# Integration of the elastoplastic mechanical behaviors of Drucker-Prager, associated (DRUCK_PRAGER) and non-aligned (DRUCK_PRAG_N_A) and postprocessings

**Summary:**

This document describes the principles of several developments concerning the elastoplastic law of behavior of Drucker-Prager in associated version (DRUCK_PRAGER) and non-aligned (DRUCK_PRAG_N_A) .

One is interested initially in integration itself of the law then, this law being lenitive, with an indicator of localization of Rice and finally with the calculation of sensitivity per direct differentiation for this law.
For the integration of the law, one uses an implicit scheme.

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# Contents

## Contents



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# 1 Introduction

The law of Drucker-Prager makes it possible to model in an elementary way the elastoplastic behavior concrete or certain grounds. Compared to the plasticity of Von-Put with isotropic work hardening, the difference lies in the presence of a term in $Tr(\boldsymbol{\sigma})$ in the formulation of the threshold and a nonworthless spherical component of the tensor of the plastic deformations.

In Code_Aster, the law exists in the associated version (`DRUCK_PRAGER`) and non-aligned (`DRUCK_PRAG_N_A`), more adapted for certain grounds because it makes it possible to better take into account dilatancy.

This note gathers the theoretical aspects several developments carried out in the code around this law: its integration according to an implicit scheme in time, an indicator of localization of Rice and the calculation of sensitivity per direct differentiation. Isotropic material is supposed. The indicator of Rice and the calculation of sensitivity do not function under the assumption of the plane constraints.

The theory and the developments were made for two types of function of work hardening: linear and parabolic, this function being in all the cases constant beyond of a cumulated plastic deformation "ultimate".

# 2 Integration of the law of behavior of Drucker-Prager

## 2.1 Notations

The mechanical constraints are counted positive in traction, the positive deformations in extension.

| | |
|---|---|
| $\boldsymbol{u}$ | displacements of the skeleton of components $u_x, u_y, u_z$ |
| $\boldsymbol{\varepsilon}=\frac{1}{2}(\nabla \boldsymbol{u}+\nabla^T \boldsymbol{u})$ | tensor of the linearized deformations |
| $\boldsymbol{e}=\boldsymbol{\varepsilon}-\frac{Tr(\boldsymbol{\varepsilon})}{3}\boldsymbol{I}$ | diverter of the deformations |
| $\varepsilon_v=Tr(\boldsymbol{\varepsilon})$ | trace of the deformations: variation of volume |
| $\boldsymbol{\varepsilon^p}$ | Tensor of the plastic deformations, |
| $\varepsilon_v^p=Tr(\boldsymbol{\varepsilon^p})$ | plastic variation of volume. |
| $\boldsymbol{e^p}$ | diverter of the plastic deformations |
| $p$ | cumulated plastic deformation |
| $\boldsymbol{\sigma}$ | Tensor of the constraints |
| $\boldsymbol{s}=\boldsymbol{\sigma}-\frac{Tr(\boldsymbol{\sigma})}{3}\boldsymbol{I}$ | diverter of the constraints |
| $\sigma_{eq}=\sqrt{\frac{3}{2}\boldsymbol{s}:\boldsymbol{s}}$ | Equivalent constraint of Von Mises |
| $I_1=Tr(\boldsymbol{\sigma})$ | trace of the constraints |
| $E_0$ | Young modulus |
| $\nu_0$ | Poisson's ratio |
| $\varphi$ | Angle of friction |
| $c$ | Cohesion |
| $\psi^0$ | Initial angle of dilatancy |

One poses $2\mu=\frac{E_0}{1+\nu_0}$ and $K=\frac{E_0}{3(1-2\nu_0)}$

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## 2.2 Formulation in associated version

### 2.2.1 Expression of the behavior

$\boldsymbol{\sigma}$ is the tensor of the constraints, which depends only on $\boldsymbol{\varepsilon}$ and its history. One considers the criterion of the Drücker-Prager type:

$$F(\boldsymbol{\sigma}, p)=\sigma_{eq}+AI_1-R(p)\leqslant 0 \quad (2.2.1\text{-}1)$$

where $A$ is a given coefficient and $R$ is a function of the cumulated plastic deformation $p$ (function of work hardening), of type linear or parabolic:

- linear work hardening

$$R(p)=\sigma_Y+h.p \quad \text{if} \quad p\in[0, p_{ultm}]$$
$$R(p)=\sigma_Y+h.p_{ultm} \quad \text{if} \quad p>p_{ultm}$$

Coefficients $h$, $p_{ultm}$ and $\sigma_Y$ are given.

(2.2.1-2)

- parabolic work hardening

$$R(p)=\sigma_Y\left(1-\left(1-\sqrt{\frac{\sigma_{Yultm}}{\sigma_Y}}\right)\frac{p}{p_{ultm}}\right)^2 \quad \text{if} \quad p\in[0, p_{ultm}]$$
$$R(p)=\sigma_{Y_{ultm}} \quad \text{if} \quad p>p_{ultm}$$

Coefficients $\sigma_{Yultm}$, $p_{ultm}$ and $\sigma_Y$ are given.

(2.2.1-3)

**Notice 1:**

*One can be given instead of $A$ and $\sigma_Y$ the binding fraction $c$ and the angle of friction $\varphi$ :*

$$A=\frac{2\sin\varphi}{3-\sin\varphi}$$
$$\sigma_Y=\frac{6c\cos\varphi}{3-\sin\varphi}$$

**Notice 2:**

*One chose in this document to privilege the variable $p$. Cumulated plastic deformation of shearing $\gamma^p=p\sqrt{3/2}$ also is very much used in soil mechanics.*

By considering an associated version one supposes that the potential of dissipation follows the same expression as that of the surface of load $F$ . The plastic flow is summarized then with:

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$$d\boldsymbol{\varepsilon}^{p}=d\lambda\frac{\partial F(\boldsymbol{\sigma},p)}{\partial\boldsymbol{\sigma}} \tag{2.2.1-4}$$

with:

$$d\lambda\geq 0 \quad ; \quad F\cdot d\lambda=0 \quad ; \quad F\leq 0 \tag{2.2.1-5}$$

The law of normality compared to the generalized force R gives the equality between the increment of cumulated plastic deformation and the increment of the multiplier $\lambda$ :

$$dp=-d\lambda\frac{\partial F(\boldsymbol{\sigma},p)}{\partial R}=d\lambda \tag{2.2.1-6}$$

### 2.2.2 Analytical resolution of the mechanical formulation

One places oneself in this chapter within the framework of finished increase. The integration of the law follows a pure implicit scheme, and the resolution is analytical. The finished increment of deformation $\Delta\boldsymbol{\varepsilon}$ known and is provided by the iteration of total Newton. One uses by convention the following notations: an index – to indicate a component at the beginning of step of loading, any index for a component at the end of the step of loading, and the operator $\Delta$ to indicate the increase in a component. The equations translating the elastic behavior are written then:

$$\boldsymbol{s}=\boldsymbol{s}^{-}+2\mu(\Delta\boldsymbol{e}-\Delta\boldsymbol{e}^{p})=\boldsymbol{s}^{e}-2\mu\Delta\boldsymbol{e}^{p} \tag{2.2.2-1}$$

$$I_1=I_1^{-}+3\mathrm{K}(\Delta\varepsilon_v-\Delta\varepsilon_v^{p})=I_1^{e}-3\mathrm{K}\Delta\varepsilon_v^{p} \tag{2.2.2-2}$$

Equations (2.2.1-4) and (2.2.1-6), taking into account (2.2.1-1), give:

$$\Delta\boldsymbol{\varepsilon}^{p}=\Delta p\left(\frac{\partial\sigma_{eq}}{\partial\boldsymbol{\sigma}}+A\frac{\partial I_1}{\partial\boldsymbol{\sigma}}\right)=\Delta p\left(\frac{3}{2}\frac{\boldsymbol{s}}{\sigma_{eq}}+A\boldsymbol{I}\right) \tag{2.2.2-3}$$

From where:

$$\Delta\varepsilon_v^{p}=3A\Delta p \tag{2.2.2-4}$$

$$\Delta\boldsymbol{e}^{p}=\frac{3}{2}\frac{\mathbf{s}}{\sigma_{eq}}\Delta p \tag{2.2.2-5}$$

If the increment $\Delta\boldsymbol{e}^{p}$ that is to say not no one, the increment of cumulated plastic deformation can be also written:

$$\Delta p=\sqrt{\frac{2}{3}\Delta\boldsymbol{e}^{p}:\Delta\boldsymbol{e}^{p}} \tag{2.2.2-6}$$

By combining the equations (2.2.2-1) and (2.2.2-5) one finds:

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$$\mathbf{s}\left(1+\frac{3\mu\Delta p}{\sigma_{eq}}\right)=\mathbf{s}^{\mathbf{e}} \quad (2.2.2\text{-}7)$$

from where:

$$\sigma_{eq}+3\mu\cdot\Delta p=\sigma_{eq}^{e} \quad (2.2.2\text{-}8)$$

what leads to:

$$\mathbf{s}\frac{\sigma_{eq}^{e}}{\sigma_{eq}}=\mathbf{s}^{\mathbf{e}} \quad (2.2.2\text{-}9)$$

By combining the equations respectively (2.2.2-7) and (2.2.2-8), and equations (2.2.2-2) and (2.2.2-4), one obtains:

$$\begin{cases}\mathbf{s}=\mathbf{s}^{\mathbf{e}}\left(1-\dfrac{3\mu}{\sigma_{eq}^{e}}\Delta p\right)\\ I_1=I_1^{e}-9KA\Delta p\end{cases} \quad (2.2.2\text{-}10)$$

By reinjecting the equation on $I_1$ and the relation $\sigma_{eq}=\sigma_{eq}^{e}-3\mu\cdot\Delta p$ in the formulation of the threshold, one obtains the scalar equation in $\Delta p$ :

$$\sigma_{eq}^{e}+AI_1^{e}-\Delta p(3\mu+9KA^2)-R(p^{-}+\Delta p)=0 \quad (2.2.2\text{-}11)$$

It is supposed that: $F\left(\boldsymbol{\sigma}^{e},p^{-}\right)>0$ .

To continue the resolution, one must now distinguish several cases:

1) **Case where** $p^{-}>p_{ultm}$
One a: $R(p^{-}+\Delta p)=R(p^{-})$
The scalar equation thus becomes: $F\left(\boldsymbol{\sigma}^{e},p^{-}\right)-\Delta p\left(3\mu+9KA^2\right)=0$
One finds:

$$\Delta p=\frac{F\left(\boldsymbol{\sigma}^{e},p^{-}\right)}{3\mu+9KA^2} \quad (2.2.2\text{-}12)$$

2) **Case where** $p^{-}\leq p_{ultm}$

2a) Linear work hardening

One a: $R(p^{-}+\Delta p)=R(p^{-})+h\Delta p$

The scalar equation thus becomes: $F\left(\boldsymbol{\sigma}^{e},p^{-}\right)-\Delta p\left(3\mu+9KA^2+h\right)=0$
One finds:

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$$\Delta p=\frac{F\left(\boldsymbol{\sigma}^{e}, p^{-}\right)}{3\mu+9KA^{2}+h} \tag{2.2.2-13}$$

2b) Parabolic work hardening

While expressing in the same way $R(p^{-}+\Delta p)$ according to $R(p^{-})$ and of $\Delta p$, it is found that the scalar equation is written:

$$F\left(\boldsymbol{\sigma}^{e}, p^{-}\right)+B\Delta p+G\Delta p^{2}=0$$

with:

$$\begin{cases} G=-\dfrac{\sigma_{Y}}{p_{ultm}^{2}}\left(1-\sqrt{\dfrac{\sigma_{Yultm}}{\sigma_{Y}}}\right)^{2} \\ B=-3\mu-9KA^{2}+\dfrac{2\sigma_{Y}}{p_{ultm}}\left(1-\sqrt{\dfrac{\sigma_{Yultm}}{\sigma_{Y}}}\right)\left(1-\left(1-\sqrt{\dfrac{\sigma_{Yultm}}{\sigma_{Y}}}\right)\dfrac{p^{-}}{p_{ultm}}\right) \end{cases}$$

The only positive root of the polynomial is:

$$\Delta p=\frac{-B-\sqrt{B^{2}-4G\cdot F\left(\boldsymbol{\sigma}^{e}, p^{-}\right)}}{2G} \tag{2.2.2-14}$$

