Modelings THHM. General information and algorithms

Summary

Modules THM of *Code_Aster* are those which treat the equations of the mechanics of the continuous mediums by using the theory of the porous environments possibly unsaturated and by considering that the mechanical, thermal and hydraulic phenomena are completely coupled. We present here, or the conservation equation equilibrium equations solved by these modules. We give a definition of the generalized constraints and generalized deformations, allowing to define way rather general what is a law of behavior THM - at least what the modules considered consider thus - and allowing to treat the nonlinear equations displayed within the framework of the algorithms of the operator STAT_NON_LINE.

Contents

1 Introduction

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We present here, or the conservation equation equilibrium equations solved by these modules. We give a definition of the generalized constraints and generalized deformations, allowing to define way rather general what is a law of behavior THM - at least what the modules considered consider thus - and allowing to treat the nonlinear equations displayed within the framework of the algorithms of the operator STAT_NON_LINE.
Laws of behavior THM strictly speaking are not developed in this document, but in the document [R7.01.11].
Chemical phenomena (transformations of the components, reactions producing of components etc…), just as the radiological phenomena are not taken into account at this stage of the development of *Code_Aster*. The mechanical, hydraulic and thermal phenomena are taken into account or not according to the behavior called upon by the user in the order STAT_NON_LINE, according to the following nomenclature:

Modeling	**Phenomena taken into account**
KIT_HM	Mechanics, hydraulics with an unknown pressure
KIT_HHM	Mechanics, hydraulics with two unknown pressures
KIT_THH	Thermics, hydraulics with two unknown pressures
KIT_THM	Thermics, mechanics, hydraulics with an unknown pressure
KIT_THHM	Thermics, mechanics, hydraulics with two unknown pressures

The document present describes the laws of conservation for the case more the general known as THHM. The simpler cases are obtained starting from the case general by simply cancelling the quantity absent.

2 Presentation of the problem: Assumptions, Notations

In this chapter, one mainly endeavours to show the porous environment and his characteristics.

2.1 Description of the porous environment

The porous environment considered is a volume made up of a more or less homogeneous solid matrix, more or less coherent (very coherent in the case of the concrete, little in the case of sand). Between the solid elements, one finds pores. One distinguishes the closed pores which do not exchange anything with their neighbors and the connected pores in which the exchanges are numerous. When one speaks about porosity, it is many connected pores about which one speaks. Inside these pores are with more the two components present at more under two phases. The system is regarded as closed.

2.2 Notations

Sizes associated with a component c present under a phase p are noted X_c^p. The index of the component c can vary 1 with 2 and that of the phase also. These components can be (and will be subscripted like such where necessary):

- w for liquid water,
- ad for the dissolved air,
- as for the dry air,
- vp for the steam.

The porous environment at the current moment is noted Ω , its border $\partial\Omega$, and it is noted $\Omega_0, \partial\Omega_0$ at the initial moment.
$\mathbf{n}$ indicate the normal in a point of $\partial\Omega$, image of the normal $\mathbf{n_0}$ with $\partial\Omega_0$. We will note $d(\partial\Omega)$ (respectively $d(\partial\Omega_0)$) the element of surface of $\partial\Omega$ (respectively $\partial\Omega_0$).

The medium is defined by:

- parameters (vector position $\mathbf{x}$, time t),
- variables (displacements, pressures, temperature),
- intrinsic sizes (forced and mass deformations, contributions, heat, enthali, flows hydraulic, thermics…).

For the solid phase, one makes the assumption of small displacements.

The various notations are clarified hereafter.

2.2.1 Descriptive variables of the medium

These are the variables whose knowledge according to time and of the place make it possible to know the state of the medium completely. These variables break up into two categories:

- geometrical variables,
- variables of thermodynamic state.

2.2.1.1 Geometrical variables

In all that follows, one adopts a Lagrangian representation compared to the skeleton (within the meaning of [bib1]) and the coordinates $\mathbf{x}=\mathbf{x}_s(t)$ are those of a material point attached to the skeleton. All the space operators of derivation are defined compared to these coordinates.

Displacements of the skeleton are noted $\mathbf{u}(\mathbf{x},t)=\begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix}$.

2.2.1.2 Thermodynamic variables of state

The thermodynamic variables are:

- pressures of the components: since we consider that there is with more the two components, there will be with more the two conservation equations of the mass, and thus by duality with more the two variables of pressure,
- the temperature of the medium $T(\mathbf{x},t)$.

2.2.1.3 Descriptive fields of the medium

The principal unknown factors, which are also the nodal unknown factors (noted $\mathbf{U}(\mathbf{x},t)$ in this document) are:

- 2 or 3 (according to the dimension of space) displacements $u_x(\mathbf{x},t), u_y(\mathbf{x},t), u_z(\mathbf{x},t)$ for modelings `KIT_HM`, `KIT_HHM`, `KIT_THM`, `KIT_THHM`,
- the temperature $T(\mathbf{x},t)$ for modelings `KIT_THH`, `KIT_THM`, `KIT_THHM`,
- two pressures $p_1(\mathbf{x},t), p_2(\mathbf{x},t)$ for modelings `KIT_HHM`, `KIT_THH`, `KIT_THHM`,
- a pressure $p_1(\mathbf{x},t)$ for modelings `KIT_HM`, `KIT_THM`.

2.2.2 Sizes

The equilibrium equations are:

- conservation of the momentum for mechanics,
- conservation lots of fluid for hydraulics,
- conservation of energy for thermics.

The equilibrium equations utilize directly the generalized constraints. The generalized constraints are connected to the deformations generalized by the laws of behavior. The generalized deformations are calculated directly starting from the variables of state and their temporal space gradients.
The laws of behavior can use additional quantities, often arranged in the internal variables. These quantities are not described in this document which strictly speaking does not treat laws of behavior.

2.2.2.1 Sizes characteristic of the heterogeneous medium

- Porosity eulérienne: φ .

  If one notes Ω_{φ} the part of volume Ω occupied by the vacuums in the current configuration, one a:

$$\varphi=\frac{\Omega_{\varphi}}{\Omega}$$ **éq 2.2.2.1 - 1**

  The definition of porosity φ is thus that of porosity eulérienne.

- The saturation of the phase p : S^{p} .

  If one notes Ω^{p} the total volume occupied by the phase p , in the current configuration, one has by definition:

$$S^{p}=\frac{\Omega^{p}}{\Omega_{j}}$$ **éq 2.2.2.1 - 2**

  This saturation is thus finally a proportion varying enters 0 and 1 . In the equations of assessment, it is clear that it is the product of porosity by saturation φS^{p} who will intervene. One can thus legitimately wonder why it is not that quantity which is taken as unknown factor. The answer comes from what it is saturation S^{p} who intervenes more simply in the laws of behavior.

- Density eulérienne of the component c in the phase p : ρ_{c}^{p} .

  If one notes M_{c}^{p} mass of the phase p component c , in volume Ω skeleton in the current configuration, one has by definition:

$$M_{c}^{p}=\int_{\Omega^{p}}\rho_{c}^{p}\,d\,\Omega^{p}=\int_{\Omega_{\varphi}}\rho_{c}^{p}S^{p}\,d\,\Omega_{\varphi}=\int_{\Omega}\rho_{c}^{p}S^{p}\varphi\,d\,\Omega$$ **éq 2.2.2.1 - 3**

Density of the phase p is simply the sum of the densities of its components:

$$\rho^p = \sum_c \rho_c^p$$

- Lagrangian homogenized density: r .
  At the moment running, mass of volume Ω , M_Ω is given by:

$$M_\Omega = \int_{\Omega_0} r \, d\Omega_0$$ **éq 2.2.2.1 - 4**

2.2.2.2 Mechanical magnitudes

- The tensor of the deformations $\boldsymbol{\varepsilon}(\mathbf{u})(\mathbf{x}, t) = \frac{1}{2}\left(\nabla \mathbf{u}^T + \nabla \mathbf{u}\right)$,
- The tensor of the constraints which are exerted on the porous environment: $\boldsymbol{\sigma}$.

This tensor breaks up into a tensor of the effective constraints plus a tensor of constraints of pressure $\boldsymbol{\sigma} = \boldsymbol{\sigma}' + \sigma_p \mathbf{I}$. $\boldsymbol{\sigma}'$ et σ_p are components of the generalized constraints. This cutting is finally rather arbitrary, but corresponds all the same to an assumption rather commonly allowed, at least for the mediums saturated with liquid.

2.2.2.3 Hydraulic sizes

- Mass contributions in components m_c^p (unit: kilogramme per cubic meter). They represent the mass of fluid brought between the initial and current moments. They belong to the generalized constraints.

$$m_c^p = J \rho_c^p \varphi S^p - \rho_c^{p_0} \varphi_0 S_0^p$$ **éq 2.2.2.3 - 1**

  The mass contributions make it possible to define the total density seen compared to the configuration of reference: $r(\mathbf{x}, t) = r_0 + m_{lq}(\mathbf{x}, t) + m_{vp}(\mathbf{x}, t) + m_{as}(\mathbf{x}, t)$, where r_0 indicate the density homogenized in an initial state.

- Hydraulic flows:

  $\mathbf{w}_c^p$ (unit: kilogramme/second/square meter) of representation eulérienne
  $\mathbf{M}_c^p$ (unit: kilogramme/second/square meter) of Lagrangian representation

One notes $\mathbf{v}_c^p$ the speed of the component c in the phase p , J Jacobien of the material transformation and $\mathbf{v}_S = \frac{d\mathbf{u}}{dt}$ the speed of the skeleton. $\rho_{c0}^p, \varphi_0, S_0^p$ the densities, porosity and saturations at the initial moment indicate. By definition:

$$\mathbf{w}_c^p = \rho_c^p \varphi S^p \left(\mathbf{v}_c^p - \mathbf{v}_s\right)$$ **éq 2.2.2.3 - 2**

The Lagrangian form of $\mathbf{w}_c^p$ noted $\mathbf{M}_c^p$ is obtained while writing:

$$\mathbf{M}_c^p . \mathbf{n}_0 d\left(\partial\Omega_0\right) = \mathbf{w}_c^p . \mathbf{n} d\left(\partial\Omega\right)$$ **éq 2.2.2.3 - 3**

Variables $m_1, \mathbf{M}_1$ and $m_2, \mathbf{M}_2$ refer each one to a component **of conservative mass**.
One poses by principle:

$$m_1 = m_1^1 + m_1^2; \quad \mathbf{M}_1 = \mathbf{M}_1^1 + \mathbf{M}_1^2$$
$$m_2 = m_2^1 + m_2^2; \quad \mathbf{M}_2 = \mathbf{M}_2^1 + \mathbf{M}_2^2$$

What we will write:

$$m_{constituant} = \sum_{\substack{nb\ phase\ du \\ constituant}} m_{constituant}^{phase}$$
$$\mathbf{M}_{constituant} = \sum_{\substack{nb\ phase\ du \\ constituant}} \mathbf{M}_{constituant}^{phase}$$

In the applications, one could for example have:

- 2 components: air and water,
- 2 phases for water,
- 1 phase for the air.

One would have then: m_1^1 et $\mathbf{M}_1^1$: contribution of mass and liquid water flow
m_1^2 et $\mathbf{M}_1^2$: contribution of mass and vapor flow
m_2^1 et $\mathbf{M}_2^1$: contribution of mass and flow of dry air
m_2^2 et $\mathbf{M}_2^2$: non-existent

- Pressures:

Since we consider that there can be two components other than the solid, there are two conservation equations of the mass, and thus two associated multipliers, i.e. two pressures p_1 et p_2. No assumption is made on what these two pressures mean p_1 et p_2. That will depend on the laws of behavior and the way of writing them. For example one can choose:

$$p_1 = \text{pression capillaire } (\text{p}(\text{gaz}) - \text{p}(\text{liquide}))$$
$$p_2 = \text{pression de gaz}(\text{ vapeur + air})$$

2.2.2.4 Thermal quantities

- Not convectée heat Q' (see further) (unit: Joule),
- Mass enthali of the components $h_c^{m\,p}$ (unit: Joule/Kelvin/kilogramme),
- Heat flow: $\mathbf{q}$ (unit: Square J/s/meter).

2.2.3 External data

- The mass force $\mathbf{F}^m$ (in practice gravity),
- Sources of heat Θ ,
- Boundary conditions relating either to variables imposed, or on imposed flows.

2.3 Derivative particulate, densities voluminal and mass

Description that we make of the medium is Lagrangian compared to the skeleton. One will find in [bib1] a definition of the concept of skeleton: “the matrix (left occluded solide+porosity) constituting the material part of the skeleton and the connected porous space of ground volume in question constitute the material point of the skeleton or particle of the skeleton”.
That is to say a an unspecified field on Ω , that is to say $\mathbf{x}_s(t)$ the punctual coordinate attached to the skeleton that we follow in his movement and is $\mathbf{x}_{fl}(t)$ the punctual coordinate attached to the fluid. One notes $\dot{a}=\frac{d^s a}{dt}$ the temporal derivative in the movement of the skeleton:

$$\dot{a}=\frac{d^s a}{dt}=\lim_{\delta t\to 0}\frac{a\left(\mathbf{x}_s(t+\delta t),t+\delta t\right)-a\left(\mathbf{x}_s(t),t\right)}{\delta t}$$

$\dot{a}$ is called particulate and often noted derivative $\frac{da}{dt}$ (for example in [bib1]). We prefer to use a notation which recalls that the configuration used to locate a particle is that of the skeleton by report to which a particle of fluid has a relative speed. For a particle of fluid the location $\mathbf{x}_s(t)$ is unspecified, i.e. that the particle of fluid which occupies the position $\mathbf{x}_s(t)$ at the moment t is not the same one as that which occupies the position $\mathbf{x}_S(t')$ at another moment t' .
That is to say then $A=\int_\Omega a\,d\Omega$ a quantity related to a density *voluminal* a , which density is it even carried *partly by the solid matter constituents and the fluids*. That is to say $a^{m\,p}_c$ density *mass* of a range by the liquid phase p component c and is a_s density *voluminal* of a bound to the solid matter constituents. All these definitions finally amount writing:

$$A=\int_\Omega a d\Omega=A_s+A_{fl}=\int_\Omega a_s d\Omega+\int_\Omega a_{fl} d\Omega=\int_\Omega\left(a_s+\sum_{p,c}\rho_c^p jS^p a^{m\,p}_c\right)d\Omega \qquad \text{éq 2.3-1}$$

While following [bib1], we note $\frac{d^{fl}A_{fl}}{dt}$ the derivative of A_{fl} if we follow Ω in the movement of the fluid and $\frac{d^s A_s}{dt}$ the derivative of A_s if we follow Ω in the movement of the skeleton.
We define then:

$$\frac{DA}{Dt}=\frac{d^s A_s}{dt}+\frac{d^{fl}A_{fl}}{dt}=\frac{d^s}{dt}\int_\Omega a_s d\Omega+\frac{d^{fl}}{dt}\int_\Omega\sum_{p,c}\rho_c^p\varphi S^p a^{m_c^p}d\Omega \qquad \text{éq 2.3-2}$$

Density $a^{m\,p}_c$ is transported with a relative speed of $\left(\mathbf{v}_c^p-\mathbf{v}_s\right)$ compared to the skeleton. Taking into account the definition of $\dot{a}=\frac{d^s a}{dt}$, and of the definition $\mathbf{w}_c^p=\rho_c^p\varphi S^p\left(\mathbf{v}_c^p-\mathbf{v}_s\right)$, it is seen easily that the total derivative of A compared to time is written finally:

$$\frac{DA}{Dt}=\int_{\Omega}\left(\dot{a}+\sum_{p,c}\operatorname{Div}\left(a^{mp}_{c}\mathbf{w}^{p}_{c}\right)\right)d\,\Omega$$

éq 2.3-3

Note:

Insofar as we made the assumption of small displacements of the skeleton, $\dot{a}=\frac{d^s a}{dt}$ can merge with the partial derivative compared to time $\frac{\partial a}{\partial t}$ and $\mathbf{v_s}$ can be regarded as worthless. In the same way, in the continuation of the note we will confuse the Lagrangian representations and eulériennes flows, $\mathbf{M}^p_c$ and $\mathbf{w}^p_c$.

