Note for the construction of scale models in dynamics

Summary:

The models used for calculations of answers in linear dynamics are increasingly large. To improve the computing times significantly, it is possible to build a subspace of reduced size which makes it possible to project the complete problem, and thus to accelerate various calculations of answer. These models of reduced components can also be used in approaches of under-structuring, and certain non-linear analyses (shocks, non-linear behavior). One also proposes a methodology allowing “to condense” the internal variables for a linear analysis of dynamics. This handbook proposes to help the users in the construction of scale models in dynamics and proposes a framework to evaluate the quality of the built scale models.

One insists on the interest of the process of orthogonalisation of the bases of projection (operator DEFI_BASE_MODALE, keyword ORTHO_BASE) allowing to eliminate the vectors too much colinéaires, resulting from the concatenation of several families of vectors. A family satisfying with the vectors almost colinéaires will lead, in worst case, with singular projected matrices and a badly posed problem.

Contents

1 Introduction

The size of the models used for calculations of answers in linear dynamics is increasingly important, and the needs on the precision for the results result in carrying out finer and more expensive calculations. Significantly to improve the computing times of the answers of these systems to the various requests, it is interesting to build a subspace of reduced size which makes it possible to project the complete problem, and thus to accelerate various calculations of answer. These models of reduced components can be used either only, or in approaches by under-structuring, for linear analyses but also for certain non-linear analyses, such as the presence of shocks. This approach is also particularly interesting when one must take account of the internal devices of specific dissipations to each structure.

The method suggested to enrich an initial subspace, built on the basis of clean mode of the system, is based on the methods of residues ([1.2]). This method allows, starting from an approximate solution of the solution, to build vectors making it possible to improve the prediction for a given problem. This method, iterative, gives very good performances with a limited number of iterations (in general, two iterations are enough).

The first iteration is in general carried out by calculating the terms called classically “static corrections”, or “static modes”. This point is developed in the first section of this document. This technique is particularly adapted for the reduction of models in the presence of external efforts very few, even if they present spatial (fluctuating pressure, gravity, etc). This approach also makes it possible to effectively deal with the problems of shocks by adopting a method of penalization (provided one is interested in the phenomena “far” from the zone from shocks). **One also gives in this part of the important indications to control the quality of the scale models built with this approach.**

In the second part, one will discuss the typical cases of the scale models of the type Craig&Bampton or McNeal, used for the approaches in dynamic under-structuring. These two methods can be seen like the generalization of enrichments by static corrections. One will also present techniques allowing to limit the size of these models, in particular for the representation of the behaviors of interfaces.

In the third part, one extends the concepts presented to the case of the structures presenting of the internal efforts such as in the deadened cases. One will discuss mainly the models of viscous damping, hysteretic and viscoelastic. The first two approaches are indeed those usually retained for the studies in dynamics. The case of damping associated with a viscoelastic law is interesting since this law introduced of the internal variables nonpresent into the initial law of behavior of material, but which can be taken into account by the means of the reduction. The typical case of the model of damping of Rayleigh, used frequently for the seismic analyses, is not presented, since in these typical cases, the clean modes of the system diagonalisent also the matrices associated with dissipation, and bases it thus built does not require particular enrichment, in-outside those associated with the external efforts.

In the fourth part, one presents, using a study poured in the base of *Code_Aster* (study n°3185: Calculation of the complex modes of a plate sandwich including a viscoelastic material), various techniques presented. These examples also make it possible to present the implementation of the methods in *Code_Aster*.

2 Reduction of model: principle of the static correction

In this part, one presents the principles of the methods of reduction of model adapted to the cases of the nondissipative structures. These principles will be then extended to the cases of the dissipative structures. One starts by recalling the relations of dynamics associated with modeling to the finite elements with the structure.

One introduces then the problem with the eigenvalues which one wishes to solve. The assumptions associated with the reduction with model are then introduced, and a method of construction of a base generating an adapted subspace is then proposed. One is interested then in the optimal reduction of this base, and in his later enrichment starting from a method of iteration on the residues.

2.1 Description and notations

Let us consider the model with the finite elements (E.F.) of a linear structure, presenting mechanisms of dissipations interns, subjected to external efforts, being able as well to be known a priori (loadings) that in front of being evaluated progressively (efforts of shocks, contact, etc). In the temporal field, the relation of balance is written

$$\left[K_e q(t) + D_v \dot{q}(t) + M \ddot{q}(t)\right] = B u(t) \quad (1)$$

where K_e is the matrix of stiffness of the structure, D_v the matrix associated with viscous dissipation, and M the matrix of mass. One notes in addition q the vector of the degrees of freedom. The external term of efforts is defined by the product of a matrix of localization of the efforts B and of a vector u specifying the temporal evolution of the excitation. This writing makes it possible to separate the space segments and temporal from an often noted effort $f(t)$. One will thus have, in this case,

$$f(t) = Bu(t) \quad (2)$$

In the presence of shocks, one will be able for example to separate the terms related to the loadings and those related to the shocks:

$$f(t) = B_{eff.ext.} u_{eff.ext.}(t) + B_{chocs} u_{chocs}(t) \quad (3)$$

B_{chocs} is the term which locates the degrees of freedom of shocks, and $u_{choc}(t)$ determine the evolution of the effort. In the simple case of a shock on the level of the degree of freedom q_c treated by penalization, one will write for example

- $u_{choc}(t) = 0$ if there is no contact
- $u_{choc}(t) = k\, q_c(t)$ if there is contact

Concerning damping, the two models considered classically in these problems are viscous damping and damping hysteretic. The viscoelastic damping is also introduced, which it is interesting to treat within the framework of the methods of reduction of model. Under these conditions, the equilibrium equation in the field of Laplace can be put in the form

$$\left[\left(K_e + j\, K_h + K_v(s)\right) + s\, D_v + s^2 M\right] q(s) = B\, u(s) \quad (4)$$

K_h is the matrix representing the behavior hysteretic, K_v the matrix representing the viscoelastic behavior. For these two models of damping, adopted modeling has direction only in the field of Laplace. Damping hysteretic, such as it is introduced, is not causal. Nevertheless, by removing this term, one finds the equations standards associated with the problem viscous and viscoelastic, licit even for $s = 0$.

2.2 Reduction of model and static correction

The construction of a scale model can be planned to solve problems of various natures, which it is the calculation of the temporal or harmonic response of a structure to a request by the use of a single scale model, or calculation of clean modes and eigenvalue by an approach by under structuring (coupling of scale models), or by dynamic condensation (reduction of whole or part of a structure). The problems of calculation of answer are put in the form (1) or (4) according to whether one considers a temporal or harmonic loading.

The problem with the eigenvalues associated, by considering the various types of damping, is put in the form

$$\left[\left(K_e + j\, K_h + K_v(\lambda_0)\right) + \lambda_0 D_v + \lambda_0^2 M\right] \psi = 0 \quad (5)$$

where them λ_0 are the complex eigenvalues, and ψ complex modes. In the case of the nondissipative model, clean modes ϕ and own pulsations ω_0 check

$$\left[K_e-\omega_0^2 M\right]\phi=0 \tag{6}$$

The tools making it possible to solve the complete problem, except for the viscoelastic case, are available in *Code_Aster*, by the means of the order CALC_MODES. However, the complex matrix jK_h is symmetrical, and nonsquare. So the algebraic properties of the matrices considered in the dissipative problem do not make it possible to implement the most effective algorithms. For complementary details on the algorithms of search for eigenvalues, one will be able to refer to [1,4,5].

2.3 Assumption of reduction of model

The resolution of the problem on the basis of complete model presented in the relation (5) is thus not possible for models of industrial size. To circumvent this difficulty, one proposes to build a scale model having the same spectral properties as the complete problem. One thus applies the existence of a base of reduction T_r who allows the reasonable representation of the behavior of the structure on the waveband of interest. Under these conditions, one thus has

$$q\approx T_r q_r \tag{7}$$

Terms q_r linear combination correspond to the amplitudes generalized of the complete problem project on the basis T_r. These amplitudes thus check, in the case general:

- for the temporal answers:

$$\left[T_r^T K_e T_r\right]q_r(t)+\left[T_r^T D_v T_r\right]\dot{q}_r(t)+\left[T_r^T M T_r\right]\ddot{q}_r(t)=T_r^T B u(t) \tag{8}$$

- for the harmonic answers:

$$T_r^T\left[\left(K_e+jK_h+K_v(s)\right)+sD_v+s^2 M\right]T_r q_r(s)=T_r^T B u(s) \tag{9}$$

The problem thus reduced must preserve the same spectral characteristics. It is thus natural to build T_r on the basis of clean mode ϕ_0 associated conservative system. **However, the clean modes are sufficient to represent correctly the answer of the system only if a very significant number of it is considered.** In practice, the calculation of a large number of modes is unrealistic for industrial problems of big size. It is thus necessary to enrich the base by projection to improve the representativeness of T_r.

2.4 Definition static correction

To effectively calculate the response of a structure to an external request, one generally chooses to project the system on a basis made up of the modes whose Eigen frequencies are in the waveband of interest of the problem. This band understands the modes of the lowest frequencies in general, but not necessarily. However, to ensure a good representation of the answer, it is often necessary to take into account a number much more significant of modes than those present in the waveband.

