SSNV221 – Hydrostatic test with a behavior
DRUCK_PRAGER linear and parabolic

Summary:

The case test proposes a purely hydrostatic loading for the associated law Drücker-Prager [R7.01.16]. The formulation of this plastic law, often used for the grounds, is made at the same time on the deviatoric and hydrostatic part; nevertheless, surface criterion presents a singularity for a purely hydrostatic state of stress. This analytical CAS-test is used to check correct work hardening in this singularity.

The test is carried out on a material point with the order SIMU_POINT_MAT. One works with imposed deformations.

One makes a test with linear work hardening (modeling A) and another with parabolic work hardening (modeling B).
1 Problem of reference

1.1 Properties of material

Rubber bands:

\[ E = 3000 \text{ MPa} \quad \text{Young modulus} \]
\[ \nu = 0,25 \quad \text{Poisson's ratio} \]

DRUCK \_ PRAGER linear (modeling A):

\[ \alpha = 0,20 \quad \text{Coefficient of dependence in pressure} \]
\[ P_{\text{ultm}} = 0,04 \quad \text{Ultimate cumulated plastic deformation} \]
\[ \sigma_Y = 6 \text{ MPa} \quad \text{Plastic constraint} \]
\[ h = 100 \text{ MPa} \quad \text{Module of work hardening} \]

DRUCK \_ PRAGER parabolic (modeling B):

\[ \alpha = 0,20 \quad \text{Coefficient of dependence in pressure} \]
\[ P_{\text{ultm}} = 0,04 \quad \text{Ultimate cumulated plastic deformation} \]
\[ \sigma_Y = 6 \text{ MPa} \quad \text{Plastic constraint} \]
\[ \sigma_{\text{ult}}^p = 10 \text{ MPa} \quad \text{Ultimate plastic constraint} \]

1.2 Loadings and boundary conditions

A voluminal deformation is imposed \( \varepsilon_v = \text{tr}(\varepsilon) \). The loading is not monotonous: one charges initially until the voluminal deformation \( \varepsilon_{v1} \), by exceeding the threshold of plasticization, then one discharges on a null level of deformation; then one still charges with the deformation \( \varepsilon_{v2} \) by thus exceeding the ultimate cumulated plastic deformation \( P_{\text{ultm}} \), beyond which one finds a perfect plasticity; one still discharges with worthless constraint (deformation equal to the plastic deformation \( \varepsilon^p_{v2} \)) and one reloads while plasticizing later on until the deformation \( \varepsilon_{v3} \). The time of loading (see Table 1.2-1) is fictitious because the plastic laws are independent of time.

<table>
<thead>
<tr>
<th>( t )</th>
<th>( \varepsilon_v )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>( \varepsilon_{v1} = 0,018 )</td>
</tr>
<tr>
<td>14</td>
<td>0</td>
</tr>
<tr>
<td>26</td>
<td>( \varepsilon_{v2} = 0,045 )</td>
</tr>
<tr>
<td>30</td>
<td>( \varepsilon^p_{v2} = 0,03667 )</td>
</tr>
<tr>
<td>40</td>
<td>( \varepsilon_{v3} = 0,06 )</td>
</tr>
</tbody>
</table>

Table 1.2-1: imposed voluminal deformation.

1.3 Initial conditions

All the components of the constraints and deformations are worthless at the beginning of the loading.
2 Reference solution

Modeling checks the behavior of the law with linear work hardening.

2.1 Method of calculating

The equations which interest us for analytical calculation are \( I_1 = \text{tr} (\boldsymbol{\sigma}) \): trace of the tensor of the constraints, \( \varepsilon_p^v \): voluminal plastic deformation:

- plastic constitutive law on the voluminal part:
  \[
  I_1 = 3K (\varepsilon_v - \varepsilon_p^v)
  \]  
  (eq 2.1-1)

- surface criterion, by posing worthless the constraint of Von Mises (\( \sigma_{eq} = 0 \)):
  \[
  F(\boldsymbol{\sigma}, p) = \alpha I_1 - R(p)
  \]  
  (eq 2.1-2)

- relation between the voluminal plastic deformation and the cumulated plastic deformation (variable intern of the plastic law):
  \[
  \varepsilon_v^p = 3\alpha p \quad \text{thus while integrating:} \quad \varepsilon_v^p = 3\alpha p
  \]  
  (eq 2.1-3)

- expression of work hardening
  - linear:
    \[
    R(p) = \sigma_y + h p \quad \text{si} \quad p \leq p_{ult}
    \]  
    \[
    R(p) = \sigma_y + h p_{ult} = \sigma_{ult} \quad \text{si} \quad p > p_{ult}
    \]  
    (eq 2.1-4)

  - parabolic:
    \[
    R(p) = \sigma_y \left( 1 - \left( \frac{\sigma_{ult}}{\sigma_y} \right)^2 \right) \quad \text{si} \quad p \leq p_{ult}
    \]  
    \[
    R(p) = \sigma_{ult} \quad \text{si} \quad p > p_{ult}
    \]  
    (eq 2.1-5)

It is observed that, as in the linear case, \( R(p) = \sigma_y \) if \( p = 0 \) and there is perfect plasticity if \( p > p_{ult} \).

