Law of Mohr-Coulomb

Summary:

This document presents the method of resolution of the law of Mohr-Coulomb in *Code_Aster*.

Contents

1 Notations

1.1 General data

$\sigma_1 \geq \sigma_2 \geq \sigma_3$	Principal constraints (in this order)
E	Young modulus
ν	Poisson's ratio
$K=\dfrac{E}{3(1-2\nu)}$	Elastic module of compressibility
$G=\dfrac{E}{2(1+\nu)}$	Elastic modulus of rigidity
φ	Natural angle of repose of material
ψ	Angle of dilatancy of material
c	Cohesion of material
$s=\sin(\varphi)$	
$t=\sin(\psi)$	
$p=\dfrac{I_1}{3}=\dfrac{trace(\boldsymbol{\sigma})}{3}$	Average constraint
$p<0$	Convention of sign for the constraint in compression
$\boldsymbol{\sigma}^e$	Tensor of elastic prediction constraints
$\boldsymbol{\varepsilon}=\boldsymbol{\varepsilon}^e+d\,\boldsymbol{\varepsilon}^p$	Tensors of the deflections total, elastic and increment of plastic deformation
$d\,\varepsilon_v^p=trace(d\,\boldsymbol{\varepsilon}^p)$	Increment of the voluminal plastic deformation
$d\,\boldsymbol{e}^p=d\,\boldsymbol{\varepsilon}^p-\dfrac{d\,\varepsilon_v^p}{3}\mathbf{1}$	Increment of the deviatoric plastic deformation
$d\,e^p=\lVert d\,\boldsymbol{e}^p\rVert=\sqrt{\dfrac{3}{2}d\,\boldsymbol{e}^p.d\,\boldsymbol{e}^p}$	Increment of deviatoric plastic deformation, or increment of equivalent deformation normalizes

1.2 Convention on the tensorial notations

Vectors of the strains and the stresses in the principal base $(\boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_3)$ are noted:

$$\boldsymbol{\varepsilon}=\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\end{Bmatrix} \text{ and } \boldsymbol{\sigma}=\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\end{Bmatrix} \quad (1)$$

The tensor of elasticity $\boldsymbol{C}$ allowing to connect $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$ in the principal base, such as $\boldsymbol{\sigma}=\boldsymbol{C}.\boldsymbol{\varepsilon}$ is written:

$$\boldsymbol{C}=\begin{bmatrix} K+\frac{4}{3}G & K-\frac{2}{3}G & K-\frac{2}{3}G \\ K-\frac{2}{3}G & K+\frac{4}{3}G & K-\frac{2}{3}G \\ K-\frac{2}{3}G & K-\frac{2}{3}G & K+\frac{4}{3}G \end{bmatrix} \quad (2)$$

With K the elastic module of compressibility and G the elastic modulus of rigidity. The strains and the stresses are of the symmetrical tensors of order two. One generally exploits this symmetry (six independent components) by representing them by vectors of dimension six resulting from the projection of these tensors in suitable bases.
The strains and the stresses given as starter and produced at exit of the resolution of the law of behavior are expressed in the orthonormal base of the symmetrical tensors of order two, noted $\bar{\boldsymbol{b}}$:

$$\bar{\boldsymbol{b}}=\begin{cases} \boldsymbol{e}_x\otimes\boldsymbol{e}_x \\ \boldsymbol{e}_y\otimes\boldsymbol{e}_y \\ \boldsymbol{e}_z\otimes\boldsymbol{e}_z \\ \dfrac{\boldsymbol{e}_x\otimes\boldsymbol{e}_y+\boldsymbol{e}_y\otimes\boldsymbol{e}_x}{\sqrt{2}} \\ \dfrac{\boldsymbol{e}_x\otimes\boldsymbol{e}_z+\boldsymbol{e}_z\otimes\boldsymbol{e}_x}{\sqrt{2}} \\ \dfrac{\boldsymbol{e}_y\otimes\boldsymbol{e}_z+\boldsymbol{e}_z\otimes\boldsymbol{e}_y}{\sqrt{2}} \end{cases} \quad (3)$$

Where $\left(\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z\right)$ represent the unit vectors of the total Cartesian base orthonormal, presumedly fixed. The condensed expression of the tensors of the strains and the stresses projected in the base $\bar{\boldsymbol{b}}$ is written:

$$\bar{\boldsymbol{\varepsilon}}=\begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \sqrt{2}\,\varepsilon_{xy} \\ \sqrt{2}\,\varepsilon_{yz} \\ \sqrt{2}\,\varepsilon_{xz} \end{pmatrix} \text{ and } \bar{\boldsymbol{\sigma}}=\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sqrt{2}\,\sigma_{xy} \\ \sqrt{2}\,\sigma_{yz} \\ \sqrt{2}\,\sigma_{xz} \end{pmatrix} \quad (4)$$

This writing reveals a term in $\sqrt{2}$ in front of the cross components. It allows:

- To express the tensor of elasticity of order four of 81 components by a tensor of order two of 36 components;
- To symmetrize this tensor of elasticity.

Indeed, while noting $\sigma_{ij}=C_{ijkl}\varepsilon_{kl}$, its form projected in the base $\bar{\boldsymbol{b}}$ becomes $\bar{\sigma}_i=\bar{C}_{ij}\bar{\varepsilon}_j$, where one has the following expression for $\bar{\boldsymbol{C}}$:

$$\bar{\boldsymbol{C}}=\begin{bmatrix} C_{xxxx} & C_{xxyy} & C_{xxzz} & \sqrt{2}C_{xxxy} & \sqrt{2}C_{xxxz} & \sqrt{2}C_{xxyz} \\ C_{xxyy} & C_{yyyy} & C_{yyzz} & \sqrt{2}C_{yyxy} & \sqrt{2}C_{yyxz} & \sqrt{2}C_{yyyz} \\ C_{zzxx} & C_{zzyy} & C_{zzzz} & \sqrt{2}C_{zzxy} & \sqrt{2}C_{zzxz} & \sqrt{2}C_{zzyz} \\ \sqrt{2}C_{xyxx} & \sqrt{2}C_{xyyy} & \sqrt{2}C_{xyzz} & 2C_{xyxy} & 2C_{xyxz} & 2C_{xyyz} \\ \sqrt{2}C_{xzxx} & \sqrt{2}C_{xzyy} & \sqrt{2}C_{xzzz} & 2C_{xzxy} & 2C_{xzxz} & 2C_{xzyz} \\ \sqrt{2}C_{yzxx} & \sqrt{2}C_{yzyy} & \sqrt{2}C_{yzzz} & 2C_{yzxy} & 2C_{yzxz} & 2C_{yzyz} \end{bmatrix} \quad (5)$$

The condensed form (**5**) is not convenient to use because of the need for handling the terms in $\sqrt{2}$ at the time of the matric operations. One prefers to him another writing, based on projection in base known as of Voigt, noted $\tilde{\boldsymbol{b}}$ and having the following expression:

$$\tilde{\boldsymbol{b}}=\begin{cases}\boldsymbol{e}_x\otimes\boldsymbol{e}_x\\ \boldsymbol{e}_y\otimes\boldsymbol{e}_y\\ \boldsymbol{e}_z\otimes\boldsymbol{e}_z\\ \boldsymbol{e}_x\otimes\boldsymbol{e}_y\\ \boldsymbol{e}_x\otimes\boldsymbol{e}_z\\ \boldsymbol{e}_y\otimes\boldsymbol{e}_z\end{cases} \quad (6)$$

The condensed expression of the tensors of the strains and the stresses projected in the base of Voigt $\tilde{\boldsymbol{b}}$ is written:

$$\tilde{\boldsymbol{\varepsilon}}=\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \varepsilon_{xy}\\ \varepsilon_{yz}\\ \varepsilon_{xz}\end{Bmatrix} \text{ and } \tilde{\boldsymbol{\sigma}}=\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \sigma_{xy}\\ \sigma_{yz}\\ \sigma_{xz}\end{Bmatrix} \quad (7)$$

This writing makes it possible to be freed from the terms in $\sqrt{2}$ in front of the crossed components, and is more convenient to use during the digital resolution of the law of behavior.