Let us consider the conservative system associated with the relation (6). While breaking up the answer of this system on the modal basis, it comes, by using the notation wrongly ∞ to indicate the size of the problem,

$$q(s)=\sum_{k=1}^{\infty}\frac{\Phi_k\Phi_k^T B u(s)}{s^2+\omega_{0k}^2} \tag{10}$$

If only they are considered N first modes of the structure, an approximate truncated solution is obtained whose validity decrease very quickly with the number of modes:

$$q(s) \approx \sum_{k=1}^{N<\infty} \frac{\Phi_k \Phi_k^T B u(s)}{s^2 + \omega_{0k}^2} \tag{11}$$

To limit the effects of truncation, one chooses a sufficient number of modes to represent the dynamic behavior, and one supplements this base of mode by including the quasi static effects of the modes truncated in the waveband of interest. In general, one considers modes up to 1.5 times the maximum frequency of interest of the problem (semi-empirical rule known as “of Hurty” [7]). Under these conditions, the Eigen frequencies of the modes apart from the band are more important than the maximum frequency of interest, one supposes then

$$s^2 \ll \omega_{0k}^2 \text{ for } k > N, \tag{12}$$

and one can thus write, by separating the terms from (10) according to their membership of the waveband of interest

$$q(s) \approx \sum_{k=1}^{N} \frac{\Phi_k \Phi_k^T B u(s)}{s^2 + \omega_{0k}^2} + \sum_{k=N+1}^{\infty} \frac{\Phi_k \Phi_k^T B u(s)}{\omega_{0k}^2} \tag{13}$$

The base of reduction associated with this model is thus made up of N first clean modes, but also of the various quasi static contributions T_s

$$T_s = \sum_{k=N+1}^{\infty} \frac{\Phi_k \Phi_k^T B}{\omega_{0k}^2} \tag{14}$$

To calculate effectively T_s, one can write the static solution q_s problem for the whole of the loading B by using the spectral decomposition of K_e^{-1} [1]. It comes then

$$q_s = \left(K_e^{-1}\right) B = \left(\sum_{k=1}^{\infty} \frac{\Phi_k \Phi_k^T}{\omega_{0k}^2} \right) B \tag{15}$$

Under these conditions, one can récrire the relation (13) by using this static answer, and it comes

$$q(s) \approx \sum_{k=1}^{N} \frac{\Phi_k \Phi_k^T B u(s)}{s^2 + \omega_{0k}^2} + K_e^{-1} B u(s) - \sum_{k=1}^{N} \frac{\Phi_k \Phi_k^T B u(s)}{\omega_{0k}^2}$$

That is to say

$$q(s) \approx \sum_{k=1}^{N} \left[\Phi_k \left(\frac{\Phi_k^T}{s^2 + \omega_{0k}^2} - \frac{\Phi_k^T}{\omega_{0k}^2} \right) B u(s) \right] + K_e^{-1} B u(s) \tag{16}$$

According to the relation (16), one can thus build an effective base of projection to calculate the answer of the dynamic problem. This base T_r is thus built around N first modes clean of the structure, and the static answer of the structure to unit efforts or to unit accelerations, applied in the

1 One has $\Phi^T K_e \Phi = \Omega^2$. If $\omega_{0k} \neq 0 \; \forall k$, then $(\Phi^T K_e \Phi)^{-1} = \Omega^{-2}$, from where $K_e^{-1} = \Phi \Omega^{-2} \Phi^T = \sum_{k=1}^{\infty} \frac{\Phi_k \Phi_k^T}{\omega_{0k}^2}$.

same way that the efforts for which one seeks the answer (cf §3 for the corresponding operators). One thus has:

$$T_r = \left[\Phi_1 \ldots \Phi_N \quad K_e^{-1} B \right] \tag{17}$$

The base of projection thus built constitutes a very good subspace for the research of the dynamic response of the structure. On the other hand, for its practical use, it is recommended of orthogonaliser the family of vectors obtained by the relation (17).

NB: Inertial correction
Of a more modest interest, one can define, in a similar way, a correction low frequencies, if the band of analysis of interest does not comprise the first modes of the structure. The relation (10) can break up in the following way:

$$q(s) \approx \sum_{k=1}^{M} \frac{\Phi_k \Phi_k^T B u(s)}{s^2} + \sum_{k=M+1}^{N} \frac{\Phi_k \Phi_k^T B u(s)}{s^2 + \omega_{0k}^2} + \sum_{k=N+1}^{\infty} \frac{\Phi_k \Phi_k^T B u(s)}{\omega_{0k}^2} \tag{18}$$

The first term corresponds to the spectral decomposition of M^{-1}, and one can thus enrich the base by projection according to the same principle, and build

$$T_r = \left[M^{-1} B \quad \Phi_1 \ldots \Phi_N \quad K_e^{-1} B \right] \tag{19}$$

2.5 Case tests illustrating the use of the static corrections

Many cases tests of *Code_Aster* use already the principles of the static correction. One can in addition note that this correction can be applied to different moment in the study:

- either it is integrated directly in the base of reduction, in accordance with the presentation which has just been made (static correction *a priori*),
- either it is superimposed after calculation with the dynamic response of the structure on the only basis of the clean modes (static correction *a posteriori*).

These two approaches are equivalent and provide obviously the same results. One will choose one or the other for questions of simplicity. One in general chooses the first approach for harmonic calculations (or frequential), whereas second is privileged in the case of transitory calculations of answers (or temporal). One can quote, among the cases tests presenting the use of the static correction, the following studies, all relative to a seismic analysis:

— Static correction *a priori,* by addition of static modes to imposed force or acceleration imposed on the base of the clean modes (constitution of a base of Ritz), or static correction *a posteriori* via the pseudo-mode (static mode with imposed acceleration) in transitory seismic analysis on modal basis:

- [V2.01.103] SDLD103 – seismic Answer transitory of a system 3 masses and 4 springs multi-supported with static correction
- [V2.01.105] SDLD105 – Response transitory of a system mass-springs to an earthquake mono-support with static correction

— Static correction *a posteriori* via the pseudo-mode (static mode with imposed acceleration) in spectral seismic analysis:

- [V2.01.030] SDLD30 – spectral seismic Answer of a system 2 masses and 3 springs multimedia
- [V2.01.301] SDLD301 – spectral seismic Answer of a system 2 masses and 3 springs multimedia (correlated or décorrélées excitations)
- [V2.02.023] SDLL23 – Beam embed-free subjected to an earthquake (spectral response)
- [V2.02.112] SDLL112 – Analysis seismic of a multimedia beam (spectral response)

2.6 Indications for the back testing of the quality of the scale model

One of the principles making it possible to validate the quality of a scale model consists in estimating, a posteriori, the variation with the balance which can exist after calculation.

2.6.1 Case general for a calculation on a scale model

To characterize this variation with balance, one uses the principle of the residue in effort. This residue in effort F_r is calculated

- for the temporal answers:

$$\left[K_e T_r\right] q_r(t) + \left[D_v T_r\right] \dot{q}_r(t) + \left[M T_r\right] \ddot{q}_r(t) - B u(t) = F_r(t) \quad (20)$$

- for the harmonic answers:

$$\left[\left(K_e + j K_h + K_v(s)\right) + s D_v + s^2 M\right] T_r q_r(s) - B u(s) = F_r(s) \quad (21)$$

One can also adopt the same approach for the calculation of the complex clean modes, detailed in section 5.4.

If the solution obtained is exact, then the residue F_r is null, by definition. In the contrary case, one can build a measurement of the variation to balance starting from the static response of the structure to this loading. One notes R_r the residue in displacement associated with each calculated complex mode. By definition, one has

$$K_e R_r = F_r \quad (22)$$

A measurement of the variation to balance is given by the elastic potential energy of the structure, that is to say

$$E_r = \frac{1}{2} \left\| R_r^T K_e R_r \right\|^2 \quad (23)$$

2.6.2 Typical case of the calculation of the modes of a generalized model

The same principle can be put in work, but in a way a little more precise, kind to reveal variations particular to various balances. Indeed, in the case of a calculation by under structuring with use of modes of interfaces, it is important to be able more precisely to determine the origin of a vague calculation. The sources of inaccuracies are:

- the bad representation of displacements to the interfaces, for each under structure,
- the dynamic behavior of the internal parts, for each under structure,
- differential displacements (separation) being able to appear with the interfaces between under structures.

The order CALC_CORR_SSD [U4.52.16] allows, on the basis of calculation of work associated with these various efforts, to build terms of corrections allowing to separately improve the behaviors of interfaces, and the dynamic behaviors internal of each one of under structures.

The details of this approach are given in the reference material on under dynamic structuring [R4.06.02]

3 Taking into account of external efforts, shocks, non-linear behaviors

The first application of the methods of reduction relates to the transitory calculation of a structure in the presence of external efforts, shocks or non-linear behaviors. The base of reduction adapted to such a calculation is directly that described with the relation (17). This base thus contains at the same time clean modes and static deformations (corrections). The family resulting from the concatenation of the two pennies families must then be orthogonalized (or orthonormal) in order to define a base.

- The clean modes of interest are calculated by the operator CALC_MODES [U4.52.02].
- The static terms of corrections can be calculated:
  - with MECA_STATIQUE [U4.51.01], or with MACRO_ELAS_MULT [U4.41.02], in the case of multiple loadings (several different requests, nodes/directions of shocks, etc);
  - or with MODE_STATIQUE [U4.52.14], keywords PSEUDO_MODE (imposed acceleration) or FORCE_NODALE (imposed force).
- These various families of vectors must then be concaténées. This operation is carried out by means of the order DEFI_BASE_MODALE [U4.64.02], by using keyword "RITZ".

