Criteria of mechanical stability

Summary:

This document presents the various criteria of mechanical stability available in *Code_Aster*. One can classify them according to two categories:

- The criterion of stability associated with the conservative systems, which arises as the generalization of the criterion of Euler based on the analysis of the matrix of reactualized total stiffness.
- The criterion of stability associated with the dissipative systems, which must take account of the constraints of irreversibility related to dissipation of energy.

These criteria are used to distinguish in the quasistatic problems, the unstable digital solutions resulting from the calculation of balance carried out in the finite element method (derivative first of worthless but derived energy second negative) of the solutions physical, stable, for which the derivative second of energy is positive.

The criteria presented in this document are directly applicable to the framework of dynamics, but as they take account neither of the matrix of mass nor of that of damping, one cannot speak about dynamic criterion of stability to the classical direction (for example, of negative or null damping becoming).

These criteria are called within the operators `STAT_NON_LINE` and `DYNA_NON_LINE`, to be able to be evaluated with each step of the nonlinear dynamic resolution incremental quasi-static or transitory.

Contents

1 Stability of a conservative system

1.1 Definition of the stability of a conservative system

The position of balance of a conservative system is known as stable if it is invariant under the effect of small disturbances. What amounts checking that the solution obtained is of course a minimum local of the energy potential or known as differently, to check that the functional calculus of Hill [bib19] is conevexe. Mathematically, that results in the checking of the positivity of the derivative second of the potential energy **Φ** at the point of balance **U**. Let us consider a small disturbance of the state of balance v, observing the boundary conditions imposed on the structure. One must always find the inequality:

$$\phi(u) \leq \phi(u+v)$$ éq 1.1 -1

The leading cause of loss of stability for a conservative mechanical structure is buckling. One is thus interested more particularly in following this concept.

Note: There exist other definitions of stability. One finds in particular stability within the meaning of Rice, criterion defines in 1975, which amounts checking the strict positivity of the eigenvalues of the acoustic tensor. However, one can have instability within the meaning of Hill before instability within the meaning of Rice. The criterion within the meaning of Hill is thus more conservative. This is why it is that which one privileges.

1.2 General concept of buckling

Buckling is a phenomenon of instability [bib6]. Its appearance can be observed in particular on slim elements of low stiffness of inflection. Beyond of a certain level of loading, the structure undergoes an important change of configuration (which can appear by sudden appearance of undulations, for example). One distinguishes two types of buckling: buckling by junction and buckling by boundary point ([bib1], [bib7], [bib8]). To describe the behavior of these two types of buckling, one considers a structure of which the parameter μ is characteristic of the loading and of which the parameter δ is characteristic of displacement.

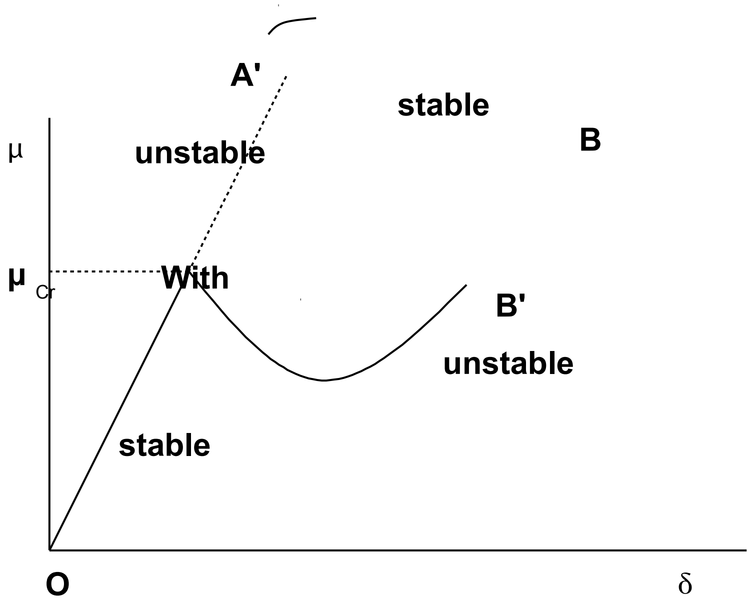

Figure 1.2-a : Buckling by junction

Between the point O and the point A, the structure admits only one family of curve (μ, δ). It can, for example to act of classical linear elasticity or elastoplasticity, where if the problem is well posed (*cf.* [§1.3]), there is the classical result of existence and unicity of the solution. On the other hand, beyond the point A , several families of curves are solutions of the problem of balance. This loss of unicity be accompanied by an instability of the initial branch (known as fundamental). The secondary branch can be stable (curved AB) or unstable (curve AB'). The load beyond which there is junction calls the critical load μ_{cr} . Buckling by junction is characterized by the fact that the mode (or direction of buckling), which initiates the secondary branch, does not generate additional work in the loading applied: mode of buckling being orthogonal to him.

An example of buckling per junction with instability of the secondary branch is in the case of a circular cylindrical hull under axial compression [bib10]. Examples of buckling per junction with stability of the secondary branch are in elastic beams in axial compression, circular rings in radial compression and rectangular plates in longitudinal compression.

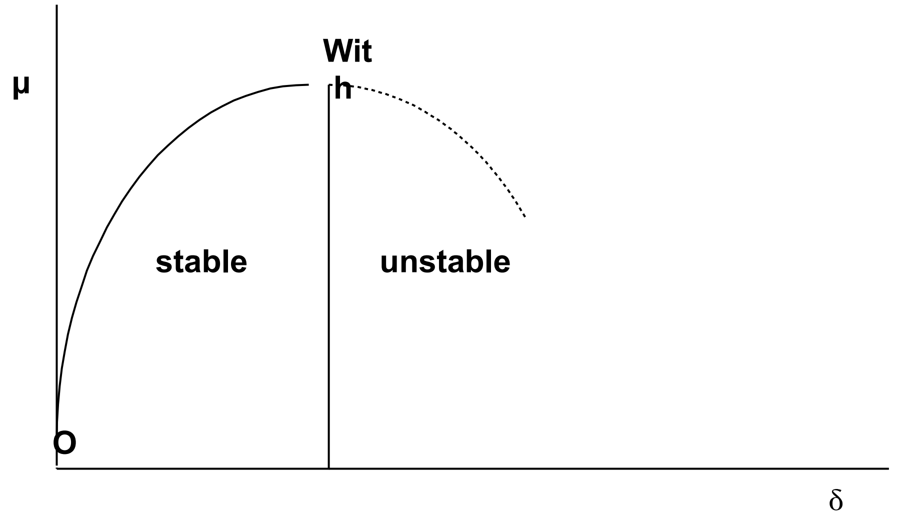

Figure 1.2-b : Buckling by boundary point

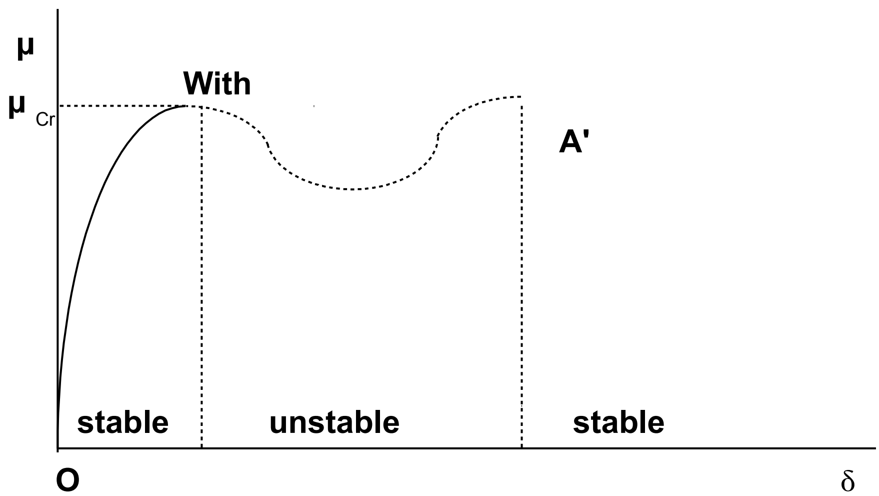

Figure 1.2-c : Buckling by boundary point with breakdown

On the figures [Figure 1.2-b] and [Figure 1.2-c], which illustrates buckling by boundary point, the structure does not admit that only one family (μ, δ) of solution of the equilibrium equations. At the point A , there is loss of stability of the solution with total loss of rigidity in the case of the figure

[Figure 1.2-b] and with a phenomenon of breakdown in the case of the figure [Figure 1.2-c] (the solution becomes again stable after a discontinuity of displacement; case of a segment of a sphere under external pressure). The point A boundary point is then called. The problem thus amounts in all the cases seeking the load from which the fundamental branch of balance becomes unstable or of dubious stability. That generally mobilizes great displacements. One can finally have the case of the ruin by plastic flow which is connected at the boundary point [Figure 1.2-b].

Code_Aster the research of the modes of linear buckling, qualified allows method of Euler. It is enough to solve a problem generalized with the eigenvalues (thanks to the operator `CALC_MODES` with the keyword `TYPE_RESU=' MODE_FLAMB'`). The two matrices arguments of the generalized problem are the matrix of rigidity and the matrix of geometrical rigidity, resulting from a linear elastic preliminary calculation (operator `MECA_STATIQUE`).

In all the cases where one cannot neglect nonthe linearities, which they, the approach Euler is geometrical or behavioral is not more valid.

We thus propose a criterion *ad hoc*, that one can regard as a generalization of the criterion of Euler on reactualized configuration. This criterion is built on the matrix of assembled tangent stiffness, which is calculated in the algorithm of the Newton type to solve the nonlinear quasi-static problems (operator `STAT_NON_LINE`) or dynamic nonlinear transients (operator `DYNA_NON_LINE`). This criterion, into nonlinear, makes it possible to treat rigorously the relations of nonlinear elastic behavior. On the other hand, the laws which present a dissipative aspect are treated rigorously only if the loading, in any point of the structure, follows a monotonous evolution (that corresponds to the assumption of Hill [bib4]).

1.3 Writing mechanical problem

This chapter aims to introduce the formalism general of structural analysis adapted to the nonlinear mechanical problem which we wish to tackle.
To start, we thus briefly will point out the setting in equation of a standard problem of structural analysis. To simplify, we place ourselves, all at least at the beginning, within the framework of the small disturbances.