2 Formulation in terms of principal constraints

This formulation is valid only under the assumption of one isotropy of material [**79**, **82**]. Indeed, this condition is necessary to guarantee that the method of radial return preserves the principal directions. Its interest lies in the fact that it simplifies the writing of the equations and authorizes of this fact of the very powerful methods of resolution (bus quasi-analytical).

The elastic behavior is purely linear.

The surface of charge is characterized by six plans within the space of principal constraints $(\sigma_1, \sigma_2, \sigma_3)$. Each one of these plans is characterized by an equation of the type:

$$F_{13}(\sigma_1, \sigma_3) = \sigma_1 - \sigma_3 + (\sigma_1 + \sigma_3)\sin\varphi - 2c\cos\varphi = 0 \quad (8)$$

Where φ and c are data material and respectively characterize the natural angle of repose and the cohesion of material.

The law is nonassociated and the plastic potential of flow G_{13} associated with the surface of load F_{13} is written in the same way:

$$G_{13}(\sigma_1, \sigma_3) = \sigma_1 - \sigma_3 + (\sigma_1 + \sigma_3)\sin\psi - 2c\cos\psi \quad (9)$$

Where ψ is a data material and characterizes the angle of dilatancy of material.

When $\psi = \varphi$, the plastic law of flow becomes associated.

A chart of the surface of load of Mohr-Coulomb within the space of principal constraints is on figure 2-1 and in the plan π on figure 2-2.

It is observed that the six plans intersect two to two following an angular edge, and meeting at the top of the cone characterized by the equation:

$$p = c\cot(\varphi) \quad (10)$$

These edges, six, as well as the top of the cone, form singularities which pose problems of digital integration, because the derivative of the surface of load are not defined in these places. One will discuss more far from the methods allowing to solve this difficulty.

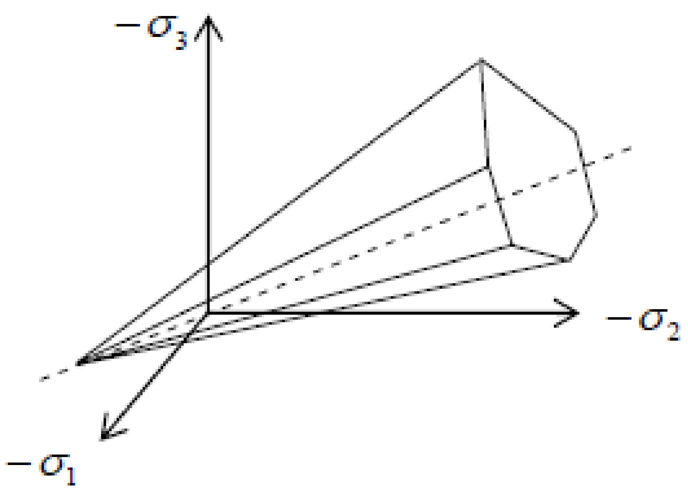

Figure 2-1: Representation of the surface of load of Mohr-Coulomb in the space three-dimensional of the principal constraints

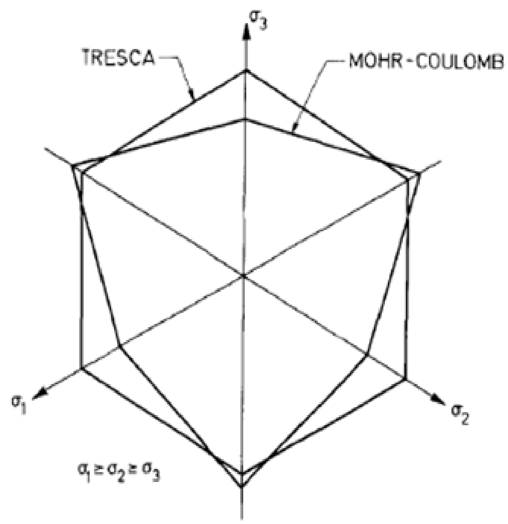

Figure 2-2: Representation of the surface of load of Mohr-Coulomb in the plan π diverters of the constraints (any vector represented in this plan corresponds to a deviatoric constraint).

3 Local integration of the law of Mohr-Coulomb

The rate of plastic deformation is given using the formula of Koiter:

$$d\boldsymbol{\varepsilon}^p=\sum_{j=1}^{m} d\lambda_j \frac{\partial G_j}{\partial \boldsymbol{\sigma}}=\sum_{j=1}^{m} d\lambda_j \boldsymbol{n}_{Gj} \quad (11)$$

Where m characterize the number of active mechanisms, equal to one, two or six following following situations:

- the final constraint is inside the surface of load, the point is regular and $m=1$;
- the final constraint is on an edge of the cone, the point is singular and $m=2$;
- the final constraint is neither inside the surface of load nor on an edge. It is then projected at the top of the cone, the point is singular and $m=6$;

The final constraint $\boldsymbol{\sigma}^+$ is calculated starting from a noted elastic prediction $\boldsymbol{\sigma}^e$ and of a correction $\boldsymbol{\sigma}^c=\boldsymbol{C}.d\boldsymbol{\varepsilon}^p$ so that:

$$\boldsymbol{\sigma}^+=\boldsymbol{\sigma}^e-d\boldsymbol{\sigma}^c=\boldsymbol{\sigma}^e-\boldsymbol{C}.d\boldsymbol{\varepsilon}^p=\boldsymbol{\sigma}^e-\sum_{j=1}^{m} d\lambda_j \boldsymbol{C}.\boldsymbol{n}_{Gj} \quad (12)$$

Plastic multipliers $d\lambda_j$ are calculated by injecting the equation (**12**) in the equation (**8**), which gives:

$$\sum_{j=1}^{m} d\lambda_j\left[(\boldsymbol{C}.\boldsymbol{n}_{Gj})_1-(\boldsymbol{C}.\boldsymbol{n}_{Gj})_3+\left((\boldsymbol{C}.\boldsymbol{n}_{Gj})_1+(\boldsymbol{C}.\boldsymbol{n}_{Gj})_3\right)s\right]=\sigma_1^e-\sigma_3^e+\left(\sigma_1^e+\sigma_3^e\right)s-2c\cos\phi \quad (13)$$

In what follows, one details the expressions corresponding to the various situations mentioned above. The procedure of resolution is recalled in the synoptic one figure 3.3-1.

3.1 Case where only one mechanism is active

One detects that the prediction activates a criterion when:

$$F_{13}(\boldsymbol{\sigma}^e)=\sigma_1^e-\sigma_3^e+\left(\sigma_1^e+\sigma_3^e\right)s-2c\cos\varphi\geq 0 \quad (14)$$

There is the following expression for the normal:

$$\boldsymbol{n}_G=\langle t+1 \quad 0 \quad t-1\rangle \quad (15)$$

And thus:

$$\boldsymbol{C}.\boldsymbol{n}_G=2\left\langle\left(K+\frac{G}{3}\right)t+G \quad \left(K-\frac{2}{3}G\right)t \quad \left(K+\frac{G}{3}\right)t-G\right\rangle \quad (16)$$

That one introduces into the equation (**13**), which becomes:

$$4d\lambda\left[G+\left(K+\frac{G}{3}\right)ts\right]=\sigma_1^e-\sigma_3^e+\left(\sigma_1^e+\sigma_3^e\right)s-2c\cos\varphi \quad (17)$$

From where the plastic multiplier is deduced:

$$d\lambda=\frac{\langle F_{13}(\boldsymbol{\sigma}^e)\rangle_+}{4\left[G+\left(K+\frac{G}{3}\right)ts\right]} \quad (18)$$

Where $\langle\rangle_+$ indicate the positive part of a size. One obtains finally:

$$\begin{Bmatrix}\sigma_1^+\\ \sigma_2^+\\ \sigma_3^+\end{Bmatrix}=\begin{Bmatrix}\sigma_1^e\\ \sigma_2^e\\ \sigma_3^e\end{Bmatrix}-\frac{\langle F_{13}(\boldsymbol{\sigma}^e)\rangle_+}{2G+2\left(K+\frac{G}{3}\right)ts}\begin{Bmatrix}\left(K+\frac{G}{3}\right)t+G\\ \left(K-\frac{2}{3}G\right)t\\ \left(K+\frac{G}{3}\right)t-G\end{Bmatrix} \tag{19}$$