In the case of nonlinear dis_visc or non-linear viscoelastic behavior dis_ecro_trac, it does not have there a finite element between the nodes concerned (just a relation of behavior). During the calculation of the reduced modal base, it can be judicious to define an element with an elastic stiffness corresponding to the tangent with the behavior of the device [R5.03.17].

The family thus built can then be orthogonalized, that is to say by the use of the keyword " ORTHO " in the same order DEFI_BASE_MODALE, that is to say by the use of a new occurrence of the order DEFI_BASE_MODALE, with the use of the keyword " ORTHO_BASE ".

The orthonormalisation can be obtained by standardisation of the orthogonal family, by the use of the order NORM_MODE [U4.52.11].

The process of orthogonalisation is especially important to eliminate the vectors too much colinéaires, resulting from the concatenation of several families of vectors. A family satisfying with the vectors almost colinéaires will lead, in worst case, with singular projected matrices and a badly posed problem.

An alternative to the use of DEFI_BASE_MODALE consist in projecting the matrices of mass and stiffness of the problem on the complete family of vectors, and calculating the modes of the problem thus reduced. The user selects then the vectors of interest to build the final base of projection. This approach, more complex to implement, makes it possible as well as possible to adjust the base of projection to calculation, but is to be held to the informed users.

NB: Note on the number of modes suitable to take into account
In the case of a problem with a known external loading, it is classically recommended to use a family of clean modes whose Eigen frequency highest is about 1.5 to 2 times the frequency of interest of calculation. If this frequency is not known a priori, it can be estimated on the basis of transform of Fourier of the excitation realized for example with CALC_FONCTION [U4.32.04]. On the other hand, in the case of a problem utilizing shocks, it is difficult to carry out a priori an estimate of the maximum frequency of interest. A classical rule consists in looking at, for the same initial loading, the evolution of "times of shocks" (lasted during which the structure is in contact with the obstacle). The convergence of times of shock in general indicates a good representativeness of the base retained for calculation.

The static use of corrections for the calculation of the answer of a structure in the presence of shocks is illustrated in the case test SDNL301 [V5.02.301]

4 Reduction of model for under dynamic structuring.

When one wishes to solve a problem of big size in dynamics of the structures, one proceeds in general by breaking up the principal structure into different under structures. One then builds a scale

model of each under structure, on the basis of his own mode. These scale models are then assembled, to build a total scale model of the complete structure. Some elements on constructions of bases adapted to the resolution of the problem by under structuring dynamic are recalled in this section. Complete presentations of these approaches, including in particular the aspects of setting in œuvre in *Code_Aster,* are available in documentations [R4.06.02] (classical dynamic under-structuring) and [R4.06.03] (cyclic dynamic under-structuring). One will be able to also refer to the references [6] to [9].

4.1 Classical methods: Craig&Bampton and MacNeal&Rubin.

In the same way that for the reduction of a single model, the reduction for the dynamic under-structuring must take into account two aspects:
- to preserve the spectral properties of the problem in the waveband of interest;
- to correctly take into account the quasi static effects of the external efforts applied to each substructure.

The first constraint is satisfied by choosing a number of modes sufficient for the substructure considered, so that the criterion of Hurty is checked.

The second constraint is checked by considering that the external efforts applied to under structure given are:
- real external efforts, applied in the interior of the field of under structure considered,
- efforts applied to under structure of interest by under adjacent structures.

The two bases of reductions classically retained for the construction of a scale model for under structuring are the bases known as “of Craig&Bampton”, and “MacNeal&Rubin” (see [8] for more details on the formulations).
The base of the type “Craig&Bampton” is built to leave:
- clean modes of the substructure, embedded on the level of the interfaces with the adjacent substructures,
- static raisings of the substructure to the unit displacements imposed successively on the degrees of freedom of the interfaces with the adjacent substructures,
- possible modes of inertial unloading, if the complete structure has rigid modes of body, or can be subjected to important accelerations of interface (case of the earthquake, for example),

The base of the type “MacNeal&Rubin” is built to leave:
- clean modes of the substructure, free on the level of the interfaces with the adjacent substructures,
- modes of rigid body, if the substructure is free when the interfaces with the others under structures are left free.
- static displacements of the substructure to unit efforts imposed successively on the degrees of freedom of the interfaces with the adjacent substructures. If the substructure presents modes of rigid body, it is necessary to regularize the calculation of static displacements, by a technique of shift of the spectrum by addition of mass to the matrix of stiffness, or calculation of the auto--balanced loads (see [6], for example)

To fix the ideas, by partitionnant the matrices of mass and stiffnesses according to the degrees of freedom of interface (subscripted i), and of the degrees of complementary freedom (subscripted c), the base of complete reduction for “Craig&Bampton” is given by

$$T_{C\&B}=\begin{bmatrix}\Phi_{cF} & -K_{cc}^{-1}K_{ci} & K_{cc}^{-1}M_{cc}\Phi_{cR}\\ 0 & I_d & 0\end{bmatrix}, \tag{24}$$

and bases it of “MacNeal&Rubin” by

$$T_{Mc\&R}=\begin{bmatrix}\Phi_{cR} & \Phi_{cL} & q_{cS}\\ \Phi_{iR} & \Phi_{iL} & q_{iS}\end{bmatrix}. \tag{25}$$

The last vectors correspond to the calculation of the response of the displacement of the structure to unit loadings imposed on the interface (in the absence of modes of rigid body for under structure)

$$\begin{bmatrix} K_{cc} & K_{ci} \\ K_{ic} & K_{ii} \end{bmatrix} \begin{bmatrix} q_{cS} \\ q_{iS} \end{bmatrix} = \begin{bmatrix} 0 \\ I_d \end{bmatrix}. \quad (26)$$

Φ_{cF} are the modes of the structure with fixed interfaces, Φ_L free modes and Φ_R modes of rigid body of under free structure to the interfaces.

These two approaches are directly available directly in *Code_Aster* by combining the results of the dedicated operators:

- DEFI_BASE_MODALE [U4.64.02]
- DEFI_INTERF_DYNA [U4.64.01]
- DEFI_BASE_MODALE [U4.64.02]
- MACR_ELEM_DYNA [U4.65.01]
- DEFI_MODELE_GENE [U4.65.02]

4.2 Use of modes of interfaces.

Nevertheless, in the case of a component presenting of the extended interfaces, the models thus reduced remain of important size. In addition, the projection of the matrices of mass and stiffness on these basis of projection makes lose the hollow character of the assembled matrices of origin. Under these conditions, it is not always interesting to build a scale model, as for the calculation of a harmonic answer, for example. On the other hand, when the number of calculations realized around the model increases, the reduction becomes interesting, especially when one can still reduce the size of the problem.

Indeed, if one remains within the framework of the dynamic low frequency, the whole of the stress patterns (or fastener) is not necessary to represent the dynamics of the problem. The wavelength of the phenomena having to forward with the interface is limited by that of the phenomenon which one can find in the fields. One can thus operate a judicious selection among the stress patterns (or of fastener).

Methods of constructions of these modes, called here “modes of interfaces”, were proposed in particular by Craig & Chang (see [10]) for the approach by under-structuring, then by Bourquin and D' Hennezel (see [11]) with a view to decomposition of field. The method suggested by Craig & Chang has the advantage of simplicity, and was adopted for the construction of the scale models in *Code_Aster* with the operator DEFI_MODELE_GENE [U4.65.02] (OPTION = 'REDUCED'). This method simply consists in projecting the matrices of mass and stiffness of the model of under structure on the basis of the stress patterns (or of fastener) and solving the problem with the eigenvalues associated:

$$T^T \left(K - \omega_r^2 M \right) T \Phi_r = 0 \quad (27)$$

with

$$T = \begin{bmatrix} -K_{cc}^{-1} K_{ci} \\ I_d \end{bmatrix} \text{ or } T = \begin{bmatrix} K_{cc} & K_{ci} \\ K_{ic} & K_{ii} \end{bmatrix}^{-1} \begin{pmatrix} 0 \\ I_d \end{pmatrix} \quad (28)$$

Modes of interfaces Φ_r are classified compared to their own pulsations, and a truncation is operated to select only the components more the low frequencies. The stress patterns (or fasteners) are then advantageously replaced by the vectors $T \Phi_r$ in the base of projection. The profit is carried out at the same time on the size of the matrix (which can be appreciably reduced if the interface is of big size), and on its topology, since, by construction, the projection of the matrices of mass and stiffness of the

system on this new basis leads to diagonal matrices. Only the rectangular matrices associated with the coupling by the mass remain full matrices.

4.3 Calculation of the modes of interface

The approach presented is not effective, since there remains necessary to build the stress patterns (or of fastener), which are of significant number when the interfaces are extended, and that the matrices should then be projected. Some solutions were suggested and tested with *Code_Aster* (see [11]).

The adopted solution consists in building a pre-conditioner of the problem to limit calculations of static raising. From the topology of the interfaces, one builds a lattice of beam topologically equivalent to the interface, and one in calculation the first modes. These modes are raised statically on the unit of the model, and these are the extend modes which are used instead of T, and allow to accelerate calculation considerably. This point is detailed in the reference material R4.06.02 (classical dynamic Under-structuring).