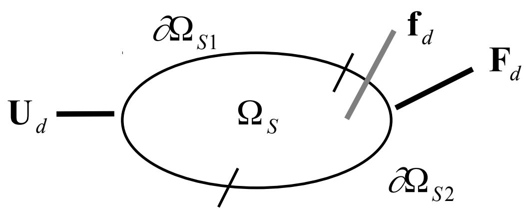

Figure 1.3-a : Representation of a problem of structural analysis

The structure Ω_S is subjected to imposed voluminal efforts $\mathbf{f}_d$, surface efforts $\mathbf{F}_d$ on the edge $\partial\Omega_{S2}$ and of imposed displacements $\mathbf{U}_d$ on the rest of the edge of Ω_S, noted $\partial\Omega_{S1}$.
The unknown factors of the problem of reference on the solid are the field of displacement $\mathbf{u}$ and the stress field of Cauchy $\boldsymbol{\sigma}$.

The solution ($\mathbf{u}, \boldsymbol{\sigma}$) problem of structure where the heating effects are neglected defines as:

To find $(\mathbf{u},\boldsymbol{\sigma})\in\mathbf{H_1}(\Omega_S)\times\mathbf{L^2}(\Omega_S)$ who checks:

- Equations of connections:

$$\mathbf{u}|_{\partial\Omega_{S1}}=\mathbf{U_d} \qquad \text{éq 1.3 - 1}$$

- Relation of behavior:

$$\boldsymbol{\sigma}=f(\boldsymbol{\varepsilon}) \text{ with } \boldsymbol{\varepsilon} \text{ who is the tensor of deformation} \qquad \text{éq 1.3 -2}$$

$$\boldsymbol{\varepsilon}=\frac{1}{2}(\nabla\mathbf{u}+\nabla\mathbf{u}^{\mathbf{T}}) \text{ in assumption of small disturbances} \qquad \text{éq 1.3 - 3}$$

If a linear elastic behavior is supposed

$$\boldsymbol{\sigma}=\mathbf{C}:\boldsymbol{\varepsilon} \qquad \text{éq 1.3 - 4}$$

- Equilibrium equations:

$$\begin{cases} \rho\,\boldsymbol{\gamma}=\nabla.\boldsymbol{\sigma}+\mathbf{f}_d \quad \text{avec}\,\boldsymbol{\gamma}=\dfrac{\mathrm{d}^2\mathbf{u}}{\mathrm{d}t^2} \\ \boldsymbol{\sigma}.\mathbf{n}\big|_{\partial\Omega_{S2}}=\mathbf{F_d} \end{cases} \qquad \text{éq 1.3 - 5}$$

1.4 Study of stability

The object of this chapter is to present the methods making it possible to determine the stability of the nonlinear quasi-static balance of a structure in a conservative system. To start, we are interested only in detection of instability, or more exactly in the loss of unicity of the solution [bib6]. Among recent work of synthesis, one can quote [bib9] or [bib7] and [bib8] which presents very complete papers on the nonlinear analysis of stability of the structures.
The calculation of the post-critical solution will not be approached.

To analyze stability, we introduce an initial configuration of reference Ω_{S0}, a current configuration Ω_S and a disturbed configuration Ω_{S1}:

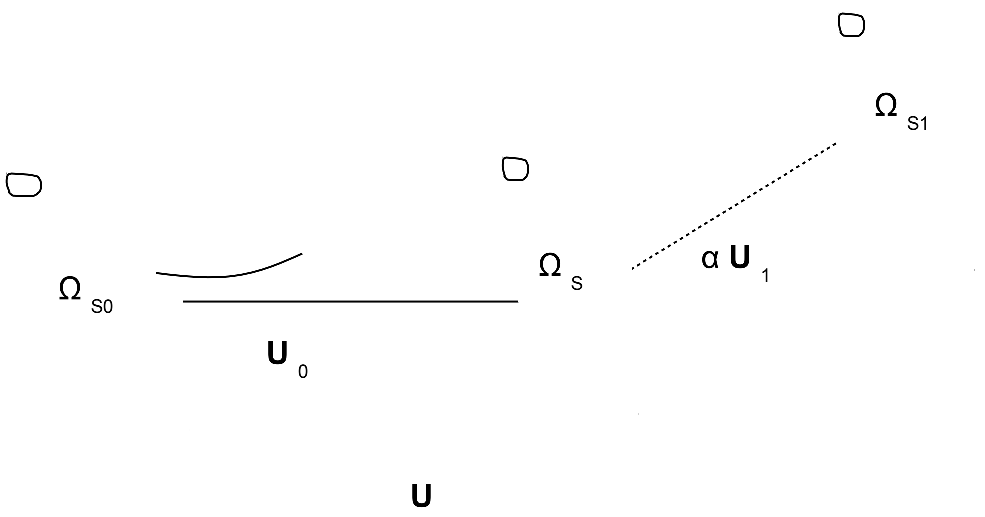

Figure 1.4-a : Definition of the various configurations

That is to say $\mathbf{u}$ the field of displacement of the points of the structure. The behavior is supposed, for the moment, linear elastic isotropic. The structure subjected to imposed displacements and efforts will become deformed and become the structure located by the current configuration Ω_S. We seek to determine a state of balance characterized by the field of displacement between the initial configuration Ω_{S0} and current configuration Ω_S, as well as a stress field of Cauchy, noted $\boldsymbol{\sigma}$, or of Piola-Kirchhoff II, noted $\boldsymbol{\pi}$:

$$\boldsymbol{\pi}=\det\mathbf{F}\,.\,\mathbf{F}^{-1}\boldsymbol{\pi}_{\mathbf{I}} \text{ with } \begin{cases}\mathbf{F}=\nabla\,\mathbf{u}+\mathbf{I} : \text{tenseur gradient de la transformation}\\ \det\mathbf{F}=\dfrac{\rho_0}{\rho}\\ \boldsymbol{\pi}_{\mathbf{I}} : \text{tenseur de Piola - Kirchhoff I}\end{cases} \Rightarrow\boldsymbol{\pi}=\frac{\rho_0}{\rho}.\,\mathbf{F}^{-1}.\,\boldsymbol{\sigma}\,.\,\mathbf{F}^{-T}$$

éq 1.4 - 1

In this expression, one sees appearing the relationship between the initial density ρ_0 and current density ρ .
The following stage is the prediction of the stability of this balance.

To this end, we will seek a criterion allowing to determine if there exists only one field of displacement balancing the efforts applied. We will suppose that the efforts increase gradually and we will seek to find as from which moment there exist two configurations Ω_S and Ω_{S1} who respect the equations of the problem: we seek a point of junction, it is - with-to say a loss of unicity of the solution. This moment will be described as moment of buckling.

1.4.1 Writing of the elastic geometrical nonlinear problem

The solution $\mathbf{u}\,,\boldsymbol{\pi}$ problem of structure without heating effects checks ([bib1], [bib7], [bib2]):

- Equations of connections:

$$\mathbf{u}\,|_{\partial\Omega_{s0}}=\mathbf{U_d} \qquad \text{éq 1.4.1-1}$$

- Elastic relation of behavior:

$$\boldsymbol{\pi}=\varphi_{,\varepsilon}(\boldsymbol{\varepsilon}) \qquad \text{éq 1.4.1-2}$$

with $\boldsymbol{\varepsilon}$ who is the tensor of deformation. If a linear elastic behavior is supposed:

$$\boldsymbol{\pi}=\mathbf{C}\,\boldsymbol{\varepsilon} \qquad \text{éq 1.4.1-3}$$

- Equilibrium equations:

$$\begin{cases}\rho\,\boldsymbol{\gamma}=\nabla\,.\,\boldsymbol{\pi}+\mathbf{f}_d \quad \text{avec}\,\boldsymbol{\gamma}=\dfrac{\mathrm{d}^2\mathbf{u}}{\mathrm{d}\,t^2}\\ \mathbf{F}\,.\,\boldsymbol{\pi}\,.\,\mathbf{n_0}\,|_{\partial\Omega_{s0}}=\mathbf{F}_d\end{cases} \qquad \text{éq 1.4.1-4}$$

The associated tensor of deformation is that of Green-Lagrange (referred with the initial configuration):

$$\begin{aligned}&\boldsymbol{\varepsilon}(\mathbf{u})=\frac{1}{2}(\mathbf{F}^T\,\mathbf{F}-\mathbf{I}) \quad \text{avec}\,\mathbf{F}=\nabla\,\mathbf{u}+\mathbf{I}\\ &\Rightarrow\boldsymbol{\varepsilon}(\mathbf{u})=\boldsymbol{\varepsilon}^{\boldsymbol{L}}(\mathbf{u})+\frac{1}{2}\,\boldsymbol{\varepsilon}^{\boldsymbol{Q}}(\mathbf{u}\,,\mathbf{u})\end{aligned} \qquad \text{éq 1.4.1-5}$$

with: $$\begin{cases} \boldsymbol{\varepsilon}^{L}(\mathbf{u})=\frac{1}{2}(\nabla\mathbf{u}+\nabla^{T}\mathbf{u}) : \text{partie linéaire} \\ \boldsymbol{\varepsilon}^{Q}(\mathbf{u},\mathbf{u})=\nabla^{T}\mathbf{u}.\nabla\mathbf{u} : \text{partie quadratique} \end{cases}$$ éq 1.4.1-6

We can now write the Principle of the Virtual Powers in geometrical nonlinear elasticity and quasi-static:

$$p^{intal\,\text{int}} - p^{ext}=0, \forall\,\mathbf{u}^{*}\text{C A 0}$$

$$\text{Avec :}\begin{cases} \mathrm{p}^{\text{int}}=\int_{\Omega_{s0}} Tr(\boldsymbol{\pi}\,\boldsymbol{\varepsilon}^{*})\,d\Omega=\int_{\Omega_{s0}} Tr\left[\left(\boldsymbol{\varepsilon}^{L}(\mathbf{u})+\frac{1}{2}\boldsymbol{\varepsilon}^{Q}(\mathbf{u},\mathbf{u})\right)\mathbf{C}(\varepsilon^{L}(\mathbf{u}^{*})+\boldsymbol{\varepsilon}^{Q}(\mathbf{u},\mathbf{u}^{*}))\right]d\Omega \\ \mathrm{p}^{\text{ext}}=\int_{\partial\Omega_{s0}} \mathbf{F}_{d}.\mathbf{u}^{*}dS+\int_{\Omega_{s0}}\mathbf{f}_{d}.\mathbf{u}^{*}d\Omega \end{cases}$$

éq 1.4.1-7

In order to obtain a discretized formulation, one can rewrite the tensor of deformation:

$$\begin{cases} \boldsymbol{\varepsilon}(\mathbf{u})=\left[\mathbf{B}^{\mathbf{L}}+\frac{1}{2}\mathbf{B}^{\mathbf{NL}}(\mathbf{u})\right].\mathbf{u} \\ \boldsymbol{\pi}=\mathbf{C}\,\boldsymbol{\varepsilon}(\mathbf{u})\,\text{avec}\,\boldsymbol{\pi}\,\text{qui est le tenseur de Piola - Kirchhoff II} \end{cases}$$ éq 1.4.1-8