If one breaks up the rate of plastic deformation into a deviatoric part $d\,\varepsilon_v^p=trace(d\,\boldsymbol{\varepsilon}^p)$ and a spherical part $d\,\boldsymbol{e}^p=d\,\boldsymbol{\varepsilon}^p-\frac{d\,\varepsilon_v^p}{3}\mathbf{1}$, one a:

$$\begin{cases} d\,\varepsilon_v^p=2\mathrm{t}\,d\,\lambda\\ d\,\boldsymbol{e}^p=d\,\lambda\left\langle \frac{t}{3}+1 \quad \frac{-2\mathrm{t}}{3} \quad \frac{t}{3}-1\right\rangle\end{cases} \tag{20}$$

That is to say the following expression of the increment of equivalent deformation:

$$d\,e^p=d\,\lambda\sqrt{t^2+3} \tag{21}$$

3.2 Case where two mechanisms are active

3.2.1 Formulation of the solution

It is checked that $\boldsymbol{\sigma}^+$ obtained starting from the preceding stage (**19**) always check:

$$\sigma_1^+\geq\sigma_2^+\geq\sigma_3^+ \tag{22}$$

If it is not the case, then it is advisable to activate two mechanisms in the following way:

$$\begin{cases}\sigma_2^+\geq\sigma_1^+\geq\sigma_3^+ \quad\Rightarrow\quad F_{13}\,et\,F_{23}\ \ actifs \quad [\textbf{LEFT}]\\ \sigma_1^+\geq\sigma_3^+\geq\sigma_2^+ \quad\Rightarrow\quad F_{13}\,et\,F_{12}\ \ actifs \quad [\textbf{RIGHT}]\end{cases} \tag{23}$$

The definition of the mechanisms `LEFT` and `RIGHT` is purely conventional, and obeys the geometrical logic represented in **figure 3.2.1-1**.

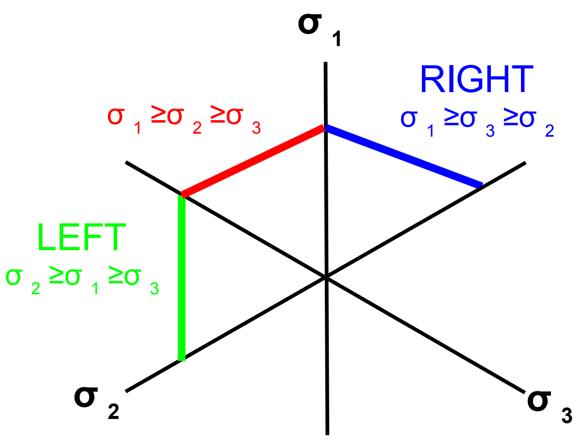

Figure 3.2.1-1: Definition of the mechanisms `LEFT` and `RIGHT`

There are the following expressions for the normals:

$$\boldsymbol{n}_G^{13}=\langle t+1 \quad 0 \quad t-1\rangle \quad \boldsymbol{n}_G^{12}=\langle t+1 \quad t-1 \quad 0\rangle \quad \boldsymbol{n}_G^{23}=\langle 0 \quad t+1 \quad t-1\rangle \tag{24}$$

Thus:

$$\begin{cases} \boldsymbol{C}.\boldsymbol{n}_G^{13}=2\left\langle \left(K+\frac{G}{3}\right)t+G \quad \left(K-\frac{2}{3}G\right)t \quad \left(K+\frac{G}{3}\right)t-G\right\rangle \\ \boldsymbol{C}.\boldsymbol{n}_G^{12}=2\left\langle \left(K+\frac{G}{3}\right)t+G \quad \left(K+\frac{G}{3}\right)t-G \quad \left(K-\frac{2}{3}G\right)t\right\rangle \\ \boldsymbol{C}.\boldsymbol{n}_G^{23}=2\left\langle \left(K-\frac{2}{3}G\right)t \quad \left(K+\frac{G}{3}\right)t+G \quad \left(K+\frac{G}{3}\right)t-G\right\rangle \end{cases} \tag{25}$$

One introduces these expressions into the equation (**13**). For the mechanism `LEFT`, one obtains the following system to solve:

$$\begin{cases} 4d\lambda^{13}\left[G+\left(K+\frac{G}{3}\right)ts\right]+2d\lambda^{23}\left[G(1-t-s)+\left(2\mathrm{K}-\frac{G}{3}\right)ts\right]=F_{13}(\boldsymbol{\sigma}^e) \\ 2d\lambda^{13}\left[G(1-t-s)+\left(2\mathrm{K}-\frac{G}{3}\right)ts\right]+4d\lambda^{23}\left[G+\left(K+\frac{G}{3}\right)ts\right]=F_{23}(\boldsymbol{\sigma}^e) \end{cases} \tag{26}$$

For the mechanism `RIGHT`, one obtains:

$$\begin{cases} 4d\lambda^{13}\left[G+\left(K+\frac{G}{3}\right)ts\right]+2d\lambda^{12}\left[G(1+t+s)+\left(2\mathrm{K}-\frac{G}{3}\right)ts\right]=F_{13}(\boldsymbol{\sigma}^e) \\ 2d\lambda^{13}\left[G(1+t+s)+\left(2\mathrm{K}-\frac{G}{3}\right)ts\right]+4d\lambda^{12}\left[G+\left(K+\frac{G}{3}\right)ts\right]=F_{12}(\boldsymbol{\sigma}^e) \end{cases} \tag{27}$$

For implementation the algorithmic, it is more pleasant to rewrite (**26**) and (27) as follows:

$$\begin{cases} A\,d\lambda^1+B^{side}d\lambda^2=F_1(\boldsymbol{\sigma}^e) \\ B^{side}d\lambda^1+A\,d\lambda^2=F_2(\boldsymbol{\sigma}^e) \end{cases} \tag{28}$$

With the following plastic multipliers:

$$d\lambda^1=d\lambda^{13} \text{ and } d\lambda^2=\begin{cases} d\lambda^{23} & [\mathbf{LEFT}] \\ d\lambda^{12} & [\mathbf{RIGHT}] \end{cases} \tag{29}$$

For surfaces of load:

$$F_1=F_{13} \text{ and } F_2=\begin{cases} F_{23} & [\mathbf{LEFT}] \\ F_{12} & [\mathbf{RIGHT}] \end{cases} \tag{30}$$

The expression of A :

$$A=4\left[G+\left(K+\frac{G}{3}\right)ts\right] \tag{31}$$

And the expression of B^{side} :

$$B^{side}=\begin{cases}2\left[G(1-t-s)+\left(2K-\frac{G}{3}\right)t\,s\right] & [\textbf{LEFT}]\\ 2\left[G(1+t+s)+\left(2K-\frac{G}{3}\right)t\,s\right] & [\textbf{RIGHT}]\end{cases} \tag{32}$$

The solution of the system of equations (**32**) exist if and only if its determinant is nonnull, that is to say:

$$det=\begin{vmatrix}A & B^{side}\\ B^{side} & A\end{vmatrix}=A^2-\left(B^{side}\right)^2\neq 0 \tag{33}$$

One can show that never arrives for "physical" values of the parameters K , G , φ and ψ . One has then as solutions:

$$\begin{cases}d\lambda^1=\dfrac{A\,F_1(\boldsymbol{\sigma}^e)-B^{side}F_2(\boldsymbol{\sigma}^e)}{det}\\ d\lambda^2=\dfrac{A\,F_2(\boldsymbol{\sigma}^e)-B^{side}F_1(\boldsymbol{\sigma}^e)}{det}\end{cases} \tag{34}$$

For the mechanism `LEFT`, one obtains finally:

$$\begin{Bmatrix}\sigma_1^+\\ \sigma_2^+\\ \sigma_3^+\end{Bmatrix}=\begin{Bmatrix}\sigma_1^e\\ \sigma_2^e\\ \sigma_3^e\end{Bmatrix}-2\,d\lambda^{13}\begin{Bmatrix}\left(K+\frac{G}{3}\right)t+G\\ \left(K-\frac{2}{3}G\right)t\\ \left(K+\frac{G}{3}\right)t-G\end{Bmatrix}-2\,d\lambda^{23}\begin{Bmatrix}\left(K-\frac{2}{3}G\right)t\\ \left(K+\frac{G}{3}\right)t+G\\ \left(K+\frac{G}{3}\right)t-G\end{Bmatrix} \tag{35}$$