3 Continuous equations

3.1 Mechanics: conservation of the momentum

We note $\boldsymbol{\sigma}$ the tensor of the constraints of Cauchy and $\mathbf{s}$ the second tensor (symmetrical) of Piola-Kirchhoff.
We note $\mathbf{P}$ the gradient of the transformation $\mathbf{x}_0=\mathbf{x}_S(0)\rightarrow\mathbf{x}_S(\mathbf{x}_0,t)$.

$$\mathbf{P}=\frac{\partial\mathbf{x}_S(\mathbf{x}_0,t)}{\partial\mathbf{x}_0}$$

One a: $\mathbf{s}=\det\mathbf{P}.\mathbf{P}^{-1}.\boldsymbol{\sigma}.\mathbf{P}^{-T}$.

The equilibrium equations mechanical are written in the configuration Ω_0 :

$$\operatorname{Div}_0(\mathbf{P}.\mathbf{s})+r\,\mathbf{F}^m=0$$

We noted Div_0 the operator of divergence compared to the variables of space $\mathbf{x}_0$ configuration Ω_0 .
Insofar as we make the assumption of small displacements and the small deformations of the skeleton, this equation can be approximate by:

$$\operatorname{Div}\boldsymbol{\sigma}+r\,\mathbf{F}^m=0$$

éq 3.1-1

We will further see we always adopt the decomposition $\boldsymbol{\sigma}=\boldsymbol{\sigma}'+\sigma_p\mathbf{I}$, where $\boldsymbol{\sigma}'$ indicate the effective constraint. It is thus with the load of the module of integration of the equilibrium equations to make the sum: $\boldsymbol{\sigma}=\boldsymbol{\sigma}'+\sigma_p\mathbf{I}$.

3.2 Hydraulics: conservation of the mass

The writing eulérienne of the conservation of the fluid mass for the component c is written:

$$\frac{d^{fl}}{dt}\int_{\Omega}\sum_{p}\rho^p_c\varphi\,S^p\,d\,\Omega=0$$

One can then apply [éq 2.3-1] while taking: $a_s=0$ and $a^{mp}_{c}=1$ and [éq 2.3-3] will give:

$$\sum_p \frac{d^s \rho_c^p \varphi S^p}{dt} + \sum_p \mathrm{Div}\left(\mathbf{w}_c^p\right) = 0$$

By using the definition of the mass contributions [éq 2.2.2.3 - 3], the definition of Lagrangian flows [éq 2.2.2.3 - 2] one finds the form Lagrangian of the conservation of the fluid mass:

$$\begin{cases} \dot{m}_1 + \mathrm{Div}_0\left(\mathbf{M}_1\right) = 0 \\ \dot{m}_2 + \mathrm{Div}_0\left(\mathbf{M}_2\right) = 0 \end{cases}$$ **éq 3.2-1**

3.3 Equation of energy

For the function thermodynamic, we adopt systematically a decomposition of the type [éq 2.3-1]. That corresponds to the fact that various energies have a whole a part carried by the solid and a part carried by the fluids. The part carried by the solid is characterized by a voluminal density whereas the parts carried by the fluid are characterized by mass densities, as we showed in the paragraph [§2.3].

Total internal energy: $E = \int_\Omega \left(e_s + \sum_{p,c} \rho_c^p \varphi S^p e_c^{mp} \right) d\Omega$ **éq 3.3.1**

Total entropy: $S = \int_\Omega \left(s_s + \sum_{p,c} \rho_c^p \varphi S^p s_c^{mp} \right) d\Omega$ **éq 3.3.2**

Total enthalpy: $H = \int_\Omega \left(h_s + \sum_{p,c} \rho_c^p \varphi S^p h_c^{mp} \right) d\Omega$ **éq 3.3.3**

Free energy: $\begin{cases} \Psi = E - T S \\ \Psi_s = e_s - T s_s \\ \Psi_c^{mp} = e_c^{mp} - T s_c^{mp} \end{cases}$ **éq 3.3.4**

Free enthalpy: $\begin{cases} G = H - T S \\ g_s = h_s - T s_s \\ g_c^{mp} = h_c^{mp} - T s_c^{mp} \end{cases}$ **éq 3.3.5**

Lastly, while noting $\dot{Q}(\Omega)$ the rate of heat received by a volume Ω, one has by definition:

$$\dot{Q}(\Omega) = \int_{\partial\Omega} \mathbf{q}.\mathbf{n}\, d\Gamma + \int_\Omega \Theta\, d\Omega$$ **éq 3.3.6**

It is pointed out finally that the enthalpy of the fluids is calculated by the formula:

$$h = e + \frac{p}{\rho}$$ **éq 3.3.7**

3.3.1 The first principle

With the definitions given higher, he is written:

$$-\dot{e} - \sum_{p,c} \mathrm{Div}\left(h_c^{mp} \mathbf{M}_c^p\right) + \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} + \sum_{p,c} \mathbf{M}_c^p . \mathbf{F}^m + \Theta - \mathrm{Div}\, q = 0$$ **éq 3.3.1-1**

This writing corresponds to the equation (22) chapter III-2-3 of [bib1], in which we neglected the terms of inertia. For the homogeneous mediums, it corresponds to the equation (31) of paragraph IV-3-2 of [bib3].

3.3.2 The second principle

Its rather well-known form is:

$$\dot{s}+\sum_{p,c}\mathrm{Div}\left(s^{mp}_{c}\mathbf{M}^{p}_{c}\right)+\mathrm{Div}\left(\frac{\mathbf{q}}{T}\right)-\frac{\Theta}{T}\geq 0$$ **éq 3.3.2-1**

By using the classical thermodynamic considerations [bib1] related to the introduction of the free enthalpy [éq 3.3.5], one shows that one must necessarily have:

$$\boldsymbol{\sigma}-\frac{\partial\Psi}{\partial\boldsymbol{\varepsilon}}=0$$ **éq 3.3.2-2**

$$g^{mp}_{c}-\frac{\partial\Psi}{\partial m^{p}_{c}}=0$$ **éq 3.3.2-3**

$$s+\frac{\partial\Psi}{\partial T}=0$$ **éq 3.3.2-4**

3.3.3 Equation of energy

Rather often, it is considered that, the transformations being reversible, the second principle provides finally an equality. Moreover, one replaces in [éq 3.3.2-1] the unknown temperature T by a value constant known as temperature of reference. It is finally about a linearization of [éq 3.3.2-1] justified if the temperature variations are “small”. Let us note that the term of transport $\sum_{p,c}\mathrm{Div}\left(s^{mp}_{c}\mathbf{M}^{p}_{c}\right)$ complicate the treatment of nonthe linearity due to the presence of the temperature in denominator of the other terms of [éq 3.3.2-1].

We work in enthalpy in order to overcome this difficulty. One leaves the equation of the first principle [éq 3.3.1-1] in which one injects the equations [éq 3.3.2-2], [éq 3.3.2-3], [éq 3.3.2-4], and the definition of the free enthalpy [éq 3.3-5] and one obtains:

$$T\,\dot{s}+\sum_{p,c}\left(h^{mp}_{c}\dot{m}^{p}_{c}-T\,s^{mp}_{c}\dot{m}^{p}_{c}\right)=-\sum_{p,c}\mathrm{Div}\left(h^{mp}_{c}\mathbf{M}^{p}_{c}\right)+\sum_{p,c}\mathbf{M}^{p}_{c}.\mathbf{F}^{m}+\Theta-\mathrm{Div}\,\mathbf{q}$$ **éq 3.3.3-1**

One poses then:

$$\delta Q'=T\,\delta s-T\sum_{p,c}s^{m^{p}_{c}}\delta m^{p}_{c}$$ **éq 3.3.3-2**

Quantity Q' the dimension of an energy per unit of volume has. It represents the heat received by the system in a transformation for which there are no contributions of heat per entry of fluid having an enthalpy. Although $\delta Q'$ is not an exact differential, we take this quantity like variable of state.

Finally, the equation of energy selected has the following form:

$$\sum_{p,c} h^{mp}_{c} \dot{m}^{p}_{c} + \dot{Q}' + \sum_{p,c} \mathrm{Div}\left(h^{mp}_{c} \mathbf{M}^{p}_{c}\right) + \mathrm{Div}\,\mathbf{q} - \sum_{p,c} \mathbf{M}^{p}_{c} . \mathbf{F}^{m} = \Theta$$ **éq 3.3.3-3**

4 Variational writing of the equilibrium equations

4.1 Mechanics

We note U_{ad} the whole of the fields of displacement kinematically acceptable, i.e. elements of $\left(H^1(\Omega)\right)^3$ checking the boundary conditions in displacement on the part of $\partial\Omega$ supporting such conditions [bib3].

The variational form of [éq 3.1-1] is:

$$\begin{cases} \boldsymbol{\sigma} = \boldsymbol{\sigma}' + \sigma_p \mathbf{I} \\ \int_\Omega \boldsymbol{\sigma} . \boldsymbol{\varepsilon}(\mathbf{v}) = \int_\Omega r\,\mathbf{F}^m . \mathbf{v}\, d\Omega + \int_{\partial\Omega} \mathbf{f}^{ext} . \mathbf{v}\, d\Gamma \qquad \forall \mathbf{v} \in U_{ad} \end{cases}$$ **éq 4.1-1**

4.2 Hydraulics

We note P_{1ad} (resp. P_{2ad}) the whole of the acceptable fields of pressure π_1 (resp. π_2), i.e. elements of $H^1(\Omega)$ checking the boundary conditions in pressure P_1 (resp. P_2) on the part of $\partial\Omega$ supporting such conditions [bib3]. The variational form of [éq 3.2-1] is:

$$\left.\begin{array}{l} -\int_\Omega \left(\dot{m}^1_1 + \dot{m}^2_1\right) . \pi_1 + \int_\Omega \left(\mathbf{M}^1_1 + \mathbf{M}^2_1\right) . \nabla \pi_1 \;\; d\Omega = \\ \qquad \int_{\partial\Omega} \left(M^1_{1ext} + M^2_{1ext}\right) . \pi_1 \;\; d\Gamma \qquad \forall \pi_1 \in P_{1ad} \\ -\int_\Omega \left(\dot{m}^1_2 + \dot{m}^2_2\right) . \pi_2 + \int_\Omega \left(\mathbf{M}^1_2 + \mathbf{M}^2_2\right) . \nabla \pi_2 \;\; d\Omega = \\ \qquad \int_{\partial\Omega} \left(M^1_{2ext} + M^2_{2ext}\right) . \pi_2 \;\; d\Gamma \qquad \forall \pi_2 \in P_{2ad} \end{array}\right\}$$ **éq 4.2-1**

who reveals scalar hydraulic flows M^{j}_{iext} on the edges.

4.3 Thermics

We note T_{ad} the whole of the acceptable fields of temperature τ, i.e. elements of $H^1(\Omega)$ checking the boundary conditions in temperature on the part of $\partial\Omega$ supporting such conditions. [bib3]. The variational form of [éq 3.3.3-3] is:

$$\begin{array}{l} \int_\Omega \dot{Q}' . \tau\, d\Omega + \sum_{p,c} \int_\Omega h^{mp}_{c} \dot{m}^{p}_{c} . \tau\, d\Omega - \int_\Omega \left(\sum_{p,c} h^{mp}_{c} \mathbf{M}^{p}_{c} + \mathbf{q}\right) . \nabla \tau\, d\Omega = \\ \qquad \int_\Omega \left(\Theta + \sum_{p,c} \mathbf{M}^{p}_{c} . \mathbf{F}^{m}\right) . \tau\, d\Omega - \int_{\partial\Omega} \left(\sum_{p,c} h^{mp}_{c} M^{p}_{cext} + q_{ext}\right) . \tau\, d\Gamma \\ \qquad\qquad \forall \tau \in T_{ad} \end{array}$$ **éq 4.3-1**

Let us note that, contrary to other presentations, and in particular [bib8] we did not inject the conservation equations of the mass, and we integrated by part the term of transport $\sum_{p,c} \mathrm{Div}\left(h_c^{mp}\mathbf{M}_c^p\right)$.

This last point has the advantage of not revealing of the derivative of a higher nature, and, contrary to revealing naturally boundary conditions relative at the entrance of heat related to hydraulic flows: $\sum_{p,c}\int_{\partial\Omega} h_c^{mp} M_{c\,ext}^p . \tau \, d\Gamma$.