2c) Final checking: Case where $(p^{-}+\Delta p)>p_{ultm}$

In the two preceding cases, once $\Delta p$ calculated, it should be checked that $p^{-}+\Delta p\leq p_{ultm}$. If this inequality is not satisfied, one has then:

$$R(p^{-}+\Delta p)=R(p_{ultm})$$

The scalar equation thus becomes:

$$F\left(\boldsymbol{\sigma}^{e}, p_{ultm}\right)-\Delta p\left(3\mu+9KA^{2}\right)=0$$

$$\Delta p=\frac{F\left(\boldsymbol{\sigma}^{e}, p_{ultm}\right)}{3\mu+9KA^{2}} \tag{2.2.2-15}$$

The principle of the analytical resolution presented above is equivalent to determine the point $\left(I_{1}, \mathbf{s}\right)$ like the projection of the point $\left(I_{1}^{e}, \mathbf{s}^{e}\right)$ on the criterion (plastic prediction rubber band-correction). This method thus comes from the law of flow approximated on a finished increment, and can be represented by the following graph:


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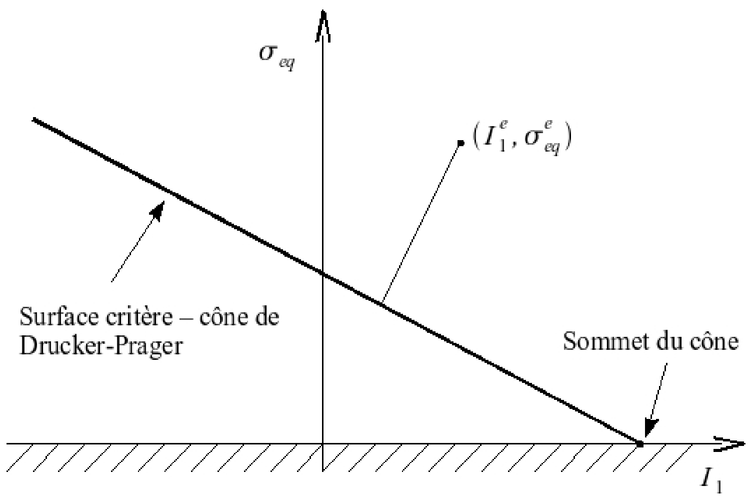


**Figure 2.2.2-1: projection on the criterion.**

**3) Projection at the top of the cone**

The integration of the law on an increment $\Delta t$ finished can be complicated when the state of stress is close to the top of the cone (see the Figure 2.2.2-1), because of the nonsmooth character of surface criterion. There are then two cases:

- case of a pure hydrostatic state,
- case of projection in an not-acceptable field.

In the typical case of one pure hydrostatic state, the derivative of the constraint of Von Mises $\sigma_{eq}$ compared to $\sigma$ is not defined. The law of flow (2.2.2-3) is unspecified (there is indeed a cone of possible normals to the criterion), and the equations (2.2.2-5), (2.2.2-7), (2.2.2-8), (2.2.2-9) cannot be written. There remains the definition of $\Delta p$ on the trail of constraints (equation 2.2.2-4). As in the case plus general, one must distinguish several cases:

1) **Case where** $p^- > p_{ultm}$ : $R(p^- + \Delta p) = R(p^-)$
The scalar equation with $\sigma_{eq} = 0$ becomes : $A I_1^e - \Delta p \cdot 9KA^2 = F(\boldsymbol{\sigma}^e, p^-) - \Delta p \cdot 9KA^2 = 0$
One finds:

$$\Delta p = \frac{I_1^e}{9KA} \quad (2.2.2\text{-}16)$$

2) **Case where** $p^- \leq p_{ultm}$

2a) Linear work hardening

One a: $R(p^- + \Delta p) = R(p^-) + h\Delta p$

The scalar equation with $\sigma_{eq} = 0$ becomes:
$A I_1^e - \Delta p \cdot 9KA^2 - R(p^-) + h\Delta p = F(\boldsymbol{\sigma}^e, p^-) - \Delta p \cdot 9KA^2 = 0$
One finds then:


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$$\Delta p=\frac{A I_1^e}{9 KA^2+h} \tag{2.2.2-17}$$

2b) Parabolic work hardening

While expressing $R(p^-+\Delta p)$ according to $R(p^-)$ and of $\Delta p$, one still finds the solution (2.2.2-14) :

with the value of B modified compared to the preceding case:

$$B=-9 KA^2+\frac{2\sigma_Y}{p_{ultm}}\left(1-\sqrt{\frac{\sigma_{Yultm}}{\sigma_Y}}\right)\left(1-\left(1-\sqrt{\frac{\sigma_{Yultm}}{\sigma_Y}}\right)\frac{p^-}{p_{ultm}}\right)$$

2c) Final checking: Case where $(p^-+\Delta p)>p_{ultm}$

In the cases 2a) and 2b), if the inequality $p^-+\Delta p\leq p_{ultm}$ is not satisfied, one a:

$$R(p^-+\Delta p)=R(p_{ultm})$$

The increment $\Delta p$ is given by the equation (2.2.2-16).

Because of the incremental resolution, it may be that the found solution is not acceptable, with $\sigma_{eq}<0$ . That can arrive when the state of stress at the moment $t^-$ is close to the top of the cone.

One then chooses to project the state of stress found by elastic prediction on the top of the cone, that is to say to refer to a state of purely hydrostatic stress. One makes a control a posteriori admissibility of the solution $(I_1, \mathbf{s})$ , and one makes possibly the correction.
In the details:

i) One brings up to date the state of stress by the means as of equations (2.2.2-12), (2.2.2-13), (2.2.2-14), (2.2.2-15).

ii) One controls that the solution $(I_1, \mathbf{s})$ found either acceptable, or that $\sigma_{eq}<0$ where, in an equivalent way, that $I_1$ maybe inside surface criterion:
$I_1\leqslant\frac{R(p)}{A}$

iii) If this condition is not checked, one imposes the checking of the criterion with $\sigma_{eq}=0$ (top of the cone): $I_1=\frac{R(p)}{A} \quad \Rightarrow A\cdot I_1-R(p)=F(\sigma, R)=0$

iv) One renews the solution with the equations then (2.2.2-16), (2.2.2-17), (2.2.2-14).

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### 2.2.3 Calculation of the tangent operator

#### 2.2.3.1 Total calculation of the tangent operator

One seeks to calculate the coherent matrix: $\frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{\varepsilon}}=\frac{\partial \mathbf{s}}{\partial \boldsymbol{\varepsilon}}+\frac{1}{3}\mathbf{I}\otimes\frac{\partial I_1}{\partial \boldsymbol{\varepsilon}}$

By deriving the system from equations (2.2-7), one obtains:

$$\begin{cases}\dfrac{\partial \mathbf{s}}{\partial \boldsymbol{\varepsilon}}=\dfrac{\partial \mathbf{s}^{\mathbf{e}}}{\partial \boldsymbol{\varepsilon}}\left(1-3\dfrac{\mu}{\sigma_{eq}^{e}}.\Delta p\right)+\dfrac{3\mu}{\left(\sigma_{eq}^{e}\right)^{2}}.\Delta p.\left(\mathbf{s}^{\mathbf{e}}\otimes\dfrac{\partial \sigma_{eq}^{e}}{\partial \boldsymbol{\varepsilon}}\right)-\dfrac{3\mu}{\sigma_{eq}^{e}}.\left(\dfrac{\mathbf{s}^{\mathbf{e}}\otimes\partial \Delta p}{\partial \boldsymbol{\varepsilon}}\right)\\ \dfrac{\partial I_1}{\partial \boldsymbol{\varepsilon}}=\dfrac{\partial I_1^{e}}{\partial \boldsymbol{\varepsilon}}-9KA\dfrac{\partial \Delta p}{\partial \boldsymbol{\varepsilon}}\end{cases}$$

**éq 2.2.3-1**

**Expression of** $\frac{\partial \mathbf{s}^{\mathbf{e}}}{\partial \boldsymbol{\varepsilon}}$

$$\frac{\partial s_{ij}^{e}}{\partial \varepsilon_{pq}}=2\mu\left(\delta_{ip}\delta_{jq}-\frac{1}{3}\delta_{ij}\delta_{pq}\right)$$

**Expression of** $\frac{\partial I_1^{e}}{\partial \varepsilon}$

$$\frac{\partial I_1^{e}}{\partial \varepsilon_{\mathrm{pq}}}=3\mathrm{K}\,\delta_{pq}$$

**Calculation of** $\frac{\partial \sigma_{eq}^{e}}{\partial \varepsilon}$

$$\frac{\partial \sigma_{\mathrm{eq}}^{e}}{\partial \varepsilon_{\mathrm{pq}}}=\frac{3\mu}{\sigma_{eq}^{e}}s_{pq}^{e}$$

**Calculation of** $\frac{\partial \Delta p}{\partial \varepsilon}$

$$\boxed{\frac{\partial \Delta p}{\partial \varepsilon_{\mathrm{pq}}}=-\frac{1}{T(\Delta p)}.\left(\frac{3\mu}{\sigma_{eq}^{e}}s_{pq}^{e}+3AK\delta_{pq}\right)}$$

with:

$$T(\Delta p)=\begin{cases}-\left(3\mu+9KA^{2}\right) & \text{dans le cas } p^{-}+\Delta p\geq p_{ultm} \quad (\text{écrouissage linéaire ou parabolique})\\ -\left(3\mu+9KA^{2}+h\right) & \text{dans le cas } p^{-}+\Delta p< p_{ultm} \quad (\text{écrouissage linéaire})\\ B+2\mathrm{G}\Delta \mathrm{p} & \text{dans le cas } p^{-}+\Delta p< p_{ultm} \quad (\text{écrouissage parabolique})\end{cases}$$

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where $B$ and $G$ have the same expression as in the paragraph [§2.2].

**Complete expression**

$$\frac{\partial\boldsymbol{\sigma}}{\partial\boldsymbol{\varepsilon}}=\left(1-\frac{3\mu}{\sigma_{eq}^{e}}\Delta p\right).\frac{\partial\mathbf{s}^{e}}{\partial\boldsymbol{\varepsilon}}+\left(\frac{3\mu}{\sigma_{eq}^{e}}\right)^{2}\left(\frac{\Delta p}{\sigma_{eq}^{e}}+\frac{1}{T}\right)\mathbf{s}^{e}\otimes\mathbf{s}^{e}+\frac{9\mu AK}{T\sigma_{eq}^{e}}(\mathbf{s}^{e}\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{s}^{e})+\left(K+\frac{9K^{2}A^{2}}{T}\right)\mathbf{I}\otimes\mathbf{I}$$

#### 2.2.3.2 Initial calculation of the tangent operator

One seeks has to express $\frac{\partial\sigma^{-}}{\partial\varepsilon^{-}}$. For that one will seek to calculate the tangent operator by a calculation of speed: $\frac{\partial\dot{\sigma}}{\partial\dot{\varepsilon}}$.

On the basis of the expression: $\dot{F}=\frac{\partial F}{\partial\dot{\boldsymbol{\sigma}}}+\frac{\partial F}{\partial p}\dot{p}=0$ it is shown that:

$$\dot{p}=\frac{3\mu}{\sigma_{eq}D}s.\dot{\varepsilon}+\frac{3AK}{D}\dot{\varepsilon}_{v}\quad\text{with}\quad D=3\mu+9KA^{2}+\frac{\partial R}{\partial p}$$

Expressions: $\dot{\boldsymbol{\sigma}}=\mathbf{H}\left(\dot{\boldsymbol{\varepsilon}}-\dot{\boldsymbol{\varepsilon}}^{p}\right)$ and $\dot{\boldsymbol{\varepsilon}}^{p}=\dot{p}\frac{\partial\mathbf{F}}{\partial\boldsymbol{\sigma}}$ it is shown then that:

$$\frac{\partial\dot{\boldsymbol{\sigma}}}{\partial\dot{\boldsymbol{\varepsilon}}}=H-\left(\frac{3\mu}{\sigma_{eq}}\boldsymbol{s}+3AK\boldsymbol{I}\right)\frac{\partial\Delta p}{\partial\boldsymbol{\varepsilon}}$$

who is not other than the form of the coherent matrix of the total system of the preceding paragraph where $\Delta p=0$.

## 2.3 Formulation in non-aligned version

The non-aligned version of the law Drucker-Prager introduced into Code_Aster does not have as a claim to model a realistic physical behavior finely. The goal is to represent most simply possible physics (coarsely) realistic, in particular in the case of the soil mechanics for which the angle of dilatancy varies with the plastic deformation.

The plastic potential is thus different from the surface of load in this new formulation. Digital integration was introduced only for the expression of behaviour with parabolic work hardening.