And for the mechanism `RIGHT` :

$$\begin{Bmatrix}\sigma_1^+\\ \sigma_2^+\\ \sigma_3^+\end{Bmatrix}=\begin{Bmatrix}\sigma_1^e\\ \sigma_2^e\\ \sigma_3^e\end{Bmatrix}-2\,d\lambda^{13}\begin{Bmatrix}\left(K+\frac{G}{3}\right)t+G\\ \left(K-\frac{2}{3}G\right)t\\ \left(K+\frac{G}{3}\right)t-G\end{Bmatrix}-2\,d\lambda^{12}\begin{Bmatrix}\left(K+\frac{G}{3}\right)t+G\\ \left(K+\frac{G}{3}\right)t-G\\ \left(K-\frac{2}{3}G\right)t\end{Bmatrix} \tag{36}$$

Or, by simplifying the writing:

$$\boldsymbol{\sigma}^+=\boldsymbol{\sigma}^e-2\,d\lambda^1\,\boldsymbol{v}_t^1-2\,d\lambda^2\,\boldsymbol{v}_t^2 \tag{37}$$

With the following plastic multipliers:

$$d\lambda^1=d\lambda^{13}\ \text{ and }\ d\lambda^2=\begin{cases}d\lambda^{23} & [\textbf{LEFT}]\\ d\lambda^{12} & [\textbf{RIGHT}]\end{cases} \tag{38}$$

And vectors:

$$\boldsymbol{v}_t^1=\begin{Bmatrix}\left(K+\frac{G}{3}\right)t+G\\ \left(K-\frac{2}{3}G\right)t\\ \left(K+\frac{G}{3}\right)t-G\end{Bmatrix} \text{ and } \boldsymbol{v}_t^2=\begin{cases}\left(\left(K-\frac{2}{3}G\right)t \quad \left(K+\frac{G}{3}\right)t+G \quad \left(K+\frac{G}{3}\right)t-G\right) & [\textbf{LEFT}]\\ \left(\left(K+\frac{G}{3}\right)t+G \quad \left(K+\frac{G}{3}\right)t-G \quad \left(K-\frac{2}{3}G\right)t\right) & [\textbf{RIGHT}]\end{cases} \tag{39}$$

In the same way, the rate of plastic deformation is written:

$$\begin{cases} d\varepsilon_v^p=2t\left(d\lambda^1+d\lambda^2\right)\\ d\boldsymbol{e}^p=d\lambda^1\left(\frac{t}{3}+1 \quad \frac{-2t}{3} \quad \frac{t}{3}-1\right)+d\lambda^2\begin{cases}\left(\frac{-2t}{3} \quad \frac{t}{3}+1 \quad \frac{t}{3}-1\right) & [\textbf{LEFT}]\\ \left(\frac{t}{3}+1 \quad \frac{t}{3}-1 \quad \frac{-2t}{3}\right) & [\textbf{RIGHT}]\end{cases}\end{cases} \tag{40}$$

That is to say the following expression of the increment of equivalent deformation:

$$d\,e^p=\sqrt{\left(\left(d\lambda^1\right)^2+\left(d\lambda^2\right)^2\right)\left(t^2+3\right)+\left(-t^2+sign\times 6t+3\right)d\lambda^1 d\lambda^2} \tag{41}$$

With $sign=\begin{cases}-1 & [\textbf{LEFT}]\\ +1 & [\textbf{RIGHT}]\end{cases}$.

3.2.2 Choice of the second mechanism

A simple criterion is proposed and purely geometrical to undoubtedly determine if the second mechanism, if it would be activated, is on the left (`LEFT`) or on the right (`RIGHT`). The vector is defined $\boldsymbol{t}_G$ perpendicular with the direction of flow as represented in figure 3.2.2-1, in the following way:

$$\begin{Bmatrix}t-1\\-2\\t+1\end{Bmatrix}=\boldsymbol{t}_G\perp\boldsymbol{n}_G=\boldsymbol{n}_G^{13}=\begin{Bmatrix}t+1\\0\\t-1\end{Bmatrix} \tag{42}$$

Line passing by 0 and parallel with $\boldsymbol{n}_G$, represented in dotted lines, characterizes the states of stresses such as $\boldsymbol{\sigma}.\boldsymbol{t}_G=0$. One thus has the following choice:

- If $\sigma.\boldsymbol{t}_G>0$ and that a second mechanism must be active, this mechanism is necessarily located on the right, that is to say the mechanism `RIGHT`;
- If $\sigma.\boldsymbol{t}_G<0$ and that a second mechanism must be active, this mechanism is necessarily located on the left, that is to say the mechanism `LEFT`;

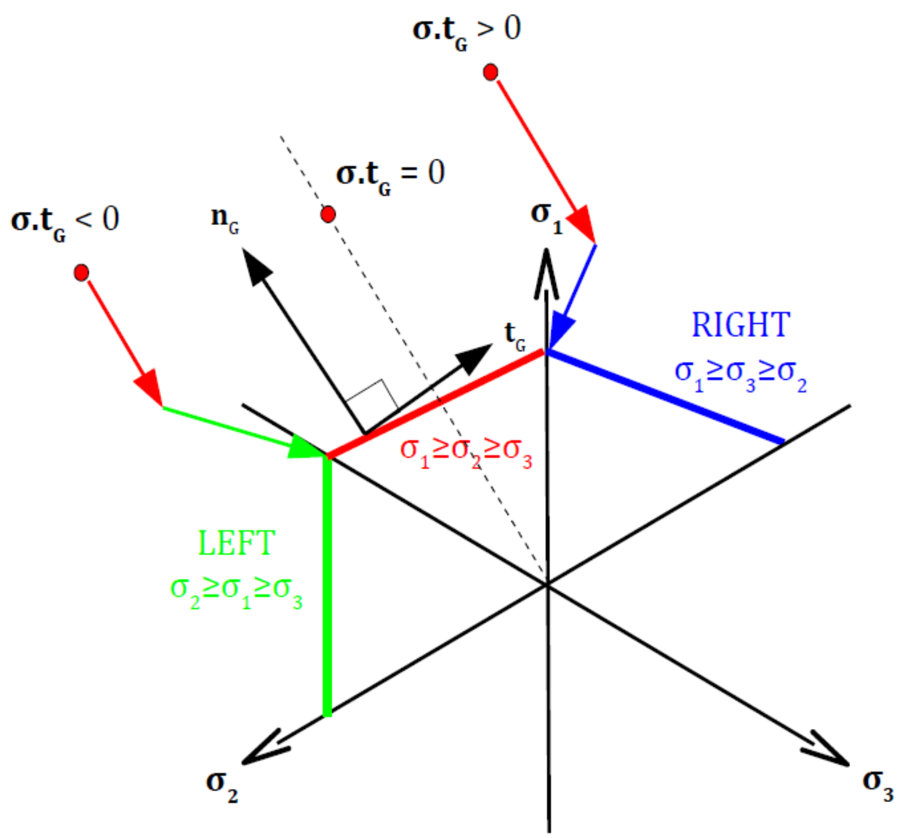

Figure 3.2.2-1: Selection criteria of the mechanisms LEFT and RIGHT

3.3 Case of projection at the top of the cone

It is checked that $\boldsymbol{\sigma}^+$ obtained starting from the preceding stage (35) or (36) always check:

$$\sigma_1^+ \geq \sigma_2^+ \geq \sigma_3^+ \quad (43)$$

If it is not the case, then it is appropriate to carry out a projection at the top of the cone of equation:

$$p_c = c \cot \varphi \quad (44)$$

One thus imposes:

$$\begin{cases} p^+ = p^e - K\, d\varepsilon_v^p := c \cot \varphi \\ \boldsymbol{\sigma}^+ := p^+ \mathbf{1} \end{cases} \quad (45)$$

In the same way, the rate of plastic deformation is written:

$$d\boldsymbol{\varepsilon}^P = d\lambda^1 \boldsymbol{n}_G^{13} + d\lambda^2 \boldsymbol{n}_G^{12} + d\lambda^3 \boldsymbol{n}_G^{23} \quad (46)$$