5 Construction of a base adapted to the dissipative problem

When the structure (or under structure) presents mechanisms of damping, the technique employed to build the scale model remains the same one, with this close the corrections to be taken into account are associated with internal efforts. One proposes to build T_d by using the first clean modes Φ calculated starting from the model of the structure not including dissipation, and an enrichment associated with the residues in displacement induced by these modes. The construction of this residue in displacement takes as a starting point the construction of the static corrections for the calculation of the response to an external request, but by taking account of the characteristics of the required answer. One presents here only the part associated with enrichment to take into account damping. In the case of a calculation of answer, it is also necessary to take into account enrichments associated with the external loadings.

5.1 Construction of a base supplements on the model of the corrections static for damping hysteretic and/or viscous.

One concentrates in this section on the two models of damping usable directly in *Code_Aster*. The construction of a fascinating scale model of account the internal variables (general viscoelasticity) will be presented in the following section.

The complex clean modes of the structure check the relation (5). By comparing the efforts associated with the dissipative part to external efforts, one can récrire the relation in the form

$$\left[K_e+\lambda_0^2 M\right]\psi=-\left[j\,K_h+\lambda_0 D_v\right]\psi \tag{29}$$

The external efforts have a particular form then, and the complex modes are compared to the answers of the conservative structure to these special efforts.

One thus seeks, in the same way that presented in the paragraph 2.4, to break up the answer on a basis built starting from the first clean modes of the structure, by complying with the rule of truncation of Hurty. By introducing the matrix of order of the viscous efforts B_v,

$$B_v=-\left(j\,K_h+\lambda_0 D_v\right) \tag{30}$$

one can write directly

$$\left[K_e+\lambda_0^2 M\right]\psi=B_v\psi \tag{31}$$

Independently of the particular nature of the term ordering the excitation (ψ), the base of relevant projection to solve this problem is form of that used to solve a problem generic than one could put in the form

$$\left[K_e + s^2 M\right] q(s) = B_v u(s) \tag{32}$$

For this particular problem, the base, noted $T_{v,}$ then takes the shape given to the paragraph 2.4, by separating the components related on the viscous efforts and the efforts hysteretic, that is to say

$$T_v = \left[\phi_1 \ldots \phi_N \quad -K_e^{-1} K_h \quad -K_e^{-1} D_v\right] \tag{33}$$

This base makes it possible to project the complete dissipative problem to consider correctly the modes complex. On the other hand, if the dissipative zones are largely distributed on the structure, matrices K_h and D_v are of big size. Size of the base T_v is then in proportion, and its calculation and its use become expensive.

5.2 Taking into account of the characteristic of the problem

To limit the size of the base of reduction to be preserved, one considers there still who one is interested only at one small portion of the spectrum of the structure, and which in fact, the wavelengths of the phenomena forwarding through dissipative materials are about those described by the first clean modes of the structure. One can thus, in a way similar to modal truncation, to truncate enrichment with the excitations which break up on the first clean modes. The corrective terms associated with the modes of higher natures can be neglected. One thus builds the base of reduction for the dissipative problem starting from the clean modes calculated for the conservative problem, and of the static corrections associated with the efforts of damping calculated for each normal mode. The base of reduction T_d reserve is thus written

$$T_d = \left[I_d \quad -K_e^{-1} K_h \quad -K_e^{-1} D_v\right]\left[\phi_1 \ldots \phi_N\right] \tag{34}$$

These terms of enrichment are also called static residues. The base thus built makes it possible to break up the free oscillations of the structure in the absence of damping with the clean modes ϕ , but also to represent the behaviors related to the presence of damping thanks to the construction of the residue in displacement around the static answer of the efforts related to dissipation.

5.3 Filtering of the base of reduction

The base T_d built in the paragraph 5.2 can be used directly to carry out the projection of the complete problem. However, certain vectors can be very colinéaires, thus involving problems of conditioning of the reduced problem. One can thus choose orthogonaliser the vectors of T_d by calculating the modes of the homogeneous problem nondissipative tiny room on this basis. Are $\left(\phi_r, \omega_r\right)$ clean modes of the reduced system

$$T_d^T \left[K_e - \omega_r^2 M\right] T_d \phi_r = 0 \tag{35}$$

The own pulsations of this reduced problem make it possible to operate a selection among the vectors of T_d to preserve only the vectors affecting for the representation of the behavior low frequency. The value of indicative truncation is defined by the “criterion of Hurty”, which consists in retaining only the vectors associated with an Eigen frequency lower than 1.5 times the maximum frequency with interest with the problem. By considering that only them N first Eigen frequencies answer this criterion, the base of projection retained for the calculation of the complex modes becomes

$$T_{dr} = T_d \phi_r \tag{36}$$

The compl modes then are calculatedexeS reduced resulting of the projection of the complete problem on this basis

$$T_{dr}^{T}\left[K_e + j\,K_h + \lambda_r D_v + \lambda_r^2 M\right] T_{dr}\,\psi_r = 0 \tag{37}$$

Complex modes ψ are then approximated by restitution on the initial degrees of freedom

$$\psi \approx T_{dr}\,\psi_r \tag{38}$$

This stage, however, is not obligatory. For the models of smalls, or when the number of modes retained for the analysis is weak (<50, to fix the ideas), it can be omitted without degradation of the results. This stage becomes necessary when the modal complex converges with difficulty.

5.4 Iterations on the residues

The base of reduction T_{dr} defined by the relations above a good starting point for the calculation of the spectrum of the dissipative complete model constitutes. However, it can prove that the quality of the got results is not satisfactory. Modes calculated on the basis T_{dr} are indeed an approximation of the exact complex modes, and the associated poles can also be different from the true poles of the system. To characterize this variation with balance, one uses the principle of the residue in effort. This residue in effort F_r is calculated for each mode complexes by

$$\left[K_e + j\,K_h + \lambda_r D_v + \lambda_r^2 M\right] T_{dr}\,\psi_r = F_r \tag{39}$$

So in the couple $\left(T_{dr}\,\psi_r, \lambda_r\right)$ one has a clean mode and an eigenvalue, then the residue F_r is null by definition. In the contrary case, one can build a measurement of the variation to balance starting from the static response of the structure to this loading. One notes R_r the residue in displacement associated with each calculated complex mode. By definition, one has

$$K_e R_r = F_r \tag{40}$$

A measurement of the variation to balance is given by the elastic potential energy of the structure, that is to say

$$E_r = \frac{1}{2}\left\| R_r^T K_e R_r \right\|^2 \tag{41}$$

If this variation exceeds a beforehand definite criterion (like a percentage of the first own pulsation, for example), then one again enriches the base by reduction by using the parts real and imaginary residues in displacements associated with the modes presenting a variation to important balance. One defines an iterative process thus such as

$$T_{dr(N+1)} = \left[T_{dr(N)} \quad \Re(R_r) \quad \Im(R_r)\right] \tag{42}$$

5.5 Reduction of models including of the internal variables (or intermediaries)

The methods of reductions also allow, with the help of some developments, to take into account modelings of behavior nonavailable in the state in *Code_Aster.* One can thus plan to carry out calculations coupling mechanics with other physics, insofar as the complementary phenomena can be described using the degrees of freedom representing the mechanical phenomena.

This case of application of the methods of reduction primarily makes it possible to present the approach in a more complete case, when the properties of material depend on time, or of the frequency of the requests. This kind of approach also makes it possible to generalize the model of damping hysteretic. In the preceding section, one considers a Young modulus and a constant rate of

loss. One can enrich this relation while applying that the Young modulus and the rate of loss are sizes which depend on the frequency. It is for example the case for the models of viscoelastic material behavior. These models give access to a causal dynamic model, which is thus usable to treat the harmonic and transitory answers of structures with viscoelastic materials.

5.5.1 Fast introduction to linear viscoelasticity

This model rests on the existence of a law of behavior making it possible to determine the state of stress according to the history of the deformations:

$$\sigma = E_\infty \epsilon(t) - \int_0^t E_v(t-\tau) \frac{\partial \epsilon(\tau)}{\partial \tau} d\tau \ , \tag{43}$$

where E_∞ represent the Young modulus high frequency, and E_v the module of relieving. This module can be represented in the temporal field by a series of Prony:

$$E_v(t) = E_r + \sum_{k=1}^{N} E_k \exp(-t/\tau_k) \ , \tag{44}$$

The Young modulus can be put in various forms [12]. In our case, one will choose the representation resulting from modeling starting from the fields of unelastic displacement [14]. One can consequently introduce internal variables ϵ_{vk} to represent the terms of relieving, the law of behavior is written then

$$\sigma = E_\infty \epsilon(t) - \sum_{k=1}^{n} E_k \epsilon_{vk}(t) \ , \tag{45}$$

where each $\epsilon_{vk}(t)$ a law of relieving follows compared to $\epsilon(t)$

$$\tau_k \dot{\epsilon}_{vk} + \epsilon_{vk} - \frac{E_\infty}{E_k} \epsilon = 0 \tag{46}$$

In the frequential field, the law of behavior is written simply

$$\sigma(s) = E(s) \epsilon(s) \ , \tag{47}$$

With

$$E(s) = E_\infty - \sum_{k=1}^{n} E_k \frac{\omega_k}{s+\omega_k} \tag{48}$$

The parameters characteristic of materials are accessible starting from tests dynamics in traction and shearing, or by identification of the nomograms provided by the manufacturers.

This form of law makes it possible to build a model of order two compatible with *Code_Aster*. On the other hand, working is not direct. To build this model, it is necessary to introduce internal variables, which establish the link between the physical degrees of freedom, impacted by the presence of a viscoelastic material, and the sizes defining the behavior. For each internal parameter, one introduces a variable q_{vk} , which describes the state of relieving of material.

