Shape functions and points of integration of the finite elements

Summarized:

One describes the geometry and topology of the finite elements established in *Code_Aster* ; for each element of reference, the statement of the shape functions and the various families of points of integration as well as the associated weights are detailed.

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1 Introduction

Into *Code_Aster*, one calls “finite element”, a triplet (phenomenon, modelization, type of mesh). There are three principal phenomena: `MECANIQUE`, `THERMAL` and `ACOUSTICS`.
There exist many modelizations; for example, for the MECHANICAL `phenomenon : 3D`, `C_PLAN`, `D_PLAN`, `AXIS`, `DKT`, `POU_D_E`,…
For a given modelization (for example `3D`) of a phenomenon (for example `MECHANICAL`), there exist several finite elements in general: an element by type of mesh supported: `HEXA8`, `HEXA20`, `PENTA6`, …

With final, there thus exists of very many finite elements (more than 500 in July 2004).

On the other hand, the types of mesh are them of reduced number: `POI1, SEG2, SEG3, SEG4, TRIA3, TRIA6, TRIA7, QUAD4, QUAD8, HEXA8, HEXA20, …, TETRA4, TETRA10.`

In general, each finite element, to carry out its elementary computations, uses the notions of interpolation function (or shape function) and diagram of integration. In general also, these shape functions and these diagrams of integration are defined on an element known as “of reference” whose geometry is defined in an often called coordinate system: (ξ, η, ζ) . The transition of the element of reference to the real element is done thanks to a geometrical transformation which uses the same interpolation functions. The element is then known as “isoparametric”. These notions are very well explained in [bib1].

The high number of finite elements of the code combined with the restricted number of the types of mesh, conduit to the fact that there exist several finite elements leaning on the same type of mesh; for example the quadrilateral with 8 nodes called `QUAD8` supports more than 60 different finite elements.

In theory, each finite element can choose its interpolation functions and its diagrams of integration as he hears it. But in practice, almost all the finite elements leaning on the same type of mesh, use the same element of reference, the same shape functions and the same diagrams of integration. The goal of this document is to describe these various elements of reference.

For each element of reference (called in the continuation of document `ELREFE`), one will indicate:

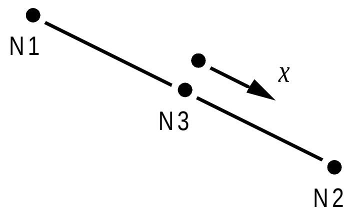
- the mesh support, the number of the nodes, their local classification and their coordinates,
- algebraical expressions of the shape functions and their derivatives first (and sometimes seconds)
- families of points of integration which one will name. For each family, one will give the number of points, their coordinates and their “weights” of integration. The sum of these weights, must give the “volume” of the element of reference. For example, the “volume” of the quadrangle of reference ($-1 \leq \xi \leq +1$, $1 < \eta < +1$) is worth 4.

2 Linear elements: SE2, SE3 and SE4

SE2 : segment with 2 nodes
many nodes : 2
many nodes tops : 2

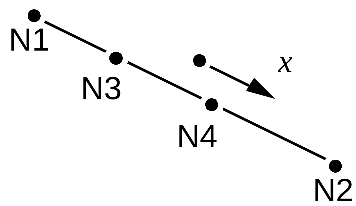
SE3 : segment with 3 nodes
many nodes : 3
many nodes tops : 2

	x
$N1$	-1.0
$N2$	1.0.0.0
$N3$	



SE4 : segment with 4 nodes
many nodes : 4
many nodes tops : 2

	x
$N1$	-1.0
$N2$	1.0
$N3$	-1. /3.
$N4$	+1. /3.