That is to say:

$$\begin{cases} d\varepsilon_v^p = 2t\left(d\lambda^1 + d\lambda^2 + d\lambda^3\right) = \dfrac{1}{K}\left(p^e - c\cot\varphi\right) \\ d\boldsymbol{e}^p = d\lambda^1 \begin{pmatrix} \dfrac{t}{3}+1 & \dfrac{-2t}{3} & \dfrac{t}{3}-1 \end{pmatrix} + d\lambda^2 \begin{pmatrix} \dfrac{t}{3}+1 & \dfrac{t}{3}-1 & \dfrac{-2t}{3} \end{pmatrix} + d\lambda^3 \begin{pmatrix} \dfrac{-2t}{3} & \dfrac{t}{3}+1 & \dfrac{t}{3}-1 \end{pmatrix} \end{cases} \quad (47)$$

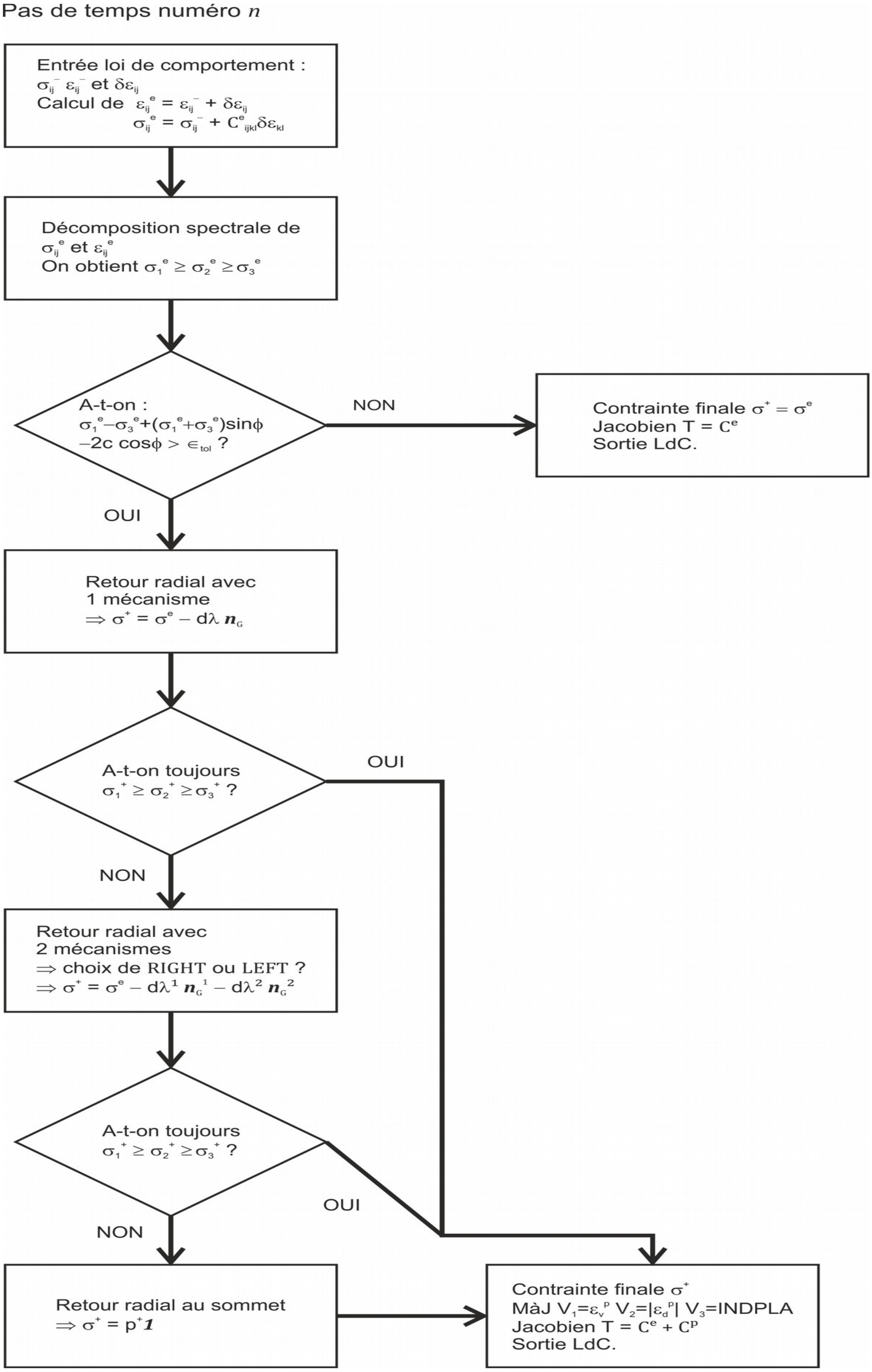

Figure 3.3-1: Synoptic of resolution of the law of Mohr-Coulomb

4 Form of the consistent tangent matrix in the principal base

4.1 Case where only one mechanism is active

The coherent tangent matrix $\boldsymbol{T}$ in the principal base is obtained by deriving the equation (**19**), which gives:

$$d\boldsymbol{\sigma}^{+}=\boldsymbol{C}.d\boldsymbol{\varepsilon}-\frac{2\,dF_{13}(\boldsymbol{\sigma}^{e})}{A}\begin{Bmatrix}\left(K+\frac{G}{3}\right)t+G\\ \left(K-\frac{2}{3}G\right)t\\ \left(K+\frac{G}{3}\right)t-G\end{Bmatrix}=\boldsymbol{C}.d\boldsymbol{\varepsilon}-\frac{2\,dF_{13}(\boldsymbol{\sigma}^{e})}{A}\boldsymbol{v}_{t} \tag{48}$$

Knowing that:

$$dF_{13}(\boldsymbol{\sigma}^{e})=d\sigma_1^e-d\sigma_3^e+\left(d\sigma_1^e+d\sigma_3^e\right)s=2\begin{Bmatrix}\left(K+\frac{G}{3}\right)s+G\\ \left(K-\frac{2}{3}G\right)s\\ \left(K+\frac{G}{3}\right)s-G\end{Bmatrix}.d\boldsymbol{\varepsilon}=2\boldsymbol{v}_{s}.d\boldsymbol{\varepsilon} \tag{49}$$

One obtains:

$$d\boldsymbol{\sigma}^{+}=\underbrace{\left(\boldsymbol{C}-\frac{4}{A}\underbrace{\boldsymbol{v}_{t}\otimes\boldsymbol{v}_{s}}_{\boldsymbol{D}}\right)}_{\boldsymbol{T}}.d\boldsymbol{\varepsilon} \tag{50}$$

Where A is given by the equation (**22**).

4.2 Case where two mechanisms are active

In the same way, the coherent tangent matrix $\boldsymbol{T}$ is obtained by deriving the equations (34) and (38), which gives:

$$d\boldsymbol{\sigma}^{+}=\underbrace{\left(\boldsymbol{C}-\frac{4}{det}\underbrace{\left(\boldsymbol{v}_t^1\otimes\boldsymbol{v}_s^1+\boldsymbol{v}_t^2\otimes\boldsymbol{v}_s^2-\frac{B^{side}}{A}\left(\boldsymbol{v}_t^1\otimes\boldsymbol{v}_s^2+\boldsymbol{v}_t^2\otimes\boldsymbol{v}_s^1\right)\right)}_{\boldsymbol{D}}\right)}_{\boldsymbol{T}}.d\boldsymbol{\varepsilon} \tag{51}$$

Where A is given by the equation (31), B^{side} by (32), det by (33) and $\boldsymbol{v}$ by (39).

4.3 Case of projection at the top of the cone

According to the equation (**45**), one has crudely $\boldsymbol{T}=\boldsymbol{0}$.

5 Form of the consistent tangent matrix in the total base

The paragraph **§4** allows to build the consistent tangent matrix in the principal base, noted $\boldsymbol{T}$. It is advisable from now on to bring back this matrix in the total base (Cartesian), that one will note $\bar{\boldsymbol{T}}$.

Notice important:

It should be noted that the construction of this consistent tangent matrix is a crucial stage at the same time for the robustness and the performance of the algorithm:

- Firstly, it is perfectly known that such a matrix allows a quadratic rate of convergence for the process of Newton;
- Secondly, this matrix gives an account of the rotation of the principal directions during an increment. Without it, the formulation of the law of Mohr-Coulomb in terms of principal constraints described in the paragraph **§2** would not be complete, since principal constraints, maintained fixed during the local integration of the law (**§3**), could not turn on the total level of the structure.