While setting out again of the relations (47) and (48), one writes

$$\sigma(s) = E_\infty \epsilon(q,s) - \sum_{k=1}^{n} E_k \frac{\omega_k}{s+\omega_k} \epsilon(q,s) = E_\infty \epsilon(q,s) - \sum_{k=1}^{n} E_k \epsilon(q_{vk},s) \tag{49}$$

The operator $\epsilon(q,s)$ is linear compared to displacements. For a law of behavior modelled by the equation (48), the link between each degree of freedom interns and the physical degree of freedom which corresponds to him and linear, controlled by the following equation:

$$q_{vk}(s)=q(s)\frac{\omega_k}{s+\omega_k} \tag{50}$$

Sizes E_∞ , E_k and ω_k are intrinsic parameters of material, associated with the evolutions of stiffnesses and the durations of relieving. Degrees of freedom q_{vk} constitute a subset of degrees of freedom associated with $k^{ième}$ couples parameters (E_k,ω_k) .

By introducing these degrees of internal freedom, and by distinguishing the purely elastic parts and the viscoelastic parts, the dynamic system is written

$$s^2\begin{bmatrix}M & 0\\ 0 & 0\end{bmatrix}\begin{Bmatrix}q\\ q_{vk}\end{Bmatrix}+s\begin{bmatrix}0 & 0\\ 0 & \frac{\alpha_k}{\omega_k}K_v\end{bmatrix}\begin{Bmatrix}q\\ q_{vk}\end{Bmatrix}+\begin{bmatrix}K_e+K_v & -\alpha_k K_v\\ -\alpha_k K_v & \alpha_k K_v\end{bmatrix}\begin{Bmatrix}q\\ q_{vk}\end{Bmatrix}=\begin{Bmatrix}F\\ 0\end{Bmatrix} \tag{51}$$

with $\alpha_k=E_k/E_\infty$

K_v corresponds to the matrix of stiffness associated with the degrees of freedom with viscoelastic materials assembled with the Young modulus E_∞ .
One can illustrate the relations describing this behavior by considering a system mass-arises for which the spring has a behavior which can be represented by n internal variables:

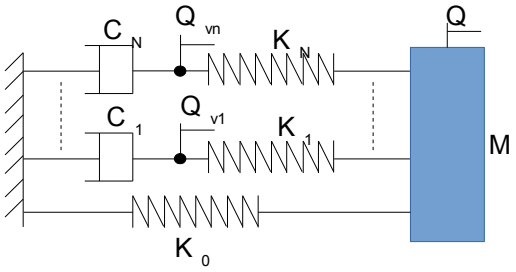

Figure 1: Representation of the viscoelastic model with internal variables for a system with a degree of freedom of displacement and N variable internal

For this simplified representation, there are the following relations

$$K_\infty=K_0+\sum_{k=1}^{n}\alpha_k K_v,\ K_k=\alpha_k K_v \text{ and } C_k=\frac{\alpha_k}{\omega_k}K_v \tag{52}$$

5.5.2 Principles of construction of a base of adapted reduction

In the state, *Code_Aster* does not allow to take into account of such laws of behavior for the linear dynamic analyses. The solution adopted to solve this problem consists in building a base of adapted reduction.

This base of reduction is built around clean modes and of the static answers of the structure to the viscoelastic efforts generated by the modes, in the same way that for the catch in depreciation account viscous or hysteretic. The various behaviors selected to build the base of reduction correspond in limiting states of the system.

To represent the quasi static behavior, it is supposed that the shock absorbers associated with the internal degrees of freedom do not work. The internal degrees of freedom quasi statically follow the

modes of the system in the absence of the shock absorbers, by supposing that the internal degrees of freedom are free. The first vectors taken into account are thus the free modes of the system which one can represent in the following way:

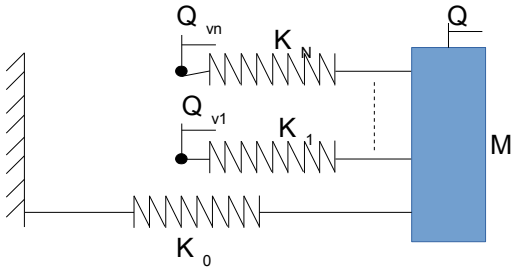

Figure 2: Calculation of the "free" modes. Representation of the system to only one degree of freedom are equivalent.

To represent the behavior ad infinitum, it is supposed that the shock absorbers associated with the internal degrees of freedom are blocked. The internal degrees of freedom thus are supposed to be embedded. The vectors taken into account are thus the free modes of the system which one can represent in the following way:

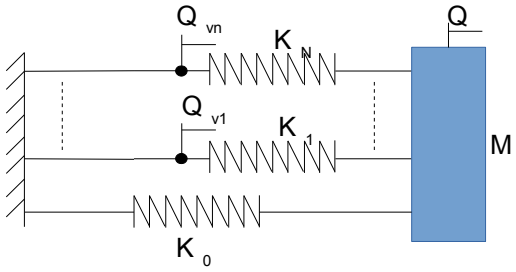

Figure 3: Calculation of the "embedded" modes. Representation of the system to only one degree of freedom are equivalent.

Lastly, to take into account the behavior of the shock absorbers associated with the internal states, one calculates the static correction associated with the movements with the definite internal states for the first modes with the structure.

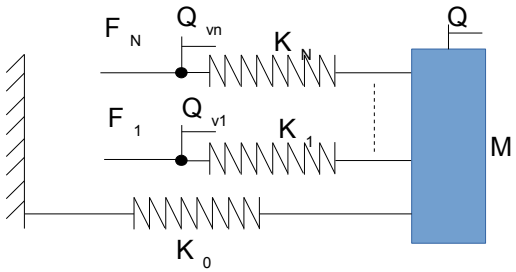

Figure 4: Calculation of the static corrections "viscoelastic". Representation of the system to only one degree of freedom are equivalent.

In the same way that one defined an enrichment associated with the degrees of freedom belonging to material having a damping hysteretic, one can associate a static term of correction, with the image of those given by the relation (33), for each couple (E_k, ω_k) internal parameters. The base of reduction T_{ve} for the viscoelastic model takes the shape then

$$[T_{ve}]=\begin{bmatrix}\Phi_l & \Phi_e & T_{pk}\\ \Phi_{lv} & 0 & T_{vk}\end{bmatrix},\ \forall k\in[1,n]\ . \tag{53}$$

Under matrices T_{pk}, T_{vk} are built starting from the static problems

$$\begin{bmatrix}K_e+K_v & -\alpha_1 K_v & -\alpha_k K_v & -\alpha_n K_v\\ -\alpha_1 K_v & \alpha_1 K_v & 0 & 0\\ -\alpha_k K_v & 0 & \alpha_k K_v & 0\\ -\alpha_n K_v & 0 & 0 & \alpha_n K_v\end{bmatrix}\begin{Bmatrix}T_{pk}\\ T_{v1k}\\ T_{vkk}\\ T_{vnk}\end{Bmatrix}=\begin{Bmatrix}0\\ 0\\ F_{vk}\\ 0\end{Bmatrix} \tag{54}$$

with the loading defined by

$$\{F_{vk}\}=\frac{\alpha_k}{\omega_k}[K_v]\{\Phi_{lv}\} \tag{55}$$

One can then solve the problem project on the basis T_{ve}, problem which takes into account the definite total viscoelastic behavior for materials.

5.5.3 Construction of the correction associated with the static efforts

Thanks to the topology of the matrix of complete stiffness, one can find a form of the opposite matrix simply. It is a matrix or each block is worth

$$\left(K_e+K_v\left(1-\sum_{k=1}^{n}\alpha_k\right)\right)^{-1}\ , \tag{56}$$

that one will note K_0^{-1}, except for the blocks on the diagonal for the internal degrees of freedom, where the block is worth $1/\alpha_k K_v^{-1}+K_0^{-1}$. It will be noted besides that K_0 corresponds to the static stiffness of the model.

For each couple of parameters (E_k, ω_k), the solution of the system (54) can put itself in the form

$$\begin{Bmatrix}T_p\\ T_{v1k}\\ T_{vkk}\\ T_{vnk}\end{Bmatrix}=\begin{bmatrix}K_0^{-1} & K_0^{-1} & K_0^{-1} & K_0^{-1}\\ K_0^{-1} & 1/\alpha_1 K_v^{-1}+K_0^{-1} & K_0^{-1} & K_0^{-1}\\ K_0^{-1} & K_0^{-1} & 1/\alpha_k K_k^{-1}+K_0^{-1} & K_0^{-1}\\ K_0^{-1} & K_0^{-1} & K_0^{-1} & 1/\alpha_n K_n^{-1}+K_0^{-1}\end{bmatrix}\begin{Bmatrix}0\\ 0\\ \alpha_k/\omega_k[K_v]\Phi_{lv}\\ 0\end{Bmatrix} \tag{57}$$

One has thus for each group of degrees of freedom of the expressions simple to calculate. For the physical degrees of freedom, it comes:

$$T_{pk}=\frac{\alpha_k}{\omega_k}K_0^{-1}K_v\Phi_{lv} \tag{58}$$

And for under - internal whole of states, one a:

$$T_{vlk}=T_{pk}\qquad\forall l\neq k \tag{59}$$

And

$$T_{vkk}=T_{pk}+\frac{1}{\omega_k}\Phi_{lv} \tag{60}$$

for the subset q_{vk}, when one is interested in the couple (E_k,ω_k)

Each vector is thus built thanks to the calculation of the answer of the free initial system, as presented on the figure 2, with a request it even built starting from the free modes of the same system.

$$[T_{ve}]=\begin{bmatrix}\Phi_l & \Phi_e & T_{p1} & T_{pk} & T_{pn}\\ \Phi_l & 0 & T_{v1} & T_{pk} & T_{pn}\\ \Phi_l & 0 & T_{p1} & T_{vk} & T_{pn}\\ \Phi_l & 0 & T_{p1} & T_{pk} & T_{vn}\end{bmatrix} \tag{61}$$

Note: Calculation of the projected matrices
After having built the blocks of the base of projection T_{ve}, it does not remain which has to build the reduced matrices, by detailing the calculation of projection.