2.1.1 Deformation in extreme cases elastic initial

This deformation is obtained for \( \varepsilon_v^p = p = 0 \).

If one poses \( F(\boldsymbol{\sigma}, p) = 0 \) (plastic evolution) one a:

\[
I_1^{el} = \frac{R(p)}{\alpha} = \frac{\sigma_y}{\alpha}
\]

\[
\varepsilon_v^{el} = \frac{1}{3K}
\]

2.1.2 Ultimate deformation

Ultimate deformation is called \( \varepsilon_v^{ult} \) that obtained for \( p = p_{ult} \).

The trace of constraints easily is found \( I_1^{ult} \) and plastic deformation \( \varepsilon_v^{ult} \) corresponding:

\[
I_1^{ult} = \frac{R(p)}{\alpha} = \frac{\sigma_{ult}}{\alpha}
\]
2.1.3 Deformation enters the yield stress and the ultimate deformation

One calculates initially the cumulated plastic deformation.

- By combining the equations (2.1-1), (2.1-2), (2.1-3) and (2.1-4) with \( F(\sigma, p) = 0 \) for \textit{linear work hardening} one a:

\[
p = \frac{3 K A \varepsilon_v - \sigma_y}{9 K \alpha^2 + h}\]  
(éq 2.1-6)

- By combining the equations (2.1-1), (2.1-2), (2.1-3) and (2.1-5) with \( F(\sigma, p) = 0 \) for \textit{parabolic work hardening} one arrives at the equation of dismantled 2:

\[
A_1 \bar{p}^2 + B_1 \bar{p} + C_1 = 0 \\
A_1 = \sigma_y |1 - \gamma|^2 \\
B_1 = 9 K \alpha^2 p_{ult} - 2 \sigma_y |1 - \gamma| \\
C_1 = \sigma_y - 3 K \alpha \varepsilon_v \varepsilon_v \\
\gamma = \sqrt{\frac{\sigma_{ult}}{\sigma_y}} \\
\bar{p} = \frac{p}{p_{ult}}
\]  
(eq 2.1-7)

One uses the equations then (2.1-3) (2.1-1) to find the plastic deformation \( \varepsilon_v^p \) and traces it constraints \( I_1 \).

If one makes discharge elastic material of way until worthless constraint, one finds a residual deformation equal to the plastic deformation; it is on the other hand necessary to charge material in compression to obtain a worthless total deflection. This second branch is also elastic, because the material of Drücker-Prager cannot plasticize in a hydrostatic state of compression. In this last case, the trace of the constraints, negative, is:

\[
I_1^c = -3 K \varepsilon_v^p
\]  
(éq 2.1-8)

2.1.4 Deformation higher than the ultimate deformation

All the quantities of interest are easily found, because the trace of constraints is known a priori and equal to \( I_1^{ult} \):

\[
\varepsilon_v^p = \varepsilon_v - \frac{I_1^{ult}}{3 K}
\]
\[ p = \frac{\varepsilon_p}{3\alpha} \]

### 2.2 Sizes and results of reference

The module of compressibility \( K \) is:

\[ K = \frac{E}{3(1-2\nu)} = 2000 \text{ MPa} \]

#### 2.2.1 Deformation in extreme cases elastic

For two modelings one finds easily:

\[ I_{1e} = 30 \text{ MPa} \]
\[ \varepsilon_{ve} = 0.005 \]

#### 2.2.2 Ultimate deformation

For two modelings one finds:

\[ I_{1u} = 50 \text{ MPa} \]
\[ \varepsilon_{vu} = 0.024 \]
\[ \varepsilon_{uu} \approx 0.03233 \]

#### 2.2.3 Deformation equalizes to 0.018 and discharges with worthless deformation

This value of deformation \( \varepsilon_{v1} = 0.018 \) is higher than the yield stress \( \varepsilon_{ve} \) and lower than \( \varepsilon_{vu} \). One calculates initially the plastic deformation cumulated with the equations (2.1-7) and (2.1-8), then plastic deformation and the trace of the constraints:

- **linear work hardening**:
  \[ p_1 = \frac{3K(A\varepsilon_{v1} - \sigma_y)}{9K\alpha^2 + h} \approx 0.019 \]
  \[ \varepsilon_{p1} = 3\alpha p_1 \approx 0.0114 \]
  \[ I_{1e} = 3K(\varepsilon_{v1} - \varepsilon_{p1}) \approx 39.51 \text{ MPa} \]

- **parabolic work hardening**:
  \[ p_1 \approx 0.0192 \]
  \[ \varepsilon_{p1} = 3\alpha p_1 \approx 0.0115 \]
  \[ I_{1e} = 3K(\varepsilon_{v1} - \varepsilon_{p1}) \approx 38.956 \text{ MPa} \]

The trace of the constraints with worthless deformation is:

- **linear work hardening**:
  \[ I_{1c} = -3K\varepsilon_{v1} \approx -68.49 \text{ MPa} \]

- **parabolic work hardening**:
  \[ I_{1c} = -3K\varepsilon_{p1} \approx -69.044 \text{ MPa} \]

Indeed, the difference between the parabolic and linear case is very weak.