In this paragraph, one describes in detail the method allowing to build $\bar{\boldsymbol{T}}$ from $\boldsymbol{T}$.

5.1 Some results on the isotropic symmetrical tensors of order two

One defines by S^3 the space of symmetrical tensors of order two in the vector space of dimension $n=3$, and tensors $\boldsymbol{Y}\in S^3$ and $\boldsymbol{X}\in S^3$ such as:

$$\boldsymbol{Y}(\boldsymbol{X}):S^3\rightarrow S^3 \tag{52}$$

The tensorial function $\boldsymbol{Y}(\boldsymbol{X})$ is known as isotropic if:

$$\boldsymbol{R}.\boldsymbol{Y}(\boldsymbol{X}).\boldsymbol{R}^t=\boldsymbol{Y}(\boldsymbol{R}.\boldsymbol{X}.\boldsymbol{R}^t) \tag{53}$$

Whatever the rotation $\boldsymbol{R}$. The assumption of isotropy implies that $\boldsymbol{Y}$ and $\boldsymbol{X}$ are coaxial, i.e. qu‘they have the same principal directions $\boldsymbol{d}_{\alpha=1,2,3}$. One notes:

$$\begin{aligned}\boldsymbol{X}&=\sum_{\alpha=1}^{3}x_\alpha\underbrace{(\boldsymbol{d}_\alpha\otimes\boldsymbol{d}_\alpha)}_{\boldsymbol{E}_\alpha}=\sum_{\alpha=1}^{3}x_\alpha\boldsymbol{E}_\alpha\\ \boldsymbol{Y}(\boldsymbol{X})&=\sum_{\alpha=1}^{3}y_\alpha(\boldsymbol{d}_\alpha\otimes\boldsymbol{d}_\alpha)=\sum_{\alpha=1}^{3}y_\alpha\boldsymbol{E}_\alpha\end{aligned} \tag{54}$$

Where $y_\alpha=y_\alpha(x_1,x_2,x_3)$ and x_α they represent eigenvalues of $\boldsymbol{Y}$ and $\boldsymbol{X}$, respectively.

5.2 Derived from an isotropic tensorial function of order two

It is supposed that the isotropic tensorial function $\boldsymbol{Y}(\boldsymbol{X})$ is differentiable compared to $\boldsymbol{X}$, and his derivative is defined $\boldsymbol{D}$ such as:

$$\boldsymbol{D}(\boldsymbol{X})\overset{\text{def}}{=}\frac{d\,\boldsymbol{Y}(\boldsymbol{X})}{d\,\boldsymbol{X}} \tag{55}$$

Applied to the equation (**41**), the following expression is obtained:

$$\boldsymbol{D}(\boldsymbol{X})=\sum_{\alpha=1}^{3}\left(\boldsymbol{E}_\alpha\otimes\frac{d\,y_\alpha}{d\,\boldsymbol{X}}+y_\alpha\frac{d\,\boldsymbol{E}_\alpha}{d\,\boldsymbol{X}}\right)=\sum_{\alpha=1}^{3}\left(y_\alpha\frac{d\,\boldsymbol{E}_\alpha}{d\,\boldsymbol{X}}+\sum_{\beta=1}^{3}\frac{\partial y_\alpha}{\partial x_\beta}\boldsymbol{E}_\alpha\otimes\frac{d\,x_\beta}{d\,\boldsymbol{X}}\right) \tag{56}$$

5.2.1 Two-dimensional case of type forced plane (C_PLAN)

In dimension two (cases C_PLAN), the characteristic equation $det(\boldsymbol{X}-x_{\alpha}\boldsymbol{I})=0$ give a quadratic equation of the eigenvalues $x_{\alpha=1,2}$ of $\boldsymbol{X}$ following type:

$$x_{\alpha}^2-I_1 x_{\alpha}+I_2=0 \text{ with } \alpha=1,2 \quad (57)$$

With:

$$\begin{cases} I_1=trace(\boldsymbol{X})=X_{11}+X_{22} \\ I_2=det(\boldsymbol{X})=X_{11}X_{22}-X_{12}X_{21} \end{cases} \quad (58)$$

The resolution of the spectral problem easily gives the following solutions for the eigenvalues:

$$\begin{cases} x_1=\dfrac{I_1+\sqrt{I_1^2-4I_2}}{2} \\ x_2=\dfrac{I_1-\sqrt{I_1^2-4I_2}}{2} \end{cases} \quad (59)$$

And clean vectors, taking account of the multiplicity of the eigenvalues:

$$\begin{cases} \boldsymbol{E}_{\alpha}=\dfrac{\boldsymbol{X}+(x_{\alpha}-I_1)\boldsymbol{I}}{2x_{\alpha}-I_1} & si \quad x_1\neq x_2 \\ \boldsymbol{E}_1=\boldsymbol{I} & si \quad x_1=x_2 \end{cases} \quad (60)$$

In particular, Carlson and Hoger show that if $x_1\neq x_2$, one a:

$$\frac{d x_{\alpha}}{d\boldsymbol{X}}=\boldsymbol{E}_{\alpha} \quad (61)$$

By using the equations (**59**), (**60**) and (**61**) in (**56**), the expression of the derivative is obtained $\boldsymbol{D}(\boldsymbol{X})$, taking account of the multiplicity of the eigenvalues:

$$\boldsymbol{D}(\boldsymbol{X})=\begin{cases} \dfrac{y_1-y_2}{x_1-x_2}\left[\boldsymbol{I}_S-\boldsymbol{E}_1\otimes\boldsymbol{E}_1-\boldsymbol{E}_2\otimes\boldsymbol{E}_2\right]+\displaystyle\sum_{\alpha=1}^{2}\sum_{\beta=1}^{2}\frac{\partial y_{\alpha}}{\partial x_{\beta}}\boldsymbol{E}_{\alpha}\otimes\boldsymbol{E}_{\beta} & si \quad x_1\neq x_2 \\ \left(\dfrac{\partial y_1}{\partial x_1}-\dfrac{\partial y_1}{\partial x_2}\right)\boldsymbol{I}_S+\dfrac{\partial y_1}{\partial x_2}\boldsymbol{I}\otimes\boldsymbol{I} & si \quad x_1=x_2 \end{cases} \quad (62)$$

With the matrix identity $\boldsymbol{I}$:

$$(\boldsymbol{I})_{ijkl}=\delta_{ik}\delta_{jl} \quad (63)$$

The matrix of transposition $(\boldsymbol{I}_t)_{ijkl}=\delta_{il}\delta_{jk}$ and symmetrization stamps it $\boldsymbol{I}_S$, such as:

$$(\boldsymbol{I}_S)_{ijkl}=\frac{1}{2}(\boldsymbol{I}+\boldsymbol{I}_t)=\frac{1}{2}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}) \quad (64)$$

Note:

It is noticed that the term $\dfrac{y_1-y_2}{x_1-x_2}\left[\boldsymbol{I}_S-\boldsymbol{E}_1\otimes\boldsymbol{E}_1-\boldsymbol{E}_2\otimes\boldsymbol{E}_2\right]$ in the derivative $\boldsymbol{D}(\boldsymbol{X})$ first equation of (**62**) express the rotation of the principal directions in the plan.