The plastic potential is the following: $G(\boldsymbol{\sigma},p)=\sigma_{eq}+\beta(p)I_{1}$

where $\beta(p)$ is a function which decrease linearly with the evolution of the plastic deformation according to the relation

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$$\beta(p)=\begin{cases}\beta(\psi^0)\left(1-\dfrac{p}{p_{ult}}\right) & \text{si } p\in[0,p_{ult}]\\ 0 & \text{si } p>p_{ult}\end{cases}$$

where $\psi^0$ indicate the initial angle of dilatancy and $\beta(\psi^0)=\dfrac{2\sin(\psi^0)}{3-\sin(\psi^0)}$ .

The plastic flow is written now

$$d\,\varepsilon_{ij}^p=dp\frac{\partial G(\boldsymbol{\sigma},p)}{\partial\sigma_{ij}}$$

knowing that one always has the criterion defining the surface of load: $F(\sigma,p)=\sigma_{eq}+AI_1-R(p)\le 0$

### 2.3.1 Analytical resolution

Method of resolution being similar to that of the chapter 2.2.2 one points out below only the expressions of the new equations

$$\begin{cases}\Delta e_{ij}^p = \dfrac{3}{2}\dfrac{s_{ij}}{\sigma_{eq}}\Delta p\\ \Delta\varepsilon_V^p = 3\beta(p)\Delta p\end{cases}$$

$$\begin{cases}s_{ij}=s_{ij}^e\left(1-3\mu\dfrac{\Delta p}{\sigma_{eq}^e}\right)\\ I_1=I_1^e-9K\beta(p)\Delta p\end{cases}$$

#### 2.3.1.1 Case where $p^- > p_{ult}$

$$\Delta p=\frac{F(\sigma^e,p^-)}{3\mu}$$

#### 2.3.1.2 Case where $p^-\le p_{ult}$

In this case $\Delta p$ is solution of a polynomial equation of the second order of which the roots will depend on the increment of deformation and the data characterizing the parameters materials. The polynomial in question is the following

$$F(\sigma^e,p^-)+C^1\Delta p+C^2\Delta p^2=0$$

where $F(\sigma^e,p^-)>0$ , and two constants $C^1$ and $C^2$ are defined by

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$$C^1 = -3\mu - 9KA\beta(p^-) + 2\frac{\sigma_Y}{p_{ult}}\left(1 - \left(1 - \sqrt{\frac{\sigma_{Yult}}{\sigma_Y}}\right)\frac{p^-}{p_{ult}}\right)\left(1 - \sqrt{\frac{\sigma_{Yult}}{\sigma_Y}}\right)$$

$$C^2 = -\frac{\sigma_Y}{p_{ult}^2}\left(1 - \sqrt{\frac{\sigma_{Yult}}{\sigma_Y}}\right)^2 + 9AK\frac{\beta(\psi^0)}{p_{ult}}$$

The root $\Delta p$ is then characterized according to the following code:

1/ if $C^2 < 0$ then $\Delta p = \frac{-C^1 - \sqrt{(C^1)^2 - 4F(\boldsymbol{\sigma}^e, p^-)C^2}}{2C^2}$

2/ if $C^2 > 0$ and $F(\boldsymbol{\sigma}^e, p^-) > \frac{(C^1)^2}{4C^2}$ then there is no solution. A recutting of the step of time is possible if the request were made in the order `STAT_NON_LINE`.

3/ if $C^2 > 0$ and $F(\boldsymbol{\sigma}^e, p^-) < \frac{(C^1)^2}{4C^2}$ and $C^1 < 0$ then the polynomial admits two solutions. One chooses smallest positive of them. $\Delta p = \frac{-C^1 - \sqrt{(C^1)^2 - 4F(\boldsymbol{\sigma}^e, p^-)C^2}}{2C^2}$

4/ if $C^2 > 0$ and $F(\boldsymbol{\sigma}^e, p^-) < \frac{(C^1)^2}{4C^2}$ and $C^1 > 0$ then there is no solution. A recutting of the step of time is possible if the request were made in the order `STAT_NON_LINE`.

### 2.3.2 Calculation of the tangent operator

The formulation is modified very little compared to the associated case: equations 2.2.3-1 become:

$$\begin{cases} \frac{\partial s}{\partial \varepsilon} = \frac{\partial s^e}{\partial \varepsilon}\left(1 - \frac{3\mu}{\sigma_{eq}^e}.\Delta p\right) + \frac{3\mu}{(\sigma_{eq}^e)^2}.\Delta p.\left(s^e \otimes \frac{\partial \sigma_{eq}^e}{\partial \varepsilon}\right) - \frac{3\mu}{\sigma_{eq}^e}.\left(s^e \otimes \frac{\partial \Delta p}{\partial \varepsilon}\right) \\ \frac{\partial I_1}{\partial \varepsilon} = \frac{\partial I_1^e}{\partial \varepsilon} - 9\mathrm{K}\left(\beta - \frac{\beta(\Psi^0)\Delta p}{p_{ult}}\right)\frac{\partial \Delta p}{\partial \varepsilon} \end{cases}$$

#### 2.3.2.1 Expression of $\frac{\partial s_{ij}^e}{\partial \varepsilon_{pq}}$

$$\frac{\partial s_{ij}^e}{\partial \varepsilon_{pq}} = 2\mu\left(\delta_{ip}\delta_{jq} - \frac{1}{3}\delta_{ij}\delta_{pq}\right)$$

#### 2.3.2.2 Expression of $\frac{\partial I_1^e}{\partial \varepsilon_{pq}}$

$$\frac{\partial I_1^e}{\partial \varepsilon_{pq}} = 3K\delta_{pq}$$

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#### 2.3.2.3 Calculation of $\frac{\partial \sigma_{eq}^e}{\partial \varepsilon_{pq}}$

$$\frac{\partial\sigma_{eq}^e}{\partial\varepsilon_{pq}}=\frac{3\mu}{\sigma_{eq}^e}s_{pq}^e$$

#### 2.3.2.4 Calculation of $\frac{\partial \Delta p}{\partial \varepsilon_{pq}}$

$$\frac{\partial \Delta p}{\partial \varepsilon_{pq}}=-\frac{1}{T(\Delta p)}.\left(\frac{3\mu}{\sigma_{eq}^e}s_{pq}^e+3\,AK\delta_{pq}\right)$$

with:

$$T(\Delta p)=\begin{cases}-3\mu & \text{si } p^-+\Delta p\geq p_{ult}\\ C^1+2C^2\,\Delta p & \text{si } p^-+\Delta p<p_{ult}\end{cases}$$

where $C^1$ and $C^2$ are constants defined in the paragraph 2.3.

#### 2.3.2.5 Complete expression

$$\frac{\partial\boldsymbol{\sigma}}{\partial\boldsymbol{\varepsilon}}=\frac{\partial\mathbf{s}}{\partial\boldsymbol{\varepsilon}}+\frac{1}{3}.\mathbf{I}\otimes\frac{\partial I_1}{\partial\boldsymbol{\varepsilon}}$$

$$\begin{cases}\frac{\partial\mathbf{s}}{\partial\boldsymbol{\varepsilon}}=\frac{\partial\mathbf{s^e}}{\partial\boldsymbol{\varepsilon}}\left(1-\frac{3\mu}{\sigma_{eq}^e}.\Delta p\right)+\frac{3\mu}{\left(\sigma_{eq}^e\right)^2}.\Delta p.\left(\mathbf{s^e}\otimes\frac{\partial\sigma_{eq}^e}{\partial\boldsymbol{\varepsilon}}\right)-\frac{3\mu}{\sigma_{eq}^e}.\left(\mathbf{s^e}\otimes\frac{\partial\Delta p}{\partial\boldsymbol{\varepsilon}}\right)\\ \frac{\partial I_1}{\partial\boldsymbol{\varepsilon}}=\frac{\partial I_1^e}{\partial\boldsymbol{\varepsilon}}-9K\left(\beta-\frac{\beta(\Psi^0)\Delta p}{p_{ult}}\right)\frac{\partial\Delta p}{\partial\boldsymbol{\varepsilon}}\end{cases}$$

$$\frac{\partial\sigma_{ij}}{\partial\varepsilon_{pq}}=\left(1-\frac{3\mu}{\sigma_{eq}^e}\Delta p\right).\frac{\partial s_{ij}^e}{\partial\varepsilon_{pq}}+\frac{1}{3}\frac{\partial I_1^e}{\partial\varepsilon_{pq}}\delta_{ij}+\frac{\partial\sigma_{eq}^e}{\partial\varepsilon_{pq}}\left(\frac{3\mu}{\left(\sigma_{eq}^e\right)^2}s_{ij}^e\Delta p\right)+\frac{\partial\Delta p}{\partial\varepsilon_{pq}}\left(-3\mu\frac{s_{ij}^e}{\sigma_{eq}^e}-3K\beta(p)\delta_{ij}+3K\frac{\beta(\psi^0)}{p_{ult}}\Delta p\delta_{ij}\right)$$

## 2.4 Internal variables of the Drucker-Prager laws associated and nonassociated

These models comprise 3 internal variables:

- $V1$ is the cumulated plastic deviatoric deformation $p$
- $V2$ is the cumulated plastic voluminal deformation $\sum\Delta\varepsilon_V^p$
- $V3$ is the indicator of state (1 if $\Delta p>0$ , 0 in the contrary case).

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# 3 Indicator of localization of Rice for the law Drucker-Prager

One defines the indicator of localization of the criterion of Rice within the framework of the law of Drucker-Prager behavior. But the definition of an indicator of localization perhaps used, a more general way, in studies in breaking process, mechanics of the damage, theory of the junction, soil mechanics and rock mechanics (and overall within the framework of materials with lenitive law of behavior).

This definite indicator a state from which the evolution of the studied mechanical system (equations, of balance, law of behavior) can lose its character of unicity. This theory allows, in other words:

1. the calculation of the possible state of initiation of the localization which is perceived like the limit of validity of calculations by classical finite elements;
2. “qualitative” determination of the angles of orientation of the zones of localization.

The criterion of localization constitutes a limit of reliability of calculations by finite elements “classical”.

## 3.1 Various ways of studying the localization

Within the framework of the studies conducted in soil mechanics, one noted a strong dependence of the digital solution according to the discretization by finite elements. It appears a concentration of high values of the plastic deformations cumulated on the level of the finite elements and it is noted that this “zone of localization” changes brutally with the refinement of the grid. This phenomenon of localization is source of digital problems and generates problems of convergences within the meaning of the finite elements.

The localization can be interpreted like an unstable, precursory phenomenon of mechanism of rupture, characterizing certain types of materials requested in the inelastic field. To study instabilities related to the localization one distinguishes, on the one hand, the material classes with behavior depend on time and on the other hand, those not depending on time. For materials with behavior independent of time, the approach commonly used is the method called by junction (it is with this method that one is interested in this note). It consists in analyzing the losses of unicity of the problem in speeds. For materials with behavior depend on time, the unicity of the problem in speeds is often guaranteed and this does not prevent the observation of instabilities at the time of their deformation. For these materials, one must then resort to other approaches. Most usually used is the approach by disturbance. This approach will not be treated in this note, but for more information to consult the notes [bib1], [bib2].

Rudnicki and Rice [bib3] showed that the study of the localization of the deformations in rock mechanics lies within the scope of the theory of the junction. This one is based on the concept of balance unstable. Rice [bib4] considers that the point of junction marks the end of the stable mode. The beginning of the localization is associated with a rheological instability of the system and this instability corresponds locally to the loss of ellipticity of the equations which control continuous incremental balance in speeds. Rice thus proposes a criterion known as of “junction by localization” which makes it possible to detect the state from which, the solution of the mathematical equations which control the problem in extreme cases considered and the evolution of the studied mechanical system (equations, of balance, law of behavior) lose their character of unicity. This theory allows the calculation of the state of initiation of the localization which is perceived like the limit of validity of calculations by classical finite elements.

## 3.2 Theoretical approach

### 3.2.1 Writing of the problem of speed

One considers a structure occupying, at one moment $t$, the open one $\Omega$ of $\Re^3$. The problem of speed consists in finding the field rates of travel $v$ when the structure is subjected at the voluminal speeds of forces $\dot{f}_d$, at the rates of imposed travel $v_d$ on a part $\partial_1\Omega$ border and at the surface speeds of efforts $\dot{F}_d$ on the complementary part $\partial_2\Omega$.
In the local writing of the problem, the field rates of travel $v$ must thus check the problem:


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1 $v$ sufficient regular and $v=v_d$ on $\partial_1\Omega$
2 Equilibrium equations:

$$\mathrm{div}[\mathbf{L}:\boldsymbol{\varepsilon}(v)]+\dot{\mathbf{f}}_d=0 \text{ on } \Omega$$

$$\mathbf{L}:\boldsymbol{\varepsilon}(v).\mathbf{n}=\dot{\mathbf{F}}_d \text{ on } \partial_2\Omega$$

$\mathbf{n}$ being the outgoing unit normal with $\partial_2\Omega$ .