One will be able in makes consider that the conditions of heat flux define directly:

$$\tilde{q}_{ext}=h_c^{mp} M_{c\,ext}^p+q_{ext}$$

5 Discretization in time

In this chapter, we are satisfied to take again the variational formulations in their applying a discretization compared to the time of type theta-diagram for thermal hydraulics and it. It is about a general method of integration of the differential equations [bib12] and [bib13].

θ is a digital parameter understood enters 0 and 1. For the linear differential equations (what is not our case…) this diagram is unconditionally stable for $\theta\geq 1/2$, it is of order 1 for $\theta\neq 1/2$ and of order 2 for $\theta=1/2$. Nevertheless, it can be preferable to use a value different from $1/2$, and this for parasitic reasons of oscillations [bib12].

Subscripted quantities by $+$ are the quantities at the end of the step of time, and those subscripted by $-$ are those of the beginning of the step of time. One notes:

$$\Delta t=t^+-t^-$$
$$a^\theta=\theta a^+ + (1-\theta) a^- \qquad \forall a$$

5.1 Mechanics

$$\begin{cases}\boldsymbol{\sigma}^+=\boldsymbol{\sigma}'^+ + \sigma_p^+\mathbf{I}\\ \int_\Omega \boldsymbol{\sigma}^+ . \boldsymbol{\varepsilon}(\mathbf{v})\, d\Omega = \int_\Omega r^+\mathbf{F}^{m+}.\mathbf{v}\, d\Omega + \int_{\partial\Omega} \mathbf{f}^{ext+}. v\, d\Gamma \qquad \forall \mathbf{v}\in U_{ad}\end{cases} \quad \text{éq 5.1-1}$$

5.2 Hydraulics

$$\left.\begin{array}{l}-\int_\Omega\left(m_1^{1^+}+m_1^{2^+}\right).\pi_1\, d\Omega+\theta\Delta t\int_\Omega\left(\mathbf{M}_1^{1^+}+\mathbf{M}_1^{2^+}\right).\nabla\pi_1\, d\Omega=\\ \quad -\int_\Omega\left(m_1^{1^-}+m_1^{2^-}\right).\pi_1\, d\Omega-(1-\theta)\Delta t\int_\Omega\left(\mathbf{M}_1^{1^-}+\mathbf{M}_1^{2^-}\right).\nabla\pi_1\, d\Omega\\ \qquad +\Delta t\int_{\partial\Omega}\left(M_{1ext}^{1^\theta}+M_{1ext}^{2^\theta}\right).\pi_1\, d\Gamma \qquad \forall\pi_1\in P_{1ad}\\ -\int_\Omega\left(m_2^{1^+}+m_2^{2^+}\right).\pi_2\, d\Omega+\theta\Delta t\int_\Omega\left(\mathbf{M}_2^{1^+}+\mathbf{M}_2^{2^+}\right).\nabla\pi_2\, d\Omega=\\ \quad -\int_\Omega\left(m_2^{1^-}+m_2^{2^-}\right).\pi_2\, d\Omega-(1-\theta)\Delta t\int_\Omega\left(\mathbf{M}_2^{1^-}+\mathbf{M}_2^{2^-}\right).\nabla\pi_2\, d\Omega\\ \qquad +\Delta t\int_{\partial\Omega}\left(M_{2ext}^{1^\theta}+M_{2ext}^{2^\theta}\right).\pi_2\, d\Gamma \qquad \forall\pi_2\in P_{2ad}\end{array}\right\} \quad \text{éq 5.2-1}$$

5.3 Thermics

$$\left.\begin{array}{c}\int_{\Omega}\left(Q'^{+}-Q'^{-}\right)\tau\, d\Omega-\theta\Delta t\int_{\Omega}\left(\sum_{p,c} h_c^{m\,p+}\mathbf{M}_c^{p+}+\mathbf{q}^{+}\right)\nabla\tau\, d\Omega\\ -\left(1-\theta\right)\Delta t\int_{\Omega}\left(\sum_{p,c} h_c^{m\,p-}\mathbf{M}_c^{p-}+\mathbf{q}^{-}\right)\nabla\tau\, d\Omega+\theta\int_{\Omega}\left(\sum_{p,c} h_c^{m\,p+}\left(m_c^{p+}-m_c^{p-}\right)\right)\tau\, d\Omega\\ +\left(1-\theta\right)\int_{\Omega} h_c^{m\,p-}\left(m_c^{p+}-m_c^{p-}\right)\tau\, d\Omega=\\ +\theta\Delta t\int_{\Omega}\sum_{p,c}\mathbf{M}_c^{p+}.\mathbf{F}^{m}\tau\, d\Omega+\left(1-\theta\right)\Delta t\int_{\Omega}\sum_{p,c}\mathbf{M}_c^{p-}.\mathbf{F}^{m}\tau\, d\Omega\\ +\Delta t\int_{\Omega}\Theta^{\theta}\tau\, d\Omega-\Delta t\int_{\partial\Omega}\left(\sum_{p,c} h_c^{m\,p^{\theta}} M_{c\,ext}^{p^{\theta}}+q_{ext}^{\theta}\right).\tau\, d\Gamma\qquad\forall\tau\in T_{ad}\end{array}\right\}\quad \text{éq 5.3-1}$$

One can again consider that the conditions of heat flux define directly:

$$\tilde{q}_{ext}^{\theta}=\sum_{p,c} h_c^{m\,p^{\theta}} M_{c\,ext}^{p^{\theta}}+q_{ext}^{\theta}$$

6 Principle of virtual work, strains and stresses generalized, laws of behavior

6.1 Generalized constraints and deformations

While referring to the variational formulations [éq 4.1-1], [éq 4.2-1] and [éq 4.3-1], it appears that one can choose:

For the generalized constraints:

$$\Sigma=\left\{\begin{array}{l}\boldsymbol{\sigma}',\boldsymbol{\sigma}_p;\\ m_1^1,\mathbf{M}_1^1,h_1^{m1};m_1^2,\mathbf{M}_1^2,h_1^{m2};\\ m_2^1,\mathbf{M}_2^1,h_2^{m1};m_2^2,\mathbf{M}_2^2,h_2^{m2};\\ Q',\mathbf{q}\end{array}\right\}\quad \text{éq 6.1-1}$$

For the generalized deformations:

$$E=\left\{\mathbf{u},\varepsilon\left(\mathbf{u}\right);p_1,\nabla p_1;p_2,\nabla p_2;T,\nabla T\right\}\quad \text{éq 6.1-2}$$

The fact is noticed that the generalized deformations contain displacements. That is due at the end $\int_{\Omega} r\,\mathbf{F}^{m}.\mathbf{v}$ variational formulation of the conservation equation of the momentum [éq 4.1-1], which term couples finally the generalized constraints and displacements because of [éq 6.3.4-1]. The generalized deformations contain the pressure and the temperature because the associated equations are parabolic.

6.2 Principle of virtual work

The whole of the nonlinear equations to solve can be put in the form:

$$\mathbf{R}\left(\mathbf{U}\right)=\mathbf{L}^{meca}\quad \text{éq 6.2-1}$$

where $\mathbf{U}$ indicate generalized displacements, i.e.: $\mathbf{U}=\{\mathbf{u}_x, \mathbf{u}_y, \mathbf{u}_z, p_1, p_2, T\}$ in the case more the general. Internal forces $\mathbf{R}$ express themselves starting from a principle of virtual work generalized. In the case of the mechanics of the continuous mediums "classical", i.e. when there is no other constituting that the solid, one is used defining the internal forces by:
$\mathbf{w}^T.\mathbf{R}=\int_\Omega \boldsymbol{\varepsilon}(\mathbf{w}).\boldsymbol{\sigma}\, d\Omega \qquad \forall\, \mathbf{w}$, field of displacement kinematically acceptable.

In this formulation, the field of deformation $\boldsymbol{\varepsilon}$ depends only on the field of space displacement and its derivative, possibly in a nonlinear way if one takes into account finished deformations. One writes symbolically:

$$\mathbf{R}=\mathbf{Q}^T\boldsymbol{\sigma}$$

The law of behavior connects the constraints $\boldsymbol{\sigma}$ with the deformations $\boldsymbol{\varepsilon}$.
Within the framework of the theory of the porous environments developed here, we try as much as possible to bring us closer to this formulation by introducing generalized constraints $\boldsymbol{\Sigma}$ and of the generalized deformations $\mathbf{E}$; The generalized deformations depend only on the field of generalized displacements $\mathbf{U}$ and of its derivative space. The operator $\mathbf{U}\rightarrow\mathbf{E}(\mathbf{U})$ is an operator of derivation compared to the field of coordinates.
The law of behavior makes it possible to calculate $\boldsymbol{\Sigma}$ according to $\mathbf{E}$.

On the other hand, we cannot write directly $\mathbf{W}^T.\mathbf{R}=\int_\Omega \mathbf{E}(\mathbf{W}).\boldsymbol{\Sigma}\, d\Omega$, for the following reasons:

- the equations which we treat are evolutionary equations in times and the derivative compared to the time of the quantities intervenes,
- the equations are nonlinear because of terms of transport related to the representation eulérienne of the fluids: these nonlinear terms appear only in the equation of thermics,
- the choice of the unknown factors makes that the nonlinear terms of transport intervene in the generalized constraints. That is to say a term of transport in the equation [éq 4.3-1], $\int_\Omega \left(\sum_{p,c} h_c^{mp}\mathbf{M}_c^p+\mathbf{q}\right).\nabla\tau\, d\Omega$, since one took as principal unknown factors for hydraulics the pressures, quantities $\mathbf{M}_c^p$ dependent on speeds of the fluids belong to the generalized constraints, just as the enthali h_c^{mp}, and the term of transport given in example is linear in deformation generalized and quadratic in generalized constraint. This made a difference with the formulation of the theory of the classical continuous mediums where the terms of great deformation are quadratic quantities of the deformations.

For all these reasons, we introduce a noted field $\overline{\boldsymbol{\Sigma}}$ such as:

$$\mathbf{R}=\mathbf{Q}^T\overline{\boldsymbol{\Sigma}} \qquad \textbf{éq 6.2-2}$$

$\mathbf{Q}^T$ is defined by:

$$\mathbf{W}^T.\mathbf{R}=\int_\Omega \mathbf{E}(\mathbf{W}).\overline{\boldsymbol{\Sigma}}\, d\Omega$$
$$\forall\,\mathbf{W} \quad \text{champ de déplacement généralisé cinématiquement admissible} \qquad \textbf{éq 6.2-3}$$

It is very seen easily and classically that $\mathbf{Q}^T$ is transposed of the operator $\mathbf{Q}$ such as:

$$\mathbf{E}=\mathbf{Q}\mathbf{U} \qquad \textbf{éq 6.1-4}$$

The field $\overline{\boldsymbol{\Sigma}}$ is a linear function of $\dot{\boldsymbol{\Sigma}}$ and nonlinear of $\boldsymbol{\Sigma}$:

$$\overline{\boldsymbol{\Sigma}}=\overline{\boldsymbol{\Sigma}}(\dot{\boldsymbol{\Sigma}},\boldsymbol{\Sigma}) \qquad \textbf{éq 6.1-5}$$

After discretization in time, $\overline{\boldsymbol{\Sigma}^+}$ will become a nonlinear function of $\boldsymbol{\Sigma}^+$ and $\boldsymbol{\Sigma}^-$:

$$\overline{\boldsymbol{\Sigma}^+}=\overline{\boldsymbol{\Sigma}^+}\left(\boldsymbol{\Sigma}^+,\boldsymbol{\Sigma}^-\right)$$ **éq 6.1-6**

Let us note finally that for algorithmic reasons (inter alia), one needs to know the derivative of the internal forces compared to generalized displacements:

$$\frac{\partial \mathbf{R}}{\partial \mathbf{U}}=\frac{\partial \mathbf{R}}{\partial \overline{\boldsymbol{\Sigma}}}\frac{\partial \overline{\boldsymbol{\Sigma}}}{\partial \boldsymbol{\Sigma}}\frac{\partial \boldsymbol{\Sigma}}{\partial \mathbf{E}}\frac{\partial \mathbf{E}}{\partial \mathbf{U}}=\mathbf{Q}^T\frac{\partial \overline{\boldsymbol{\Sigma}}}{\partial \boldsymbol{\Sigma}}\frac{\partial \boldsymbol{\Sigma}}{\partial \boldsymbol{\Sigma}}\mathbf{Q}$$

It is clear that $\frac{\partial \overline{\boldsymbol{\Sigma}}}{\partial \boldsymbol{\Sigma}}$ depends only on the form of the equilibrium equations.

6.3 Laws of behavior

A law of behavior will be simply defined like an unspecified relation between generalized constraints and generalized deformations. The internal variables are defined like fields necessary to the calculation of the constraints, whose evolution is given by the laws of behavior, but which do not intervene directly in the equilibrium equations.

Moreover, we consider that the laws of behavior are written in incremental form and that they are local. While noting $\boldsymbol{\chi}$ the internal variables, a law of behavior is thus a relation:

$$\boldsymbol{\Sigma},\boldsymbol{\chi},\dot{\mathbf{E}}\rightarrow\dot{\boldsymbol{\Sigma}},\dot{\boldsymbol{\chi}}$$

After discretization in time, the law of behavior becomes a relation:

$$\boldsymbol{\Sigma}^-,\boldsymbol{\chi}^-,\mathbf{E}^-,\mathbf{E}^+\rightarrow\boldsymbol{\Sigma}^+,\boldsymbol{\chi}^+$$

The law of behavior will have to also provide the only term which in the expression $\frac{\partial \mathbf{R}}{\partial \mathbf{U}}=\mathbf{Q}^T\frac{\partial \overline{\boldsymbol{\Sigma}}}{\partial \boldsymbol{\Sigma}}\frac{\partial \boldsymbol{\Sigma}}{\partial \mathbf{E}}\mathbf{Q}$ depends on it, namely $\frac{\partial \boldsymbol{\Sigma}}{\partial \mathbf{E}}$. Finally a relation of behavior is a relation:

$$\boldsymbol{\Sigma}^-,\boldsymbol{\chi}^-,\mathbf{E}^-,\mathbf{E}^+\rightarrow\boldsymbol{\Sigma}^+,\boldsymbol{\chi}^+,\frac{\partial \boldsymbol{\Sigma}^+}{\partial \mathbf{E}^+}$$ **éq 6.3-1**

In the following paragraphs, we specify certain aspects of the laws of behavior by distinguishing the mechanical, hydraulic and thermal contributions.