The power of the internal efforts becomes:

$$\mathbf{P}^{int}=\int_{\Omega_{SO}} Tr\left[\pi.[\mathbf{B}^{\mathbf{L}}+\mathbf{B}^{\mathbf{NL}}(\mathbf{u})]^{T}\mathbf{u}^{*}\right]d\Omega$$ éq 1.4.1-9

By taking account of the relation of behavior [éq 1.4.1-3]:

$$\mathbf{P}^{int}=\int_{\Omega_{s0}} Tr\left[[\mathbf{B}^{\mathbf{L}}+\frac{1}{2}\mathbf{B}^{\mathbf{NL}}(\mathbf{u})]^{T}\mathbf{C}[\mathbf{B}^{\mathbf{L}}+\mathbf{B}^{\mathbf{NL}}(\mathbf{u})]\mathbf{u}.\mathbf{u}^{*}\right]d\Omega$$ éq 1.4.1-10

After discretization by the finite elements, one can put this equation in matric form:

$$\mathbf{u}^{*}.[\mathbf{K_0}+\mathbf{K}^{\mathbf{L}}(\mathbf{u})+\mathbf{K}^{\mathbf{Q}}(\mathbf{u})].\mathbf{u}=\mathbf{P}^{\mathbf{ext}}$$ éq 1.4.1-11

The matrix $\mathbf{K}^{\mathbf{L}}$ is symmetrical and there are the following expressions:

$$\begin{cases} \mathbf{K_0}=\int_{\Omega_{s0}} \mathbf{B}^{L^T}\mathbf{C}\mathbf{B}^{L}d\Omega \\ \mathbf{K}^{L}=\int_{\Omega_{s0}}\left[\frac{1}{2}\mathbf{B}^{NL}(\mathbf{u})^{T}\mathbf{C}\mathbf{B}^{L}+\mathbf{B}^{L^T}\mathbf{C}\mathbf{B}^{L}(\mathbf{u})\right]d\Omega \\ \mathbf{K}^{Q}=\frac{1}{2}\int_{\Omega_{s0}}\mathbf{B}^{NL}(\mathbf{u})^{T}\mathbf{C}\mathbf{B}^{NL}d\Omega \end{cases}$$ éq 1.4.1-12

One obtains directly what precedes the writing in matric form by balance:

$$[\mathbf{K_0}+\mathbf{K}^{\mathbf{L}}(\mathbf{u})+\mathbf{K}^{\mathbf{Q}}(\mathbf{u})].\mathbf{u}=\mathbf{F}^{\mathbf{ext}}$$ éq 1.4.1-13

That is to say still, in an equivalent way:

$$\mathbf{F}^{int}=\mathbf{F}^{\mathbf{ext}} \quad \text{avec} \quad \mathbf{F}^{int}=\int_{\Omega_{S0}} [\mathbf{B}^{\mathbf{L}}+\mathbf{B}^{\mathbf{NL}}(\mathbf{u})]^t \boldsymbol{\pi}\, d\, OMEGA \qquad \text{éq 1.4.1-14}$$

We can just as easily formulate the Principle of the Virtual Powers starting from the state of stress of Cauchy and of the tensor of deformation of Almansi (thus on the current configuration). One obtains then:

$$\int_{\Omega_S} Tr(\boldsymbol{\sigma}\,\boldsymbol{\varepsilon}(\mathbf{u}^*))\, d\,\Omega=\int_{\partial\Omega_S} \mathbf{F}_d.\mathbf{u}^*\, dS+\int_{\Omega_S} \mathbf{f}_d.\mathbf{u}^*\, d\,\Omega \qquad \text{éq 1.4.1-15}$$

That one can also put in the following form, after discretization:

$$\int_{\Omega_S} \mathbf{B}^{\mathbf{T}}\boldsymbol{\sigma}\, d\,\Omega=\mathbf{F}^{\text{int}}=\mathbf{F}^{ext} \qquad \text{éq 1.4.1-16}$$

That is to say still, by supposing the elastic relation of behavior:

$$\mathbf{K}\,\mathbf{u}=\mathbf{F}^{ext} \quad \text{avec}\, \mathbf{K}=\int_{\Omega_S} \mathbf{B}^{\mathbf{T}}\mathbf{C}\,\mathbf{B}\, d\,\Omega \qquad \text{éq 1.4.1-17}$$

The integrals of these equations are calculated on current volume Ω_S who depends, of course, of the field of solution displacement $\mathbf{u}$. In the same way, the operator $\mathbf{B}$ must be calculated on the current configuration Ω_S and not on the initial configuration Ω_{S0}, as it was the case previously.

1.4.2 Study of stability into nonlinear geometrical

One will seek if there exists a second field of displacement kinematically acceptable which checks the equilibrium equations: one thus seeks to know if there will be junction.

This second field will be written as the sum of a disturbance added to the first solution, is: $\mathbf{u}=\alpha\,\mathbf{u_1}$, with α who is a very small reality and which one will make tend towards 0. The field $\mathbf{u_1}$ is selected kinematically acceptable to 0.

The Principle of the Virtual Powers will be then written for this new field.

The field of deformation is put in the form:

$$\boldsymbol{\varepsilon}(\mathbf{u}+\alpha\,\mathbf{u_1})=\boldsymbol{\varepsilon}(\mathbf{u})+\alpha\left[\boldsymbol{\varepsilon}^{L}(\mathbf{u_1})+\frac{1}{2}(\boldsymbol{\varepsilon}^{Q}(\mathbf{u_1},\mathbf{u}))\right]+\frac{\alpha^2}{2}\boldsymbol{\varepsilon}^{q}(\mathbf{u_1},\mathbf{u_1}) \qquad \text{éq 1.4.2-1}$$

The virtual deformations are given by:

$$\boldsymbol{\varepsilon}_1^*=\boldsymbol{\varepsilon}^{L}(\mathbf{u}^*)+\boldsymbol{\varepsilon}^{q}(\mathbf{u},\mathbf{u}^*)+\alpha\,\boldsymbol{\varepsilon}^{Q}(\mathbf{u_1},\mathbf{u}^*)=\boldsymbol{\varepsilon}(\mathbf{u}^*)+\alpha\,\boldsymbol{\varepsilon}^{Q}(\mathbf{u_1},\mathbf{u}) \qquad \text{éq 1.4.2-2}$$

In the same way, if we choose Ω_{S0} like configuration of reference, the constraints become:

$$\boldsymbol{\pi}_1=\boldsymbol{\pi}+\alpha\,\mathbf{C}\left[\boldsymbol{\varepsilon}^{L}(\mathbf{u_1})+\frac{1}{2}(\boldsymbol{\varepsilon}^{Q}(\mathbf{u},\mathbf{u_1})+\boldsymbol{\varepsilon}^{Q}(\mathbf{u_1},\mathbf{u}))\right]+\frac{\alpha^2}{2}\mathbf{C}\,\boldsymbol{\varepsilon}^{Q}(\mathbf{u_1},\mathbf{u_1}) \qquad \text{éq 1.4.2-3}$$

We can now express the Principle of the Virtual Powers for the field of disturbed displacement. Let us take as assumptions that the imposed forces do not depend on displacement and that the initial configuration is selected like reference.

$$\begin{cases} P_1^{int}=P^{int} \\ +\alpha\left[\int_{\Omega_{S0}} Tr\left(\boldsymbol{\pi}\ \boldsymbol{\varepsilon}^Q(\mathbf{u_1},\mathbf{u}^*)\right)d\Omega+\int_{\Omega_{S0}} Tr\left[\boldsymbol{\varepsilon}\ (\mathbf{u}^*)\mathbf{C}\left(\boldsymbol{\varepsilon}^L(\mathbf{u_1})+\frac{1}{2}(\boldsymbol{\varepsilon}^Q(\mathbf{u},\mathbf{u_1})+\boldsymbol{\varepsilon}^Q(\mathbf{u_1},\mathbf{u}))\right)\right]d\Omega\right]+o(\alpha) \\ P_1^{ext}=P^{ext} \\ P_1^{int}-P_1^{ext}=0 \end{cases}$$

éq 1.4.2-4

For α sufficient small, it will be enough that the term proportional to α in the expression [éq 1.4.2-4] is null so that the Principle of the Virtual Powers is checked for the field $\mathbf{u}=\alpha\mathbf{u_1}$. In this case, there will not be thus more unicity of the solution, which will translate the loss of stability of the system.

When the imposed efforts do not depend on the geometrical configuration, the study of stability is thus stated like:

Knowing the actual position, i.e the field of displacement $\mathbf{u}$ kinematically acceptable and the stress field $\boldsymbol{\pi}$, if there exists a field of displacement $\mathbf{u_1}$ kinematically acceptable to 0 and such as, for any displacement $\mathbf{u}^*$ kinematically acceptable to 0, one has:

$$\int_{\Omega_{S0}} Tr(\boldsymbol{\pi}\ \boldsymbol{\varepsilon}^Q(\mathbf{u_1},\mathbf{u}^*))d\Omega$$
$$+\int_{\Omega_{S0}} Tr\left[\boldsymbol{\varepsilon}^L(\mathbf{u}^*)\mathbf{C}\boldsymbol{\varepsilon}^L(\mathbf{u_1})+\boldsymbol{\varepsilon}^Q(\mathbf{u},\mathbf{u}^*)\mathbf{C}\varepsilon^L(\mathbf{u_1})+\varepsilon^L(\mathbf{u}^*)\mathbf{C}\boldsymbol{\varepsilon}^Q(\mathbf{u},\mathbf{u_1})+\boldsymbol{\varepsilon}^Q(\mathbf{u},\mathbf{u}^*)\mathbf{C}\boldsymbol{\varepsilon}^Q(\mathbf{u},\mathbf{u_1})\right]d\Omega$$
$$=0$$

éq 1.4.2-5

Then the problem considered is unstable.

One can express this condition of junction in matric form by introducing, moreover, the geometrical matrix of stiffness $\mathbf{K}(\boldsymbol{\pi})$ who discretizes the first term of it:

$$\begin{aligned}&\forall\ \mathbf{u}^*\mathrm{CA}\ 0, \mathbf{u}^{*T}\mathbf{K_t}\mathbf{u_1}=0\\ &\text{Avec}\quad \mathbf{K}_T=\mathbf{K}_0+\mathbf{K}^L(\mathbf{u})+\mathbf{K}^Q(\mathbf{u})+\mathbf{K}(\boldsymbol{\pi})\ \ \text{qui est la raideur tangente}\end{aligned}$$

éq 1.4.2-6

If one writes the condition of junction on the current configuration Ω_S, then one a:

$$\forall\ \mathbf{u}^*\mathrm{CA}\ 0, \mathbf{u}^{*T}[\boldsymbol{K}+\boldsymbol{K}(\boldsymbol{\sigma})]\mathbf{u_1}=0$$

éq 1.4.2-7

The constraint to be considered is then the constraint of Cauchy and all the integrals are evaluated on the current field Ω_S.