•Conditions of compatibility (one limits oneself here to the small disturbances):

$$\boldsymbol{\varepsilon}(v)=\frac{1}{2}\left[\nabla\mathbf{v}+(\nabla\mathbf{v})^T\right]$$

where the operator $\mathbf{L}$ is defined in a general way for the laws of behavior written in incremental form by the relation:

$$\dot{\boldsymbol{\sigma}}=\mathbf{L}(\boldsymbol{\varepsilon},\mathbf{V}):\dot{\varepsilon}$$

with:

$$\mathbf{L}=\begin{cases}\mathbf{E} & \text{si } F<0 \text{ ou } F=0 \text{ et } \dfrac{\mathbf{b}:\mathbf{E}:\dot{\boldsymbol{\varepsilon}}}{h}\leq 0\\ \mathbf{H}=\quad \mathbf{E}-\dfrac{(\mathbf{E}:\mathbf{a})\otimes(\mathbf{b}:\mathbf{E})}{h} & \text{si } F=0 \text{ et } \dfrac{\mathbf{b}:\mathbf{E}:\dot{\boldsymbol{\varepsilon}}}{h}>0\end{cases}$$

where $\boldsymbol{\sigma}$ is the constraint, $\boldsymbol{\varepsilon}$ total deflection, $\mathbf{V}$ a set of internal variables and $F$ surface threshold of plasticity. Expressions of $a, b, E$ and H depend on the formulation of the law of behavior.

### 3.2.2 Results of existence and unicity, Loss of ellipticity

We give in this chapter some results without demonstrations. The reference for these demonstrations however is specified.

A sufficient condition of existence and unicity of the preceding problem is: $\dot{\sigma}:\dot{\varepsilon}>0$ . This inequality can be interpreted like a definition, in the three-dimensional case, of not-softening. The demonstration is made by Hill [bib5] for standard materials and by Benallal [bib1] for the materials not-standards.

The loss of ellipticity corresponds to the moment for which the operator $\mathbf{N.H.N}$ becomes singular for a direction $\mathbf{N}$ in a point of the structure. This condition is equivalent to the condition: $\det(\mathbf{N.H.N})=0$ . It is the condition of "junction continues"[1] within the meaning of Rice also called acoustic tensor. Rice and Rudnicki [bib3] show that this condition of loss of ellipticity of the local problem speed is a requirement with the "continuous or discontinuous" junction[2] for the solid. The boundary conditions do not play any part, only the law of behavior defines the conditions of localization (threshold of localization and orientation of the surface of localization.

The continuous junctions thus provide the lower limit of the range of deformation for which the discontinuous junctions can occur.

### 3.2.3 Analytical resolution for the case 2d.

One poses $\mathrm{N}=(\mathrm{N}_1,\mathrm{N}_2,0)$ with $\mathrm{N}_1^2+\mathrm{N}_2^2=1$

1 In a junction continues, a plastic deformation occurs inside and outside the zone of localization and one has the same law of behavior inside and outside the band.

2 In a discontinuous junction, there is on both sides of the band a continuity of displacement but there is not the same behavior. An elastic discharge occurs with external of the zone of localization, while a loading and an elastoplastic deformation continue occur inside.

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One has then: $\mathbf{N.H.N}=\begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & C \end{bmatrix}$ where Ortiz [bib6] shows that:

$$C=N_1^2 H_{1313}+N_2^2 H_{2323}>0$$
$$A_{11}=N_1^2 H_{1111}+N_1 N_2 (H_{1112}+H_{1211})+N_2^2 H_{1212}$$
$$A_{22}=N_1^2 H_{1212}+N_1 N_2 (H_{1222}+H_{2212})+N_2^2 H_{2222}$$
$$A_{12}=N_1^2 H_{1112}+N_1 N_2 (H_{1122}+H_{1212})+N_2^2 H_{1222}$$
$$A_{21}=N_1^2 H_{1211}+N_1 N_2 (H_{1212}+H_{2211})+N_2^2 H_{2212}$$

It is thus enough to study the sign of $\det(A)$ as specified by Doghri [bib7]:

$$\det(A)=a_0 N_1^4+a_1 N_1^3 N_2+a_2 N_1^2 N_2^2+a_3 N_1 N_2^3+a_4 N_2^4$$

with:

$$a_0=H_{1111} H_{1212}-H_{1112} H_{1211}$$
$$a_1=H_{1111}(H_{1222}+H_{2212})-H_{1112} H_{2211}-H_{1122} H_{1211}$$
$$a_2=H_{1111} H_{2222}+H_{1112} H_{1222}+H_{1211} H_{2212}-H_{1122} H_{1212}-H_{1122} H_{2211}-H_{1212} H_{2211}$$
$$a_3=H_{2222}(H_{1112}+H_{1211})-H_{1122} H_{2212}-H_{1222} H_{2211}$$
$$a_4=H_{1212} H_{2222}-H_{1222} H_{2212}$$

One poses then $N_1=\cos\theta$ and $N_2=\sin\theta$ with $\theta\in]-\frac{\pi}{2};\ +\frac{\pi}{2}]$. Two cases then are distinguished:

- if $\theta=+\frac{\pi}{2}$ then $\det(A)=0$ if $a_4=0$ ;
- if $\theta\neq+\frac{\pi}{2}$ then $\det(A)=0$ if $f(x)=a_4 x^4+a_3 x^3+a_2 x^2+a_1 x+a_0=0$ with $x=\tan\theta$ .

### 3.2.4 Calculation of the roots

To solve a polynomial of degree N (like that definite above, where n=4) one proposes to use the method known as "Companion Matrix Polynomial". The principle of this method consists in seeking the eigenvalues of the matrix (of Hessenberg type) of order N associated with the polynomial. If the polynomial is considered $P(x)=x^n+a_{n-1} x^{n-1}+\ldots+a_k x^k+\ldots+a_1 x+a_0$ . To seek the roots of this polynomial amounts seeking the eigenvalues of the matrix:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -a_0 \\ 1 & 0 & 0 & 0 & 0 & -a_1 \\ 0 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 1 & 0 & 0 & -a_k \\ 0 & 0 & 0 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 1 & -a_{n-1} \end{bmatrix}$$

This indicator is calculated by the option `INDL_ELGA` of `CALC_CHAMP` [U4.81.04]. It produces in each point of integration 5 components: the first is the indicator of localization being worth 0 if

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$det(N.H.N) > 0$ (not of localization), and being worth 1 if not, which corresponds has a possibility of localization. The other components provide the directions of localization.

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# 4 Calculations of sensitivity

The analysis of sensitivity relates only to the version associated with the formulation described with the chapter 2.2.

## 4.1 Sensitivity to the data materials

### 4.1.1 The direct problem

We place ourselves in this part within the framework of the resolution of non-linear calculations.
In *Code_Aster*, any non-linear static calculation is solved incrémentalement. It thus requires with each step of load $i \in \{1, I\}$ the resolution of the système of non-linear equation:

$$\begin{cases} R(u_i, t_i) + B^t \lambda_i = L_i \\ \mathbf{Bu}_i = u_i^d \end{cases}$$

**éq 4.1.1-1**

with

$$(\mathbf{R}(\mathbf{u_i}, t_i))_k = \int_\Omega \boldsymbol{\sigma}(\mathbf{u_i}) : \boldsymbol{\varepsilon}(\mathbf{w_k}) \, d\Omega$$

**éq 4.1.1-2**

- $\mathbf{w_k}$ is the function of form of $k$ ième degree of freedom of the modelled structure,
- $(\mathbf{R}(\mathbf{u_i}, t_i))$ is the vector of the nodal forces.

The resolution of this system is done by the method of Newton-Raphson:

$$\begin{cases} \mathbf{K}_i^n \delta\mathbf{u}_i^{n+1} + \mathbf{B}^t \delta\lambda_i^{n+1} = \mathbf{L}_i - \mathbf{R}(\mathbf{u}_i^n, t_i) + \mathbf{B}^t \lambda_i^n \\ \mathbf{B}\delta u_i^{n+1} = -\mathbf{B}u_{i-1}^n \end{cases}$$

**éq 4.1.1-3**

where $\mathbf{K_i^n} = \frac{\partial \mathbf{R}}{\partial \mathbf{u}}\Big|_{(u_i^n, t_i)}$ is the tangent matrix with the step of load $i$ and with the iteration of Newton $n$ .

The solution is thus given by:

$$\begin{cases} \mathbf{u}_i = \mathbf{u}_{i-1} + \sum_{n=0}^{N} \delta\mathbf{u}_i^n \\ \lambda_i = \lambda_{i-1} + \sum_{n=0}^{N} \delta\lambda_i^n \end{cases}$$

**éq 4.1.1-4**

with $N$ , the iteration count of Newton which was necessary to reach convergence.

### 4.1.2 Derived calculation

#### 4.1.2.1 Preliminaries

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Within the framework of the calculation of sensitivity, it is necessary to insist on the dependences of a size compared to the others. We will thus clarify that the results of preceding calculation depend of a parameter $\Phi$ given (elastic limit, Young modulus, density,…) and that in the following way:

$$u_i = u_i(\Phi)\ ,\ \lambda_i = \lambda_i(\Phi)\ .$$

But that is not sufficient. Also we place ourselves within the framework of an incremental calculation with law of behavior of the Drucker-Prager type. If one considers the interdependences of the parameters on an algorithmic level, one can write:

$$\mathbf{R} = \mathbf{R}(\boldsymbol{\sigma}_{\mathbf{i-1}}(\Phi),\ p_{i-1}(\Phi), \Delta\mathbf{u}(\Phi))$$

$$\boldsymbol{\sigma}_{\mathbf{i}} = \boldsymbol{\sigma}_{\mathbf{i-1}}(\Phi) + \Delta\boldsymbol{\sigma}(\sigma_{i-1}(\Phi),\ p_{i-1}(\Phi),\ \Delta\mathbf{u}(\Phi),\ \Phi)$$

$$p_i = p_{i-1}(\Phi) + \Delta p(\boldsymbol{\sigma}_{\mathbf{i-1}}(\Phi),\ p_{i-1}(\Phi),\ \Delta\mathbf{u}(\Phi),\ \Phi)$$

Where $\Delta\mathbf{u}$ is the increment of displacement to convergence with the step of load $i$ .

Let us specify the direction of the notations which we will use for the derivative:

- $\dfrac{\partial X}{\partial Y}$ indicate the partial derivative **explicit** of $X$ compared to $Y$ ,
- $X_{,Y}$ indicate the variation **total** of $X$ compared to $Y$ .

#### 4.1.2.2 Derivation of balance

Taking into account the preceding remarks, let us express the total variation of [éq 2.1-1] compared to $\Phi$ :

$$\begin{cases} \dfrac{\partial\mathbf{R}}{\partial\Phi} + \dfrac{\partial\mathbf{R}}{\partial\Delta\mathbf{u}}\cdot\Delta\mathbf{u}_{,\Phi} + \dfrac{\partial\mathbf{R}}{\partial\boldsymbol{\sigma}_{i-1}}\cdot\boldsymbol{\sigma}_{i-1,\Phi} + \dfrac{\partial\mathbf{R}}{\partial p_{i-1}}\cdot p_{i-1,\Phi} + \mathbf{B}^t\boldsymbol{\lambda}_{i,\Phi} & = & 0 \\ \mathbf{B}\Delta\mathbf{u}_{,\Phi} & = & -\mathbf{B}\mathbf{u}_{i-1,\Phi} \end{cases} \quad \textbf{éq 4.1.2.2 - 1}$$

Let us notice that here $\dfrac{\partial\mathbf{R}}{\partial\Phi}=0$ : $\mathbf{R}$ does not depend explicitly on $\Phi$ but implicitly as we will see it in detail in the continuation.

That is to say:

$$\begin{cases} \mathbf{K}_i^N\Delta\mathbf{u}_{,\Phi} + \mathbf{B}^t\boldsymbol{\lambda}_{i,\Phi} & = & -\mathbf{R}_{,\Phi}\big|_{\Delta\mathbf{u}\neq\Delta\mathbf{u}(\Phi)} \\ \mathbf{B}\Delta\mathbf{u}_{,\Phi} & = & -\mathbf{B}\mathbf{u}_{i-1,\Phi} \end{cases} \quad \textbf{éq 4.1.2.2 - 2}$$

Where

- $\mathbf{K}_\mathbf{i}^\mathbf{N}$ is the last tangent matrix used to reach convergence in the iterations of Newton,
- $\mathbf{R}_{,\Phi}\big|_{\Delta u\neq\Delta u(\Phi)}$ is the total variation of $\mathbf{R}$ , without taking account of the dependence of $\Delta\mathbf{u}$ compared to $\boldsymbol{\Phi}$ .