6.3.1 Mechanical law of behavior

6.3.1.1 General writing

The internal variables are noted $\boldsymbol{\chi}$. A law of behavior of mechanics, within the framework THM is written:

$$\begin{cases}\boldsymbol{\sigma}^+=\boldsymbol{\sigma}^+\left(\boldsymbol{\varepsilon}^+,p_1^+,p_2^+,T^+;\boldsymbol{\varepsilon}^-,p_1^-,p_2^-,T^-,\boldsymbol{\sigma}^-,\boldsymbol{\chi}^-\right)\\ \boldsymbol{\chi}^+=\boldsymbol{\chi}^+\left(\boldsymbol{\varepsilon}^+,p_1^+,p_2^+,T^+;\boldsymbol{\varepsilon}^-,p_1^-,p_2^-,T^-,\boldsymbol{\sigma}^-,\boldsymbol{\chi}^-\right)\end{cases}$$ **éq 6.3.1.1 - 1**

6.3.1.2 Case of the effective constraints

In the case of the assumption of the effective constraints, there is the decomposition: $\boldsymbol{\sigma}=\boldsymbol{\sigma}'+\sigma_p\mathbf{I}$ where $\boldsymbol{\sigma}'$ is the tensor of the effective constraints and σ_p is a scalar.

The internal variables are separate in two parts: mechanical internal variables $\boldsymbol{\chi}_\sigma$ and hydraulic internal variables $\boldsymbol{\chi}_H$. The mechanical law of behavior is divided then into two laws, **whose first can be an already existing law within the usual framework of thermomechanics.**

$$\begin{cases} \boldsymbol{\sigma}'^{+}=\boldsymbol{\sigma}'^{+}\left(\boldsymbol{\varepsilon}^{+},T^{+};\boldsymbol{\varepsilon}^{-},T^{-},\boldsymbol{\sigma}'^{-},\boldsymbol{\chi}'^{-}_{\sigma}\right) \\ \boldsymbol{\chi}'^{+}_{\sigma}=\boldsymbol{\sigma}'^{+}_{\sigma}\left(\boldsymbol{\varepsilon}^{+},T^{+};\boldsymbol{\varepsilon}^{-},T^{-},\boldsymbol{\sigma}'^{-},\boldsymbol{\chi}'^{-}_{\sigma}\right) \end{cases}$$

éq 6.3.1.2 - 1

$$\begin{cases} \boldsymbol{\sigma}^{+}_{p}=\boldsymbol{\sigma}^{+}_{p}\left(p_1^{+},p_2^{+};p_1^{-},p_2^{-},\boldsymbol{\chi}_H^{-}\right) \\ \boldsymbol{\chi}^{+}_{H}=\boldsymbol{\chi}^{+}_{H}\left(p_1^{+},p_2^{+};p_1^{-},p_2^{-},\boldsymbol{\chi}_H^{-}\right) \end{cases}$$

éq 6.3.1.2 - 2

The dependences indicated by the equations [éq 6.3.1.2 - 1] and [éq 6.3.1.2 - 2] do not have a theoretical justification a priori. It is simply a question of showing the most general possible dependences from the point of view of the data-processing programming. One notices in this decomposition that the dependence compared to thermics was left in the effective constraints; typically, it is thought that the laws on the effective constraints are written as in classical thermomechanics:

$$\boldsymbol{\sigma}'^{+}=\boldsymbol{\sigma}'^{+}\left(\boldsymbol{\varepsilon}^{+}-\alpha^{+}T^{+}\mathbf{I},\boldsymbol{\varepsilon}^{-}-\alpha^{-}T^{-}\mathbf{I},\boldsymbol{\sigma}'^{-},\boldsymbol{\chi}_{\sigma}^{-}\right)$$

6.3.1.3 Choice of the constraints

Because of rather frequent use of the assumption of the effective constraints, one decides that the vector of the constraints for the mechanical part contains in all the cases the tensor of the effective constraints $\boldsymbol{\sigma}'$ and the scalar σ_p. In the case general where the assumption of the effective constraints is not retained, one will have simply: $\sigma_p=0$. It is thus with the load of the module of integration of the equilibrium equations (and not of the laws of behavior) to make the sum: $\boldsymbol{\sigma}^{+}=\boldsymbol{\sigma}'^{+}+\sigma_p^{+}\mathbf{I}$.

6.3.2 Hydraulics

The hydraulic law of behavior provides the following relations:

$$\begin{cases} \left.\begin{array}{l} m_c^{p+}=m_c^{p+}\left(\boldsymbol{\varepsilon}^{+},p_1^{+},p_2^{+},T^{+};\boldsymbol{\varepsilon}^{-},p_1^{-},p_2^{-},T^{-},m_c^{p-},\mathbf{M}_c^{p-},\boldsymbol{\chi}_H^{-}\right) \\ \mathbf{M}_c^{p+}=\mathbf{M}_c^{p+}\left(\begin{array}{l}\boldsymbol{\varepsilon}^{+},p_1^{+},\nabla p_1^{+},p_2^{+},\nabla p_2^{+},T^{+},\nabla T^{+}; \\ \boldsymbol{\varepsilon}^{-},p_1^{-},\nabla p_1^{-},p_2^{-},\nabla p_2^{-}; \\ T^{-},\nabla T^{-},\mathbf{M}_c^{p-},\boldsymbol{\chi}_H^{-};\mathbf{F}^{m+}\end{array}\right)\end{array}\right\} \begin{array}{l}\forall c \\ \forall p\end{array} \\ \boldsymbol{\chi}_H^{+}=\boldsymbol{\chi}_H^{+}\left(\boldsymbol{\varepsilon}^{+},p_1^{+},p_2^{+},T^{+};\boldsymbol{\varepsilon}^{-},p_1^{-},p_2^{-},T^{-},m_c^{p-},\boldsymbol{\chi}_H^{-}\right) \end{cases}$$

éq 6.3.2-1

It is noticed that the field of gravity is a data of the hydraulic law of behavior because the evolution of the vector of flow follows relations of the type:

$$\mathbf{M}=\lambda_H\rho^{fl}\left[-\nabla P+\rho^{fl}\mathbf{F}^{m}\right]$$

6.3.3 Thermal law of behavior

The laws of behavior give:

$$\begin{cases} Q'^{+}=Q'^{+}\left(\boldsymbol{\varepsilon}^{+},p_1^{+},p_2^{+},T^{+};\boldsymbol{\varepsilon}^{-},p_1^{-},p_2^{-},T^{-},Q'^{-}\right) \\ h_c^{mp+}=h_c^{mp+}\left(\boldsymbol{\varepsilon}^{+},p_1^{+},p_2^{+},T^{+};\boldsymbol{\varepsilon}^{-},p_1^{-},p_2^{-},T^{-},h_c^{mp-}\right) \qquad \forall c \,\text{et}\, \forall p \\ \mathbf{q}^{+}=\mathbf{q}^{+}\left(\boldsymbol{\varepsilon}^{+},p_1^{+},p_2^{+},T^{+},\nabla T^{+};\boldsymbol{\varepsilon}^{-},p_1^{-},p_2^{-},T^{-},\nabla T^{-},\mathbf{q}^{-}\right) \\ \chi_T^{+}=\chi_T^{+}\left(\boldsymbol{\varepsilon}^{+},p_1^{+},p_2^{+},T^{+},\nabla T^{+};\boldsymbol{\varepsilon}^{-},p_1^{-},p_2^{-},T^{-},\nabla T^{-},\boldsymbol{\chi}_T^{-}\right) \end{cases}$$ **éq 6.3.3-1**

Let us note that we introduced possible internal variables related to thermics.

6.3.4 Homogenized density

By definition, the homogenized density, which intervenes in the equilibrium equation of mechanics [éq 3.1-1] is given by:

$$r^{+}=r_0+m_1^{1+}+m_1^{2+}+m_2^{1+}+m_2^{2+}$$ **éq 6.3.4-1**

This equation is not a law of behavior, but she belongs to the conservation equations. She is intégrée in the module of calculation of the equilibrium equations, the modules of calculation of the laws of behavior not having to treat it.

6.3.5 Operator of mass

This operator is appealable by the option `MASS_MECA` for all the elements concerned with modelings containing of the hydromechanics in 3D and in `D_PLAN`, that is to say: `'THM'`, `'THHM'`, `'HHM'`, `'HH2M'`, `'THH2M'`. It thus makes it possible to treat a dynamic analysis. It understands only terms associated with the degrees of freedom in translation, that is to say respectively: `'DX'` , `'DY'` in 2D, and `'DX'` , `'DY'` , `'DZ'` in 3D.

These terms are evaluated in each element before assembly starting from the functions of form of the isoparametric elements $\mathbf{N}$ described in the document [R3.01.00] according to the expression:

$$M_{ij}^{e}=\int_{\Omega_e}\rho\,\mathbf{N}_i.\,\mathbf{N}_j\,d\,\Omega$$

The preceding expression contains the homogenized density ρ initial (correspondent with r_0 in the paragraph [§6.3.4]) entered behind the operand `RHO` keyword `THM_DIFFU` order `DEFI_MATERIAU`. Indeed, one judges for the current dynamic applications of short time (of the standard seismic loading) that the modification of the density is sufficiently weak, cf [bib15]. Thus one avoids an update of the matrix of mass.

Elementary terms M_{ij}^{e} are supplemented during their assembly by worthless terms associated with the degrees of additional freedom specific to coupled modeling. It results from it during their expansion in the matrix of assembled mass a shift of degrees of freedom related to the possible presence of the components `'PRESS1'` , `'PRESS2'` , and `'TEMPE'` in modeling.

7 Algorithm of resolution

7.1 Nonlinear algorithm of resolution of the equilibrium equations

In the case Général of modeling (variable coefficients, desaturation, convection) the variational problem presented above [éq 4.1-1] with [éq 4.3-1] is nonlinear compared to the fields of displacement, pressure and temperature. After discretization by finite elements, one obtains a nonlinear matric system. The matrix of the tangent operator contains moreover one term nonsymmetrical treaty as tel. One uses in all the cases of modeling the nonlinear solvor `STAT_NON_LINE` of *Code_Aster* resting on a method of Newton-Raphson, described in [bib5]. The vectorial functional calculus is introduced:

$$\mathbf{F}(\mathbf{U})=\mathbf{R}(\mathbf{U})-\mathbf{L}^{meca}$$ **éq 7.1-1**

The associated tangent operator is noted: $D\mathbf{F}=\frac{\partial \mathbf{F}}{\partial \mathbf{U}}$.

For the modules `THM`, objects of this note, the operator $\mathbf{L}^{meca}$ does not depend on generalized displacements. All the terms depending on generalized displacements were introduced into $\mathbf{R}$, and it is precisely for this reason that displacements are found in the generalized deformations. Let us note on this subject the very particular treatment of the term $\int_\Omega r\,\mathbf{F}^m.\mathbf{v}$ equation [éq 4.1-1].

According to [éq 6.3.4-1], $\int_\Omega r\,\mathbf{F}^m.\mathbf{v}\,d\Omega=\int_\Omega\left(r_0+m_1^{1+}+m_1^{2+}+m_2^{1+}+m_2^{2+}\right)\mathbf{F}^m.\mathbf{v}\,d\Omega$.

We chose to divide this term into two:
The term $\int_\Omega r_0\mathbf{F}^m.\mathbf{v}\,d\Omega$ is a contribution to $\mathbf{L}^{meca}$ if the user informed the operand `GRAVITY` loading used (defined by the order `AFFE_CHAR_MECA`), whereas the term $\int_\Omega\left(m_1^{1+}+m_1^{2+}+m_2^{1+}+m_2^{2+}\right)\mathbf{F}^m.\mathbf{v}\,d\Omega$, which depends on the generalized constraints is a contribution to $\mathbf{R}$.

7.2 Passage des nodal values with the values at the points of Gauss

As in all the codes of finite elements, the terms are calculated by loop on the elements and buckles on the points of gauss. While noting $\mathbf{R}_g^{el}$ and $D\mathbf{F}_g^{el}$ values at the point of Gauss g element el nodal forces and of the tangent operator, and w_g^{el} the weight of integration related to this point of Gauss, one a:

$$\mathbf{R}(\mathbf{U})=\sum_{el}\sum_{g} w_g^{el}\,\mathbf{R}_g^{el}(\mathbf{U})$$ **éq 7.2-1**

$$D\mathbf{F}(\mathbf{U})=\sum_{el}\sum_{g} w_g^{el}\,D\mathbf{F}_g^{el}(\mathbf{U})$$ **éq 7.2-2**

Let us note then $\mathbf{U}^{el}$ the vector of the nodal unknown factors on a finite element el . One can thus have:

$$\text{par exemple } \mathbf{U}^{el}=\left.\begin{matrix} u\\ v\\ w\\ p_1\\ p_2\\ T\end{matrix}\right\}\text{noeud 1}\quad \left.\begin{matrix} u\\ v\\ w\\ p_1\\ p_2\\ T\end{matrix}\right\}\text{noeud 2}\quad \left.\begin{matrix} u\\ v\\ w\\ p_1\\ p_2\\ T\end{matrix}\right\}\text{noeud 3}.$$

Let us note too $\mathbf{E}_g^{el}$ the vector of the deformations generalized at the point of Gauss g element el and $\mathbf{\Sigma}_g^{el}$ the vector of constraints generalized for the point of Gauss g element el. In the most complete case one has as follows:

$$\mathbf{E}_g^{el}=\begin{pmatrix} \mathbf{u}\\ \boldsymbol{\varepsilon}(u)\\ p_1\\ \nabla p_1\\ p_2\\ \nabla p_2\\ T\\ \nabla T\end{pmatrix}_g^{el} ;\mathbf{\Sigma}_g^{el}=\begin{pmatrix} \boldsymbol{\sigma}'\\ \boldsymbol{\sigma}_p\\ m_1^1\\ \mathbf{M}_1^1\\ h^{m_1^1}\\ m_1^2\\ \mathbf{M}_1^2\\ h^{m_1^2}\\ m_2^1\\ \mathbf{M}_2^1\\ h^{m_2^1}\\ m_2^2\\ \mathbf{M}_2^2\\ h^{m_2^2}\\ Q'\\ \mathbf{q}\end{pmatrix}_g^{el}$$

The functions of form of the finite elements make it possible to calculate the matrix then $\mathbf{Q}_g^{el}$ of passage of the nodal unknown factors to the deformations generalized at the points of Gauss defined by:

$$\mathbf{E}_g^{el}=\mathbf{Q}_g^{el}.\mathbf{U}_g^{el}$$ **éq 7.2-3**

7.3 Vectors and matrices according to the options

The presentations of the two following paragraphs are made in the case more the general where one has an equation of mechanics, two equations of hydraulics and an equation of thermics. Indices g and el from now on are omitted, but it is clear that what is described applies to each point of Gauss of each element.