For the matrix of reduced mass $\tilde{M}$, one has quickly

$$\tilde{M}=T_{ve}^T\begin{bmatrix}M & 0\\ 0 & 0\end{bmatrix}T_{ve}=\begin{bmatrix}\Phi_l^T\\ \Phi_e^T\\ T_{p1}^T\\ T_{pk}^T\\ T_{pn}^T\end{bmatrix}[M]\begin{bmatrix}\Phi_l & \Phi_e & T_{p1} & T_{pk} & T_{pn}\end{bmatrix} \tag{62}$$

The calculation of $\tilde{M}$ is thus realized by simple projection of the matrix of mass of the problem on the basis containing the free modes, the embedded modes, and the corrections T_{pk}.

In the same way, the construction of the reduced matrix of damping $\tilde{C}$ is rather simple, because of its topology. One thus has directly

$$\tilde{C}=T_{ve}^T\sum_{k=1}^{n}\left(\frac{\alpha_k}{\omega_k}\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & K_v & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\right)T_{ve}=\sum_{k=1}^{n}\left(\frac{\alpha_k}{\omega_k}\left[\tilde{C}_k\right]\right) \tag{63}$$

Where $\tilde{C}_k$ is given simply by

$$\left[\tilde{C}_k\right]=\frac{\alpha_k}{\omega_k}\begin{bmatrix}\Phi_{lv}^T\\ 0\\ T_{p1}^T\\ T_{vk}^T\\ T_{pn}^T\end{bmatrix}[K_v]\begin{bmatrix}\Phi_{lv} & 0 & T_{p1} & T_{vk} & T_{pn}\end{bmatrix} \tag{64}$$

For the matrix $\tilde{K}$, after simplifications, it comes

$$\left[\tilde{K}\right]=\begin{bmatrix}\Phi_l^T\\ \Phi_e^T\\ T_{pl}^T\\ T_{pk}^T\\ T_{pn}^T\end{bmatrix}\left[K_0\right]\left[\Phi_l \quad \Phi_e \quad T_{pl} \quad T_{pk} \quad T_{pn}\right]+\sum_{k=1}^{n}\left(\alpha_k\begin{bmatrix}0\\ \Phi_e^T\\ 0\\ \frac{-1}{\omega_k}\Phi_l^T\\ 0\end{bmatrix}\left[K_v\right]\left[0 \quad \Phi_e \quad 0 \quad \frac{-1}{\omega_k}\Phi_l \quad 0\right]\right) \tag{65}$$

5.5.4 Implementation data-processing

Details for the setting in œuvre of these techniques in *Code_Aster* are presented in this section. Indeed, *Code_Aster* allows to solve the reduced dissipative problem, but its construction is not direct. Indeed, the addition of the internal states complicates the construction of the scale model more, since one cannot directly assemble the matrices for the complete problem, not than one cannot directly assemble the vectors associated with the base with reduction, in particular the corrections associated with the viscoelastic efforts defined by the relations (54) and (55). It is thus necessary initially to build the vectors of the base of projection, then to then build the projected matrices.

The adopted solution consists in assembling various models, in order to be able to solve several successive systems, and to build the matrices reduced for each under system by simple summation. An example detailed for the setting in œuvre of these methods is presented with the study n°3185 present in the base of the studies.

5.5.4.1 Definition of materials

To construct the whole of these calculations with *Code_Aster*, it is necessary to assemble the matrix of mass M, as well as the various matrices of stiffnesses K_v and K_0, essential to calculations of the modes in the various configurations, like to calculations of the corrective terms.

Consequently, it is necessary to define several “under-materials”, all attached to viscoelastic starting material, to assemble these matrices. For each materials, one will be able to use two definitions. The first, which corresponds to the quasi static behavior, used for the calculation of K_0, and a second, used for the calculation of K_v. In the example of study 3185 presents in the base, one uses the ISD 112. The two materials are thus defined by:

```
ISD112_0=DEFI_MATERIAU (…) #-- For the assembly of K0
ISD112_v=DEFI_MATERIAU (…) #-- For the assembly of Kv
```

The various Young moduli beforehand will have been declared E_k and frequencies ω_k defining the behavior of material:

```
Ek= [E_0, E_1, E_2, E_3, E_4, E_5]
E_I=sum (Ek)
omk= [om1, om2, om3, om4, om5]
for i1 in arranges (len (omk)):
    ak [i1] =Ek [i1+1] /E_I
```

However, by construction for our problem, the components of K_v associated with degrees of freedom not belonging to viscoelastic material must be worthless. To satisfy this condition, one will define other materials in double. A first definition of material supplements will be used for the assembly of K_0, and a second, where the term of stiffness will be selected very weak, quite lower than the value of

E_∞ reserve to assemble K_v. In the example, the selected material is steel. There will be thus two definitions for materials:

```
ACIER_0=DEFI_MATERIAU (ELAS=_F (E=2.1E11, NU=0.3, RHO=7800.,
AMOR_HYST=0.0,),)
```

Then one will define a second material for which the properties of stiffnesses are worthless:

```
ACIER_v=DEFI_MATERIAU (ELAS=_F (E=0., NU=0.3, RHO=7800., AMOR_HYST=0.0,),)
```

The various elementary matrices of the problems are then built on the basis of these various material. One definite a first field fascinating material of account the whole of the properties for the static problem

```
CHMAT_0=AFFE_MATERIAU (…, AFFE= (_F (GROUP_MA=…, MATER=ACIER_0,),
       _F (GROUP_MA=…, MATER=ISD112_0,),),)
```

and a second for the assembly of K_v :

```
CHMAT_v=AFFE_MATERIAU (…, AFFE= (_F (GROUP_MA=…, MATER=ACIER_v,),
       _F (GROUP_MA=…, MATER=ISD112_v,),),)
```

5.5.4.2 Calculation of free" and "embedded" modes the "

Once the defined materials, it is enough to assemble the various matrices of rigidity, of mass.

```
KELEM_0=CALC_MATR_ELEM (OPTION=' RIGI_MECA',…, CHAM_MATER=CHMAT_0,…,)
KELEM_v=CALC_MATR_ELEM (OPTION=' RIGI_MECA',…, CHAM_MATER=CHMAT_v,…,)
MELEM=CALC_MATR_ELEM (OPTION=' MASS_MECA',…, CHAM_MATER=CHMAT_0,…,)

K_0=ASSE_MATRICE (MATR_ELEM=KELEM_0, NUMÉRIQUE_DDL=…,)
K_v=ASSE_MATRICE (MATR_ELEM=KELEM_v, NUMÉRIQUE_DDL=…,)
M=ASSE_MATRICE (MATR_ELEM=MELEM, NUMÉRIQUE_DDL=…,)
```

The matrix K_e+K_v is obtained by linear combination:

```
#-- to have Que+Kv = KB + Kv * (nap alpha_k between 1 and N)
K_Ev=COMB_MATR_ASSE (COMB_R= (_F (MATR_ASSE=K_0, COEF_R=1.,),
                        _F (MATR_ASSE=K_v, COEF_R=sum (ak),),),)
```

One can thus now calculate the "free" modes and the "embedded" modes.

5.5.4.3 Calculation of the correction T_p associated with the physical degrees of freedom

The whole of the terms T_{pk} and T_{vk}, $\forall k$, is built by linear combination of the clean modes "free", and of the corrective term $K_0^{-1}K_v\Phi_l$. This last term is obtained by looping on the vectors of Φ_l :

```
for N in arranges (Nb):
   #-- Extraction of the modes --#
   Phi_l [N] =CRÉA_CHAMP (TYPE_CHAM=' nœu_DEPL_R',
      NUME_DDL=Num,
      OPERATION=' EXTR',
      RESULTAT=Mo_lib,
      NOM_CHAM=' DEPL',
      NUME_MODE=n+1,)

   #-- calculation of Kv*Phi --#
   KvPhi [N] =PROD_MATR_CHAM (MATR_ASSE=K_v, CHAM_NO=Phi_l [N],)

   #-- transformation into loading --#
   Fv [N] =AFFE_CHAR_MECA (MODELE=MODELE, VECT_ASSE=KvPhi [N],)
   nom_cas_p=' Tp_ '+str (n+1); #-- Physical DDL
```

```
        #-- Calculation of the KB \ (Kv*Phi) --#
        yew n==0:
            Tp=MACRO_ELAS_MULT (MODELE=MODELE,
               CHAM_MATER=CHMAT_0,
                NUME_DDL=Num,
               CHAR_MECA_GLOBAL=COND_LIM,
               CAS_CHARGE= (_F (NOM_CAS=nom_cas_p, CHAR_MECA=Fv [N],),),
            )
        else:
            Tp=MACRO_ELAS_MULT (reuse=Tp,
               MODELE=MODELE,
               CHAM_MATER=CHMAT_0,
                NUME_DDL=Num,
               CHAR_MECA_GLOBAL=COND_LIM,
               CAS_CHARGE= (_F (NOM_CAS=nom_cas_p, CHAR_MECA=Fv [N],),),
            )
```