The problem lies now in the calculation of $\mathbf{R}_{,\Phi}\big|_{\Delta u\neq\Delta u(\Phi)}$ .

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**Note:**

*In [éq 4.1.2.2 - 2], one used the fact that* $\mathbf{K}_{\mathbf{i}}^{\mathbf{N}}=\frac{\partial \mathbf{R}(\mathbf{u}_{\mathbf{i}}, t_i)}{\partial \Delta \mathbf{u}}$ *whereas in [éq 4.1.1-3] one defined it by* $\mathbf{K}_{\mathbf{i}}^{\mathbf{N}}=\frac{\partial \mathbf{R}(\mathbf{u}_{\mathbf{i}}, t_i)}{\partial \mathbf{u}_{\mathbf{i}}^{\mathbf{N}}}$ *. There is well equivalence of these two definitions in measurement where* $\mathbf{u}_i = \mathbf{u}_{i-1} + \Delta\mathbf{u}$ *and that* $\mathbf{R}$ *depends indeed on* $\Delta \mathbf{u}$ *(and as well sure of* $\boldsymbol{\sigma}_{\mathbf{i-1}}$ *and* $p_{i-1}$ *).*

**Note:**

*If one derives compared to* $\Phi$ *directly [éq 4.1.1-3], one finds* $\mathbf{K}^n \frac{\partial \mathbf{u}^{n+1}}{\partial \Phi} + \mathbf{B}^t \boldsymbol{\lambda},_\Phi = -\mathbf{R},_{\Phi/\Delta\mathbf{u}\neq\Delta\mathbf{u}/\Phi} - \mathbf{K}^n,_\Phi \delta\mathbf{u}^{n+1}$ *. What is the same thing* ***with convergence*** *and reveals that the error on* $\frac{\partial \mathbf{u}}{\partial \Phi}$ *depends on* $\mathbf{K}^{-1}\mathbf{K},_\Phi$ *.*

#### 4.1.2.3 Calculation of the derivative of the law of behavior

In the continuation, by preoccupation with a clearness, we will give up the indices $i-1$ .
According to [éq 4.1.1-2], one can rewrite $\mathbf{R},_\Phi|_{\Delta u\neq \Delta u(\Phi)}$ in the form:

$$\mathbf{R},_\Phi|_{\Delta u\neq\Delta u(\Phi)}=\int_\Omega \left(\boldsymbol{\sigma},_\Phi+\Delta\boldsymbol{\sigma},_\Phi|_{\Delta\mathbf{u}\neq\Delta\mathbf{u}(\Phi)}\right):\boldsymbol{\varepsilon}(\mathbf{w}_{\mathbf{k}})\,d\Omega \quad \textbf{éq 4.1.2.3 - 1}$$

One must thus calculate $\Delta\boldsymbol{\sigma},_\Phi|_{\Delta\mathbf{u}\neq\Delta\mathbf{u}(\Phi)}$ . With this intention, we will use the expressions which intervene in the digital integration of the law of behavior.

#### 4.1.2.4 Case of linear elasticity

Within the framework of linear elasticity, the law of behavior is expressed by:

$$\begin{cases}\Delta\tilde{\boldsymbol{\sigma}}=2\mu.\tilde{\boldsymbol{\varepsilon}}(\Delta\mathbf{u})\\ Tr(\Delta\boldsymbol{\sigma})=3\mathrm{K}.Tr(\boldsymbol{\varepsilon}(\Delta\mathbf{u}))\end{cases}$$

or:

$$\Delta\boldsymbol{\sigma}=2\mu.\tilde{\boldsymbol{\varepsilon}}(\Delta\mathbf{u})+K.Tr(\boldsymbol{\varepsilon}(\Delta\mathbf{u})).\mathbf{Id} \quad \textbf{éq 4.1.2.4 - 1}$$

where $\mathbf{Id}$ is the tensor identity of order 2.

Then, by calculating the total variation of [éq 4.1.2.4 - 1] compared to $\Phi$ , one obtains:

$$\Delta\boldsymbol{\sigma},_\Phi=2\mu,_\Phi.\tilde{\boldsymbol{\varepsilon}}(\Delta\mathbf{u})+K,_\Phi.Tr(\boldsymbol{\varepsilon}(\Delta\mathbf{u})).\mathbf{Id}+2\mu.\tilde{\boldsymbol{\varepsilon}}(\Delta\mathbf{u},_\Phi)+K.Tr(\boldsymbol{\varepsilon}(\Delta\mathbf{u},_\Phi)).\mathbf{Id} \quad \textbf{éq 4.1.2.4 - 2}$$

That is to say:

$$\Delta\boldsymbol{\sigma},_\Phi|_{\Delta\mathbf{u}\neq\Delta\mathbf{u}(\Phi)}=2\mu,_\Phi.\tilde{\boldsymbol{\varepsilon}}(\Delta\mathbf{u})+K,_\Phi.Tr(\boldsymbol{\varepsilon}(\Delta\mathbf{u})).\mathbf{Id} \quad \textbf{éq 4.1.2.4 - 3}$$

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#### 4.1.2.5 Case of the elastoplasticity of the Drucker-Prager type

The law of behavior of the Drucker-Prager type is written:

$$\begin{cases} \boldsymbol{\varepsilon}(\Delta\mathbf{u}) - \mathbf{S}:\boldsymbol{\sigma} = \dfrac{3}{2}\cdot\Delta p\cdot\dfrac{\tilde{\boldsymbol{\sigma}}+\Delta\tilde{\boldsymbol{\sigma}}}{(\boldsymbol{\sigma}+\Delta\boldsymbol{\sigma})_{eq}} \\ (\boldsymbol{\sigma}+\Delta\boldsymbol{\sigma})_{eq} + A\cdot Tr(\boldsymbol{\sigma}+\Delta\boldsymbol{\sigma}) \leq R(p+\Delta p) \end{cases}$$ éq 4.1.2.5 - 1

where $\mathbf{S}$ is the tensor of the elastic flexibilities and $R$ is the criterion of plasticity defined by:

in the case of a linear work hardening:

$$R(p)=h\cdot p+\sigma^y \text{ pour } 0\leq p\leq p_{ultm}$$
$$R(p)=h\cdot p_{ultm} \text{ pour } p\geq p_{ultm}$$

in the case of a parabolic work hardening:

$$R(p)=\sigma^y\cdot\left(1-\left(1-\sqrt{\frac{\sigma^y_{\text{ultm}}}{\sigma^y}}\right)\cdot\frac{p}{p_{\text{ultm}}}\right)^2 \text{ pour } 0\leq p\leq p_{ultm}$$
$$R(p)=\sigma^y_{ultm} \text{ pour } p\geq p_{ultm}$$

In digital terms, this law of behavior is integrated using an algorithm of radial return: one makes an elastic prediction (noted $\boldsymbol{\sigma}^{\mathbf{e}}$ ) that one corrects if the threshold is violated. One thus writes:

$$\begin{cases} \Delta\tilde{\boldsymbol{\sigma}} = 2\mu.\tilde{\boldsymbol{\varepsilon}}(\Delta\mathbf{u}) - 3\mu\cdot\dfrac{\Delta p}{\sigma^e_{eq}}\cdot\tilde{\boldsymbol{\sigma}}^e \\ Tr(\Delta\boldsymbol{\sigma}) = 3K\cdot Tr(\boldsymbol{\varepsilon}(\Delta\mathbf{u})) - 9K\cdot A\cdot\Delta p \\ \Delta p = \text{solution de } \sigma^e_{eq} - (3\mu+9K\cdot A^2)\cdot\Delta p + A\cdot Tr(\sigma^e) - R(p^- + \Delta p) = 0 \end{cases}$$ éq 4.1.2.5 - 2

We will distinguish two cases.

**1 er case** : $\Delta p=0$
What amounts saying that at the time of these step of load, the point of Gauss considered did not see an increase in its plasticization. One finds oneself then in the case of linear elasticity:

$$\Delta\boldsymbol{\sigma}_{,\Phi}\big|_{\Delta\mathbf{u}\neq\Delta\mathbf{u}(\Phi)} = 2\mu_{,\Phi}.\tilde{\boldsymbol{\varepsilon}}(\Delta\mathbf{u}) + K_{,\Phi}.Tr(\boldsymbol{\varepsilon}(\Delta\mathbf{u})).\mathbf{Id}$$ éq 4.1.2.5 - 3

**2 eme case** : $\Delta p>0$
Taking into account the dependences between variables in [éq 4.1.2.5 - 1], one can write:


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$$\begin{cases} \Delta\boldsymbol{\sigma},_{\Phi} = \dfrac{\partial\Delta\boldsymbol{\sigma}}{\partial\Phi} + \dfrac{\partial\Delta\boldsymbol{\sigma}}{\partial\boldsymbol{\sigma}}\cdot\boldsymbol{\sigma},_{\Phi} + \dfrac{\partial\Delta\boldsymbol{\sigma}}{\partial p}\cdot p,_{\Phi} + \dfrac{\partial\Delta\boldsymbol{\sigma}}{\partial\boldsymbol{\varepsilon}(\Delta\mathbf{u})}\cdot\boldsymbol{\varepsilon}(\Delta\mathbf{u},_{\Phi}) \\ \Delta p,_{\Phi} = \dfrac{\partial\Delta p}{\partial\Phi} + \dfrac{\partial\Delta p}{\partial\boldsymbol{\sigma}}\cdot\boldsymbol{\sigma},_{\Phi} + \dfrac{\partial\Delta p}{\partial p}\cdot p,_{\Phi} + \dfrac{\partial\Delta p}{\partial\boldsymbol{\varepsilon}(\Delta\mathbf{u})}\cdot\boldsymbol{\varepsilon}(\Delta\mathbf{u}),_{\Phi} \end{cases}$$

**éq 4.1.2.5 - 4**

Moreover, in agreement with the algorithmic integration of the law, we will separate parts deviatoric and hydrostatic.

$$\begin{cases} \Delta\boldsymbol{\sigma},_{\Phi}\Big|_{\Delta\mathbf{u}\neq\Delta\mathbf{u}(\Phi)} = \dfrac{\partial\Delta\tilde{\boldsymbol{\sigma}}}{\partial\Phi} + \dfrac{1}{3}\cdot\dfrac{\partial Tr(\Delta\boldsymbol{\sigma})}{\partial\Phi}\cdot\mathbf{Id} \\ \quad + \dfrac{\partial\Delta\tilde{\boldsymbol{\sigma}}}{\partial\boldsymbol{\sigma}}\cdot\boldsymbol{\sigma},_{\Phi} + \dfrac{1}{3}\cdot\dfrac{\partial Tr(\Delta\boldsymbol{\sigma})}{\partial\boldsymbol{\sigma}}\cdot\mathbf{Id}\cdot\boldsymbol{\sigma},_{\Phi} \\ \quad + \dfrac{\partial\Delta\tilde{\boldsymbol{\sigma}}}{\partial p}\cdot p,_{\Phi} + \dfrac{1}{3}\cdot\dfrac{\partial Tr(\Delta\boldsymbol{\sigma})}{\partial p}\cdot\mathbf{Id}\cdot p,_{\Phi} \\ \Delta p,_{\Phi}\Big|_{\Delta\mathbf{u}\neq\Delta\mathbf{u}(\Phi)} = \dfrac{\partial\Delta p}{\partial\Phi} + \dfrac{\partial\Delta p}{\partial\boldsymbol{\sigma}}\cdot\boldsymbol{\sigma},_{\Phi} + \dfrac{\partial\Delta p}{\partial p}\cdot p,_{\Phi} \end{cases}$$