7.3.1 Residue or nodal force: options RAPH_MECA and FULL_MECA

One distributes the terms of the variational formulation according to the following principle:
If $\mathbf{E}_g^{*el}$ indicate a virtual field of deformation, $\mathbf{E}_g^{*el}=\left(\mathbf{v},\boldsymbol{\varepsilon}(\mathbf{v}),\pi_1,\nabla\pi_1,\pi_2,\nabla\pi_2,\tau,\nabla\tau\right)$ calculated starting from a vector of virtual nodal displacements $\mathbf{U}^{*el}$, one can define: $\mathbf{E}_g^{*el\,T}.\overline{\boldsymbol{\Sigma}}_g^{el}(\mathbf{U})=\overline{\Sigma}_1\mathbf{v}+\overline{\Sigma}_2(\mathbf{v})+\overline{\Sigma}_3\pi_1+\overline{\Sigma}_4\nabla\pi_1+\overline{\Sigma}_5\pi_2+\overline{\Sigma}_6\nabla\pi_2+\overline{\Sigma}_7\tau+\overline{\Sigma}_8\nabla\tau$. The discrete variational formulations then are taken again [éq 5.1-1], [éq 5.2-1], [éq 5.3-1], and the integrals there are replaced $\int_\Omega f\,d\,\Omega$ by $\sum_{el}\sum_g w_g^{el}f_g^{el}$ for all the integral ones f. One distinguishes the terms multiplying respectively $\mathbf{v}$, $\boldsymbol{\varepsilon}(\mathbf{v})$, π_1 , $\nabla\pi_1$, π_2 , $\nabla\pi_2$, τ and $\nabla\tau$, and one finds:

Index	$\overline{\Sigma}$	associated with
1	$-\left(m_1^{1+}+m_1^{2+}+m_2^{1+}+m_2^{2+}\right)\mathbf{F}^{m+}$	$\mathbf{v}$
2	$\boldsymbol{\sigma}'^{+}+\sigma_p^{+}\mathbf{I}$	$\boldsymbol{\varepsilon}(\mathbf{v})$
3	$-m_1^{1+}-m_1^{2+}+m_1^{1-}+m_1^{2-}$	π_1
4	$\theta\Delta t\left(\mathbf{M}_1^{1+}+\mathbf{M}_1^{2+}\right)+(1-\theta)\Delta t\left(\mathbf{M}_1^{1-}+\mathbf{M}_1^{2-}\right)$	$\nabla\pi_1$
5	$-m_2^{1+}-m_2^{2+}+m_2^{1-}+m_2^{2-}$	π_2
6	$\theta\Delta t\left(\mathbf{M}_2^{1+}+\mathbf{M}_2^{2+}\right)+(1-\theta)\Delta t\left(\mathbf{M}_2^{1-}+\mathbf{M}_2^{2-}\right)$	$\nabla\pi_2$
7	$-Q'^{+}+Q'^{-}$ $-\left(\theta h_1^{m1+}+(1-\theta)h_1^{m1-}\right)\left(m_1^{1+}-m_1^{1-}\right)-\left(\theta h_1^{m2+}+(1-\theta)h_1^{m2-}\right)\left(m_1^{2+}-m_1^{2-}\right)$ $-\left(\theta h_2^{m1+}+(1-\theta)h_2^{m1-}\right)\left(m_2^{1+}-m_2^{1-}\right)-\left(\theta h_2^{m2+}+(1-\theta)h_2^{m2-}\right)\left(m_2^{2+}-m_2^{2-}\right)$ $+\Delta t\,\theta\left(\mathbf{M}_1^{1+}+\mathbf{M}_1^{2+}+\mathbf{M}_2^{1+}+\mathbf{M}_2^{2+}\right).\mathbf{F}^m+\Delta t(1-\theta)\left(\mathbf{M}_1^{1-}+\mathbf{M}_1^{2-}+\mathbf{M}_2^{1-}+\mathbf{M}_2^{2-}\right).\mathbf{F}^m$	τ
8	$+\theta\Delta t\left(h_1^{m1+}\mathbf{M}_1^{1+}+h_1^{m2+}\mathbf{M}_1^{2+}+h_2^{m1+}\mathbf{M}_2^{1+}+h_2^{m2+}\mathbf{M}_2^{2+}+\mathbf{q}^{+}\right)+$ $+(1-\theta)\Delta t\left(h_1^{m1-}\mathbf{M}_1^{1-}+h_1^{m2-}\mathbf{M}_1^{2-}+h_2^{m1-}\mathbf{M}_2^{1-}+h_2^{m2-}\mathbf{M}_2^{2-}+\mathbf{q}^{-}\right)$	$\nabla\tau$

Note:

In the first term $\overline{\boldsymbol{\Sigma}}_1$ do not appear the term $-r_0\mathbf{F}^m$ because it is put in the external loading $\mathbf{L}^{meca}$ and calculated by the option of calculation of the loading external of gravity.

By using the definition [éq 7.2-1] of $\mathbf{R}_g^{el}$, one a:

$\mathbf{U}^{*el^T}.\mathbf{R}_g^{el}=\mathbf{E}_g^{*el^T}.\overline{\mathbf{\Sigma}}_g^{el}$, which still gives us:

$\mathbf{R}_g^{el}=\mathbf{Q}_g^{el^T}.\overline{\mathbf{\Sigma}}_g^{el}$

This last equality is only the local form on the level of a point of Gauss of [éq 6.2-2].

7.3.2 Tangent operator: options FULL_MECA, RIGI_MECA_TANG

In what follows, if $\mathbf{X}$ indicate a vector of components X^i and $\mathbf{Y}$ a vector of components Y^j, $\left[\frac{\partial X}{\partial Y}\right]$ a matrix will indicate of which the element occupying the line i and the column j is $\frac{\partial X^i}{\partial Y^j}$.

To calculate the tangent operator, the following quantities will be calculated:

$[\mathbf{DRDE}]=$

DR1U	DR1E	DR1P1	DR1GP1	DR1P2	DR1GP2	DR1T	DR1GT
DR2U	DR2E	DR2P1	DR2GP1	DR2P2	DR2GP2	DR2T	DR2GT
DR3U	DR3E	DR3P1	DR3GP1	DR3P2	DR3GP2	DR3T	DR3GT
DR4U	DR4E	DR4P1	DR4GP1	DR4P2	DR4GP2	DR4T	DR4GT
DR5U	DR5E	DR5P1	DR5GP1	DR5P2	DR5GP2	DR5T	DR5GT
DR6U	DR6E	DR6P1	DR6GP1	DR6P2	DR6GP2	DR6T	DR6GT
DR7U	DR7E	DR7P1	DR7GP1	DR7P2	DR7GP2	DR7T	DR7GT
DR8U	DR8E	DR8P1	DR8GP1	DR8P2	DR8GP2	DR8T	DR8GT

Where one noted:

$$DRiU=\frac{\partial \mathbf{F}_i}{\partial \mathbf{u}} \quad DRiGP1=\frac{\partial \mathbf{F}_i}{\partial \nabla p_1}$$
$$DRiE=\frac{\partial \mathbf{F}_i}{\partial \boldsymbol{\varepsilon}} \quad DRiGP2=\frac{\partial \mathbf{F}_i}{\partial \nabla p_2}$$
$$DRiP1=\frac{\partial \mathbf{F}_i}{\partial p_1} \quad DRiT=\frac{\partial \mathbf{F}_i}{\partial T}$$
$$DRiP2=\frac{\partial \mathbf{F}_i}{\partial p_2} \quad DRiDT=\frac{\partial \mathbf{F}_i}{\partial \nabla T}$$

To do these calculations one considers that the laws of behavior provide, for the corresponding options, all the derivative following:

$$
\mathbf{D\Sigma\,DE}=\begin{bmatrix}
\frac{\partial \boldsymbol{\sigma}'}{\mathbf{u}} & \frac{\partial \boldsymbol{\sigma}'}{\partial \boldsymbol{\varepsilon}} & \frac{\partial \boldsymbol{\sigma}'}{\partial p_1} & \frac{\partial \boldsymbol{\sigma}'}{\partial \nabla p_1} & \frac{\partial \boldsymbol{\sigma}'}{\partial p_2} & \frac{\partial \boldsymbol{\sigma}'}{\partial \nabla p_2} & \frac{\partial \boldsymbol{\sigma}'}{\partial T} & \frac{\partial \boldsymbol{\sigma}'}{\partial \nabla T} \\
\frac{\partial \sigma_p}{\partial \mathbf{u}} & \frac{\partial \sigma_p}{\partial \boldsymbol{\varepsilon}} & \frac{\partial \sigma_p}{\partial p_1} & \frac{\partial \sigma_p}{\partial \nabla p_1} & \frac{\partial \sigma_p}{\partial p_2} & \frac{\partial \sigma_p}{\partial \nabla p_2} & \frac{\partial \sigma_p}{\partial T} & \frac{\partial \sigma_p}{\partial \nabla T} \\
\frac{\partial m_1^1}{\partial \mathbf{u}} & \frac{\partial m_1^1}{\partial \boldsymbol{\varepsilon}} & \frac{\partial m_1^1}{\partial p_1} & \frac{\partial m_1^1}{\partial \nabla p_1} & \frac{\partial m_1^1}{\partial p_2} & \frac{\partial m_1^1}{\partial \nabla p_2} & \frac{\partial m_1^1}{\partial T} & \frac{\partial m_1^1}{\partial \nabla T} \\
\frac{\partial \mathbf{M}_1^1}{\partial \mathbf{u}} & \frac{\partial \mathbf{M}_1^1}{\partial \boldsymbol{\varepsilon}} & \frac{\partial \mathbf{M}_1^1}{\partial p_1} & \frac{\partial \mathbf{M}_1^1}{\partial \nabla p_1} & \frac{\partial \mathbf{M}_1^1}{\partial p_2} & \frac{\partial \mathbf{M}_1^1}{\partial \nabla p_2} & \frac{\partial \mathbf{M}_1^1}{\partial T} & \frac{\partial \mathbf{M}_1^1}{\partial \nabla T} \\
\frac{\partial h^{m_1^1}}{\partial \mathbf{u}} & \frac{\partial h^{m_1^1}}{\partial \boldsymbol{\varepsilon}} & \frac{\partial h^{m_1^1}}{\partial p_1} & \frac{\partial h^{m_1^1}}{\partial \nabla p_1} & \frac{\partial h^{m_1^1}}{\partial p_2} & \frac{\partial h^{m_1^1}}{\partial \nabla p_2} & \frac{\partial h^{m_1^1}}{\partial T} & \frac{\partial h^{m_1^1}}{\partial \nabla T} \\
\frac{\partial m_1^2}{\partial \mathbf{u}} & \frac{\partial m_1^2}{\partial \boldsymbol{\varepsilon}} & \frac{\partial m_1^2}{\partial p_1} & \frac{\partial m_1^2}{\partial \nabla p_1} & \frac{\partial m_1^2}{\partial p_2} & \frac{\partial m_1^2}{\partial \nabla p_2} & \frac{\partial m_1^2}{\partial T} & \frac{\partial m_1^2}{\partial \nabla T} \\
\frac{\partial \mathbf{M}_1^2}{\partial \mathbf{u}} & \frac{\partial \mathbf{M}_1^2}{\partial \boldsymbol{\varepsilon}} & \frac{\partial \mathbf{M}_1^2}{\partial p_1} & \frac{\partial \mathbf{M}_1^2}{\partial \nabla p_1} & \frac{\partial \mathbf{M}_1^2}{\partial p_2} & \frac{\partial \mathbf{M}_1^2}{\partial \nabla p_2} & \frac{\partial \mathbf{M}_1^2}{\partial T} & \frac{\partial \mathbf{M}_1^2}{\partial \nabla T} \\
\frac{\partial h^{m_1^2}}{\partial \mathbf{u}} & \frac{\partial h^{m_1^2}}{\partial \boldsymbol{\varepsilon}} & \frac{\partial h^{m_1^2}}{\partial p_1} & \frac{\partial h^{m_1^2}}{\partial \nabla p_1} & \frac{\partial h^{m_1^2}}{\partial p_2} & \frac{\partial h^{m_1^2}}{\partial \nabla p_2} & \frac{\partial h^{m_1^2}}{\partial T} & \frac{\partial h^{m_1^2}}{\partial \nabla T} \\
\frac{\partial m_2^1}{\partial \mathbf{u}} & \frac{\partial m_2^1}{\partial \boldsymbol{\varepsilon}} & \frac{\partial m_2^1}{\partial p_1} & \frac{\partial m_2^1}{\partial \nabla p_1} & \frac{\partial m_2^1}{\partial p_2} & \frac{\partial m_2^1}{\partial \nabla p_2} & \frac{\partial m_2^1}{\partial T} & \frac{\partial m_2^1}{\partial \nabla T} \\
\frac{\partial \mathbf{M}_2^1}{\partial \mathbf{u}} & \frac{\partial \mathbf{M}_2^1}{\partial \boldsymbol{\varepsilon}} & \frac{\partial \mathbf{M}_2^1}{\partial p_1} & \frac{\partial \mathbf{M}_2^1}{\partial \nabla p_1} & \frac{\partial \mathbf{M}_2^1}{\partial p_2} & \frac{\partial \mathbf{M}_2^1}{\partial \nabla p_2} & \frac{\partial \mathbf{M}_2^1}{\partial T} & \frac{\partial \mathbf{M}_2^1}{\partial \nabla T} \\
\frac{\partial h^{m_2^1}}{\partial \mathbf{u}} & \frac{\partial h^{m_2^1}}{\partial \boldsymbol{\varepsilon}} & \frac{\partial h^{m_2^1}}{\partial p_1} & \frac{\partial h^{m_2^1}}{\partial \nabla p_1} & \frac{\partial h^{m_2^1}}{\partial p_2} & \frac{\partial h^{m_2^1}}{\partial \nabla p_2} & \frac{\partial h^{m_2^1}}{\partial T} & \frac{\partial h^{m_2^1}}{\partial \nabla T} \\
\frac{\partial m_2^2}{\partial \mathbf{u}} & \frac{\partial m_2^2}{\partial \boldsymbol{\varepsilon}} & \frac{\partial m_2^2}{\partial p_1} & \frac{\partial m_2^2}{\partial \nabla p_1} & \frac{\partial m_2^2}{\partial p_2} & \frac{\partial m_2^2}{\partial \nabla p_2} & \frac{\partial m_2^2}{\partial T} & \frac{\partial m_2^2}{\partial \nabla T} \\
\frac{\partial \mathbf{M}_2^2}{\partial \mathbf{u}} & \frac{\partial \mathbf{M}_2^2}{\partial \boldsymbol{\varepsilon}} & \frac{\partial \mathbf{M}_2^2}{\partial p_1} & \frac{\partial \mathbf{M}_2^2}{\partial \nabla p_1} & \frac{\partial \mathbf{M}_2^2}{\partial p_2} & \frac{\partial \mathbf{M}_2^2}{\partial \nabla p_2} & \frac{\partial \mathbf{M}_2^2}{\partial T} & \frac{\partial \mathbf{M}_2^2}{\partial \nabla T} \\
\frac{\partial h^{m_2^2}}{\partial \mathbf{u}} & \frac{\partial h^{m_2^2}}{\partial \boldsymbol{\varepsilon}} & \frac{\partial h^{m_2^2}}{\partial p_1} & \frac{\partial h^{m_2^2}}{\partial \nabla p_1} & \frac{\partial h^{m_2^2}}{\partial p_2} & \frac{\partial h^{m_2^2}}{\partial \nabla p_2} & \frac{\partial h^{m_2^2}}{\partial T} & \frac{\partial h^{m_2^2}}{\partial \nabla T} \\
\frac{\partial Q'}{\partial \mathbf{u}} & \frac{\partial Q'}{\partial \boldsymbol{\varepsilon}} & \frac{\partial Q'}{\partial p_1} & \frac{\partial Q'}{\partial \nabla p_1} & \frac{\partial Q'}{\partial p_2} & \frac{\partial Q'}{\partial \nabla p_2} & \frac{\partial Q'}{\partial T} & \frac{\partial Q'}{\partial \nabla T} \\
\frac{\partial \mathbf{q}}{\partial \mathbf{u}} & \frac{\partial \mathbf{q}}{\partial \boldsymbol{\varepsilon}} & \frac{\partial \mathbf{q}}{\partial p_1} & \frac{\partial \mathbf{q}}{\partial \nabla p_1} & \frac{\partial \mathbf{q}}{\partial p_2} & \frac{\partial \mathbf{q}}{\partial \nabla p_2} & \frac{\partial \mathbf{q}}{\partial T} & \frac{\partial \mathbf{q}}{\partial \nabla T}
\end{bmatrix}
$$

