of the effect speed by viscoplasticity" - Thesis (1/15/98).
- G. HEINFLING "Contribution to the digital modeling of the behavior of the concrete and the reinforced concrete structures under thermomechanical requests at high temperature" - Thesis (1/14/98)
- R.T. ROCKAFELLAR "Convex analysis" Princeton University Press 1972
- B. HALPHEN and NGUYEN QUOC HIS, "On generalized standard materials" Newspaper of Mechanical Flight 14 n° 1.1975
- NR. TARDIEU, I. VAUTIER, E. LORENTZ "quasi static nonlinear Algorithm" reference document Aster [R5.03.01].

7 Features and checking

This document relates to the law of behavior `BETON_DOUBLE_DP` (keyword `BEHAVIOR` of `STAT_NON_LINE`) and its associated material `BETON_DOUBLE_DP` (order `DEFI_MATERIAU`).

This law of behavior is checked by the cases following tests:

SSNP116	Coupling creep/cracking - uniaxial Traction	[V6.03.116]
SSNV143	Biaxial traction with the law of behavior `BETON_DOUBLE_DP`	[V6.04.143]
SSNV150	Triaxial traction with the law of behavior `BETON_DOUBLE_DP`	[V6.04.150]
SSNV151	Traction/Compression with the law of behavior `BETON_DOUBLE_DP`	[V6.04.151]

8 Description of the versions of the document

Version Aster	Author (S) Organization (S)	Description of the modifications
5	***C. CHAVANT***, *EDF-R&D/AMA* *B.CIREE* *CS-SI*	Initial text