1.4.2.1 Stability condition of a nonlinear elastic balance

It comes immediately, that if there exists a state such that the tangent matrix $\mathbf{K_T}$ defined above is singular, we will have displayed a field of displacement well $\mathbf{u}_1$ not no one which shows the loss of unicity of the solution of the mechanical problem. This field of displacement is the mode of buckling. One can notice that the condition of junction is well checked, whatever the standard and it sign of $\mathbf{u}_1$:

in this direction, one thus speaks about mode of buckling, like direction, because one limited oneself in [éq 1.4.2-4] to the first order in α .

1.4.2.2 Case of small displacements: load of Euler

When displacements can be qualified the small ones before buckling, one can confuse the initial configuration with the current geometry. Matrices $\mathbf{K}^L$ and $\mathbf{K}^Q$ can then be neglected. Moreover, the constraint $\boldsymbol{\pi}$ can be confused with the usual constraint $\boldsymbol{\sigma}$; the equations of buckling are written then:

$$[\mathbf{K}_0+\mathbf{K}(\boldsymbol{\sigma})]\mathbf{u}_1=0 \qquad \text{éq 1.4.2.2 - 1}$$

It is advisable to notice that the matrix $\mathbf{K}(\boldsymbol{\sigma})$ is proportional to $\boldsymbol{\sigma}$ and thus with the loading applied to the structure. If one multiplies the constraint by λ , one obtains:

$$[\mathbf{K}_0+\lambda\mathbf{K}(\boldsymbol{\sigma})]\mathbf{u}_1=0 \qquad \text{éq 1.4.2.2 - 2}$$

This equation immediately makes think of a problem generalized with the eigenvalues, of the same type as in the case of the research of the modes of vibration, which is written:

$$[\mathbf{K}_0-\omega^2\mathbf{M}]\mathbf{v}_1=0 \qquad \text{éq 1.4.2.2 - 3}$$

The matrix $\mathbf{K}(\boldsymbol{\sigma})$ is replaced by the matrix of mass M , and one sees appearing the own pulsation ω , whereas $\mathbf{v}_1$ is the associated mode of vibration.

If one wishes to study buckling under a loading of which only a part is controlled (variable part of the loading), by a principle of superposition, the contribution, constant, loading not controlled must be added at the end $\mathbf{K}_0$ and only the constraint generated by the controlled loading will be in the term in λ . Formally, the following problem is thus posed:

$$[\mathbf{K}_0+\mathbf{K}(\boldsymbol{\sigma}_{cte})+\lambda\,\mathbf{K}(\boldsymbol{\sigma}_{var})]\,\mathbf{u}_1$$

$$\text{Avec :}\begin{cases}\boldsymbol{\sigma}_{cte}\ \text{: contrainte générée par le chargement non piloté}\\ \boldsymbol{\sigma}_{var}\text{: contrainte générée par le chargement piloté}\end{cases} \qquad \text{éq 1.4.2.2 - 4}$$

The two stress fields are obtained by resolution of two linear problems, one for the loading not controlled, the other for the controlled part of the total loading (*cf.* [U2.08.04] and [bib17]).

1.4.2.3 Typical case of the imposed forces depend on the geometry

Example of the following pressures:
When the external forces depend on the configuration, that involves that the work of the external forces intervenes under the stability condition. Let us take the example of a pressure applied to the structure. This pressure will be supposed to be constant during buckling: in other words, the value of pressure does not change during displacement.
This assumption corresponds to two types of real problems. The first type is that where the volume of the fluid imposing the pressure on the structure is very large in front of the variations of volume generated by the displacement of the solid. The problems of pressure tanks inboard, where displacements of walls are considerable compared to dimensions of the structure itself, thus do not return within this framework.

The second case corresponds to the existence of a source of fluid which makes it possible to keep the pressure with a constant value. It is not then necessary any more to worry about the amplitude of the displacement of the solid.
The value of pressure being taken fixes, the variation of the normal in the course of time is to be taken into account. This variation is due to the field of displacement which modifies the surface of the structure. In the same way, if one reasons in terms of resultant and thus of integral, the element of surface can also change surface. Consequently, the resultant of the compressive forces will vary and it is advisable to take account of it.

It is however difficult to display the existence of a potential. This is why one is reduced to the conservative case.

We see quickly that the power of the efforts, expressed on the current configuration, associated with a pressure is given by the following equation (see for example [bib11]):

$$\mathrm{P}_{pression}^{ext}=\int_{\partial\Omega_{SP}} p\left[\mathbf{n}+\alpha\frac{dS_1}{dS}\mathbf{n_1}\right].\mathbf{u}^* dS \qquad \text{éq 1.4.2.3 - 1}$$

In this equation, we notice that the power of the external efforts is modified in displacement $\alpha\mathbf{u_1}$. We will have then:

$$\mathrm{P}_1^{ext}=\mathrm{P}^{ext}+\int_{\partial\Omega_{SP}} p\,\alpha\,\mathbf{n_1}.\mathbf{u}^* dS_1 \qquad \text{éq 1.4.2.3 - 2}$$

Finally, the matrix $\mathbf{K}_{\mathrm{T}}$ is enriched by an additional term, function of the pressure:

$$\mathbf{K}_T=\mathbf{K}_0+\mathbf{K}^L(\mathbf{u})+\mathbf{K}^Q(\mathbf{u})+\mathbf{K}(\boldsymbol{\pi})+\mathbf{K}(p) \qquad \text{éq 1.4.2.3 - 3}$$

If one writes the operators on the current geometry, one leads to:

$$\mathbf{K}_T=\mathbf{K}+\mathbf{K}(\boldsymbol{\sigma})+\mathbf{K}(p) \qquad \text{éq 1.4.2.3 - 4}$$

When we are in the presence of following compressive forces, same methods that those presented previously will be able to apply to calculate the buckling loads: it will be enough to supplement the matrix $\mathbf{K}_{\mathrm{T}}$ with the new term $\mathbf{K}(p)$. One can show that the matrix $\mathbf{K}(p)$ is symmetrical if the compressive forces do not work on the “edge” of the model.

1.4.2.4 Vibrations under prestressing

Same methodology can also under investigation apply vibrations of the structure in the current configuration Ω_S . This structure is prestressed and deformed. It is enough to write the Principle of the Virtual Powers nonlinear geometrical [éq 1.4.1-7] by taking account of the effects of inertia and by injecting the assumption there that displacements are of the periodic functions of the type:

$$\mathbf{u}_1(t)=\mathbf{v}_1\sin(\omega t) \qquad \text{éq 1.4.2.4 - 1}$$

It results from this:

$$[\mathbf{K}_0+\mathbf{K}^L(\mathbf{u})+\mathbf{K}^Q(\mathbf{u})+\mathbf{K}(\boldsymbol{\pi})+\mathbf{K}(p)-\omega^2\mathbf{M}]\mathbf{v}_1=0 \qquad \text{éq 1.4.2.4 - 2}$$

First of all, we notice, in this equation, that when we have a critical condition then the Eigen frequency of vibration of the structure corresponding to the mode of buckling is worthless.
Moreover, we observe that the Eigen frequencies of the structure charged are different from those of the initial structure for two reasons:

1. The own pulsation ω is modified by prestressing p : it is the principal effect which is used, for example, to grant a violin. The tension of the cord exploits the height of the corresponding note, therefore on its Eigen frequency.
2. A second effect is the variation of the frequency by modification of the geometry: the geometrical matrix of starting stiffness $\mathbf{K}_0$ is replaced by the matrix of stiffness on the current geometry: $\mathbf{K}_0+\mathbf{K}^L+\mathbf{K}^Q$. What causes to modify the vibratory equations.

The operator DYNA_NON_LINE allows to carry out vibratory analyses on the current nonlinear configuration (keyword MODE_VIBR), but without taking into account of prestressing for the moment.

1.5 Implementation in the code

In any rigour, in order to make sure of the analysis of stability of a nonlinear quasi-static calculation, it is necessary to use the criterion of stability *ad hoc* with each step of incremental calculation. Any criterion of nonlinear stability must thus be intrinsically the least expensive possible in time CPU and place memory.
Speaking Algorithmiquement, it appears judicious to establish the call to the criterion inside even of the routine corresponding to the operator STAT_NON_LINE [bib15]. Indeed, the principle of call to each step puts up badly with a completely outsourced call to the incremental method of resolution of the nonlinear mechanical problem.

1.6 Criterion of Euler

This criterion (cf [§ 1.4.2.2]) requires only the resolution of a linear static problem, then the construction and the assembly of the geometrical matrix of stiffness. This one and stamps it assembled stiffness are then to pass like argument of a solvor [bib12] for the problem to the eigenvalues [éq 1.4.2.2 - 2].

At exit one thus recovers the modes of buckling and the critical loads corresponding. For more details, the user will be able usefully to consult the document [U2.08.04] [bib17].

1.7 Nonlinear criterion

1.7.1 Impact on the operator STAT_NON_LINE

Let us start by briefly pointing out the operation of the incremental method of resolution of the nonlinear problems of structure [bib15].