#### 2.2.4 Loading until deformation EGA to 0.045 and 0.06

One reloads material up to the values of deformation \( \varepsilon_{v2} = 0.045 \) and \( \varepsilon_{v3} = 0.06 \), higher than \( \varepsilon_{uu} \).

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The results are the same ones for two modelings.

For $\epsilon_{v_2} = 0.045$ :

$$\epsilon_{v_2}^p = \epsilon_{v_2} - \frac{I_1^{ult}}{3K} \approx 0.03667$$

$$p_2 = \frac{\epsilon_{v_2}^p}{3} \approx 0.0611$$

Following the elastic discharge (until worthless constraint), one finds $\epsilon_r = \epsilon_{v_2}^p \ , \ p = p_2$.

For $\epsilon_{v_3} = 0.06$ :

$$\epsilon_{v_3}^p = \epsilon_{v_3} - \frac{I_1^{ult}}{3K} \approx 0.05167$$

$$p_3 = \frac{\epsilon_{v_3}^p}{3} \approx 0.0861$$

2.2.5 Stress-strain curves

In the Figures (2.2.5-a) and (2.2.5-b) one represents the curve $\left( \epsilon_r, I_1 \right)$ for linear and parabolic work hardening. In red are the points tested by the CAS-test.

![Figure 2.2.5-a: stress-strain curves for linear work hardening.](image-url)
2.3 Uncertainties on the solution

The solution is analytical.

2.4 Bibliographical references

3 Modeling A

3.1 Characteristics of modeling

The test is carried out on a material point with the order SIMU_POINT_MAT. One works with imposed deformations. Work hardening is linear.

3.2 Sizes and results of reference

<table>
<thead>
<tr>
<th>Not on Figure 2.2.5-a</th>
<th>Checked quantity</th>
<th>Value of reference</th>
<th>Type of reference</th>
<th>Tolerance (relative)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Trace of the constraints</td>
<td>$I_1^{1} = 39.51$ MPa</td>
<td>ANALYTICAL</td>
<td>$10^{-6}$ %</td>
</tr>
<tr>
<td>2</td>
<td>Trace of the constraints</td>
<td>$I_1^{1c} = -68.49$ MPa</td>
<td>ANALYTICAL</td>
<td>$10^{-6}$ %</td>
</tr>
<tr>
<td>3 or 4</td>
<td>Spherical part of the plastic deformation</td>
<td>$\varepsilon_{v2}^{p} = 0.03667$</td>
<td>ANALYTICAL</td>
<td>$10^{-6}$ %</td>
</tr>
<tr>
<td>3 or 5</td>
<td>Trace of constraints</td>
<td>$I_1^{ult} = 50$ MPa</td>
<td>ANALYTICAL</td>
<td>$10^{-6}$ %</td>
</tr>
<tr>
<td>5</td>
<td>Spherical part of the plastic deformation</td>
<td>$\varepsilon_{v3}^{p} = 0.051667$</td>
<td>ANALYTICAL</td>
<td>$10^{-6}$ %</td>
</tr>
</tbody>
</table>
4 Modeling B

4.1 Characteristics of modeling

The test is carried out on a material point with the order `SIMU_POINT_MAT`. One works with imposed deformations. Work hardening is parabolic.

4.2 Sizes and results of reference

<table>
<thead>
<tr>
<th>Not on Figure 2.2.5-b</th>
<th>Checked quantity</th>
<th>Value of reference</th>
<th>Type of reference</th>
<th>Tolerance (relative)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 or 2</td>
<td>Spherical part of the plastic deformation</td>
<td>$\varepsilon_p^{v_2}=0.03667$</td>
<td>ANALYTICAL</td>
<td>$10^{-6}$ %</td>
</tr>
<tr>
<td>1 or 3</td>
<td>Trace of constraints</td>
<td>$I_{ult}^{1}=50\ MPa$</td>
<td>ANALYTICAL</td>
<td>$10^{-6}$ %</td>
</tr>
<tr>
<td>3</td>
<td>Spherical part of the plastic deformation</td>
<td>$\varepsilon_p^{v_3}=0.051667$</td>
<td>ANALYTICAL</td>
<td>$10^{-6}$ %</td>
</tr>
</tbody>
</table>

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5 Summary of the results

The results of the CAS-test are satisfactory, *Code_Aster* reproduced the analytical results with a high precision.