Note:

In these expressions, the derivative compared to $\mathbf{u}$ *are all worthless, but we keep the writing taking into account the definition of the matrices* $\mathbf{Q}_g^{el}$ *that we adopted.*

The call to the laws of behavior will provide the pieces of the matrix $D\Sigma\mathbf{DE}$ according to the equations present:

$$[\mathbf{DMECDE}]=\begin{bmatrix}\frac{\partial \boldsymbol{\sigma}'}{\partial \boldsymbol{\varepsilon}}\\ \frac{\partial \sigma_p}{\partial \varepsilon}\end{bmatrix};[\mathbf{DMECP1}]=\begin{bmatrix}\frac{\partial \boldsymbol{\sigma}'}{\partial p_1} & \frac{\partial \boldsymbol{\sigma}'}{\partial \nabla p_1}\\ \frac{\partial \sigma_p}{\partial p_1} & \frac{\partial \sigma_p}{\partial \nabla p_1}\end{bmatrix};[\mathbf{DMECP}2]=\begin{bmatrix}\frac{\partial \boldsymbol{\sigma}'}{\partial p_2} & \frac{\partial \boldsymbol{\sigma}'}{\partial \nabla p_2}\\ \frac{\partial \sigma_p}{\partial p_2} & \frac{\partial \sigma_p}{\partial \nabla p_2}\end{bmatrix}[\mathbf{DMECDT}]=\begin{bmatrix}\frac{\partial \boldsymbol{\sigma}}{\partial T} & \frac{\partial \boldsymbol{\sigma}}{\partial \nabla T}\\ \frac{\partial \sigma_p}{\partial T} & \frac{\partial \sigma_p}{\partial \nabla T}\end{bmatrix}$$

$$[\mathbf{DP11DE}]=\begin{bmatrix}\frac{\partial m_1^1}{\partial \boldsymbol{\varepsilon}}\\ \frac{\partial \mathbf{M}_1^1}{\partial \boldsymbol{\varepsilon}}\\ \frac{\partial h^{m_1^1}}{\partial \boldsymbol{\varepsilon}}\end{bmatrix};[\mathbf{DP11P1}]=\begin{bmatrix}\frac{\partial m_1^1}{\partial p_1} & \frac{\partial m_1^1}{\partial \nabla p_1}\\ \frac{\partial \mathbf{M}_1^1}{\partial p_1} & \frac{\partial \mathbf{M}_1^1}{\partial \nabla p_1}\\ \frac{\partial h^{m_1^1}}{\partial p_1} & \frac{\partial h^{m_1^1}}{\partial \nabla p_1}\end{bmatrix};[\mathbf{DP11P2}]=\begin{bmatrix}\frac{\partial m_1^1}{\partial p_2} & \frac{\partial m_1^1}{\partial \nabla p_2}\\ \frac{\partial \mathbf{M}_1^1}{\partial p_2} & \frac{\partial \mathbf{M}_1^1}{\partial \nabla p_2}\\ \frac{\partial h^{m_1^1}}{\partial p_2} & \frac{\partial h^{m_1^1}}{\partial \nabla p_2}\end{bmatrix}[\mathbf{DP11DT}]=\begin{bmatrix}\frac{\partial m_1^1}{\partial T} & \frac{\partial m_1^1}{\partial \nabla T}\\ \frac{\partial \mathbf{M}_1^1}{\partial T} & \frac{\partial \mathbf{M}_1^1}{\partial \nabla T}\\ \frac{\partial h^{m_1^1}}{\partial T} & \frac{\partial h^{m_1^1}}{\partial \nabla T}\end{bmatrix}$$

$$[\mathbf{DP12DE}]=\begin{bmatrix}\frac{\partial m_1^2}{\partial \boldsymbol{\varepsilon}}\\ \frac{\partial \mathbf{M}_1^2}{\partial \boldsymbol{\varepsilon}}\\ \frac{\partial h^{m_1^2}}{\partial \boldsymbol{\varepsilon}}\end{bmatrix};[\mathbf{DP12P1}]=\begin{bmatrix}\frac{\partial m_1^2}{\partial p_1} & \frac{\partial m_1^2}{\partial \nabla p_1}\\ \frac{\partial \mathbf{M}_1^2}{\partial p_1} & \frac{\partial M_1^2}{\partial \nabla p_1}\\ \frac{\partial h^{m_1^2}}{\partial p_1} & \frac{\partial h^{m_1^2}}{\partial \nabla p_1}\end{bmatrix};[\mathbf{DP12P2}]=\begin{bmatrix}\frac{\partial m_1^2}{\partial p_2} & \frac{\partial m_1^2}{\partial \nabla p_2}\\ \frac{\partial \mathbf{M}_1^2}{\partial p_2} & \frac{\partial \mathbf{M}_1^2}{\partial \nabla p_2}\\ \frac{\partial h^{m_1^2}}{\partial p_2} & \frac{\partial h^{m_1^2}}{\partial \nabla p_2}\end{bmatrix}[\mathbf{DP12DT}]=\begin{bmatrix}\frac{\partial m_1^2}{\partial T} & \frac{\partial m_1^2}{\partial \nabla T}\\ \frac{\partial \mathbf{M}_1^2}{\partial T} & \frac{\partial \mathbf{M}_1^2}{\partial \nabla T}\\ \frac{\partial h^{m_1^2}}{\partial T} & \frac{\partial h^{m_1^2}}{\partial \nabla T}\end{bmatrix}$$

$$[\mathbf{DP21DE}]=\begin{bmatrix}\frac{\partial m_2^1}{\partial \boldsymbol{\varepsilon}}\\ \frac{\partial \mathbf{M}_2^1}{\partial \boldsymbol{\varepsilon}}\\ \frac{\partial h^{m_2^1}}{\partial \boldsymbol{\varepsilon}}\end{bmatrix};[\mathbf{DP21P1}]=\begin{bmatrix}\frac{\partial m_2^1}{\partial p_1} & \frac{\partial m_2^1}{\partial \nabla p_1}\\ \frac{\partial \mathbf{M}_2^1}{\partial p_1} & \frac{\partial \mathbf{M}_2^1}{\partial \nabla p_1}\\ \frac{\partial h^{m_2^1}}{\partial p_1} & \frac{\partial h^{m_2^1}}{\partial \nabla p_1}\end{bmatrix};[\mathbf{DP21P2}]=\begin{bmatrix}\frac{\partial m_2^1}{\partial p_2} & \frac{\partial m_2^1}{\partial \nabla p_2}\\ \frac{\partial \mathbf{M}_2^1}{\partial p_2} & \frac{\partial \mathbf{M}_2^1}{\partial \nabla p_2}\\ \frac{\partial h^{m_2^1}}{\partial p_2} & \frac{\partial h^{m_2^1}}{\partial \nabla p_2}\end{bmatrix}[\mathbf{DP21DT}]=\begin{bmatrix}\frac{\partial m_2^1}{\partial T} & \frac{\partial m_2^1}{\partial \nabla T}\\ \frac{\partial \mathbf{M}_2^1}{\partial T} & \frac{\partial \mathbf{M}_2^1}{\partial \nabla T}\\ \frac{\partial h^{m_2^1}}{\partial T} & \frac{\partial h^{m_2^1}}{\partial \nabla T}\end{bmatrix}$$

$$[\mathbf{DP22DE}]=\begin{bmatrix}\frac{\partial m_2^2}{\partial \boldsymbol{\varepsilon}}\\ \frac{\partial \mathbf{M}_2^2}{\partial \boldsymbol{\varepsilon}}\\ \frac{\partial h^{m_2^2}}{\partial \boldsymbol{\varepsilon}}\end{bmatrix};[\mathbf{DP22P1}]=\begin{bmatrix}\frac{\partial m_2^2}{\partial p_1} & \frac{\partial m_2^2}{\partial \nabla p_1}\\ \frac{\partial \mathbf{M}_2^2}{\partial p_1} & \frac{\partial \mathbf{M}_2^2}{\partial \nabla p_1}\\ \frac{\partial h^{m_2^2}}{\partial p_1} & \frac{\partial h^{m_2^2}}{\partial \nabla p_1}\end{bmatrix};[\mathbf{DP22P2}]=\begin{bmatrix}\frac{\partial m_2^2}{\partial p_2} & \frac{\partial m_2^2}{\partial \nabla p_2}\\ \frac{\partial \mathbf{M}_2^2}{\partial p_2} & \frac{\partial \mathbf{M}_2^2}{\partial \nabla p_2}\\ \frac{\partial h^{m_2^2}}{\partial p_2} & \frac{\partial h^{m_2^2}}{\partial \nabla p_2}\end{bmatrix}[\mathbf{DP22DT}]=\begin{bmatrix}\frac{\partial m_2^2}{\partial T} & \frac{\partial m_2^2}{\partial \nabla T}\\ \frac{\partial \mathbf{M}_2^2}{\partial T} & \frac{\partial \mathbf{M}_2^2}{\partial \nabla T}\\ \frac{\partial h^{m_2^2}}{\partial T} & \frac{\partial h^{m_2^2}}{\partial \nabla T}\end{bmatrix}$$

$$[\mathbf{DTDE}]=\begin{bmatrix}\frac{\partial Q'}{\partial \boldsymbol{\varepsilon}}\\ \frac{\partial \mathbf{q}}{\partial \boldsymbol{\varepsilon}}\end{bmatrix};[\mathbf{DTDP1}]=\begin{bmatrix}\frac{\partial Q'}{\partial p_1} & \frac{\partial Q'}{\partial \nabla p_1}\\ \frac{\partial \mathbf{q}}{\partial p_1} & \frac{\partial \mathbf{q}}{\partial \nabla p_1}\end{bmatrix};[\mathbf{DTDP2}]=\begin{bmatrix}\frac{\partial Q'}{\partial p_2} & \frac{\partial Q'}{\partial \nabla p_2}\\ \frac{\partial \mathbf{q}}{\partial p_2} & \frac{\partial \mathbf{q}}{\partial \nabla p_2}\end{bmatrix}[\mathbf{DTDT}]=\begin{bmatrix}\frac{\partial Q'}{\partial T} & \frac{\partial Q'}{\partial T}\\ \frac{\partial \mathbf{q}}{\partial T} & \frac{\partial \mathbf{q}}{\partial T}\end{bmatrix}$$