1.7.1.1 Algorithm of STAT_NON_LINE

The index will be used *I* (like "moment") to note the number of an increment of load and the exhibitor *N* (like "Newton") to note the number of the iteration of Newton in progress. The algorithm used in the operator STAT_NON_LINE can then be written schematically in the following way:

$(\mathbf{u}_0, \boldsymbol{\lambda}_0)$ and $\boldsymbol{\sigma}_0$ known

Buckle over moments t_i (or increments of load): loading $\mathbf{L}_i = \mathbf{L}(t_i)$

- $(\mathbf{u}_{i-1}, \boldsymbol{\lambda}_{i-1})$ known
- Prediction: calculation of $\Delta\,\mathbf{u}_i^0$ and $\Delta\,\boldsymbol{\lambda}_i^0$
- Buckle on iterations of Newton: calculation of a continuation $(\Delta\,\mathbf{u}_i^n, \Delta\boldsymbol{\lambda}_i^n)$
  - $(\mathbf{u}_i^n, \boldsymbol{\lambda}_i^n)$ and $(\Delta\,\mathbf{u}_i^n, \Delta\boldsymbol{\lambda}_i^n)$ known
  - Calculation of the matrices and vectors associated with the following loads
  - Expression of the relation of behavior
    - calculation of the constraints $\boldsymbol{\sigma}_i^n$ and of the internal variables $\boldsymbol{\alpha}_i^n$ starting from the values $\boldsymbol{\sigma}_{l-1}$ and $\boldsymbol{\alpha}_{l-1}$ with preceding balance (t_{i-1}) and of the increment of displacement $\Delta\,\mathbf{u}_\mathbf{i}^\mathbf{n} = \mathbf{u}_i^n - \mathbf{u}_{i-1}$ since this balance
    - calculation of the "nodal forces": $\mathbf{Q}^\mathbf{T}\boldsymbol{\sigma}_i^n + \mathbf{B}^T\boldsymbol{\lambda}_i^n$
    - possible calculation of the matrix of tangent stiffness: $\mathbf{K}_i^n = \mathbf{K}(\mathbf{u}_i^n)$
  - Calculation of the direction of research $(\Delta\,\mathbf{u}_i^{n+1}, \Delta\,\boldsymbol{\lambda}_i^{n+1})$ by resolution of a linear system
  - Iterations of linear research: ρ
  - Actualization of the variables and their increments:
    $$\begin{cases}\mathbf{u}_i^{n+1} = \mathbf{u}_i^n + \rho\,\boldsymbol{\delta}\,\mathbf{u}_i^{n+1}\\ \boldsymbol{\lambda}_i^{n+1} = \boldsymbol{\lambda}_i^n + \rho\,\boldsymbol{\delta}\,\boldsymbol{\lambda}_i^{n+1}\end{cases} \text{et} \begin{cases}\Delta\,\mathbf{u}_i^{n+1} = \Delta\,\mathbf{u}_i^n + \rho\,\boldsymbol{\delta}\,\mathbf{u}_i^{n+1}\\ \Delta\boldsymbol{\lambda}_i^{n+1} = \Delta\,\boldsymbol{\lambda}_i^n + \rho\,\boldsymbol{\delta}\,\boldsymbol{\lambda}_i^{n+1}\end{cases}$$
  - Test of convergence
- Filing of the results at the moment t_i

$$\begin{cases}\mathbf{u}_i=\mathbf{u}_{i-1}+\Delta\,\mathbf{u}_i\\ \boldsymbol{\lambda}_i=\boldsymbol{\lambda}_{i-1}+\Delta\,\boldsymbol{\lambda}_i\\ \boldsymbol{\sigma}_i\\ \boldsymbol{\alpha}_i\end{cases}$$

It is noticed that there are three levels of overlapping loops: a loop external on the steps of time, a loop of iterations (qualified the total ones) of Newton and possible subloops for linear research (if she is asked by the user) and certain relations of behavior requiring of the iterations (known as interns), for example for elastoplasticity in plane constraints.

If one chooses the criterion based on the assembled tangent matrix, it is necessary to have this matrix reactualized for each step where one wants to analyze stability.

It is the case when one uses a method of the Newton type, and not a modified method of the Newton type.

One leads then to the following algorithm:

$(\mathbf{u}_0, \boldsymbol{\lambda}_0)$ and $\boldsymbol{\sigma}_0$ known

Buckle over moments t_i (or increments of load): loading $\mathbf{L}_i = \mathbf{L}(t_i)$

- $(\mathbf{u}_{i-1}, \boldsymbol{\lambda}_{i-1})$ known
- Prediction: calculation of $\Delta \mathbf{u}_i^0$ and $\Delta \boldsymbol{\lambda}_i^0$
- Buckle on iterations of Newton: calculation of a continuation $(\Delta \mathbf{u}_i^n, \Delta \boldsymbol{\lambda}_i^n)$
  - $(\mathbf{u}_i^n, \boldsymbol{\lambda}_i^n)$ and $(\Delta \mathbf{u}_i^n, \Delta \boldsymbol{\lambda}_i^n)$ known
  - Calculation of the matrices and vectors associated with the following loads
  - Expression of the relation of behavior
    - calculation of the constraints $\boldsymbol{\sigma}_i^n$ and of the internal variables $\boldsymbol{\alpha}_i^n$ starting from the values $\boldsymbol{\sigma}_{l-1}$ and $\boldsymbol{\alpha}_{l-1}$ with preceding balance (t_{i-1}) and of the increment of displacement $\Delta \mathbf{u}_\mathbf{i}^\mathbf{n} = \mathbf{u}_i^n - \mathbf{u}_{i-1}$ since this balance
    - calculation of the "nodal forces": $\mathbf{Q}^\mathbf{T} \boldsymbol{\sigma}_i^n + \mathbf{B}^T \boldsymbol{\lambda}_i^n$
    - possible calculation of the matrix of tangent stiffness: $\mathbf{K}_i^n = \mathbf{K}(\mathbf{u}_i^n)$
  - Calculation of the direction of research $(\Delta \mathbf{u}_i^{n+1}, \Delta \boldsymbol{\lambda}_i^{n+1})$ by resolution of a linear system
  - Iterations of linear research: ρ
  - Actualization of the variables and their increments:

$$\begin{cases} \mathbf{u}_i^{n+1} = \mathbf{u}_i^n + \rho \, \boldsymbol{\delta} \, \mathbf{u}_i^{n+1} \\ \boldsymbol{\lambda}_i^{n+1} = \boldsymbol{\lambda}_i^n + \rho \, \boldsymbol{\delta} \, \boldsymbol{\lambda}_i^{n+1} \end{cases} \text{et} \begin{cases} \Delta \mathbf{u}_i^{n+1} = \Delta \mathbf{u}_i^n + \rho \, \boldsymbol{\delta} \, \mathbf{u}_i^{n+1} \\ \Delta \boldsymbol{\lambda}_i^{n+1} = \Delta \boldsymbol{\lambda}_i^n + \rho \, \boldsymbol{\delta} \, \boldsymbol{\lambda}_i^{n+1} \end{cases}$$

  - Test of convergence
- Filing of the results at the moment t_i

$$\begin{cases} \mathbf{u}_i = \mathbf{u}_{i-1} + \Delta \mathbf{u}_i \\ \boldsymbol{\lambda}_i = \boldsymbol{\lambda}_{i-1} + \Delta \boldsymbol{\lambda}_i \\ \boldsymbol{\sigma}_i \\ \boldsymbol{\alpha}_i \end{cases}$$

- Criterion of stability, function of the reactualized tangent stiffness: $\mathbf{K}_i^n = \mathbf{K}(\mathbf{u}_i^n)$

The criterion is calculated at the end of the step, just after filing. It thus has like arguments the quantities converged with the current step. Moreover, this choice of position of call makes it possible to take account correctly following loadings, since their calculation is done at the time of the iterations of Newton. The criterion could not thus be called before the end of these iterations.

1.7.1.2 Impact on the structure of data result of `STAT_NON_LINE`

The call of the nonlinear criterion of stability will induce the resolution of a problem to the eigenvalues. The result of this calculation will be thus a set of couples critical load/mode of buckling.
The critical loads are scalars and the associated modes are fields of displacement, which will come to enrich the structure of data result by `STAT_NON_LINE`.

1.7.2 Characteristics related to the tensor of deformation

In the code, it is advisable to distinguish two large families from description of the deformations.

On the one hand the linearized tensor corresponds to the case of the small disturbances (argument SMALL keyword DEFORMATION), but also with the case of the small disturbances reactualized (Lagrangian reactualized with each step of incremental calculation: argument PETIT_REAC keyword DEFORMATION).

The tensor of deformation is written then (like [éq 1.3-3]):

$$\varepsilon = \frac{1}{2}\left(\nabla \mathbf{u} + \nabla^T \mathbf{u}\right) \qquad \text{éq 1.7.2-1}$$

The use of PETIT_REAC imply a resolution of the balance of the structure on its current geometry with a tensor of deformations linearized. One thus calculates the increment of deformation compared to the position $\mathbf{X}$, with displacement $\mathbf{u}$ and with the increment of displacement $\Delta\,\mathbf{u}$ in the following way:

$$\Delta\,\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial \Delta u_i}{\partial (X+u)_j} + \frac{\partial \Delta u_j}{\partial (X+u)_i}\right) \qquad \text{éq 1.7.2-2}$$

In addition, the code proposes tensors of deformation of the Green-Lagrange type (GROT_GDEP) for the treatment of great displacements (and the rotations finished for certain elements of structure) but under assumption of small deformations. The tensor used is the following classical tensor [éq 1.4.1-5]:

$$\varepsilon_{ij}(u) = \frac{1}{2}\left(u_{i,j} + u_{j,i} = u_{k,i} u_{k,j}\right) \qquad \text{éq 1.7.2-3}$$

The keyword GROT_GDEP applies to modelings beam, hull or 3D.

Lastly, the framework of modeling in great transformations most complete accessible in *Code_Aster* is resulting from the theory of Simo and corresponds to the keyword SIMO_MIEHE. He takes into account great rotations and the great deformations since the law of behavior is written in great deformations. For more precise details on the basic differences between the various types of deformations, the documentation [bib16] of *Code_Aster* present in detail modeling SIMO_MIEHE.

Code_Aster does not allow calculations in configuration eulerienne: as with the tensor of Almansi, for example. All the tensors of deformation available are of Lagrangian type.

The basic difference, as for the writing of the criterion, is between the linearized deformations (SMALL and PETIT_REAC) and deformations GROT_GDEP and SIMO_MIEHE.
Indeed, *Code_Aster* need has to make its search for balance of the tangent matrix. This one is written according to the equation ([§ 2.2.2.1] of documentation on STAT_NON_LINE [bib15]):

$$\mathbf{K}_T = \mathbf{Q}^T : \frac{\partial \boldsymbol{\sigma}}{\partial \mathbf{u}} + \frac{\partial \mathbf{Q}^T}{\partial \mathbf{u}} : \boldsymbol{\sigma} \qquad \text{éq 1.7.2-4}$$

However, $\mathbf{Q}^T : \frac{\partial \boldsymbol{\sigma}}{\partial \mathbf{u}}$ corresponds at the end classic of material rigidity and $\frac{\partial \mathbf{Q}^T}{\partial \mathbf{u}} : \boldsymbol{\sigma}$ corresponds at the end of geometrical rigidity which is present only in great displacements. Thus the criterion of buckling (formally assimilable to [éq 1.4.2.2 - 2]): $(\mathbf{K} + \lambda\,\mathbf{K}(\boldsymbol{\sigma}))\boldsymbol{\nu} = 0$ is valid only in small deformations, since the geometrical term of rigidity is regarded as negligible in the tangent matrix. One then can, with reason, to make a classical research of the eigenvalues and clean vectors of standard buckling of Euler.

On the other hand in great transformation, the evaluation of this criterion by the same method is problematic for two reasons: on the one hand, in the tangent matrix, the geometrical term of rigidity is already calculated and, on the other hand, the matrix $\mathbf{K}(\boldsymbol{\sigma})$ that it would possibly be necessary to add is obtained under *Code_Aster* in small deformations. For these reasons, it is necessary to evaluate in a way different the criterion according to the type of tensor of deformation requested by the user.