5.5.4.4 Construction of the bases of complete projections

The elementary “bricks” for the calculation of the reduced matrices are now built, it does not remain which has to build the various modal bases. Here the principal stages of this construction are summarized.

```
    #-------------------------------------------------------------#
    #-- construction of the bases
    #
    # bases 1 Tc1 = [Phi_l Phi_e Tp1… Tpn] - calculation of Mg and kg
    # base 2 Tc2 (K) = [Phi_l 0 Tp1 Tvk Tpn] - calculation of Ck
    # base 3 Tc3 (K) = [0 Phi_e 0 - Phi_l/omk 0] - calculation of Kk
    #-------------------------------------------------------------#

    Zero=CRÉA_CHAMP (…, ADZE = (_F (CHAM_GD = Phi_l [0],
          ALL = ' OUI',
          CUMUL=' OUI',
          COEF_R = 0.0),),)

    #-- Beginning of construction of Tc1
    #-- Extraction of the modes “embed”: Phi_e
    #-- Extraction of the vectors of Tp: Tp_n
#-- Construction of identically worthless modes: Mo_zero
for N in arranges (Nb):
    Phi_e [N] =CRÉA_CHAMP (…, OPERATION=' EXTR',
       RESULTAT=Mo_enc,
       NUME_MODE=n+1,)

    nom_cas_p=' Tp_ ‘+str (n+1);

    Tp_n [N] =CRÉA_CHAMP (…, OPERATION=' EXTR',
       RESULTAT=Tp,
       NOM_CAS=nom_cas_p,)

    Mo_zero=CRÉA_RESU (…,
       OPERATION=' AFFE',
       TYPE_RESU=' MODE_MECA',
       NOM_CHAM=' DEPL',
       AFFE=_F (CHAM_GD=Zero),)

    Tc1=CRÉA_RESU (…,
       OPERATION=' AFFE',
```

```
        TYPE_RESU=' MULT_ELAS',
        NOM_CHAM=' DEPL',
        AFFE=_F (CHAM_GD=Phi_e [N]),)


   #-- Beginning of construction of Tc2 (K) and Tc3 (K)
   for K in arranges (len (ak)): #-- buckle on the internal states
      for N in arranges (Nb):
         Tc2 [K] =CRÉA_RESU (…, OPERATION=' AFFE',
                TYPE_RESU=' MULT_ELAS',
                NOM_CHAM=' DEPL',
                AFFE=_F (CHAM_GD=Zero),)

         Tc3 [K] =CRÉA_RESU (…, OPERATION=' AFFE',
                TYPE_RESU=' MULT_ELAS',
                NOM_CHAM=' DEPL',
                AFFE=_F (CHAM_GD=Phi_e [N]),)

   #-- Complement of Tc1 --#
   for K in arranges (len (ak)):
      for N in arranges (Nb):
         Temp= CREA_CHAMP (…, OPERATION= 'ADZE',
                ADZE = (_F (CHAM_GD = Tp_n [N],
                ALL = ' OUI',
                CUMUL=' OUI',
                COEF_R = ak [K] /omk [K]),),)

         Tc1=CRÉA_RESU (…, OPERATION=' AFFE',
                TYPE_RESU=' MULT_ELAS',
                NOM_CHAM=' DEPL',
                AFFE=_F (CHAM_GD=Temp),
         )

         TO DESTROY (CONCEPT=_F (NOM= ('temp',),),)

      #-- Complement of Tc2 (K) and Tc3 (K) --#
      for K in arranges (len (ak)):
         for L in arranges (len (ak)):
            yew k==l:
               coef_mo = 1/omk (K)
            else:
               coef_mo=0.

            for N in arranges (Nb):
               #-- Tc2 [K]
               Temp= CREA_CHAMP (…, OPERATION= 'ADZE',
                      ADZE = (
                          _F (…, CHAM_GD = Phi_l [N], COEF_R = coef_mo,),
                          _F (…, CHAM_GD = Tp_n [N], COEF_R = ak [K] /omk
                          [K]),),),)

               Tc2 [K] =CRÉA_RESU (…, OPERATION=' AFFE',
                      TYPE_RESU=' MULT_ELAS',
                      AFFE=_F (CHAM_GD=Temp),)

               TO DESTROY (CONCEPT=_F (NOM= ('temp',),),)

               #-- Tc3 [K]
               Temp= CREA_CHAMP (…, OPERATION= 'ADZE',
```

```
                ADZE = (…, _F (CHAM_GD = Phi_l [N], COEF_R = - coef_mo),),)

        Tc3 [K] =CRÉA_RESU (…, OPERATION=' AFFE',
              TYPE_RESU=' MULT_ELAS', NOM_AFFE=_F (CHAM_GD=Temp),)

        TO DESTROY (CONCEPT=_F (NOM= ('temp',),),)


    #-- Definition of the bases of projections
    BM_Tc1=DEFI_BASE_MODALE (RITZ= (_F (MODE_MECA=Mo_lib,),
           _F (MULT_ELAS=Tc1,),), NUME_REF=Num,)

    BM_Tc2 [K] =DEFI_BASE_MODALE (RITZ= (_F (MODE_MECA=Mo_lib,),
           _F (MULT_ELAS=Tc2 [K],),), NUME_REF=Num,)

    BM_Tc3 [K] =DEFI_BASE_MODALE (RITZ= (_F (MODE_MECA=Mo_zero,),
           _F (MULT_ELAS=Tc3 [K],),), NUME_REF=Num,)


    #-- Construction of Ck by projection of Kv on Tc2 (K)
    nom_C=' Ck_ '+ str (K)
    PROJ_BASE (BASE=BM_Tc2 [K], NB_VECT=3*Nb, PROFIL=' PLEIN',
           MATR_ASSE_GENE= (_F (MATRICE=CO (nom_C), MATR_ASSE=K_v,),),
    )
    #-- Construction of Kk by projection of Kv on Tc2 (K)
    nom_K=' Kk_ '+ str (K)
    PROJ_BASE (BASE=BM_Tc3 [K], NB_VECT=3*Nb, PROFIL=' PLEIN',
           MATR_ASSE_GENE= (_F (MATRICE=CO (nom_K), MATR_ASSE=K_v,),),)
```

It only remains to summon the whole of the elementary matrices Ck, and to add all them Ck with the matrix Kg, and has to solve the reduced problem. However, so that the classification of all these matrices are coherent, it is advisable to proceed starting from matrices all projected on the same basis. One thus projects three matrices on $\mathrm{BM\_Tc1}$, in order to then build the matrices of mass, damping and stiffness reduced:

```
PROJ_BASE (BASE=BM_Tc1,
       NB_VECT=3*Nb,
       PROFIL=' PLEIN',
       MATR_ASSE_GENE= (
           _F (MATRICE=CO ('M_g'), MATR_ASSE=M,),
           _F (MATRICE=CO ('K_g'), MATR_ASSE=K_0,),
           _F (MATRICE=CO ('C_g'), MATR_ASSE=K_v,),),)

Num_gene=NUMÉRIQUE_DDL_GENE (BASE=BM_Tc1, NB_VECT= (2+len (ak))*Nb,)
```

The sum of these matrices is carried out after extraction of the various projected matrices of the associated concepts:

```
Kt=K_g.EXTR_MATR_GENE ()
Ct=0*C_g.EXTR_MATR_GENE ()

for K in arranges (len (ak)):
   exec ('Ct=Ct+ak [K] /omk [K] *Ck_' + str (K) + '.EXTR_MATR_GENE ()')
   exec ('Kt=Kt+ak [K] *Kk_' + str (K) + '.EXTR_MATR_GENE ()')
```

Then reintroduction of the matrices in the concepts:

```
C_g.RECU_MATR_GENE (Ct)
K_g.RECU_MATR_GENE (Kt)
```

```
Mo_c_g=CALC_MODES(MATR_RIGI=K_g,
    MATR_MASS=M_g,
    MATR_AMOR=C_g,
    TYPE_RESU=' DYNAMIQUE',
    OPTION=' TOUT',
    SOLVEUR_MODAL=_F (METHODE=' QZ'),
    CALC_FREQ=_F (SEUIL_FREQ=1e-6),
    VERI_MODE=_F (STOP_ERREUR=' NON'),
)
```

5.5.4.5 Comments

This approach for the moment is reserved to the experienced users, since the reduction of model in the presence of internal states can lead to singular behaviors. This presentation makes it possible nevertheless to show the opportunities given by *Code_Aster* in term of modularity.
The search for eigenvalues can be done only with certain parameter settings of the operator of search for eigenvalues: **CALC_MODES** [U4.52.02].
The reference material [R5.01.02] specifies the types of with problems which one can deal and possible parameter settings.
However, for the calculation of the clean modes of the scale model, it is recommended to use the full solvor (METHODE= 'QZ') by searching all the eigenvalues (OPTION=' TOUT'). It will be noted in addition that the choices of designs of CALC_MODES make that the real poles are filtered, and are not displayed in the results.