**éq 4.1.2.5 - 5**

And thus, one calculates:

$$\boxed{\dfrac{\partial\Delta\boldsymbol{\sigma}}{\partial\Phi}}$$

$$\frac{\partial\Delta\tilde{\boldsymbol{\sigma}}}{\partial\Phi} = \frac{\partial 2\mu}{\partial\Phi}\cdot\tilde{\boldsymbol{\varepsilon}}(\Delta\mathbf{u}) - \frac{\partial 3\mu}{\partial\Phi}\cdot\frac{\Delta p}{\sigma^e_{eq}}\cdot\tilde{\boldsymbol{\sigma}}^e - 3\mu\cdot\frac{\frac{\partial\Delta p}{\partial\Phi}}{\sigma^e_{eq}}\cdot\tilde{\boldsymbol{\sigma}}^e + 3\mu\cdot\frac{\Delta p\cdot\frac{\partial\sigma^e_{eq}}{\partial\Phi}}{{\sigma^e_{eq}}^2}\cdot\tilde{\boldsymbol{\sigma}}^e - 3\mu\cdot\frac{\Delta p}{\sigma^e_{eq}}\cdot\frac{\partial\tilde{\boldsymbol{\sigma}}^e}{\partial\Phi}$$

$$\frac{\partial Tr(\Delta\boldsymbol{\sigma})}{\partial\Phi} = \frac{\partial 3K}{\partial\Phi}\cdot Tr(\boldsymbol{\varepsilon}(\Delta\mathbf{u})) - \frac{\partial 9K}{\partial\Phi}\cdot A\cdot\Delta p - 9K\cdot\frac{\partial A}{\partial\Phi}\cdot\Delta p - 9K\cdot A\cdot\frac{\partial\Delta p}{\partial\Phi}$$

$$\boxed{\dfrac{\partial\Delta\boldsymbol{\sigma}}{\partial\sigma}}$$

$$\frac{\partial\Delta\tilde{\boldsymbol{\sigma}}}{\partial\boldsymbol{\sigma}} = -3\mu\cdot\frac{\frac{\partial\Delta p}{\partial\sigma}}{\sigma^e_{eq}}\otimes\tilde{\boldsymbol{\sigma}}^e + 3\mu\cdot\frac{\Delta p}{{\sigma^e_{eq}}^2}\cdot\frac{\partial\sigma^e_{eq}}{\partial\boldsymbol{\sigma}}\otimes\tilde{\boldsymbol{\sigma}}^e - 3\mu\cdot\frac{\Delta p}{\sigma^e_{eq}}\cdot\mathbf{J}$$

where $\mathbf{J}$ is the operator deviatoric defined by: $\mathbf{J}:\boldsymbol{\sigma}=\tilde{\boldsymbol{\sigma}}$

$$\frac{\partial Tr(\Delta\boldsymbol{\sigma})}{\partial\boldsymbol{\sigma}} = -9K\cdot A\cdot\frac{\partial\Delta p}{\partial\boldsymbol{\sigma}}$$

$$\boxed{\dfrac{\partial\Delta\sigma}{\partial p}}$$

$$\frac{\partial\Delta\tilde{\boldsymbol{\sigma}}}{\partial p} = -\frac{3\mu}{\sigma^e_{eq}}\cdot\frac{\partial\Delta p}{\partial p}\cdot\tilde{\boldsymbol{\sigma}}^e$$

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$$\frac{\partial Tr(\Delta\boldsymbol{\sigma})}{\partial p} = -9 \cdot K \cdot \mathrm{A} \cdot \frac{\partial \Delta p}{\partial p}$$

$$\boxed{\Delta p_{,\Phi}}$$

The fact is used that: $(\boldsymbol{\sigma}+\Delta\boldsymbol{\sigma})_{eq} = (\boldsymbol{\sigma}+\Delta\boldsymbol{\sigma})^{e}_{eq} - 3\mu \cdot \Delta p$

$$\Delta p_{,\Phi} = \frac{1}{3\mu} \cdot ((\boldsymbol{\sigma}+\Delta\boldsymbol{\sigma})^{e}_{eq,\Phi} - (\boldsymbol{\sigma}+\Delta\boldsymbol{\sigma})_{eq,\Phi} - \frac{\partial 3\mu}{\partial \Phi} \cdot \Delta p)$$

**Note:**

*In these calculations were or must be used the following results:*

$$\frac{\partial \tilde{\boldsymbol{\sigma}}^{e}}{\partial \Phi} = \frac{\partial 2\mu}{\partial \Phi} \cdot \tilde{\boldsymbol{\varepsilon}}(\Delta\mathbf{u})$$

*Tensor of order 2*

$$\frac{\partial \sigma^{e}_{eq}}{\partial \Phi} = \frac{3}{2} \cdot \frac{(\frac{\partial 2\mu}{\partial \Phi} \cdot \tilde{\boldsymbol{\varepsilon}}(\Delta\mathbf{u})) : (\tilde{\boldsymbol{\sigma}} + 2\mu . \tilde{\boldsymbol{\varepsilon}}(\Delta\mathbf{u}))}{\sigma^{e}_{eq}}$$

*Scalar*

$$\frac{\partial \sigma^{e}_{eq}}{\partial \boldsymbol{\sigma}} = \frac{3}{2} \cdot \frac{\tilde{\boldsymbol{\sigma}}^{e}}{\sigma^{e}_{eq}}$$

*Tensor of order 2*

$$\frac{\partial \tilde{\boldsymbol{\sigma}}^{e}}{\partial \boldsymbol{\sigma}} = \mathbf{J}$$

*Tensor of order 4*

$$\frac{\partial Tr(\sigma^{e})}{\partial \boldsymbol{\sigma}} = Id$$

*Tensor of order 2*

$$\frac{\partial Tr(\sigma^{e})}{\partial \Phi} = \frac{\partial 3K}{\partial \Phi} \cdot Tr(\boldsymbol{\varepsilon}(\Delta\mathbf{u}))$$

*Scalar*

One must also calculate the derivative partial of the increment of plastic deformation cumulated compared to the parameters materials, the constraints and the cumulated plastic deformation (cf Annexes)

Those are obtained by deriving the equation solved to calculate the increment of plastic deformation cumulated during direct calculation.

#### 4.1.2.6 Calculation of the derivative of displacement

Once $\Delta\boldsymbol{\sigma}_{,\Phi}|_{\Delta\mathbf{u}\neq\Delta\mathbf{u}(\Phi)}$ calculated, one can constitute the second member $\mathbf{R}_{,\Phi}|_{\Delta\mathbf{u}\neq\Delta\mathbf{u}(\Phi)}$ while using [éq 4.1.2.3 - 1]. One then solves the system [éq 4.1.2.2 - 2] and one obtains the increment of derived displacement compared to $\Phi$ .

#### 4.1.2.7 Calculation of the derivative of the other sizes

Now that one has $\Delta\mathbf{u}_{,\Phi}$ , one must calculate the derivative of the other sizes. One separates two more cases:

**Linear elasticity**
According to [éq 4.1.2.5 - 1], one as follows calculates the derivative of the increment of constraint:

$$\Delta\boldsymbol{\sigma}_{,\Phi} = \Delta\boldsymbol{\sigma}_{,\Phi}|_{\Delta\mathbf{u}\neq\Delta\mathbf{u}(\Phi)} + 2\mu . \tilde{\boldsymbol{\varepsilon}}(\Delta\mathbf{u}_{,\Phi}) + K . Tr(\boldsymbol{\varepsilon}(\Delta\mathbf{u}_{,\Phi})) . \mathbf{Id}$$

The increment of cumulated plastic deformation, as for him, does not see evolution:

$$\Delta p_{,\Phi} = 0$$

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**Elastoplasticity of the Drucker-Prager type**

If $\Delta p = 0$ , the preceding case is found.
If not, one obtains according to [éq 4.1.2.5 - 2]:

$$\Delta \boldsymbol{\sigma}_{,\Phi} = \Delta \boldsymbol{\sigma}_{,\Phi}\big|_{\Delta \mathbf{u} \neq \Delta \mathbf{u}(\Phi)} + \frac{\partial \Delta \boldsymbol{\sigma}}{\partial \boldsymbol{\varepsilon}(\Delta \mathbf{u})} : \boldsymbol{\varepsilon}(\Delta \mathbf{u}_{,\Phi})$$

And for the cumulated plastic deformation, one uses the following relation:

$$(\boldsymbol{\sigma} + \Delta\boldsymbol{\sigma})_{eq} = (\boldsymbol{\sigma} + \Delta\boldsymbol{\sigma})^{e}_{eq} - 3\mu \cdot \Delta p$$

This one enables us to write that:

$$\Delta p_{,\Phi} = \frac{1}{3\mu} \cdot \left( (\boldsymbol{\sigma} + \Delta\boldsymbol{\sigma})^{e}_{eq,\Phi} - (\boldsymbol{\sigma} + \Delta\boldsymbol{\sigma})_{eq,\Phi} - \frac{\partial 3\mu}{\partial \Phi} \cdot \Delta p \right)$$

The significant equivalent constraints are calculated as follows:

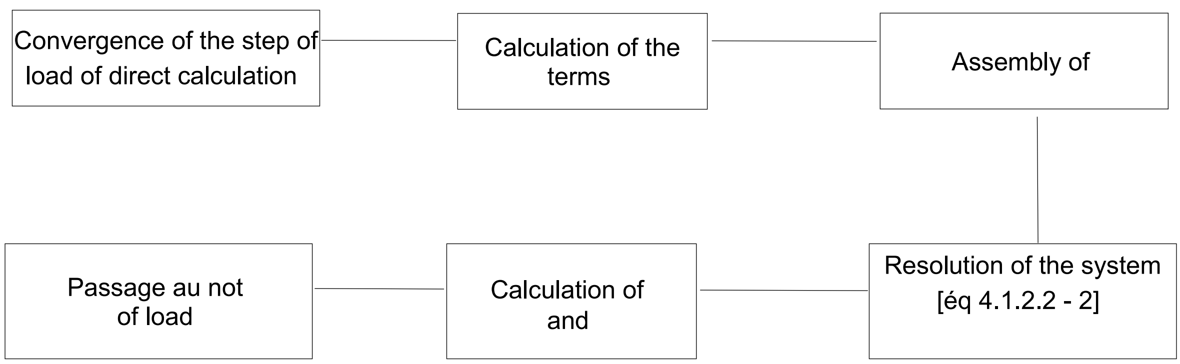
$$(\boldsymbol{\sigma} + \Delta\boldsymbol{\sigma})^{e}_{eq,\Phi} = \frac{3}{2(\boldsymbol{\sigma} + \Delta\boldsymbol{\sigma})^{e}_{eq}} \cdot \left(\tilde{\boldsymbol{\sigma}}_{,\Phi} + \frac{\partial 2\mu}{\partial \Phi} \cdot \tilde{\boldsymbol{\varepsilon}}(\Delta \mathbf{u}) + 2\mu \cdot \tilde{\boldsymbol{\varepsilon}}(\Delta \mathbf{u}_{,\Phi})\right) : \left(\tilde{\boldsymbol{\sigma}} + 2\mu \cdot \tilde{\boldsymbol{\varepsilon}}(\Delta \mathbf{u})\right)$$

$$(\boldsymbol{\sigma} + \Delta\boldsymbol{\sigma})_{eq,\Phi} = \frac{3}{2(\boldsymbol{\sigma} + \Delta\boldsymbol{\sigma})_{eq}} \cdot \left(\tilde{\boldsymbol{\sigma}}_{,\Phi} + \Delta\tilde{\boldsymbol{\sigma}}_{,\Phi}\right) : \left(\tilde{\boldsymbol{\sigma}} + \Delta\tilde{\boldsymbol{\sigma}}\right)$$

Once all these calculations are finished, all the derived sizes are reactualized and one passes to the step of load according to.

#### 4.1.2.8 Synthesis

To summarize the preceding paragraphs, one represents the various stages of calculation by the following diagram:




## 4.2 Sensitivity to the loading

The approach is here rather close to that of the preceding paragraph. We develop it nevertheless entirely in a preoccupation with a clearness, so that this paragraph can be read independently.

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### 4.2.1 The direct problem: expression of the loading

Until now we expressed the direct problem in the form:

$$\begin{cases} \mathbf{R}(\mathbf{u}_i, t_i) + \mathbf{B}^t \lambda_i & = \mathbf{L}_i \\ \mathbf{B}\mathbf{u}_i & = \mathbf{u}_i^d \end{cases}$$

**éq 4.2.1-1**

The loadings are gathered with the second member and understand the imposed forces $\mathbf{L_i}$ and imposed displacements $\mathbf{u_i^d}$ .

Let us suppose that the loading in imposed force $\mathbf{L_i}$ depends on a scalar parameter $\alpha$ in the following way:

$$\mathbf{L_i}(\alpha) = \mathbf{L_i^1} + \mathbf{L_i^2}(\alpha)$$

**éq 4.2.1-2**

Where

- $\mathbf{L_i^1}$ is a vector independent of $\alpha$ ,
- $\mathbf{L_i^2}$ depends linearly on $\alpha$ .

One wishes to calculate the sensitivity of the results of direct calculation to a variation of the parameter $\alpha$ .