5.2.2 Two-dimensional case of plane deformations type (D_PLAN) and axisymmetric (AXIS)

The direction out-plan $\alpha=3$ being fixed, the expression of the derivative $\boldsymbol{D}(\boldsymbol{X})$ is obtained starting from the preceding case. Indeed, by isolating the term $\alpha=3$ in the equation (**56**), there is the following expression:

$$\boldsymbol{D}(\boldsymbol{X})=\underbrace{\sum_{\alpha=1}^{2}\left(y_\alpha\frac{d\boldsymbol{E}_\alpha}{d\boldsymbol{X}}+\sum_{\beta=1}^{2}\frac{\partial y_\alpha}{\partial x_\beta}\boldsymbol{E}_\alpha\otimes\frac{d x_\beta}{d\boldsymbol{X}}\right)}_{\boldsymbol{D}_{2D}(\boldsymbol{X})}+\underbrace{\sum_{\alpha=1}^{2}\frac{\partial y_\alpha}{\partial x_3}\boldsymbol{E}_\alpha\otimes\frac{d x_3}{d\boldsymbol{X}}+\sum_{\beta=1}^{3}\frac{\partial y_3}{\partial x_\beta}\boldsymbol{E}_3\otimes\frac{d x_\beta}{d\boldsymbol{X}}}_{\boldsymbol{D}_3(\boldsymbol{X})} \quad (65)$$

Where $\boldsymbol{D}_{2D}(\boldsymbol{X})$ is given by the equation (**56**). The complementary term $\boldsymbol{D}_3(\boldsymbol{X})$ is written, by taking account of the multiplicity of the eigenvalues, in the following way:

$$\boldsymbol{D}_3(\boldsymbol{X})=\begin{cases}\sum_{\alpha=1}^{2}\left(\frac{\partial y_\alpha}{\partial x_3}\boldsymbol{E}_\alpha\otimes\boldsymbol{E}_3+\frac{\partial y_3}{\partial x_\alpha}\boldsymbol{E}_3\otimes\boldsymbol{E}_\alpha\right)+\frac{\partial y_3}{\partial x_3}\boldsymbol{E}_3\otimes\boldsymbol{E}_3 & si \quad x_1\neq x_2\\ \frac{\partial y_1}{\partial x_3}\boldsymbol{I}_p\otimes\boldsymbol{E}_3+\frac{\partial y_3}{\partial x_1}\boldsymbol{E}_3\otimes\boldsymbol{I}_p+\frac{\partial y_3}{\partial x_3}\boldsymbol{E}_3\otimes\boldsymbol{E}_3 & si \quad x_1=x_2\end{cases} \quad (66)$$

Where $\boldsymbol{I}_p$ is the matrix of the orthogonal projection of $\boldsymbol{I}_S$ in the plan $(\boldsymbol{e}_x,\boldsymbol{e}_y)$:

$$\boldsymbol{I}_p=\begin{cases}\frac{1}{2}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}) & \text{si } i,j,k,l\in\{1,2\}\\ 0 & \text{sinon}\end{cases} \quad (67)$$

5.2.3 Three-dimensional case

In dimension three, the characteristic equation $det(\boldsymbol{X}-x_\alpha\boldsymbol{I})=0$ give a cubic equation of the eigenvalues $x_{\alpha=1,2,3}$ of $\boldsymbol{X}$ following type:

$$x_\alpha^3-I_1x_\alpha^2+I_2x_\alpha-I_3=0 \text{ with } \alpha=1,2,3 \quad (68)$$

With:

$$\begin{cases}I_1=trace(\boldsymbol{X})\\ I_2=\frac{1}{2}\left[trace(\boldsymbol{X})^2-trace(\boldsymbol{X}^2)\right]\\ I_3=det(\boldsymbol{X})\end{cases} \quad (69)$$

The resolution of the spectral problem easily gives the following solutions for the eigenvalues:

$$\begin{cases}x_1=-2\sqrt{Q}\cos\left(\frac{\theta}{3}\right)+\frac{I_1}{3}\\ x_2=-2\sqrt{Q}\cos\left(\frac{\theta+2\pi}{3}\right)+\frac{I_1}{3}\\ x_3=-2\sqrt{Q}\cos\left(\frac{\theta-2\pi}{3}\right)+\frac{I_1}{3}\end{cases} \quad (70)$$

Where Q and θ are given by:

$$Q=\frac{I_1^2-3I_2}{9} \text{ and } \theta=\cos^{-1}\left(\frac{R}{\sqrt{Q^3}}\right) \tag{71}$$

With:

$$R=\frac{-2I_1^3+9I_1I_2-27I_3}{54} \tag{72}$$

And clean vectors, by taking account of the multiplicity of the eigenvalues:

$$\begin{cases} \boldsymbol{E}_\alpha=\dfrac{x_\alpha}{2x_\alpha^3-I_1x_\alpha^2+I_3}\left[\boldsymbol{X}^2+(x_\alpha-I_1)\boldsymbol{X}+\dfrac{I_3}{x_\alpha}\boldsymbol{I}\right] & si \quad x_1\neq x_2\neq x_3 \\ \boldsymbol{E}_\beta=\boldsymbol{I}-\boldsymbol{E}_\alpha & si \quad x_\alpha\neq x_\beta \\ \boldsymbol{E}_1=\boldsymbol{I} & si \quad x_1=x_2=x_3 \end{cases} \tag{73}$$

In the second equation of (**73**), $\boldsymbol{E}_\alpha$ is calculated with the assistance the first equation. Without giving the intermediate stages of calculation, the derivative $\boldsymbol{D}(\boldsymbol{X})$, by taking account of the multiplicity of the eigenvalues, is written finally:

$$\boldsymbol{D}(\boldsymbol{X})=\begin{cases} \displaystyle\sum_{\alpha=1}^{3}\frac{y_\alpha}{(x_\alpha-x_\beta)(x_\alpha-x_\gamma)}\left[\frac{d\boldsymbol{X}^2}{d\boldsymbol{X}}-(x_\beta+x_\gamma)\boldsymbol{I}_S\right. \\ \left.-(2x_\alpha-x_\beta-x_\gamma)\boldsymbol{E}_\alpha\otimes\boldsymbol{E}_\alpha-(x_\beta-x_\gamma)(\boldsymbol{E}_\beta\otimes\boldsymbol{E}_\beta-\boldsymbol{E}_\gamma\otimes\boldsymbol{E}_\gamma)\right] & si \quad x_1\neq x_2\neq x_3 \\ \displaystyle+\sum_{a=1}^{3}\sum_{b=1}^{3}\frac{\partial y_a}{\partial x_b}\boldsymbol{E}_a\otimes\boldsymbol{E}_b & \\ s_1\dfrac{d\boldsymbol{X}^2}{d\boldsymbol{X}}-s_2\boldsymbol{I}_S-s_3\boldsymbol{X}\otimes\boldsymbol{X}+s_4\boldsymbol{X}\otimes\boldsymbol{I}+s_5\boldsymbol{I}\otimes\boldsymbol{X}-s_6\boldsymbol{I}\otimes\boldsymbol{I} & si \quad x_\alpha\neq x_\beta=x_\gamma \\ \left(\dfrac{\partial y_1}{\partial x_1}-\dfrac{\partial y_1}{\partial x_2}\right)\boldsymbol{I}_S+\dfrac{\partial y_1}{\partial x_2}\boldsymbol{I}\otimes\boldsymbol{I} & si \quad x_1=x_2=x_3 \end{cases} \tag{74}$$

Where (α,β,γ) corresponds to a cyclic permutation of $(1,2,3)$. $\boldsymbol{I}$ and $\boldsymbol{I}_S$ are given by the equations (**63**) and (**64**), respectively. By noticing that $\boldsymbol{X}$ is a tensor symmetrical, care should be taken to apply itsymmetrical operator of derivation for the evaluation of $\frac{d\boldsymbol{X}^2}{d\boldsymbol{X}}$, which gives the following form:

$$\begin{aligned} \left(\frac{d\boldsymbol{X}^2}{d\boldsymbol{X}}\right)_{ijkl} &=\frac{d(X_{im}X_{mj})}{dX_{kl}}=\frac{1}{2}(\delta_{ik}\delta_{lm}+\delta_{il}\delta_{km})X_{mj}+\frac{X_{im}}{2}(\delta_{mk}\delta_{jl}+\delta_{ml}\delta_{kj}) \\ &=\frac{1}{2}(\delta_{ik}X_{lj}+\delta_{il}X_{kj}+\delta_{jl}X_{ik}+\delta_{kj}X_{il}) \end{aligned} \tag{75}$$