In addition, by deriving the expression from the residue compared to the constraints, one defines:

$$\mathbf{D}\overline{\boldsymbol{\Sigma}}\,\mathbf{D}\boldsymbol{\Sigma}=\begin{bmatrix}
\frac{\partial \overline{\Sigma}_1}{\partial \boldsymbol{\sigma}'} & \frac{\partial \overline{\Sigma}_1}{\partial \sigma_p} & \frac{\partial \overline{\Sigma}_1}{\partial m_1^1} & \frac{\partial \overline{\Sigma}_1}{\partial \mathbf{M}_1^1} & \frac{\partial \overline{\Sigma}_1}{\partial h^{m_1^1}} & \frac{\partial \overline{\Sigma}_1}{\partial m_1^2} & \frac{\partial \overline{\Sigma}_1}{\partial \mathbf{M}_1^2} & \frac{\partial \overline{\Sigma}_1}{\partial h^{m_1^2}} & \frac{\partial \overline{\Sigma}_1}{\partial m_2^1} & \frac{\partial \overline{\Sigma}_1}{\partial \mathbf{M}_2^1} & \frac{\partial \overline{\Sigma}_1}{\partial h^{m_2^1}} & \frac{\partial \overline{\Sigma}_1}{\partial m_2^2} & \frac{\partial \overline{\Sigma}_1}{\partial \mathbf{M}_2^2} & \frac{\partial \overline{\Sigma}_1}{\partial h^{m_2^2}} & \frac{\partial \overline{\Sigma}_1}{\partial Q'} & \frac{\partial \overline{\Sigma}_1}{\partial \mathbf{q}} \\
\frac{\partial \overline{\Sigma}_2}{\partial \boldsymbol{\sigma}'} & \frac{\partial \overline{\Sigma}_2}{\partial \sigma_p} & \frac{\partial \overline{\Sigma}_2}{\partial m_1^1} & \frac{\partial \overline{\Sigma}_2}{\partial \mathbf{M}_1^1} & \frac{\partial \overline{\Sigma}_2}{\partial h^{m_1^1}} & \frac{\partial \overline{\Sigma}_2}{\partial m_1^2} & \frac{\partial \overline{\Sigma}_2}{\partial \mathbf{M}_1^2} & \frac{\partial \overline{\Sigma}_2}{\partial h^{m_1^2}} & \frac{\partial \overline{\Sigma}_2}{\partial m_2^1} & \frac{\partial \overline{\Sigma}_2}{\partial \mathbf{M}_2^1} & \frac{\partial \overline{\Sigma}_2}{\partial h^{m_2^1}} & \frac{\partial \overline{\Sigma}_2}{\partial m_2^2} & \frac{\partial \overline{\Sigma}_2}{\partial \mathbf{M}_2^2} & \frac{\partial \overline{\Sigma}_2}{\partial h^{m_2^2}} & \frac{\partial \overline{\Sigma}_2}{\partial Q'} & \frac{\partial \overline{\Sigma}_2}{\partial \mathbf{q}} \\
\frac{\partial \overline{\Sigma}_3}{\partial \boldsymbol{\sigma}'} & \frac{\partial \overline{\Sigma}_3}{\partial \sigma_p} & \frac{\partial \overline{\Sigma}_3}{\partial m_1^1} & \frac{\partial \overline{\Sigma}_3}{\partial \mathbf{M}_1^1} & \frac{\partial \overline{\Sigma}_3}{\partial h^{m_1^1}} & \frac{\partial \overline{\Sigma}_3}{\partial m_1^2} & \frac{\partial \overline{\Sigma}_3}{\partial \mathbf{M}_1^2} & \frac{\partial \overline{\Sigma}_3}{\partial h^{m_1^2}} & \frac{\partial \overline{\Sigma}_3}{\partial m_2^1} & \frac{\partial \overline{\Sigma}_3}{\partial \mathbf{M}_2^1} & \frac{\partial \overline{\Sigma}_3}{\partial h^{m_2^1}} & \frac{\partial \overline{\Sigma}_3}{\partial m_2^2} & \frac{\partial \overline{\Sigma}_3}{\partial \mathbf{M}_2^2} & \frac{\partial \overline{\Sigma}_3}{\partial h^{m_2^2}} & \frac{\partial \overline{\Sigma}_3}{\partial Q'} & \frac{\partial \overline{\Sigma}_3}{\partial \mathbf{q}} \\
\frac{\partial \overline{\Sigma}_4}{\partial \boldsymbol{\sigma}'} & \frac{\partial \overline{\Sigma}_4}{\partial \sigma_p} & \frac{\partial \overline{\Sigma}_4}{\partial m_1^1} & \frac{\partial \overline{\Sigma}_4}{\partial \mathbf{M}_1^1} & \frac{\partial \overline{\Sigma}_4}{\partial h^{m_1^1}} & \frac{\partial \overline{\Sigma}_4}{\partial m_1^2} & \frac{\partial \overline{\Sigma}_4}{\partial \mathbf{M}_1^2} & \frac{\partial \overline{\Sigma}_4}{\partial h^{m_1^2}} & \frac{\partial \overline{\Sigma}_4}{\partial m_2^1} & \frac{\partial \overline{\Sigma}_4}{\partial \mathbf{M}_2^1} & \frac{\partial \overline{\Sigma}_4}{\partial h^{m_2^1}} & \frac{\partial \overline{\Sigma}_4}{\partial m_2^2} & \frac{\partial \overline{\Sigma}_4}{\partial \mathbf{M}_2^2} & \frac{\partial \overline{\Sigma}_4}{\partial h^{m_2^2}} & \frac{\partial \overline{\Sigma}_4}{\partial Q'} & \frac{\partial \overline{\Sigma}_4}{\partial \mathbf{q}} \\
\frac{\partial \overline{\Sigma}_5}{\partial \boldsymbol{\sigma}'} & \frac{\partial \overline{\Sigma}_5}{\partial \sigma_p} & \frac{\partial \overline{\Sigma}_5}{\partial m_1^1} & \frac{\partial \overline{\Sigma}_5}{\partial \mathbf{M}_1^1} & \frac{\partial \overline{\Sigma}_5}{\partial h^{m_1^1}} & \frac{\partial \overline{\Sigma}_5}{\partial m_1^2} & \frac{\partial \overline{\Sigma}_5}{\partial \mathbf{M}_1^2} & \frac{\partial \overline{\Sigma}_5}{\partial h^{m_1^2}} & \frac{\partial \overline{\Sigma}_5}{\partial m_2^1} & \frac{\partial \overline{\Sigma}_5}{\partial \mathbf{M}_2^1} & \frac{\partial \overline{\Sigma}_5}{\partial h^{m_2^1}} & \frac{\partial \overline{\Sigma}_5}{\partial m_2^2} & \frac{\partial \overline{\Sigma}_5}{\partial \mathbf{M}_2^2} & \frac{\partial \overline{\Sigma}_5}{\partial h^{m_2^2}} & \frac{\partial \overline{\Sigma}_5}{\partial Q'} & \frac{\partial \overline{\Sigma}_5}{\partial \mathbf{q}} \\
\frac{\partial \overline{\Sigma}_6}{\partial \boldsymbol{\sigma}'} & \frac{\partial \overline{\Sigma}_6}{\partial \sigma_p} & \frac{\partial \overline{\Sigma}_6}{\partial m_1^1} & \frac{\partial \overline{\Sigma}_6}{\partial \mathbf{M}_1^1} & \frac{\partial \overline{\Sigma}_6}{\partial h^{m_1^1}} & \frac{\partial \overline{\Sigma}_6}{\partial m_1^2} & \frac{\partial \overline{\Sigma}_6}{\partial \mathbf{M}_1^2} & \frac{\partial \overline{\Sigma}_6}{\partial h^{m_1^2}} & \frac{\partial \overline{\Sigma}_6}{\partial m_2^1} & \frac{\partial \overline{\Sigma}_6}{\partial \mathbf{M}_2^1} & \frac{\partial \overline{\Sigma}_6}{\partial h^{m_2^1}} & \frac{\partial \overline{\Sigma}_6}{\partial m_2^2} & \frac{\partial \overline{\Sigma}_6}{\partial \mathbf{M}_2^2} & \frac{\partial \overline{\Sigma}_6}{\partial h^{m_2^2}} & \frac{\partial \overline{\Sigma}_6}{\partial Q'} & \frac{\partial \overline{\Sigma}_6}{\partial \mathbf{q}} \\
\frac{\partial \overline{\Sigma}_7}{\partial \boldsymbol{\sigma}'} & \frac{\partial \overline{\Sigma}_7}{\partial \sigma_p} & \frac{\partial \overline{\Sigma}_7}{\partial m_1^1} & \frac{\partial \overline{\Sigma}_7}{\partial \mathbf{M}_1^1} & \frac{\partial \overline{\Sigma}_7}{\partial h^{m_1^1}} & \frac{\partial \overline{\Sigma}_7}{\partial m_1^2} & \frac{\partial \overline{\Sigma}_7}{\partial \mathbf{M}_1^2} & \frac{\partial \overline{\Sigma}_7}{\partial h^{m_1^2}} & \frac{\partial \overline{\Sigma}_7}{\partial m_2^1} & \frac{\partial \overline{\Sigma}_7}{\partial \mathbf{M}_2^1} & \frac{\partial \overline{\Sigma}_7}{\partial h^{m_2^1}} & \frac{\partial \overline{\Sigma}_7}{\partial m_2^2} & \frac{\partial \overline{\Sigma}_7}{\partial \mathbf{M}_2^2} & \frac{\partial \overline{\Sigma}_7}{\partial h^{m_2^2}} & \frac{\partial \overline{\Sigma}_7}{\partial Q'} & \frac{\partial \overline{\Sigma}_7}{\partial \mathbf{q}} \\
\frac{\partial \overline{\Sigma}_8}{\partial \boldsymbol{\sigma}'} & \frac{\partial \overline{\Sigma}_8}{\partial \sigma_p} & \frac{\partial \overline{\Sigma}_8}{\partial m_1^1} & \frac{\partial \overline{\Sigma}_8}{\partial \mathbf{M}_1^1} & \frac{\partial \overline{\Sigma}_8}{\partial h^{m_1^1}} & \frac{\partial \overline{\Sigma}_8}{\partial m_1^2} & \frac{\partial \overline{\Sigma}_8}{\partial \mathbf{M}_1^2} & \frac{\partial \overline{\Sigma}_8}{\partial h^{m_1^2}} & \frac{\partial \overline{\Sigma}_8}{\partial m_2^1} & \frac{\partial \overline{\Sigma}_8}{\partial \mathbf{M}_2^1} & \frac{\partial \overline{\Sigma}_8}{\partial h^{m_2^1}} & \frac{\partial \overline{\Sigma}_8}{\partial m_2^2} & \frac{\partial \overline{\Sigma}_8}{\partial \mathbf{M}_2^2} & \frac{\partial \overline{\Sigma}_8}{\partial h^{m_2^2}} & \frac{\partial \overline{\Sigma}_8}{\partial Q'} & \frac{\partial \overline{\Sigma}_8}{\partial \mathbf{q}}
\end{bmatrix}$$

All these quantities not being inevitably calculated, one will note, for i from 1 to 8:

$$[\mathbf{D}\overline{\Sigma}\,i\,\mathbf{D}\sigma]=\left[\frac{\partial \overline{\Sigma}_i}{\partial \boldsymbol{\sigma}'},\frac{\partial \overline{\Sigma}_i}{\partial \sigma_p}\right] \qquad [\mathbf{D}\overline{\Sigma}\,i\,\mathbf{DP}21]=\left[\frac{\partial \overline{\Sigma}_i}{\partial m_2^1},\frac{\partial \overline{\Sigma}_i}{\partial \mathbf{M}_2^1},\frac{\partial \overline{\Sigma}_i}{\partial h^{m_2^1}}\right]$$

$$[\mathbf{D}\overline{\Sigma}\,i\,\mathbf{DP}11]=\left[\frac{\partial \overline{\Sigma}_i}{\partial m_1^2},\frac{\partial \overline{\Sigma}_i}{\partial \mathbf{M}_1^1},\frac{\partial \overline{\Sigma}_i}{\partial h^{m_1^2}}\right] \qquad [\mathbf{D}\overline{\Sigma}\,i\,\mathbf{DP}22]=\left[\frac{\partial \overline{\Sigma}_i}{\partial m_2^2},\frac{\partial \overline{\Sigma}_i}{\partial \mathbf{M}_2^2},\frac{\partial \overline{\Sigma}_i}{\partial h^{m_2^2}}\right]$$

$$[\mathbf{D}\overline{\Sigma}\,i\,\mathbf{DP}12]=\left[\frac{\partial \overline{\Sigma}_i}{\partial m_1^2},\frac{\partial \overline{\Sigma}_i}{\partial \mathbf{M}_1^1},\frac{\partial \overline{\Sigma}_i}{\partial h^{m_1^2}}\right] \qquad [\mathbf{D}\overline{\Sigma}\,i\,\mathbf{DT}]=\left[\frac{\partial \overline{\Sigma}_i}{\partial Q'},\frac{\partial \overline{\Sigma}_i}{\partial \mathbf{q}}\right]$$

It is then clear that:

$$\mathbf{D}\overline{\boldsymbol{\Sigma}}\,\mathbf{DE}=\mathbf{D}\overline{\boldsymbol{\Sigma}}\,\mathbf{D}\boldsymbol{\Sigma}\,.\,\mathbf{D}\boldsymbol{\Sigma}\,\mathbf{DE}$$

And the contribution of the point of Gauss to the tangent matrix $D\mathbf{F}_g^{el}$ is obtained by:

$$D\mathbf{F}_g^{el}=\mathbf{Q}_g^{el^T}.\mathbf{D}\overline{\Sigma}\,\mathbf{DE}.\mathbf{Q}_g^{el}$$

7.4 Total algorithm

The algorithm becomes then:

<u>Initializations:</u>

Calculation of $\mathbf{L}^{meca^+}$ (option CHAR_MECA)

Calculation of $D\mathbf{F}^-$ (option RIGI_MECA_TANG)

Calculation of $\Delta\mathbf{U}_0$ by: $D\mathbf{F}^-.\Delta\mathbf{U}_0=\mathbf{L}^{meca^+}-\mathbf{L}^{meca^-}$

Iterations of balance of Newton
Buckle elements el
Buckle points of Gauss G
calculation $\mathbf{Q}_g^{el}$
calculation $\mathbf{E}_g^{el^-}=\mathbf{Q}_g^{el}.\mathbf{U}^{el^-}$ and $\mathbf{E}_g^{el^+}=\mathbf{Q}_g^{el}.\mathbf{U}^{el^+}$
calculation of: $\mathbf{\Sigma}_{gn}^{el^+},\boldsymbol{\alpha}_g^{el^+},\dfrac{\partial\mathbf{\Sigma}_{gn}^{el^+}}{\partial\mathbf{E}_{gn}^{el^+}}$ (according to option) from $\mathbf{E}_g^{el^-},\mathbf{\Sigma}_g^{el^-},\boldsymbol{\alpha}_g^{el^-},\mathbf{E}_{gn}^{el^+}$
calculation of $\overline{\mathbf{\Sigma}}_{gn}^{el^+}$ from $\mathbf{\Sigma}_{gn}^{el^+}$; $\mathbf{R}_{gn}^{el^+}=\mathbf{Q}_g^{el^T}.\overline{\mathbf{\Sigma}}_{gn}^{el^+}$
calculation of $\dfrac{\partial\overline{\mathbf{\Sigma}}_{gn}^{el^+}}{\partial\mathbf{\Sigma}_{gn}^{el^+}}$ from $\mathbf{\Sigma}_{gn}^{el^+}$; $D\mathbf{F}_{gn}^{el^+}=\mathbf{Q}_g^{el^T}.\dfrac{\partial\overline{\mathbf{\Sigma}}_{gn}^{el^+}}{\partial\mathbf{\Sigma}_{gn}^{el^+}}.\dfrac{\partial\mathbf{\Sigma}_{gn}^{el^+}}{\partial\mathbf{E}_{gn}^{el^+}}.\mathbf{Q}_g^{el}$ (according to option)