### 4.2.2 The derived problem

#### 4.2.2.1 Derivation of balance

As in the preceding chapter, by taking account of the dependences between the various fields, one derives balance [éq 4.2.1-1] by report $\alpha$ :

$$\begin{cases} \dfrac{\partial \mathbf{R}}{\partial \alpha} + \dfrac{\partial \mathbf{R}}{\partial \Delta \mathbf{u}} \cdot \Delta \mathbf{u}_{,\alpha} + \dfrac{\partial \mathbf{R}}{\partial \boldsymbol{\sigma}_{i-1}} \cdot \boldsymbol{\sigma}_{i-1,\alpha} + \dfrac{\partial \mathbf{R}}{\partial p_{i-1}} \cdot p_{i-1,\alpha} + \mathbf{B}^t \boldsymbol{\lambda}_{i,\alpha} & = \mathbf{L}_i^2(1) \\ \mathbf{B}\Delta \mathbf{u}_{,\alpha} & = -\mathbf{B}\mathbf{u}_{i-1,\alpha} \end{cases}$$

**éq 4.2.2.1 - 1**

One used the fact that $\mathbf{L_i^2}$ depends linearly on $\alpha$ .

That is to say:

$$\begin{cases} \mathbf{K}_i^N \Delta \mathbf{u}_{,\alpha} + \mathbf{B}^t \boldsymbol{\lambda}_{i,\alpha} & = \mathbf{L}_i^2(1) - \mathbf{R}_{,\alpha}\Big|_{\Delta \mathbf{u} \neq \Delta \mathbf{u}(\alpha)} \\ \mathbf{B}\Delta \mathbf{u}_{,\alpha} & = -\mathbf{B}\mathbf{u}_{i-1,\alpha} \end{cases}$$

**éq 4.2.2.1 - 2**

Where

- $\mathbf{K_i^N}$ is the last tangent matrix used to reach convergence in the iterations of Newton,

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- $\mathbf{R}{,_\alpha}|_{\Delta\mathbf{u}\neq\Delta\mathbf{u}(\alpha)}$ is the total variation of $\mathbf{R}$, without taking account of the dependence of $\Delta\mathbf{u}$ compared to $\alpha$.

The problem lies like previously in the calculation of $\mathbf{R}{,_\alpha}|_{\Delta\mathbf{u}\neq\Delta\mathbf{u}(\alpha)}$.

#### 4.2.2.2 Calculation of the derivative of the law of behavior

According to [éq 4.1.1-2], one can rewrite $\mathbf{R}{,_\alpha}|_{\Delta\mathbf{u}\neq\Delta\mathbf{u}(\alpha)}$ in the form:

$$\mathbf{R}{,_\alpha}|_{\Delta\mathbf{u}\neq\Delta\mathbf{u}(\alpha)}=\int_\Omega\left(\boldsymbol{\sigma}{,_\alpha}+\Delta\boldsymbol{\sigma}{,_\alpha}|_{\Delta\mathbf{u}\neq\Delta\mathbf{u}(\alpha)}\right):\boldsymbol{\varepsilon}(\mathbf{w_k})\,d\Omega \qquad \textbf{éq 4.2.2.2 - 1}$$

With this intention, we will use the expressions which intervene in the digital integration of the law of behavior to calculate $\Delta\boldsymbol{\sigma}{,_\alpha}|_{\Delta\mathbf{u}\neq\Delta\mathbf{u}(\alpha)}$.

#### 4.2.2.3 Case of linear elasticity

Within the framework of linear elasticity, the law of behavior is expressed by:

$$\Delta\boldsymbol{\sigma}=2\mu.\tilde{\boldsymbol{\varepsilon}}(\Delta\mathbf{u})+K.Tr(\boldsymbol{\varepsilon}(\Delta\mathbf{u})).\mathbf{Id} \qquad \textbf{éq 4.2.2.3 - 1}$$

where $\mathbf{Id}$ is the tensor identity of order 2.

Then, by calculating the total variation of [éq 4.2.2.3 - 1] compared to $\alpha$, one obtains:

$$\begin{aligned}\Delta\sigma{,_\alpha}&=2\mu{,_\alpha}.\tilde{\varepsilon}(\Delta u)+K{,_\alpha}.Tr(\varepsilon(\Delta u)).\mathbf{Id}+2\mu.\tilde{\varepsilon}(\Delta u{,_\alpha})+K.Tr(\varepsilon(\Delta u{,_\alpha})).\mathbf{Id}\\&=0.\qquad +0.\qquad +2\mu.\tilde{\varepsilon}(\Delta u{,_\alpha})+K.Tr(\varepsilon(\Delta u{,_\alpha})).\mathbf{Id}\end{aligned} \qquad \textbf{éq 4.2.2.3 - 2}$$

That is to say:

$$\Delta\boldsymbol{\sigma}{,_\alpha}|_{\Delta\mathbf{u}\neq\Delta\mathbf{u}(\alpha)}=0.$$

#### 4.2.2.4 Case of the elastoplasticity of the Drucker-Prager type

As previously, we will distinguish two cases.

**1 er case** : $\Delta p=0$
What amounts saying that at the time of these step of load, the point of Gauss considered did not see an increase in its plasticization. One finds oneself then in the case of linear elasticity:

$$\Delta\boldsymbol{\sigma}{,_\alpha}|_{\Delta\mathbf{u}\neq\Delta\mathbf{u}(\alpha)}=0.$$

**2 eme case** : $\Delta p>0$
Taking into account the dependences between variables, one can write:

$$\begin{cases}\Delta\sigma{,_\alpha} = \dfrac{\partial\Delta\sigma}{\partial\alpha}+\dfrac{\partial\Delta\sigma}{\partial\sigma}\cdot\sigma{,_\alpha}+\dfrac{\partial\Delta\sigma}{\partial p}\cdot p{,_\alpha}+\dfrac{\partial\Delta\sigma}{\partial\varepsilon(\Delta u)}\cdot\varepsilon(\Delta u{,_\alpha})\\[2mm] \Delta p{,_\alpha} = \dfrac{\partial\Delta p}{\partial\alpha}+\dfrac{\partial\Delta p}{\partial\sigma}\cdot\sigma{,_\alpha}+\dfrac{\partial\Delta p}{\partial p}\cdot p{,_\alpha}+\dfrac{\partial\Delta p}{\partial\varepsilon(\Delta u)}\cdot\varepsilon(\Delta u){,_\alpha}\end{cases}$$

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Moreover, in agreement with the algorithmic integration of the law, we will separate parts deviatoric and hydrostatic.

$$\begin{cases} \Delta\sigma,_{\alpha}\big|_{\Delta u\neq\Delta u(\alpha)} & = \dfrac{\partial\Delta\tilde{\sigma}}{\partial\alpha} + \dfrac{1}{3}\cdot\dfrac{\partial Tr(\Delta\sigma)}{\partial\alpha}\cdot\mathbf{Id} \\ & + \dfrac{\partial\Delta\tilde{\sigma}}{\partial\sigma}\cdot\sigma,_{\alpha} + \dfrac{1}{3}\cdot\dfrac{\partial Tr(\Delta\sigma)}{\partial\sigma}\cdot\mathbf{Id}\cdot\sigma,_{\alpha} \\ & + \dfrac{\partial\Delta\tilde{\sigma}}{\partial p}\cdot p,_{\alpha} + \dfrac{1}{3}\cdot\dfrac{\partial Tr(\Delta\sigma)}{\partial p}\cdot\mathbf{Id}\cdot p,_{\alpha} \\ \Delta p,_{\alpha}\big|_{\Delta u\neq\Delta u(\alpha)} & = \dfrac{\partial\Delta p}{\partial\alpha} + \dfrac{\partial\Delta p}{\partial\sigma}\cdot\sigma,_{\alpha} + \dfrac{\partial\Delta p}{\partial p}\cdot p,_{\alpha} \end{cases}$$

And thus, one calculates:

$$\boxed{\frac{\partial\Delta\boldsymbol{\sigma}}{\partial\alpha}}$$

Insofar as one does not have dependence clarifies $\Delta\boldsymbol{\sigma}$ compared to $\alpha$ , one obtains:

$$\frac{\partial\Delta\tilde{\boldsymbol{\sigma}}}{\partial\alpha}=0.$$
$$\frac{\partial Tr(\Delta\boldsymbol{\sigma})}{\partial\alpha}=0.$$

$$\boxed{\frac{\partial\Delta\boldsymbol{\sigma}}{\partial\boldsymbol{\sigma}}}$$

$$\frac{\partial\Delta\tilde{\sigma}}{\partial\sigma}=-3\mu\cdot\frac{\dfrac{\partial\Delta p}{\partial\sigma}}{\sigma_{eq}^{e}}\otimes\tilde{\sigma}^{e}+3\mu\cdot\frac{\Delta p}{\sigma_{eq}^{e^2}}\cdot\frac{\partial\sigma_{eq}^{e}}{\partial\sigma}\otimes\tilde{\sigma}^{e}-3\mu\cdot\frac{\Delta p}{\sigma_{eq}^{e}}\cdot J$$

where $\mathbf{J}$ is the operator deviatoric.

$$\frac{\partial Tr(\Delta\sigma)}{\partial\sigma}=-9\mathrm{K}\cdot A\cdot\frac{\partial\Delta p}{\partial\sigma}$$

$$\boxed{\frac{\partial\Delta\boldsymbol{\sigma}}{\partial p}}$$

$$\frac{\partial\Delta\tilde{\sigma}}{\partial p}=-\frac{3\mu}{\sigma_{eq}^{e}}\cdot\frac{\partial\Delta p}{\partial p}\cdot\tilde{\sigma}^{e}$$

$$\frac{\partial Tr(\Delta\sigma)}{\partial p}=-9\cdot K\cdot A\cdot\frac{\partial\Delta p}{\partial p}$$

$$\boxed{\Delta p,_{\alpha}}$$

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The fact is used that: $(\sigma+\Delta\sigma)_{eq}=(\sigma+\Delta\sigma)_{eq}^{e}-3\mu\cdot\Delta p$

$$\Delta p_{,\alpha}=\frac{1}{3\mu}\cdot\left((\sigma+\Delta\sigma)_{eq}^{e,\alpha}-(\sigma+\Delta\sigma)_{eq_{,\alpha}}-\frac{\partial 3\mu}{\partial\alpha}\cdot\Delta p\right)$$

One will refer again to the remark at the end of [§ 4.1.2.5] for the sizes whose calculation was not here detailed.

#### 4.2.2.5 Calculation of the derivative of displacement

Once $\Delta\boldsymbol{\sigma},_{\alpha}|_{\Delta\mathbf{u}\neq\Delta\mathbf{u}(\alpha)}$ calculated, one can constitute the second member $\mathbf{R},_{\alpha}|_{\Delta\mathbf{u}\neq\Delta\mathbf{u}(\alpha)}$ . One then solves the system [éq 4.2.2.1 - 1] and one obtains the increment of derived displacement compared to $\alpha$ .

#### 4.2.2.6 Calculation of the derivative of the other sizes

Now that one has $\Delta\mathbf{u},_{\alpha}$ , one must calculate the derivative of the other sizes. One separates two more cases:

**Linear elasticity**
According to [éq 4.2.2.3 - 1], one as follows calculates the derivative of the increment of constraint:

$$\Delta\boldsymbol{\sigma},_{\alpha}=0.+2\mu.\tilde{\boldsymbol{\varepsilon}}(\Delta\mathbf{u},_{\alpha})+K.Tr(\boldsymbol{\varepsilon}(\Delta\mathbf{u},_{\alpha})).\mathbf{Id}$$

The increment of cumulated plastic deformation, as for him, does not see evolution:

$$\Delta p,_{\alpha}=0$$

**Elastoplasticity of the type Drucker Prager**
If $\Delta p=0$ , the preceding case is found.
If not, one obtains:

$$\Delta\sigma,_{\alpha}=\Delta\sigma,_{\alpha}|_{\Delta u\neq\Delta u(\alpha)}+\frac{\partial\Delta\sigma}{\partial\varepsilon(\Delta u)}:\varepsilon(\Delta u,_{\alpha})$$

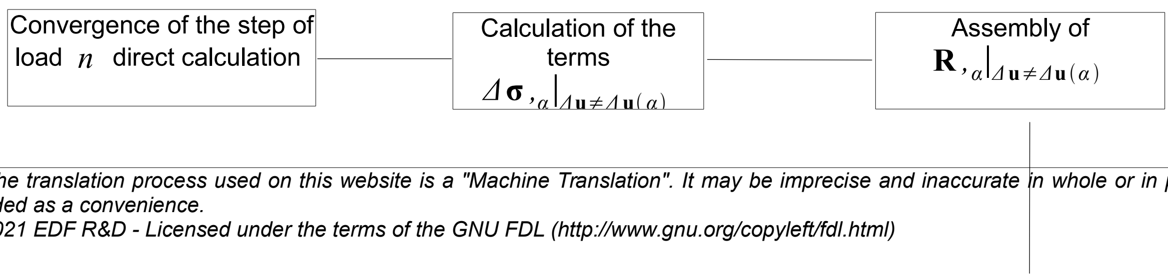
And for the cumulated plastic deformation:

$$\Delta p_{,\alpha}=\frac{1}{3\mu}\cdot\left((\sigma+\Delta\sigma)_{eq}^{e,\alpha}-(\sigma+\Delta\sigma)_{eq_{,\alpha}}-\frac{\partial 3\mu}{\partial\alpha}\cdot\Delta p\right)$$

Once all these calculations are finished, all the derived sizes are reactualized and one passes to the step of load according to.