If one made the choice of a description eulerienne, the development of a criterion of the type reactualized Euler would be facilitated on the level of the calculation of the term $\mathbf{K}(\boldsymbol{\sigma})$, whatever the tensor of deformation.

1.7.2.1 In linearized deformations: SMALL and PETIT_REAC

As we said previously, this case does not pose major problems. It is enough to calculate the geometrical matrix of rigidity and to make a classical search for modes and eigenvalues, of type Euler [éq 1.4.2.2 - 2]:

$$(\mathbf{K}+\lambda\,\mathbf{K}(\boldsymbol{\sigma}))\boldsymbol{\nu}=0 \qquad \text{éq 1.7.2.1 - 1}$$

$\mathbf{K}$ is the tangent matrix reactualized at the end of the step of time.
In this case, one can thus speak indeed about criterion of the type reactualized Euler.
As one is in small deformations, the matrix of geometrical rigidities is proportional to the loading. Therefore, when the critical coefficient is obtained λ, it is enough to multiply it by the real load with the step of current time to obtain the critical load of buckling. The case $\lambda=1$ thus corresponds to the loss of stability.
Certain finite elements like the hulls DKT do not allow the calculation of the geometrical matrix of rigidity, contrary to the elements of the type COQUE_3D, for example.

1.7.2.2 In great displacements: GROT_GDEP and SIMO_MIEHE

The classical method does not apply any more in this case. Indeed, *Code_Aster* calculate like tangent matrix the matrix of material rigidity plus the geometrical matrix of rigidity (and possibly, the contribution due to the following pressures).
One in the manners of checking buckling then is to only make a research of the eigenvalues of the tangent matrix. If one of the eigenvalues is negative, it is that the matrix became singular and that an instability occurred between the moment when all its eigenvalues were positive and moment when one of it became negative.
The problem to be treated is thus slightly different since in the case of small deformations (SMALL and PETIT_REAC), there is the following system to solve [éq 1.7.2.1 - 1]: , $(\mathbf{K}+\lambda\,\mathbf{I})\boldsymbol{\nu}=0$ whereas in the case GROT_GDEP and SIMO_MIEHE it is necessary to solve:

$$(\mathbf{K}+\lambda\,\mathbf{I})\boldsymbol{\nu}=0 \qquad \text{éq 1.7.2.2 - 1}$$

With $\mathbf{I}$ who is the matrix identity and λ is, this time, of physical size equivalent to $\mathbf{K}$, whereas in the case of small deformations, the eigenvalue λ is adimensional (from where its direct interpretation as a multiplying coefficient of the loading).

One of the defects inherent in this method compared to more classical research explained higher [§1.4.2] is that one can have forecasts of buckling only when one approaches "close" the critical load, even when one exceeds it. Far from this load, the first found eigenvalue does not have really physical meaning since nonlinearities can appear between the step running and the calculated critical load. The coefficient report criticizes on load at the moment i is thus different from that at the moment $i+1$ whereas in small deformations this report remains constant.
Moreover, for all the steps of time, all the eigenvalues and clean vectors except lowest do not have any physical meaning since, for a clean couple vector eigenvalue $(\mathbf{V}_i, \lambda_i)$, one a:

$$(\mathbf{K}(\mathbf{u})+\mathbf{K}(\boldsymbol{\sigma}))\mathbf{V}_i=\lambda_i\mathbf{V}_i \qquad \text{éq 1.7.2.2 - 2}$$

This has clear direction only as from the moment when $\lambda_i \to 0$, in which case one finds the critical load and the clean vector criticizes associated.
Always compared to criterion of Euler (reactualized [éq 3.2.2.1 - 1] or not [éq 2.3.2.2 - 2]), one notices that the eigenvalue of the problem [éq 1.7.2.2 - 1]: $(\mathbf{K}+\lambda\mathbf{I})\boldsymbol{\nu}=0$ is not adimensionnée. It results from this a greater difficulty from interpretation as for knowing if the value is “small” or not. In other words, when can one say that one is close to a junction?
To define a relevant interval and of use general, in order to limit the vicinity of an instability, it would be interesting of adimensionner the eigenvalues.

1.7.2.3 Case of mixed modelings

Like *Code_Aster* allows to assign several types of deformations to the same structure, it is necessary to consider the case where one uses several types of tensors of deformation in same calculation.
The differentiation of the various elementary matrices being of no utility, it is appropriate to be solved to slice at the total level between a method or the other. One chose to extract the values and clean vectors from the tangent matrix without adding geometrical matrices of stiffnesses. All occurs as if the structure were in deformation of the Green-Lagrange type from the point of view of the criterion. Indeed, let us consider an unspecified solid made up of two parts I and II. On part I, the tensor of deformation which was adopted is the linearized tensor SMALL and on part II that of Green - Lagrange. The tangent matrix resulting from the assembly of the two submatrices becomes:

$$\begin{bmatrix} \mathbf{K}_I & * & 0 \\ * & * & * \\ 0 & * & \mathbf{K}_{II}+\mathbf{K}_{II}(\boldsymbol{\sigma}) \end{bmatrix} \quad \text{éq 1.7.2.3 - 1}$$

The spangled terms represent the nodes common to both parts and are thus a linear combination of the values of the two matrices. In this configuration, it appears that none the solutions is satisfactory but that less penalizing a search for “type Green - Lagrange” [§ is to make1.7.2.2] i.e. to use $(\mathbf{K}+\lambda\mathbf{I})\boldsymbol{\nu}=0$ [éq 1.7.2.2 - 1].

This solution not being exact but nevertheless the only able one to be carried out simply, it is envisaged to add a message of alarm informing the user whom the got results are not guaranteed due to the juxtaposition of several types of tensors of deformations.

1.7.3 Improvement of the performances of the criterion

During resolution incremental of problem quasi-static nonlinear, in ideal and if it is admitted that the discretization in time is sufficiently fine, he would be necessary to make an analysis of stability to each step of calculation. With each step, that induces the resolution of a problem to the eigenvalues, certainly limited in search of some modes. The analysis of stability thus brings an important overcost CPU, with a nonlinear calculation already being able to be long.

The idea is to call on the resolution of a problem to the eigenvalues only when it is really necessary, therefore when the current configuration is “close” to an instability. If one can define this vicinity by a preset interval, then one can call on a test of Sturm [bib12].
This test makes it possible to know if there exists at least an eigenvalue on the interval of research. In the affirmative, one will be able to then carry out modal research. In the contrary case, one continues the quasi-static incremental resolution, without solving problem with the eigenvalues.
The cost of a test of Sturm is notably lower than the cost of research of the critical loads.
The interval of research for the test of Sturm can, either to be given by the user, or to have a value by default in the code.

In the case of a criterion of Euler reactualized (case of the small deformations [§1.7.2.1]), where the problem to be solved is written: $(\mathbf{K}+\lambda\mathbf{K}(\boldsymbol{\sigma}))\boldsymbol{\nu}=0$ [éq 1.7.2.1 - 1], the interval of research must be centered on the eigenvalue $\lambda=1$ (which corresponds to value -1 for the algorithm of CALC_MODES, because he solves in fact a problem of the type: $\mathbf{K}\,\mathbf{v}=\mu\,\mathbf{K}(\boldsymbol{\sigma})\,\mathbf{v}$)).

The terminals of the interval are the terminals of the multiplying coefficient of the loading, therefore adimensional quantities, which are function of the safety coefficients and the evaluation of uncertainties for the problem given. The test of Sturm is implemented within this framework.
In the specific case adapted to the tensor of Green-Lagrange [§1.7.2.2], where one solves: $(\mathbf{K}+\lambda\,\mathbf{I})\,\mathbf{v}=0$ [éq 1.7.2.2 - 1], the interval is centered on 0. Moreover, the terminals of the interval of test, contrary to the preceding case, are not adimensionnées [§1.7.2.2]. It is thus more difficult to identify relevant and general values (for the case of the default values). The test of Sturm is not currently established for this case.

1.8 Generalization with dynamics

We will not approach here the framework of the criteria of dynamic instability (negative damping…). It is just a question of announcing that the nonlinear criterion presented here can completely apply directly in nonlinear dynamics. It will then detect any potential buckling of the structure, within the meaning of the singularity of the total matrix of reactualized tangent stiffness.
In order to be exhaustive in terms of analysis of stability on a nonlinear dynamic study, the user should use two criteria:

- a criterion of buckling (criterion on the stiffness),
- a dynamic criterion (criterion on damping or the total quadratic linearized problem [bib14], for example).

For the moment, the criterion of buckling on the stiffness (identical to that of STAT_NON_LINE) is only available in DYNA_NON_LINE.

Modeling coupled fluid-structure (U, p,Φ) [R4.02.02], which is available in DYNA_NON_LINE, requires some adaptations of use of the non-linear criterion of stability. Indeed, this coupled formulation generates a matrix of intrinsically singular stiffness total assembled on all the fluid degrees of freedom, which makes it incompatible with the research method of eigenvalues used for the analysis of stability. One can however circumvent the problem by correcting the problem assembled (matrix of stiffness and geometrical stiffness if need be) thanks to the use of two specific keywords. The analysis of stability relates then to the degrees of freedom structures alone.

1.9 Validation of the developments

The cases tests of validation are: SSNL126 and SSLL105D.
More precisely, the cases tests SSNL126 treat the case of a beam fixed at an end and subjected to a compression at the other end. Modeling is three-dimensional, with elastoplastic relation of behaviour to linear isotropic work hardening. Two representations kinematics are presented:

- modeling a: linearized deformations,
- modeling b: deformations of Green-lagrange.

The case test SSLL105D is based on a problem of beam in L , of which one studies elastic buckling. The finite elements are of standard beam.

1.10 Extension of the criterion of buckling to Traitement elastoplastic behavior

Far from any exhaustiveness, we will present only the simplest approaches here, for their easy establishment in the code.

When the structure functions in an elastoplastic mode, buckling is affected by the loss of resistance due to plasticity [bib2]. The modification comes from the relation of behavior during additional displacement $\alpha \mathbf{u}_1$.

The constraint becomes, in incremental form:

$$\Delta \boldsymbol{\pi}_1 = \Delta \boldsymbol{\pi} + \alpha \mathbf{C}_T [\Delta \boldsymbol{\varepsilon}^L(\mathbf{u}) + \Delta \varepsilon^Q(\mathbf{u},\mathbf{u})] + \frac{\alpha^2}{2} \mathbf{C}_T \Delta \boldsymbol{\varepsilon}^Q(\mathbf{u}_1,\mathbf{u}_1) \quad 1.11\text{-}1$$

In this expression, the matrix of behavior is the tangent matrix $\mathbf{C}_T$. The choice of this matrix is not immediate: indeed, the matrix depends on $\alpha \mathbf{u}_1$ and is thus not known as long as the mode is unknown. One can, for example, to discharge during buckling if the mode develops in a direction and to charge if it develops in the opposite direction. It is thus necessary to make an assumption for the behavior during plastic buckling. To start, we will apply the assumption of Hill [bib4] who leaves the principle that the structure continues to plastically charge during buckling.