6 Case test for *Code_Aster*

To implement the calculation of complex modes on a scale model, a simple example is introduced. The calculation of the modes on this structure will be carried out with *Code_Aster.* One will compare the results got by using the complete model initially, then various scale models. One will detail then the standard command file allowing to carry out this calculation, while insisting on the points not approached in official documentation.

6.1 Note on the studies presented

The command files used for calculations of the complex modes of the scale models are available in the base of the studies, under the number of study 3185. They do not present a manner optimized to carry out calculations, in particular in the calculation of the residues and the construction of the bases of projection. On this point, it is pointed out that developments were studied to optimize this procedure.

However, it is a question of illustrating the approach to make it possible to the user to reproduce and adapt methodology.
The principal points of interest are clarified. It is mainly about:

- The definition of materials: One must indeed define “doubly” materials to carry out the calculation of the residues
- It is necessary to assign a damping hysteretic with each material, under sorrow that Ci is not correctly taken into account. For nondissipative materials, it is thus necessary to fix a damping hysteretic no one.
- Options for the solvor of clean modes: This last must be used with the option “ CENTER ” only, even if the required modes are the first of the structure the other options are not supported by the solvor.

6.2 Digital example with damping hysteretic

The example retained for this study is a square plate of 1 meter of dimensioned and 6 centimetres thickness, composed of a material sandwich viscoelastic steel/material/steel. The layers linear, elastic and isotropic, are stuck perfectly and thickness 2 centimetres. The model finite elements is presented

on Figure 6.2-a. This plate is embedded on an edge, and one is interested in the first 10 modes of this structure. The Young modulus of viscoelastic material is not realistic. It was adjusted to obtain reasonable depreciation, in spite of the thickness of the layer. These properties are recalled in Table 6.2-1.

Material	Young modulus (Pa)	Poisson's ratio	Density ($kg.m^{-3}$)	Rate of loss
Steel	$2,1.10^{11}$	0.3	7800	0.0
Visco	$1,5.10^{10}$	0.49	1400	1.0

Table 6.2-1 : Properties of materials used for calculation.

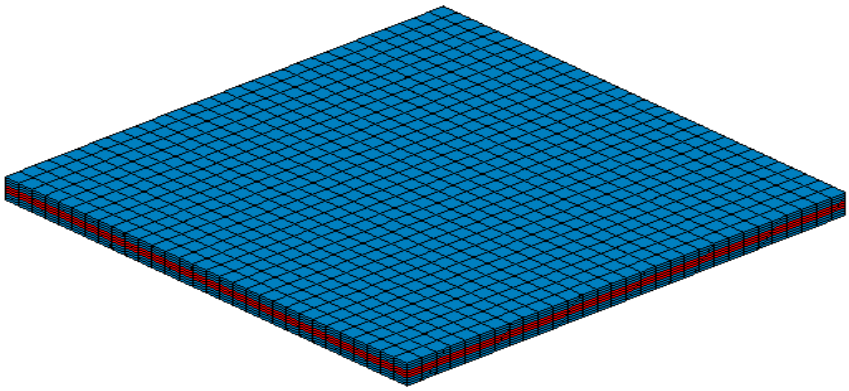

Figure 6.2-a : Model E.F of the structure of reference.

The model is built by using linear voluminal elements, with a total of 9610 nodes, are 27900 degrees of freedom.

The first 10 modes of the deadened structure are calculated while using *Code_Aster*. In each case, one only carries out calculation on a scale model built starting from the first 20 normal modes (bases T_{20}), and a scale model built starting from the first 10 normal modes nouveau riche with the residues associated with these 10 normal modes (bases T_{10+10}). A calculation of reference realized with the nonsmall-scale complete model is also carried out. The bases of reductions used are thus

$$T_{20}=[\Phi_1 \cdots \Phi_{20}] \tag{66}$$

and

$$T_{10+10}=\left[[\Phi_1 \cdots \Phi_{10}] \quad K_e^{-1} K_h [\Phi_1 \cdots \Phi_{10}]\right] \tag{67}$$

The results got with the various models are presented in Table 6.2-2 and it Table 6.2-3 .

Complete	Base T_{20}	Base T_{10+10}	Relative error complete/modes (%)	Relative error complete/residues (%)
61.84	61.39	61.84	-0.73	0
138.71	135.24	138.7	-2.5	-0.01
357.37	345.84	357.36	-3.23	0
449.34	436.52	449.29	-2.85	-0.01
485.45	465.42	485.41	-4.13	-0.01
533.39	533.27	533.4	-0.02	0

803.6	764.05	803.49	-4.92	-0.01
935.99	886.65	935.76	-5.27	-0.02
998.4	949.07	998.24	-4.94	-0.02
1053.3	995.94	1053.2	-5.45	-0.01

Table 6.2-2 : Eigen frequencies (in Hz) of the model sandwich.

The frequencies calculated starting from the scale model on 10 modes and 10 residues are almost identical to those calculated starting from the complete model. The frequencies obtained starting from the scale model on the basis of normal mode the first 20 are reasonable, but present nevertheless a significant error.

Complete	Base T_{20}	Base T_{10+10}	Complete/modes	Complete/residues
1.4	2.16	1.4	54.29	0
3.78	6.07	3.74	60.58	-1.06
4.95	8.07	4.93	63.03	-0.4
4.3	6.9	4.27	60.47	-0.7
6.55	10.28	6.51	56.95	-0.61
1.91	1.91	1.9	0	-0.52
8.33	12.71	8.27	52.58	-0.72
9.3	13.9	9.29	49.46	-0.11
8.08	12.55	8.06	55.32	-0.25
9.35	14.12	9.21	51.02	-1.5

Table 6.2-3 : Depreciation (in %) of the model sandwich.

The calculation of reduced depreciation is, on the other hand, much more clear-cut. Results got with the scale model on the normal modes only, except for mode 6, led to a very clear over-estimate of damping. The integration of the residues in the base of reduction of the model improves the results very clearly. The maximum error on calculation, compared to the model of reference, is indeed just higher than the percent, for a base of projection of equivalent size in both cases.

6.3 Digital example with viscoelastic damping describes by internal variables

The same geometry was taken again to build this study. However, it is not possible to build case test of reference, using the nonsmall-scale model, with *Code_Aster*. The results of reference, calculated on the complete model, were thus got with *Matlab* and *Structural Dynamic Toolbox*. The characteristics of materials used are presented in the table 6.3-1.

Material	Young modulus (Pa)	Poisson's ratio	Density ($kg.m^{-3}$)	Relaxation time (s)
Steel	$2{,}1.10^{11}$	0.3	7800	∞
Visco	$4{,}53.10^{6}$	0.49	1400	∞
	$3{,}51.10^{6}$	0.49		$5{,}58.10^{-2}$
	$1{,}395.10^{7}$	0.49		$3{,}20.10^{-3}$
	$4{,}04.10^{7}$	0.49		$3{,}01.10^{-4}$
	$1{,}20.10^{8}$	0.49		$4{,}58.10^{-5}$
	$7{,}57.10^{8}$	0.49		$5{,}93.10^{-6}$

Table 6.3-1 : Properties of materials used.

The results got with the complete model and the scale model are presented in the table 6.3-2. The scale model was built on the basis of clean mode the first ten of the nondissipative structure.

Complete		Reduced	
Fréq. (Hz)	Amor. (%)	Fréq. (Hz)	Amor. (%)

25.26	7.00	25.26	7.00
48.41	10.60	47.96	10.85
150.61	8.55	150.59	8.55
174.11	11.39	174.07	11.40
186.38	8.69	186.17	8.88
309.68	10.10	309.64	10.11
426.46	8.04	426.36	8.08
438.48	9.80	439.97	10.31
463.72	8.93	463.77	8.98
523.22	0.04	523.22	0.04

Table 6.3-2 : Comparison of the results – Frequencies and depreciation.

The got results are very good, and the error for the first ten modes is very weak, whether it is the frequencies or depreciation.

7 Conclusions on the use of the methods of reduction

One briefly presented the principles of the reduction of model based on the initial construction of under space, then of his enrichment starting from calculations of residues. All these techniques rest on the principle of the static correction, and thus on the introduction, in the base of reduction, of an estimate of the response of the structure to various efforts. These efforts can be external requests, as in the case of the classical static correction, but also of the efforts applied by adjacent substructures, like in the case of the methods of under structuring, or of the internal efforts, like in the case of the dissipative structures.

These techniques were illustrated within the particular framework of the calculation of complex modes for a strongly dissipative structure presenting damping material. Damping was initially represented by using a behavior hysteretic, then a viscoelastic behavior describes by internal variables. A simple case test of strongly dissipative sandwich structure was introduced. The results got for calculations of complex modes are very good, and show the relevance of these techniques. Interest of these techniques in the case of approaches by understructuring (classical [R4.06.02], cyclic [R4.06.03], or decomposition of field [11]) was largely shown besides in addition.

The methods presented are conceived to facilitate calculation on important models, or the calculations repeated (harmonic answer, for example) on models of average/big size (several ten a thousand of degrees of freedom). When the size of the model is low, and under the assumptions of damping hysteretic (Nota bene: since it is not possible in the case more the general with Aster), one will be able to thus privilege complete calculation. On the other hand, for means/large models including a dissipative material, even if the zone where damping is important is of small, the whole of the problem will have to be treated like a complex problem. It thus becomes extremely interesting to build a scale model on the basis of clean mode of the real problem, enriched by the residues, or to adopt an approach by under structuring.

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