#### 4.2.2.7 Synthesis

To summarize the preceding paragraphs, one represents the various stages of calculation by the following diagram:

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Passage au not of load $n+1$ — Calculation of $\Delta \boldsymbol{\sigma},_{\alpha}$ and $\Delta p,_{\alpha}$ — Resolution of the system [éq 4.2.2.1 - 1] $\Rightarrow \Delta \mathbf{u},_{\alpha}$

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# 5 Features and checking

The law of behavior can be defined by the keywords DRUCK_PRAG and DRUCK_PRAG_N_A for the non-aligned version (order STAT_NON_LINE, keyword factor BEHAVIOR). They are associated with materials DRUCK_PRAG and DRUCK_PRAG_FO (order DEFI_MATERIAU).

The law HOEK_BROWN is checked by the cases following tests:

| | | |
|---|---|---|
| SSND104 | [V6.08.104] | Validation of the behavior DRUCK_PRAG_N_A |
| SSNP124 | [V6.03.124] | Biaxial test drained with a behavior DRUCK_PRAGER polishing substance |
| SSNP125 | Non-existent documentation | Validation of the option INDL_ELGA for the behavior DRUCK_PRAGER |
| SSNV168 | [V6.04.168] | Triaxial compression test drained with a behavior DRUCK_PRAGER polishing substance |
| WTNA101 | [V7.33.101] | Triaxial compression test not-drained with a behavior DRUCK_PRAGER polishing substance |
| WTNP114 | [V7.32.114] | Case test of reference for the calculation of the mechanical deformations |

The tests according to specifically check the calculation of sensitivity to the parameters of the law:

| | | |
|---|---|---|
| SENSM12 | [V1.01.190] | Plate under pressure in plane deformations (plasticity of DRUCK_PRAGER) |
| SENSM13 | [V1.01.192] | Triaxial compression test with the model of the type 3D |
| SENSM14 | [V1.01.193] | Cavity 2D calculation of sensitivity (Law DRUCK_PRAGER) |

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# 7 Description of the versions of the document

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| Version Aster | Author (S)<br>Organization (S) | Description of the modifications |
|---|---|---|
| 7.4 | R.FERNANDES, P. OF BONNIERES, C.CHAVANT EDF R & D/AMA | Initial text |
| 9.4 | R.FERNANDES | Addition of the nonassociated model |

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# Annexe 1 Calculation of the derivative partial of $\Delta p$

## A1.1 Calculation of the derivative partial of the increment of plastic deformation in the case of a linear work hardening

$R(p)=h\cdot p+\sigma^{y}$ for $0\leq p<p_{ultm}$

$$\Delta p=\frac{\sigma_{eq}^{e}+\mathrm{A}\cdot Tr(\boldsymbol{\sigma}^{e})-h\cdot p^{-}-\sigma^{Y}}{9K\cdot\mathrm{A}^{2}+3\mu+h}$$

thus:

$$\frac{\partial\Delta p}{\partial\Phi}=\frac{1}{9K.\mathrm{A}^{2}+3\mu+h}\cdot(\frac{\partial\sigma_{eq}^{e}}{\partial\Phi}+\frac{\partial\mathrm{A}}{\partial\Phi}\cdot Tr(\boldsymbol{\sigma}^{e})+\mathrm{A}\cdot\frac{\partial Tr(\boldsymbol{\sigma}^{e})}{\partial\Phi}-\frac{\partial h}{\partial\Phi}\cdot p^{-}-\frac{\partial\sigma^{y}}{\partial\Phi}$$
$$-\Delta p\cdot(9\cdot\frac{\partial K}{\partial\Phi}\cdot\mathrm{A}^{2}+18\cdot K\cdot A\cdot\frac{\partial\mathrm{A}}{\partial\Phi}+\frac{\partial 3\mu}{\partial\Phi}+\frac{\partial h}{\partial\Phi}))$$

$$\frac{\partial\Delta p}{\partial\boldsymbol{\sigma}}=\frac{1}{3\mu+9K\cdot\mathrm{A}^{2}+h}\cdot(A\cdot\frac{\partial Tr(\boldsymbol{\sigma}^{e})}{\partial\boldsymbol{\sigma}}+\frac{\partial\sigma_{eq}^{e}}{\partial\boldsymbol{\sigma}})$$

$$\frac{\partial\Delta p}{\partial p}=-h\cdot\frac{1}{3\mu+9K\cdot\mathrm{A}^{2}+h}$$

$R(p)=h\cdot p_{ultm}+\sigma^{y}$ for $p>p_{ultm}$

$$\Delta p=\frac{\sigma_{eq}^{e}+\mathrm{A}\cdot Tr(\boldsymbol{\sigma}^{e})-h\cdot p_{ultm}-\sigma^{Y}}{9K\cdot\mathrm{A}^{2}+3\mu}$$

thus:

$$\frac{\partial\Delta p}{\partial\Phi}=\frac{1}{9K\cdot\mathrm{A}^{2}+3\mu}\cdot(\frac{\partial\sigma_{eq}^{e}}{\partial\Phi}+\frac{\partial\mathrm{A}}{\partial\Phi}\cdot Tr(\boldsymbol{\sigma}^{e})+\mathrm{A}\cdot\frac{\partial Tr(\boldsymbol{\sigma}^{e})}{\partial\Phi}-\frac{\partial h}{\partial\Phi}\cdot p_{ultm}-h\cdot\frac{\partial p_{ultm}}{\partial\Phi}-\frac{\partial\sigma^{y}}{\partial\Phi}$$
$$-\Delta p\cdot(9\cdot\frac{\partial K}{\partial\Phi}\cdot\mathrm{A}^{2}+18\cdot K\cdot A\cdot\frac{\partial\mathrm{A}}{\partial\Phi}+\frac{\partial 3\mu}{\partial\Phi}))$$

$$\frac{\partial\Delta p}{\partial\boldsymbol{\sigma}}=\frac{1}{3\mu+9K\cdot\mathrm{A}^{2}}(A\cdot\frac{\partial Tr(\boldsymbol{\sigma}^{e})}{\partial\boldsymbol{\sigma}}+\frac{\partial\sigma_{eq}^{e}}{\partial\boldsymbol{\sigma}})$$

$$\frac{\partial\Delta p}{\partial p}=0$$

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## A1.2 Calculation of the derivative partial of the increment of plastic deformation in the case of a parabolic work hardening

$$R(p)=\sigma^y\cdot(1-(1-\sqrt{\frac{\sigma^y_{ultm}}{\sigma^y}})\cdot\frac{p}{p_{ultm}})^2 \text{ for } 0\le p<p_{ultm}$$

$$\frac{\partial\sigma^e_{eq}}{\partial\Phi}-(\frac{\partial 3\mu}{\partial\Phi}+9\mathrm{A}^2.\frac{\partial K}{\partial\Phi}+18K.\mathrm{A}.\frac{\partial \mathrm{A}}{\partial\Phi}).\Delta p-(3\mu+9K.\mathrm{A}^2).\frac{\partial\Delta p}{\partial\Phi}+\frac{\partial\mathrm{A}}{\partial\Phi}.Tr(\sigma^e)+\mathrm{A}.\frac{\partial Tr(\sigma^e)}{\partial\Phi}$$
$$-\frac{\partial\sigma^y}{\partial\Phi}.(1-(1-\sqrt{\frac{\sigma^y_{ultm}}{\sigma^y}}).\frac{p^-+\Delta p}{p_{ultm}})^2$$
$$-2\sigma^y.(1-(1-\sqrt{\frac{\sigma^y_{ultm}}{\sigma^y}}).\frac{p^-+\Delta p}{p_{ultm}}).$$
$$(\frac{\partial\sigma^y_{ultm}}{\partial\Phi}.\frac{p^-+\Delta p}{2p_{ultm}.\sqrt{\sigma^y_{ultm}.\sigma^y}}-\frac{\partial\sigma^y}{\partial\Phi}.\frac{p^-+\Delta p}{2p_{ultm}.\sigma^y}.\sqrt{\frac{\sigma^y_{ultm}}{\sigma^y}}+\frac{\partial p_{ultm}}{\partial\Phi}.(1-\sqrt{\frac{\sigma^y_{ultm}}{\sigma^y}}).\frac{p^-+\Delta p}{p_{ultm}^{\ 2}}-\frac{\partial\Delta p}{\partial\Phi}\frac{1-\sqrt{\frac{\sigma^y_{ultm}}{\sigma^y}}}{p_{ultm}})$$
$$=0$$

$$\frac{\partial\sigma^e_{eq}}{\partial\boldsymbol{\sigma}}-(3\mu+9K\cdot\mathrm{A}^2)\cdot\frac{\partial\Delta p}{\partial\boldsymbol{\sigma}}+A\cdot\frac{\partial Tr(\boldsymbol{\sigma}^e)}{\partial\boldsymbol{\sigma}}+2\sigma^y\cdot(1-(1-\sqrt{\frac{\sigma^y_{ultm}}{\sigma^y}})\cdot\frac{p^-+\Delta p}{p_{ultm}})\cdot\frac{1-\sqrt{\frac{\sigma^y_{ultm}}{\sigma^y}}}{p_{ultm}}\cdot\frac{\partial\Delta p}{\partial\boldsymbol{\sigma}}=0$$

$$-(3\mu+9K\cdot\mathrm{A}^2)\cdot\frac{\partial\Delta p}{\partial p}+2\sigma^y\cdot(1-(1-\sqrt{\frac{\sigma^y_{ultm}}{\sigma^y}})\frac{p^-+\Delta p}{p_{ultm}})\cdot\frac{1-\sqrt{\frac{\sigma^y_{ultm}}{\sigma^y}}}{p_{ultm}}\cdot(1+\frac{\partial\Delta p}{\partial p})=0$$

$$R(p)=\sigma^y_{ultm} \text{ for } p>p_{ultm}$$

$$\frac{\partial\Delta p}{\partial\Phi}=\frac{1}{3\mu+9K\cdot\mathrm{A}^2}(\frac{\partial\sigma^e_{eq}}{\partial\Phi}-(\frac{\partial 3\mu}{\partial\Phi}+\frac{\partial 9K}{\partial\Phi}\cdot\mathrm{A}^2+18K\cdot\frac{\partial\mathrm{A}}{\partial\Phi}\cdot A)\cdot\Delta p+\frac{\partial\mathrm{A}}{\partial\Phi}\cdot Tr(\boldsymbol{\sigma}^e)+\mathrm{A}\cdot\frac{\partial Tr(\boldsymbol{\sigma}^e)}{\partial\Phi}-\frac{\partial\sigma^y_{ultm}}{\partial\Phi})$$

$$\frac{\partial\Delta p}{\partial\boldsymbol{\sigma}}=\frac{1}{3\mu+9K\cdot\mathrm{A}^2}(\frac{\partial\sigma^e_{eq}}{\partial\boldsymbol{\sigma}}+A\cdot\frac{\partial Tr(\boldsymbol{\sigma}^e)}{\partial\boldsymbol{\sigma}})$$

$$\frac{\partial\Delta p}{\partial p}=0$$

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## A1.3 Case of projection at the top of the cone

The principle of the analytical resolution consists in determining the effective constraints like the projection of the elastic constraints on the criterion.

It may be that there is no solution.

If the condition $\Delta p \le \frac{\sigma_{eq}^{e}}{3\mu}$ is not respected, it is necessary to find the constraints effective by projection at the top of the cone $\Delta p = \frac{\sigma_{eq}^{e}}{3\mu}$.

In this case, one obtains:

$$\frac{\partial \Delta p}{\partial \Phi} = \frac{1}{3\mu} \cdot \left( \frac{\partial \sigma_{eq}^{e}}{\partial \Phi} - \Delta p \cdot \frac{\partial 3\mu}{\partial \Phi} \right)$$

$$\frac{\partial \Delta p}{\partial \boldsymbol{\sigma}} = \frac{1}{3\mu} \cdot \frac{\partial \sigma_{eq}^{e}}{\partial \boldsymbol{\sigma}}$$

$$\frac{\partial \Delta p}{\partial p} = 0$$

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