Calculation of $\delta\mathbf{U}_{n+1}$ by:
$D\mathbf{F}_n^+.\delta\mathbf{U}_{n+1}=-\mathbf{R}_n^++\mathbf{L}^{meca^+}$
Actualization :
$\Delta\mathbf{U}_{n+1}=\Delta\mathbf{U}_n+\rho\,\delta\mathbf{U}_{n+1}$
If test convergence OK
fine Newton: no next time
If not
$n=n+1$

8 The option FORC_NODA

On the level of the continuous equations, the option FORC_NODA corresponds to the calculation of the operator $\mathbf{R}=\mathbf{Q}^T\overline{\mathbf{\Sigma}}$. At the discrete level, the option FORC_NODA come down to calculate the vector $\mathbf{R}_g^{el}=\mathbf{Q}_g^{el^T}.\overline{\mathbf{\Sigma}}_g^{el}$.
As we already noted that $\overline{\mathbf{\Sigma}}$ depends not only on $\mathbf{\Sigma}$, but also of $\dot{\mathbf{\Sigma}}$, one should not be astonished to see appearing the step of time Δt and constraints at the same time at time + and time $^{-}$
The algorithm of Newton-Raphson of the order STAT_NON_LINE use the option FORC_NODA for the calculation of the prediction at the beginning of each step of time. It is not thus pain-killer to correctly calculate all the terms for this option, including those which depend on the step of time. We illustrate this question for a simple example corresponding to the only equation of hydraulics.
That is to say a simplified version of the hydraulic equation:

$$-\int_\Omega \frac{dm}{dt}.p^*d\Omega+\int_\Omega \mathbf{M}.\nabla p^*d\Omega=M_{ext}.p^* \qquad \textbf{éq 8-1}$$

After discretization in time:

$$-\int_\Omega \Delta m.p^*d\Omega+\Delta t\int_\Omega\left(\theta\mathbf{M}^++(1-\theta)\mathbf{M}^-\right).\nabla p^*d\Omega=M_{ext}^\theta.p^*$$

Revealing $\Delta\mathbf{M}=\mathbf{M}^+-\mathbf{M}^-$ and $\Delta p=p^+-p^-$, and writing a law of behavior: $\Delta m=N\Delta p$, one finds:

$$-\int_\Omega N\Delta p.p^*d\Omega+\Delta t\int_\Omega\Delta\mathbf{M}.\nabla p^*d\Omega=M_{ext}^\theta p^*-\Delta t\int_\Omega\mathbf{M}^-.\nabla p^*d\Omega$$

By definition the phase of prediction STAT_NON_LINE is written:

$$\mathbf{K}_0\Delta\mathbf{u}^0=F_{ext}^1-\mathbf{Q}^T\boldsymbol{\sigma}_0$$

It is then clear that one must take $\int_\Omega \mathbf{Q}^T \boldsymbol{\sigma}_0 p^* d\Omega = -\Delta t \int_\Omega \mathbf{M}^- \nabla p^* d\Omega$

9 Space discretization

Finite elements THM of *Code_Aster* are mixed elements, in the sense that they have at the same time unknown factors of displacements, pressures and temperatures. A choice of discretization where displacements, the pressures and the temperatures are interpolated with the same order of approximation led to oscillations, especially for choices of step of too small times compared to the discretization in space One will consult on this subject for example [bib10]. This problem is also in keeping with the manner of calculating the matrix known as of mass, and one will be able to consult on this subject [bib14]. We give in addition in appendix, to illustrate our matter, the solution for the first step of time of a problem of mono consolidation dimensional with an interpolation $P1P1$. It is seen that for a small step of time, it is very oscillating.
For this reason, quadratic elements THM are elements in $P2P1$, i.e. the interpolation of displacements is quadratic and that of the temperatures and pressures is linear. We nevertheless kept all the unknown factors on all the nodes, including the nodes mediums, but we imposed in the calculation of the matrices of rigidity that the pressure of a node medium of segment is equal to the half summons nodes top of the segment to which it belongs.
In addition, in the programming, we took account of the following property:
That is to say s a node top, w_s^1 its function of form as pertaining to a linear element (for example QUAD4), and w_s^2 its function of form as pertaining to a quadratic element (for example QUAD8). That is to say na the number of edges having s like end and w_{ma}^2 the function of form of the quadratic interpolation attached to a node medium of edge, one has the relation then:

$$w_s^1 = w_s^2 + \sum_{ma=1}^{na} \frac{w_{ma}^2}{2}$$

This said, and including in interpolation $P2P1$, conditions of nonoscillation exist on the step of time. [bib10] the relation gives:

$$\Delta t > \frac{\Delta x^2}{20 C_v}$$

where C_v is the coefficient of consolidation: $C_v = \frac{kE(1-\nu)}{\rho_{lq}(1+\nu)(1-2\nu)}$, k being the permeability measured in m/s.

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11 Description of the versions of the document

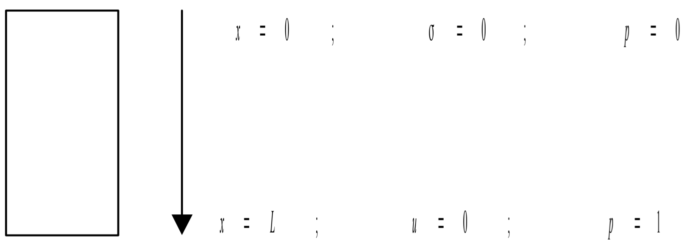
Version Aster	Author (S) Organization (S)	Description of the modifications
5	C.Chavant EDF-R&D/AMA	Initial text.
11	S.Granet EDF-R&D/AMA	Modifications: working; and expression of the conservation of the momentum in dynamics, § 6.3.5.
12	F.Voldoire EDF-R&D/AMA	Corrections of several equations: 4.2.1, 5.2.1, 5.3.1, 6.3.2-1 and §8.

Annexe 1 Problem mono dimensional P1P1

A unidimensional problem of consolidation is considered whose unknown factors vary only according to the only variable of space x .
A rectangular field length L , is filled with a porous material of coefficients of Lamé λ and μ , of coefficient of biot b and of module of biot $N=\frac{1}{M}$ and of hydraulic conductivity λ_h . The density of the fluid is noted ρ .

One notes σ the constraint σ_{xx} , u displacement in the direction x , $\varepsilon=\frac{\partial u}{\partial x}$ deformation, p pressure, m the mass contribution in fluid, $\mathbf{M}$ the flow of fluid.

The boundary conditions are:
in $x=0$: $\sigma=0\,;p=0$
in $x=L$: $u=0\,;p=1$ for $t>0$
Initial conditions in $t=0$ are $\sigma=u=p=0$.



The setting in equation gives:
Mechanical balance and linear elasticity: $\sigma=(\lambda+2\mu)\epsilon-bp=0$

Conservation of the mass: $\frac{\partial m}{\partial t}+\frac{\partial \mathbf{M}}{\partial x}=0$

Law of Darcy: $\mathbf{M}=-\lambda_h\rho\frac{\partial p}{\partial x}$

Couplings: $\frac{m}{\rho}=Np+b\epsilon$

We make a discretization in implicit time **and we are interested in calculation of the first step of time** :

The system of two equations is obtained:

$$\begin{cases}\sigma=(\lambda+2\mu)\epsilon(u)-bp=0\\ Np+b\epsilon-\lambda_h\Delta t\Delta\,p=0\end{cases}$$

The mixed variational formulation of this problem is:

$$\begin{cases}\int_\Omega\left[(\lambda+2\mu)\epsilon(u)\epsilon(u^*)-bp\,\epsilon(u^*)\right]=0 & \forall u^*\\ \int_\Omega\left[Npp^*+b\epsilon(u)\,p^*+\lambda_h\Delta t\nabla p\nabla p^*\right]=0 & \forall p^*\end{cases}$$

éq Year 1-1

Concerning the space discretization, we cut out the mediums in n finite elements. Nodes of the element i are i and $i+1$. One notes u_e^1 and p_e^1 the displacement and pressure of the first node of the element e , u_e^2 and p_e^2 the displacement and pressure of its second node. It is supposed that one uses finite elements P1P1, i.e. that displacements as the pressures are interpolated linearly. The discretization of the first equation gives then:

$$\sum_e \frac{u_e^{*2}-u_e^{*1}}{\Delta x}\left[(\lambda+2\mu)\frac{u_e^2-u_e^1}{\Delta x}-b\frac{p_e^1+p_e^2}{2}\right]=0$$

From where one deduces:

$$\left(u_e^2-u_e^1\right)=\frac{b\Delta x}{\lambda+2\mu}\frac{p_e^1+p_e^2}{2} \qquad \forall e$$

éq Year 1-2

The discretization of the second equation gives:

$$\sum_e \frac{p_e^{*2}+p_e^{*1}}{2}\left[b\frac{u_e^2-u_e^1}{\Delta x}+N\frac{p_e^1+p_e^2}{2}\right]+\lambda_h\Delta t\sum_e \frac{p_e^{*2}-p_e^{*1}}{\Delta x}\frac{p_e^2-p_e^1}{\Delta x}=0$$

In there bearing [éq Year 1-2], one finds:

$$\sum_e \frac{p_e^{*2}+p_e^{*1}}{2}\left(N+\frac{b^2}{\lambda+2\mu}\right)\frac{p_e^1+p_e^2}{2}+\lambda_h\frac{\Delta t}{\Delta x^2}\sum_e\left(p_e^{*2}-p_e^{*1}\right)\left(p_e^2-p_e^1\right)=0$$

éq Year 1-3

So now, we make tighten the step of time towards zero with step of constant space, $\lambda_h\frac{\Delta t}{\Delta x^2}<<\left(N+\frac{b^2}{\lambda+2\mu}\right)$, and [éq Year 1-3] is reduced to:

$$\sum_e\left(p_e^{*2}+p_e^{*1}\right)\left(p_e^1+p_e^2\right)=0$$

One introduces a total classification of the nodes and unknown factors of pressure:

$$p_j^1=p_j \;;\quad p_j^2=p_{j+1}$$

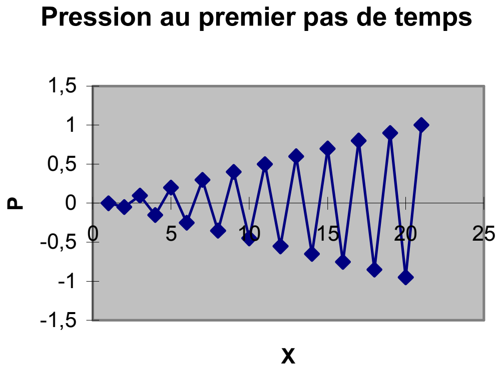
One can see that this whole of relations gives finally:

$$\begin{cases} p_1=0 \\ p_i+2p_{i+1}+p_{i+2}=0 \qquad \forall i\in[1,n-1] \\ p_{n+1}=1 \end{cases}$$

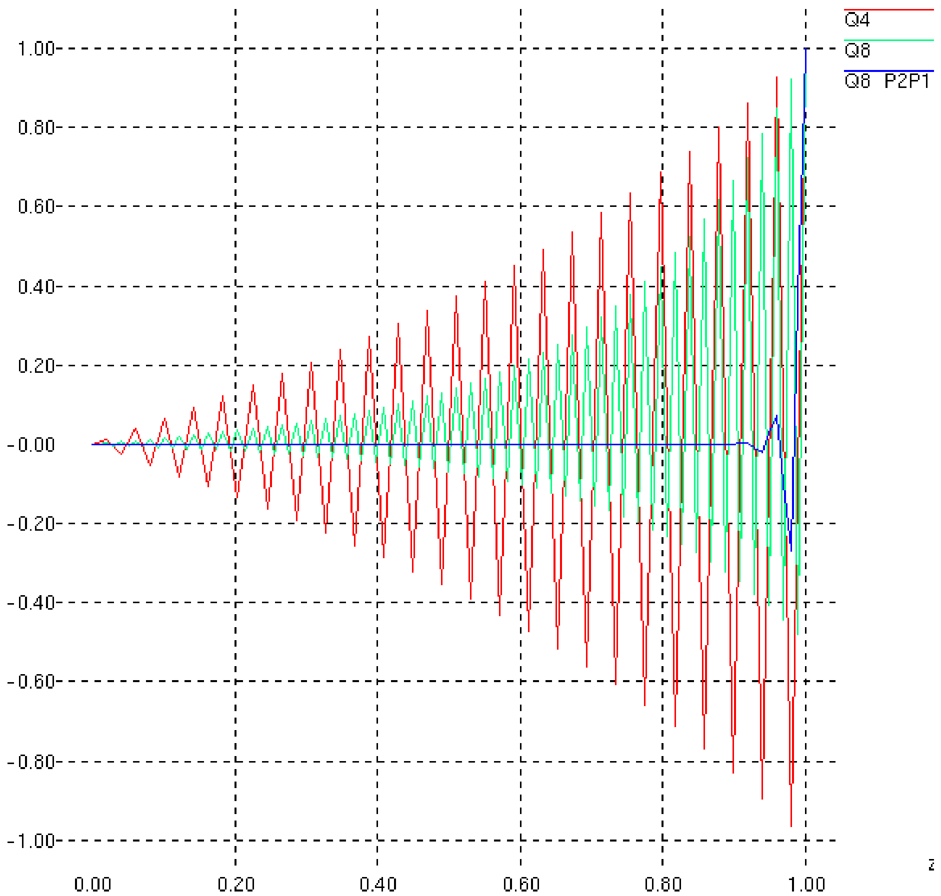
The solution of this continuation is:

$$\begin{cases} p_{2j-1} = \dfrac{j-1}{n} \\ p_{2j} = -\dfrac{2j-1}{2n} \end{cases}$$

What gives the distribution of following pressure:



As an indication, we give Ci below a comparison between digital results got and elements $P1P1$, $P2P2$ and $P2P1$.



It is seen that the element $P2P1$ do not remove the oscillation, but attenuates it appreciably.