Let us consider an elastoplastic law of type Von Mises. We define the three modules: E who is the Young modulus, E_T the tangent module, and the secant module. These modules are recalled on the following figure:

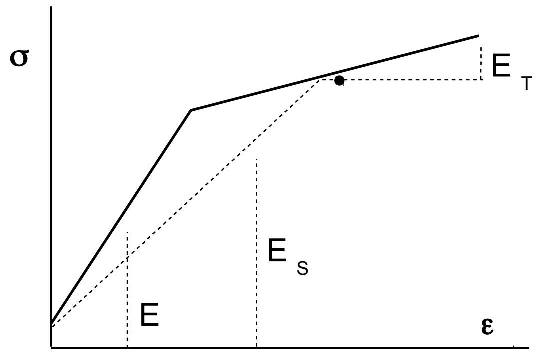

Figure 1.11.5-a: Representation of the various modules on a traction diagram 1D

Then we propose three possible methodologies.

The assumption of the tangent module simply consists in replacing the Young modulus by the tangent module in the relation of behavior. One obtains then:

$$\mathbf{C}_T = \frac{E}{E_T} \mathbf{C} \quad 1.11\text{-}2$$

This method is very rudimentary, but it is always pessimistic, which can constitute an advantage, if one places oneself from the point of view of dimensioning.

The method used usually consists in using the tangent matrix of incremental calculation (operator `STAT_NON_LINE` [bib15]). We thus have the following equation in the case of the plasticity of Von Mises [bib16]:

$$\mathbf{C}_T = \mathbf{C} \left[\mathbf{I} - \frac{\mathbf{A}[\boldsymbol{\sigma}^D \otimes \boldsymbol{\sigma}^{D^T}] \mathbf{A} \mathbf{C}}{h + \frac{\boldsymbol{\sigma}^{D^T} \mathbf{A} \mathbf{A} \boldsymbol{\sigma}^D}{\|\boldsymbol{\sigma}^D\|_{VM}}} \right]$$

$$\text{Avec}\begin{cases}\boldsymbol{\sigma}^{\boldsymbol{D}} : \text{vecteur déviateur des contraintes}\\ \mathbf{A} : \text{matrice intervenant dans la norme de VonMises}\left(\|\boldsymbol{\sigma}^{\boldsymbol{D}}\|_{VM}=\sqrt{\boldsymbol{\sigma}^{\boldsymbol{D}^T}\mathbf{A}\,\boldsymbol{\sigma}^{\boldsymbol{D}}}\right)\\ h : \text{pente plastique définie par } h=\dfrac{E\,.\,E^T}{E\text{ - }E^T}\end{cases} \quad 1.11\text{-}3$$

This method is perfectly rigorous only in nonlinear elasticity or if the assumption of Hill is respected: it does not make it possible to predict the junctions in the ways of loading. As soon as the relation of behavior is dissipative (see chapter **2**), the critical loads calculated will not be exact that if one can check that the loading is monotonous, in any point of the structure (Hill [bib4]).

The most realistic method consists in using the theory finished of the deformation only to calculate the load of plastic buckling. The tangent matrix of behavior is given by the equation below:

$$\mathbf{C}_T=\left[\left(\frac{1}{E_T}-\frac{1}{E_S}\right)\frac{\mathbf{A}[\boldsymbol{\sigma}^{\boldsymbol{D}}\otimes\boldsymbol{\sigma}^{\boldsymbol{D}^T}]\mathbf{A}}{\|\boldsymbol{\sigma}^{\boldsymbol{D}}\|_{VM}}+\mathbf{C}^{-1}+\left(\frac{1}{E_S}-\frac{1}{E}\right)\mathbf{A}\right]^{-1} \quad 1.11\text{-}4$$

Compared to the method based on the matrix of tangent stiffness [éq 1.11-3], this criterion requires the construction and the assembly of a specific total matrix. This expensive operation comes to weigh down the incremental resolution.

For considerations of general information and minimization of the development cost and cost of calculation (CPU and memory), we thus choose the criterion based on the tangent module [éq 1.11-3].

1.11 Conclusion

Code_Aster offer two criteria of stability, within the meaning of buckling, for the structural analyses:

1. On the one hand, whenever a linearized approach is enough, one can apply a criterion of the type Euler ([bib13] and [bib18]), by call to an operator of resolution of the problem to the eigenvalues generalized (`CALC_MODES` with the keyword `TYPE_RESU=' MODE_FLAMB'`).

2. In addition, for all the cases where it is essential to take account of nonthe linearities, which they to the relation of behavior or the great transformations, the user is due can employ an adapted criterion, of type generalized Euler. The call of this criterion is done during the incremental resolution of the quasi-static problem (operator `STAT_NON_LINE` [bib15]).

With each step of time, the criterion is based on the resolution of a problem to the eigenvalues [bib13] on the matrices of brought up to date total stiffnesses. This criterion, which is declined in two different forms, according to the tensor of deformation chosen, is based on a linearization around the step of current calculation. It accepts any type of tensor of deformation, as any type of relation of behavior for which one is able to build the matrix of total stiffness, at every moment. Moreover, the selected criterion is perfectly rigorous in the case of the relations of nonlinear elastic behavior, and can be wide with the case of elastoplasticity associated with the assumption with Hill [bib4].

2 Stability of a dissipative system

2.1 Definition of the stability of a dissipative system

When one is interested in dissipative phenomena (case of plastic or fragile materials…) one adds in the expression of total energy *Φ* a term representing the dissipated part. Irreversible degradations are then associated with quantities, generally scalar, like the damage or plasticity. The criterion of buckling presented previously does not take account of the irreversibility. It is sufficient but nonnecessary to justify stability [bib19]. When the criterion of buckling is reached, the unicity of the solution is not guaranteed any more, but one cannot conclude directly on stability. One notes *U* the variable of drifting state of the reversible part of energy and *α* the variable of drifting state of the irreversible part of the studied mechanical phenomenon. The criterion of stability then consists in checking the positivity of the derivative second of energy in the direction of the increase in variable *α* [bib20]. Let us consider an acceptable disturbance (*v,B≥0*). The criterion of stability then amounts checking that one always finds the inequality:

$$\phi(u,\alpha)\leq\phi(u+v,\alpha+b)$$ éq 2 .1 - 1

From the mathematical point of view, that amounts checking that the function *Φ* carry out a local minimum in (*U* , *α*).

One is interested more particularly in the case of the models of damage. One considers (*U* , *α*), one state structure Ω checking balance:

$$\forall v\in C^0, \int_\Omega \left(\frac{\partial\phi}{\partial u}(u,\alpha).v\right)dx=0$$ éq 2 .1 - 2

and the criterion of damage, taking account of the irreversibility of the variable of state *α:*

$$\frac{\partial\phi}{\partial\alpha}(u,\alpha)\geq 0 \; et \; \forall\beta\in C^0\geq 0, \int_\Omega \left(\frac{\partial\phi}{\partial\alpha}(u,\alpha).\beta\right)dx=0$$ éq 2 .1 - 3

O N obtains quickly, with a development of Taylor to order 2 [bib20], equivalence between criterion 2 .1-1 and the following criterion, writing on the derivative second of *Φ:*

$$D^2\phi(u,\alpha)\geq 0$$ éq 2 .1 - 4

where D^2 is the operator of derived second.

2.2 Écriture within the framework of the finite element method

From the point of view of the finite elements and by preserving the preceding notations, this criterion is written like the positivity of the quotient of Rayleigh under constraints of inequalities according to:

$$\forall(\boldsymbol{v},\boldsymbol{b}\geq\boldsymbol{0})\neq\boldsymbol{0},\; Q_{rc}=\frac{(\boldsymbol{v},\boldsymbol{b})^t.\boldsymbol{K}^T(\boldsymbol{v},\boldsymbol{b})}{(\boldsymbol{v},\boldsymbol{b})^T.(\boldsymbol{v},\boldsymbol{b})}\geq 0$$ éq 2 . 2 - 1

The most classical way to check that a function is positive consists in calculating its minimum and making sure of its positivity. One presents in the part which follows, the algorithm of minimization under constraints of inequalities programmed in *Code_Aster.*

2.3 Algorithm of optimization under constraints of inequalities

Among the algorithms available in the literature, that which proved to be most robust and also simplest to program is the method of the powers, to which one adds, with each iteration, the projection of the degrees of freedom $\boldsymbol{b}$ on the whole of the positive values, kind to check the imposed unilateral constraint [bib21]. The algorithm is built in two stages, in the following way:

1. The method of the powers is very much used in mathematics applied for the research of the maximum eigenvalues of a matrix ***M*** . A shift on the greatest eigenvalue $\boldsymbol{\lambda_m}$ **: NR** = $\boldsymbol{\lambda_m I_D - M}$ then allows to carry out the research of the smallest clean modes. One presents below the algorithm in his initial form. Maybe, in the form used for the research of the maximum eigenvalue of a symmetrical matrix ***A:***

Soit P la projection d'un vecteur sur sa partie positive.
Initalisation : $(\boldsymbol{v},\boldsymbol{b}\geq \boldsymbol{0})_0/\|(\boldsymbol{v},\boldsymbol{b}\geq \boldsymbol{0})_0\|=1$.
Pour $k\geq 1$ Faire :
$(\boldsymbol{w},\boldsymbol{c})=\boldsymbol{A}(\boldsymbol{v},\boldsymbol{b})_{k-1}$,
$(\boldsymbol{v},\boldsymbol{b})_k=\left(\boldsymbol{w},P(\boldsymbol{c})\right)$,
$(\boldsymbol{v},\boldsymbol{b})_k=(\boldsymbol{v},\boldsymbol{b})_k/\|(\boldsymbol{v},\boldsymbol{b})_k\|$
Si : $\|(\boldsymbol{v},\boldsymbol{b})_k-(\boldsymbol{v},\boldsymbol{b})_{k-1}\|<\epsilon$ alors Fin
Sinon : $k=k+1$
Fin Si
Fin Pour

Figure 2.3-a: Diagram of the algorithm of research of the maximum under constraint for a matrix *With*

2. Its limit of reliability is at the neighbourhoods of the thousand of degrees of freedom. To free themselves from this limit and to be able to deal with industrial problems, one uses the method of reduction available in Sorensen [bib22], who consists in projecting the problem on a basis made up of *N* smaller clean modes of the structure. By taking account of this stage one récrit the algorithm in the following way, where K is always the tangent operator, Q is the operator of projection and B_N projection in reduced space :

Réduction : $\boldsymbol{K}^T=\boldsymbol{Q}^T \boldsymbol{B}_n \boldsymbol{Q}$
Initialisation : $(\boldsymbol{v},\boldsymbol{b}\geq\boldsymbol{0})_0$; $z_0=\boldsymbol{Q}(\boldsymbol{v},\boldsymbol{b})_0/\left(\boldsymbol{Q}\|(\boldsymbol{v},\boldsymbol{b})_0\|\right)=1$,
Pour $k\geq 1$ Faire :
$(\boldsymbol{w},\boldsymbol{c})=\boldsymbol{Q}^T \boldsymbol{B}_n \boldsymbol{z}_{k-1}$,
$(\boldsymbol{v},\boldsymbol{b})_k=\left(\boldsymbol{w},P(\boldsymbol{c})\right)$,
$\boldsymbol{z}_k=\boldsymbol{Q}(\boldsymbol{v},\boldsymbol{b})_k$,
$\boldsymbol{z}_k=\boldsymbol{z}_k/\|\boldsymbol{z}_k\|$
Si : $\|\boldsymbol{z}_k-\boldsymbol{z}_{k-1}\|<\epsilon$ alors Fin
Sinon déflation
Fin si
Fin pour

Figure 2.3-b: Diagram of the algorithm of search for maximum under constraints of inequality with method of projection

2.4 Implementation in the code

The study of stability is started with the call of the order `CRIT_STAB` of the operator `STAT_NON_LINE`, under the condition `TYPE = 'STABILITY'`. Sizes, or degrees of freedom, checking the unilateral condition of irreversibility are declared in a list `DDL_STAB= ('','',…)`.