Lastly, expressions of $s_{i=1,6}$ are the following ones:

$$
\begin{aligned}
s_1 &= \frac{y_\alpha - y_\gamma}{(x_\alpha - x_\gamma)^2} + \frac{1}{x_\alpha - x_\gamma}\left(\frac{\partial y_\gamma}{\partial x_\beta} - \frac{\partial y_\gamma}{\partial x_\gamma}\right)\\
s_2 &= 2\,x_\gamma \frac{y_\alpha - y_\gamma}{(x_\alpha - x_\gamma)^2} + \frac{x_\alpha + x_\gamma}{x_\alpha - x_\gamma}\left(\frac{\partial y_\gamma}{\partial x_\beta} - \frac{\partial y_\gamma}{\partial x_\gamma}\right)\\
s_3 &= 2\frac{y_\alpha - y_\gamma}{(x_\alpha - x_\gamma)^3} + \frac{1}{(x_\alpha - x_\gamma)^2}\left(\frac{\partial y_\alpha}{\partial x_\gamma} + \frac{\partial y_\gamma}{\partial x_\alpha} - \frac{\partial y_\alpha}{\partial x_\alpha} - \frac{\partial y_\gamma}{\partial x_\gamma}\right)\\
s_4 &= 2\,x_\gamma \frac{y_\alpha - y_\gamma}{(x_\alpha - x_\gamma)^3} + \frac{1}{x_\alpha - x_\gamma}\left(\frac{\partial y_\alpha}{\partial x_\gamma} - \frac{\partial y_\gamma}{\partial x_\beta}\right) + \frac{x_\gamma}{(x_\alpha - x_\gamma)^2}\left(\frac{\partial y_\alpha}{\partial x_\gamma} + \frac{\partial y_\gamma}{\partial x_\alpha} - \frac{\partial y_\alpha}{\partial x_\alpha} - \frac{\partial y_\gamma}{\partial x_\gamma}\right)\\
s_5 &= 2\,x_\gamma \frac{y_\alpha - y_\gamma}{(x_\alpha - x_\gamma)^3} + \frac{1}{x_\alpha - x_\gamma}\left(\frac{\partial y_\gamma}{\partial x_\alpha} - \frac{\partial y_\gamma}{\partial x_\beta}\right) + \frac{x_\gamma}{(x_\alpha - x_\gamma)^2}\left(\frac{\partial y_\alpha}{\partial x_\gamma} + \frac{\partial y_\gamma}{\partial x_\alpha} - \frac{\partial y_\alpha}{\partial x_\alpha} - \frac{\partial y_\gamma}{\partial x_\gamma}\right)\\
s_6 &= 2\,x_\gamma^2 \frac{y_\alpha - y_\gamma}{(x_\alpha - x_\gamma)^3} + \frac{x_\alpha x_\gamma}{(x_\alpha - x_\gamma)^2}\left(\frac{\partial y_\alpha}{\partial x_\gamma} + \frac{\partial y_\gamma}{\partial x_\alpha}\right) - \frac{x_\gamma^2}{(x_\alpha - x_\gamma)^2}\left(\frac{\partial y_\alpha}{\partial x_\alpha} + \frac{\partial y_\gamma}{\partial x_\gamma}\right) - \frac{x_\alpha + x_\gamma}{x_\alpha - x_\gamma}\frac{\partial y_\gamma}{\partial x_\beta}
\end{aligned}
\tag{76}
$$

Where (α, β, γ) corresponds to a cyclic permutation of $(1, 2, 3)$.

Note:

It is noticed that the following term (all but commonplace):

$$
\begin{aligned}
\sum_{\alpha=1}^{3} \frac{y_\alpha}{(x_\alpha - x_\beta)(x_\alpha - x_\gamma)}\Bigg[&\frac{d\boldsymbol{X}^2}{d\boldsymbol{X}} - (x_\beta + x_\gamma)\boldsymbol{I}_S - (2\,x_\alpha - x_\beta - x_\gamma)\boldsymbol{E}_\alpha \otimes \boldsymbol{E}_\alpha\\
&-(x_\beta - x_\gamma)(\boldsymbol{E}_\beta \otimes \boldsymbol{E}_\beta - \boldsymbol{E}_\gamma \otimes \boldsymbol{E}_\gamma)\Bigg]
\end{aligned}
\tag{77}
$$

Who appears in the derivative $\boldsymbol{D}(\boldsymbol{X})$ first equation of (**74**) express the rotation of the principal directions in space three-dimensional.

5.2.4 Application to the case of Mohr-Coulomb

The transposition of the preceding formulas to the numeric work implementation deserves some precise details. There are first of all the following correspondences:

- $\boldsymbol{X} = \tilde{\boldsymbol{\varepsilon}}^{pred}$ and $x_\alpha = \varepsilon_\alpha^{pred}$;
- $\boldsymbol{Y} = \tilde{\boldsymbol{\sigma}}^{+}$ and $y_\alpha = \sigma_\alpha^{+}$;
- $\boldsymbol{E}_\alpha = \tilde{\boldsymbol{d}}_\alpha^{pred} \otimes \tilde{\boldsymbol{d}}_\alpha^{pred}$;
- $(\boldsymbol{T})_{\alpha\beta} = \dfrac{\partial y_\alpha}{\partial x_\beta}$ is the consistent tangent matrix in the principal base calculated in the paragraph **§4** ;

The notation ^{pred} indicate that one works with "predicted" sizes given as starter by the process of Newton, the notation $^{+}$ with sizes resulting from the local resolution of the law of behavior, and the notation $\tilde{}$ with the base of Voigt. It will be noted that the predicted principal directions $\tilde{\boldsymbol{d}}_\alpha^{pred}$ are fixed during the local resolution, which is coherent with the assumption of isotropy adopted (see the explanations of the paragraph **§5.1**).

Having all this information at the conclusion of the local resolution of the law of behavior, one from of deduced the consistent tangent matrix $\bar{\boldsymbol{T}} = \bar{\boldsymbol{D}}$ expressed in the base of projection $\bar{\boldsymbol{b}}$ defined in the paragraph **§1.2 :**

- The equation (**62**) in the two-dimensional case in plane constraints (`C_PLAN`);
- The equation (**66**) in the two-dimensional case in deformation planes (`D_PLAN`) or axisymmetric (`AXIS`);
- The equation (**74**) in the three-dimensional case (3D);

The second important information relates to the convention of writing of the various tensors. Indeed, by preoccupation with general information, a notation used for the tensors in all the paragraph **§5** is the classical notation, revealing of the tensors until the order four. This writing is unsuitable with the digital resolution, where one prefers to use condensed notations made possible by the fact that one works with tensors symmetrical of order two (constraints and deformations are it always). One distinguishes two forms from notations condensed correspondent at two bases of projection (see **§1.2**):

- The orthonormal base $\bar{b}$ symmetrical tensors of order two. It is in this base that the constraints and the deformations are given to the entry and the exit of the local resolution of the behavior;
- The base known as of Voigt $\tilde{b}$, much more convenient to use at the time of the local digital resolution of the behavior because it avoids having to handle coefficients in $\sqrt{2}$ at the time of the matric operations;

The diagram of resolution is summarized in figure 5.2.4-1.

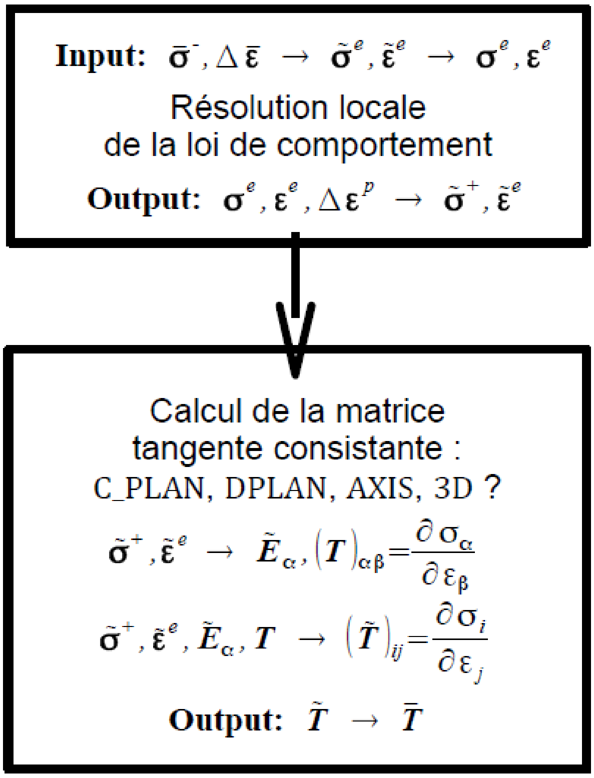

Figure 5.2.4-1: Process of resolution of the law of Mohr-Coulomb. Description of the writing in the various bases of projection.

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