2.5 Example of application: Case of the bar in uniform traction

Principal the case of reference found in the bibliography is the study of the stability of the homogeneous solution of a bar damaged under the effect of a uniform loading of traction [bib23]:

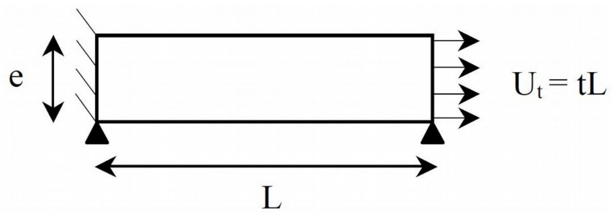

Figure 2 . 5 - has : Representation of the bar in uniform traction

2.5.1 Analytical results of stability

It is shown [bib23] that there exist two types of solutions to the studied problem:

- The homogeneous solution, uniformly damaged.
- The localised solutions which concentrate the damage of the bar on a precise zone.

The study of stability of the homogeneous solution thus amounts checking that the reached energy levels, by considering small disturbances of the solution having the form of a localization, are always higher than that obtained starting from the homogeneous solution.

On the basis of the following formulation in gradient of damage of energy [bib24]:

$$\varphi=\int_\Omega\left(\frac{1}{2}(1-\alpha)^2 E_0\varepsilon(u)^2+\frac{{\sigma_M}^2}{E_0}\alpha+\frac{E_0 l^2}{2}\nabla\boldsymbol{\alpha}.\nabla\boldsymbol{\alpha}\right)dx \qquad \text{éq 2.5.1 - 1}$$

the comparison of the energy levels then makes it possible to plot the diagram of stability of the homogeneous solution according to the relationship between the length of the bar *L*, the length interns model *L* and of loading applied U_T [bib23]:

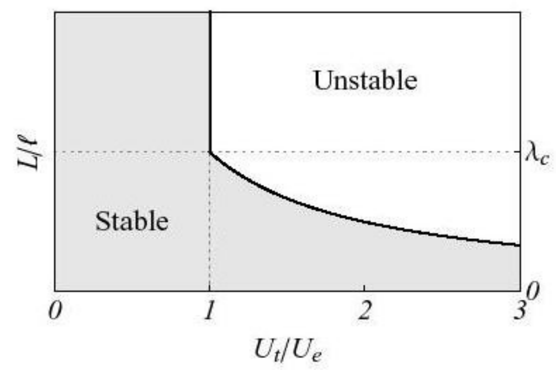

Figure 2 . 5 . 1 has : Analytical diagram of stability of the bar in traction

2.5.2 Results of stability got with *Code_Aster*

By using the algorithm of optimization developed in the code (2.3), one finds a diagram of stability similar to 5% near on the loading from which instability is detected [bib25] :

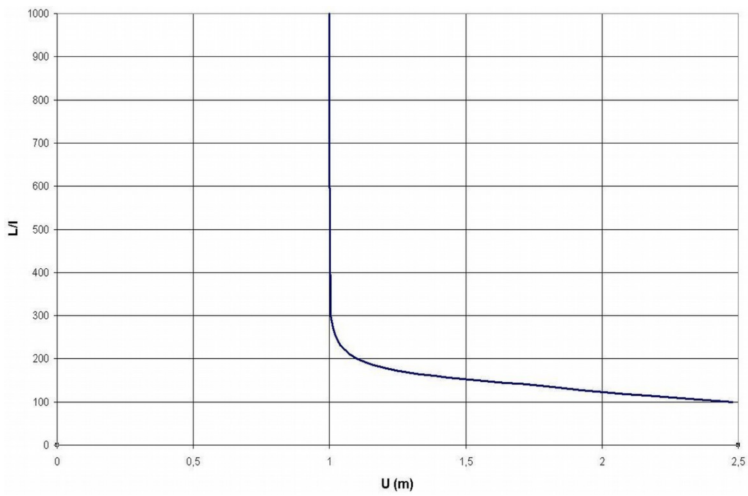

Figure 2 . 5 . 2 has : Diagram of stability of the bar in traction obtained with *Code_Aster*

Parametric studies on the influence amongst calculated Eigen frequencies and on which one rests for the study of stability shows that it is necessary to take it about thirty for discretized problems with more or less 100,000 ddls but that it becomes important to consider of it a good hundred for the problems of larger sizes. What involves more important costs of calculation CPU, the algorithm being greedy in time. This is why it is really started only when the study of buckling shows that the unicity of the solution is not assured any more.

When the algorithm shows that the criterion of stability is not checked any more (CHAR_STAB negative) , the vector minimizing the quotient of Rayleigh (éq 2.2-1) is called direction of instability. One finds it like result under the nomination MODE_STAB (by analogy with MODE_FLAMB for the criterion of buckling) . The disturbance of the current solution by this direction of instability destabilizes it and makes it possible to fork towards a stable solution. In the example presented here, the direction of instability is the localis ation of the damage on one of the two ends of the bar.

2.6 Conclusion

Code_Aster allows to carry out studies of stability on dissipative problems such as the problems of plasticity or damage. The algorithm used is based on the method of the powers to which the projection of the degrees of freedom is added, in space respecting the unilateral constraints of irreversibility. Its application is really put in work only when the criterion of unicity is violated. In the contrary case, unicity is sufficient to guarantee stability. If one detects with a step of time of calculation finite elements the loss of stability of the solution, the algorithm provides the direction to be added like disturbance to find a stable forked solution.

3 Bibliography

- [1] C. CHAVANT, A. COMBESCURE, J. DEVOS, A. HOFFMANN, Y. MEZIERE: “Elastic and plastic buckling of the thin hulls”, Courses IPSI, 1982.
- [2] S. DURING, A. COMBESCURE: “Analysis of junction in great plastic deformations élasto -: elementary formulation and validation”, Report interns n°190, LMT - Cachan, 1997.
- [3] NR. GREFFET: “Coupled simulation fluid-structure applied to the problems of nonlinear instability under flow”, Doctorate, LMT, ENS-Cachan, 2001.
- [4] R. HILL: “With general theory for uniqueness and stability in elastic-plastic solids”, J. Mech. Phys. Solids, vol. 6,236-249, 1958.
- [5] T.J.R. HUGHES, W.K. LIU, I. LEVIT: “Nonlinear dynamic finite element analysis of shells, Nonlinear Finite Analysis Element in Structural Mechanics”, W. Wunderlich, E. Stein & K.J. Bathe editors, Berlin, Springer, 151-168, 1981.
- [6] G. LOOSS: “Elementary stability and junction theory”, Springer-Verlag, 1990.
- [7] LIGHT A., A. COMBESCURE, MR. POTIER-FERRY: “Junction, buckling, stability in Mechanics of the structures”, Course IPSI, 1998.
- [8] LIGHT A.: “Junction, buckling, stability in mechanics of the structures”, EDF-DER HI - 74/98/024/0 Notes.
- [9] J. SHI: “Structural Computing critical points and secondary paths in nonlinear stability analysis by finite element method”, Computer & Structures, vol. 58, n°1, 203-220, 1996.
- [10] J.C. WOHLEVER, T.J. HEALEY: “With group theoretic approach to the total junction analysis of year axially compressed cylindrical Shell”, comp. Meth. In Applied Mech. And Engrg., vol. 122,315-349, 1995.
- [11] “Efforts external of pressure in great displacements” [R3.03.04].
- [12] “vibroacoustic Elements” [R4.02.02].
- [13] “Algorithm of resolution for the generalized problem” [R5.01.01].
- [14] “Calculation algorithm of the problem quadratic of eigenvalues” [R5.01.02].
- [15] “quasi-static nonlinear Algorithm” [R5.03.01].
- [16] “Integration of the elastoplastic relations” [R5.03.02].
- [17] “Model of Rousselier in great deformations” [R5.03.06].
- [18] “Note of calculation to buckling” [U2.08.04].
- [19] Q-S. NGUYEN: “Stability and nonlinear mechanics”, HERMES Science Publications, 2000.
- [20] A. BENALLAL and J-J. MARIGO: “Junction and stability resulting in gradient theories with softening”, Modeling Simul. MATER Sci. Eng., 15: 283-195, 2007.
- [21] SEEGER A. PINTO DA COSTA and A.: “Numerical resolution of cone-constrained eigenvalue problems”, Computational and Applied Mathematics, 28(1): 37-61, 2009.
- [22] “Solveurs modal and resolution of the generalized problem (GEP)” [R5.01.01].

• [23] K. PHAM, H. AMOR, J-J. MARIGO and C. MAURINI: "Gradient ramming models and to their uses to approximate brittle fracture", International Newspaper of Damage Mechanics, vol. 20, No 4: 618-652, 2011.

• [24] "Law of endommagemet regularized quadratic ENDO_CARRE" [R5.03.26]

• [25] "Validation of the algorithm of optimization under constraint of inequalities of option DDL_STAB" [V6.02.138].

4 Description of the versions of the document

Version Aster	**Author (S) Organization (S)**	**Description of the modifications**
6	*NR. GREFFET, J.M. PROIX, L. SALMONA EDF*R & D /AMA	Initial text
08/04/12	*NR. GREFFET EDF-R&D/AMA,*	Generalization with dynamics