shape functions of the segment with 2 nodes:

$$w_1(x)=0.5(1-x) \quad w_2(x)=0.5(1+x)$$

shape functions of the segment with 3 nodes:

$$w_1(x)=-0.5(1-x)x \quad w_2(x)=0.5(1+x)x \quad w_3(x)=(1+x)(1-x)$$

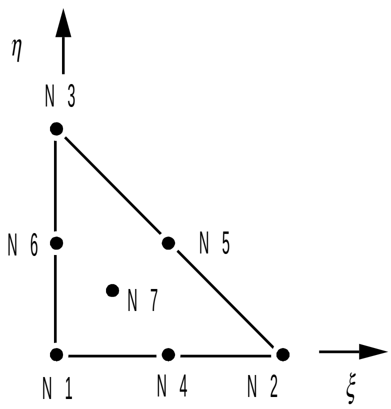
shape functions of the segment with 4 nodes:

$$w_1(x)=\frac{16}{9}(1-x)\left(x+\frac{1}{3}\right)\left(x-\frac{1}{3}\right)$$
$$w_2(x)=-\frac{16}{9}(1+x)\left(\frac{1}{3}-x\right)\left(x+\frac{1}{3}\right)$$
$$w_3(x)=\frac{16}{27}(x-1)(x+1)\left(x-\frac{1}{3}\right)$$
$$w_4(x)=-\frac{16}{27}(x-1)(x+1)\left(x+\frac{1}{3}\right)$$

Nb of pts of intégr.	Not	x	Weights
1	1.0.0.2.0		
2	1	0.577350269189626	1.0
	2	-0.577350269189626	1.0
3	1	-0.774596669241	0.55555…
	2.0.0		0.88888…
	3	0.774596669241	0.55555…
4	1	0.339981043584856	0.652145154862546
	2	-0.339981043584856	0.652145154862546
	3	0.861136311594053	0.347854845137454
	4	-0.861136311594053	0.347854845137454

3 surface elements

3.1 Triangles: ELREFE TR3, TR6, TR7



Coordinated of the nodes:

	ξ	η
N1	0.0.0.0	
N2	1.0.0.0	
N3	0.0.1.0	
N4	0.5.0.0	
N5	0.5.0.5	
N6	0.0.0.5	
N7	1/3	1/3

Family	**Not**	ξ	η	**Weight**
FPG1	1	1/3	1/3	1/2
FPG3	1	1/6	1/6	1/6
	2	2/3	1/6	1/6
	3	1/6	2/3	1/6
FPG4	1	1/5	1/5	25/(24*4)
	2	3/5	1/5	25/(24*4)
	3	1/5	3/5	25/(24*4)
	4	1/3	1/3	-27/(24*4)
FPG6	1	B	B	P2
	2	1 - 2 B	B	P2
	3	B	1 - 2 B	P2
	4	has	1 - 2 P1	5
	has		has	P1
	6	1 - 2 has		P1
COT3	1	1/2	1/2	1/6
	2	0	1/2	1/6
	3	1/2	0	1/6

With P1 = 0.11169079483905, P2 = 0.0549758718227661,
A = 0.445948490915965, B = 0.091576213509771

Family	Not	ξ	η	Weight
FPG7	1	1/3	1/3	9/80
	2	A	A	P1
	3	1-2A	A	P1
	4	A	1-2A	P1
	5	B	B	P2
	6	1-2B	B	P2
	7	B	1-2B	P2

With A = 0.470142064105115
B = 0.101286507323456
P1 = 0.066197076394253
P2 = 0.062969590272413

Family	Not	ξ	η	Weight
FPG12	1	A	A	P1
	2	1-2A	A	P1
	3	A	1-2A	P1
	4	B	B	P2
	5	1-2B	B	P2
	6	B	1-2B	P2
	7	C	D	P3
	8	D	C	P3
	9	1-C-D	C	P3
	10	1-C-D	D	P3
	11	C	1-C-D	P3
	12	D	1-C-D	P3

With A = 0.063089014491502
B = 0.249286745170910
C = 0.310352451033785
D = 0.053145049844816
P1 = 0.025422453185103
P2 = 0.058393137863189
P3 = 0.041425537809187

TR3 : triangle with 3 nodes
many nodes : 3
many nodes tops : 3

shape functions and derived first of the triangle with 3 nodes:

$\{N\}$	$\{\partial N/\partial \xi\}$	$\{\partial N/\partial \eta\}$
$1-\xi-\eta$	-1	-1
ξ	1	0
η	0	1

TR6 : triangle with 6 nodes
many nodes : 6
many nodes tops : 3

shape functions, derived first of the triangle with 6 nodes:

$\{N\}$	$\{\partial N/\partial \xi\}$	$\{\partial N/\partial \eta\}$
$-(1-\xi-\eta)(1-2(1-\xi-\eta))$	$1-4(1-\xi-\eta)$	$1-4(1-\xi-\eta)$
$-\xi(1-2\xi)$	$-1+4\xi$	0
$-\eta(1-2\eta)$	0	$-1+4\eta$
$4\xi(1-\xi-\eta)$	$4(1-2\xi-\eta)$	-4ξ
$4\xi\eta$	4η	4ξ
$4\eta(1-\xi-\eta)$	-4η	$4(1-\xi-2\eta)$

second derivative of the triangle with 6 nodes:

$\{\partial^2 N/\partial \xi^2\}$	$\{\partial^2 N/\partial \xi \partial \eta\}$	$\{\partial^2 N/\partial \eta^2\}$
4	4	4
4	0	0
0	0	4
-8	-4	0
0	4	0
0	-4	-8

TR7 : triangle with 7 nodes
many nodes : 7
many nodes tops : 3

shape functions of the triangle with 7 nodes:

$\{N\}$
$1-3(\xi+\eta)+2(\xi^2+\eta^2)+7\xi\eta-3\xi\eta(\xi+\eta)$
$\xi(-1+2\xi+3\eta-3\eta(\xi+\eta))$
$\eta(-1+2\xi+3\eta-3\xi(\xi+\eta))$
$4\xi(1-\xi-4\eta+3\eta(\xi+\eta))$
$4\xi\eta(-2+3(\xi+\eta)$
$4\eta(1-4\xi-\eta+3\xi(\xi+\eta))$
$27\xi\eta(1-\xi-\eta)$

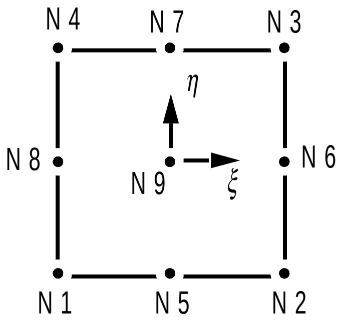
derived first from the triangle with 7 nodes:

$\{\partial N/\partial\xi\}$	$\{\partial N/\partial\eta\}$
$-3+4\xi+7\eta-6\xi\eta-3\eta^2$	$-3+7\xi+4\eta-6\xi\eta-3\xi^2$
$-1+4\xi+3\eta-6\xi\eta-3\eta^2$	$3\xi(1-\xi-2\eta)$
$3\xi(1-2\eta-\xi)$	$-1+3\xi+4\eta-6\xi\eta-3\xi^2$
$4(1-2\xi-4\eta+6\xi\eta+3\eta^2)$	$4\xi(-4+3\xi+6\eta)$
$4\eta(-2+6\xi+3\eta)$	$4\xi(-2+3\xi+6\eta)$
$4\eta(-4+6\xi+3\eta)$	$4(-1-4\xi-2\eta+6\xi\eta+3\xi^2)$
$27\eta(1-2\xi-\eta)$	$27\xi(1-\xi-2\eta)$

second derivative of the triangle with 7 nodes:

$\{\partial^2 N/\partial\xi^2\}$	$\{\partial^2 N/\partial\xi\,\partial\eta\}$	$\{\partial^2 N/\partial\eta^2\}$
$4-6\eta$	$7-6\xi-6\eta$	$4-6\xi$
$4-6\eta$	$3-6\xi-6\eta$	-6ξ
-6η	$3-6\xi-6\eta$	$4-6\xi$
$4(-2+6\eta)$	$4(-4+6\xi+6\eta)$	24ξ
24η	$4(-2+6\xi+6\eta)$	24ξ
24η	$4(-4+6\xi+6\eta)$	$4(-2+6\xi)$
-54η	$27(1-2\xi-2\eta)$	-54ξ

3.2 Quadrangles: `ELREFE QU4`, `QU8`, `QU9`



Coordinated of the nodes:

	ξ	η
N1	-1.0	-1.0
N2	1.0	-1.0
N3	1.0.1.0	
N4	-1.0	1.0.0.0
N5		-1.0
N6	1.0.0.0.0.0	
N7		1.0
N8	-1.0	0.0.0.0.0.0
N9		

Family	Not	ξ	η	Weight
FPG1	1	0	0	4
FPG4	1	- has	- has	1.0
	2	has	- has	1.0
	3	has		1.0
	4	- has		1.0
		has = $1/\sqrt{3}$		
FPG9	1	- has	- has	25/81
	2	has	- has	25/81
	3	has		25/81
	4	- has		25/81
	5.0.0		- has	40/81
	6	has	0.0	40/81
	7.0.0		has	40/81
	8	- has	0.0	40/81
	9.0.0.0.0			64/81
		a= 0.774596669241483		

QU4 : quadrangle with 4 nodes
many nodes : 4
many nodes tops : 4

shape functions, derived first and seconds of the quadrangle with 4 nodes:

$\{N\}$	$\{\partial N/\partial\xi\}$	$\{\partial N/\partial\eta\}$
$(1-\xi)(1-\eta)/4$	$-(1-\eta)/4$	$-(1-\xi)/4$
$(1+\xi)(1-\eta)/4$	$(1-\eta)/4$	$-(1+\xi)/4$
$(1+\xi)(1+\eta)/4$	$(1+\eta)/4$	$(1+\xi)/4$
$(1-\xi)(1+\eta)/4$	$-(1+\eta)/4$	$(1-\xi)/4$

$\{\partial^2 N/\partial\xi^2\}$	$\{\partial^2 N/\partial\xi\,\partial\eta\}$	$\{\partial^2 N/\partial\eta^2\}$
0	1/4	0
0	-1/4	0
0	1/4	0
0	-1/4	0

QU8 : quadrangle with 8 nodes
many nodes : 8
many nodes tops : 4

shape functions and derived first of the quadrangle with 8 nodes:

$\{N\}$	$\{\partial N/\partial\xi\}$	$\{\partial N/\partial\eta\}$
$(1-\xi)(1-\eta)(-1-\xi-\eta)/4$	$(1-\eta)(2\xi+\eta)/4$	$(1-\xi)(\xi+2\eta)/4$
$(1+\xi)(1-\eta)(-1+\xi-\eta)/4$	$(1-\eta)(2\xi-\eta)/4$	$-(1+\xi)(\xi-2\eta)/4$
$(1+\xi)(1+\eta)(-1+\xi+\eta)/4$	$(1+\eta)(2\xi+\eta)/4$	$(1+\xi)(\xi+2\eta)/4$
$(1-\xi)(1+\eta)(-1-\xi+\eta)/4$	$-(1+\eta)(-2\xi+\eta)/4$	$(1-\xi)(-\xi+2\eta)/4$
$(1-\xi^2)(1-\eta)/2$	$-\xi(1-\eta)$	$-(1-\xi^2)/2$
$(1+\xi)(1-\eta^2)/2$	$(1-\eta^2)/2$	$-\eta(1+\xi)$
$(1-\xi^2)(1+\eta)/2$	$-\xi(1+\eta)$	$(1-\xi^2)/2$
$(1-\xi)(1-\eta^2)/2$	$-(1-\eta^2)/2$	$-\eta(1-\xi)$

second derivative of the quadrangle with 8 nodes:

$\{\partial^2 N/\partial\xi^2\}$	$\{\partial^2 N/\partial\xi\,\partial\eta\}$	$\{\partial^2 N/\partial\eta^2\}$
$(1-\eta)/2$	$(1-2\xi-2\eta)/4$	$(1-\xi)/2$
$(1-\eta)/2$	$-(1+2\xi-2\eta)/4$	$(1+\xi)/2$
$(1+\eta)/2$	$(1+2\xi+2\eta)/4$	$(1+\xi)/2$
$(1+\eta)/2$	$-(1-2\xi+2\eta)/4$	$(1-\xi)/2$
$-1+\eta$	ξ	0
0	$-\eta$	$-1-\xi$
$-1-\eta$	$-\xi$	0
0	η	$-1+\xi$

QU9 : quadrangle with 9 nodes
many nodes : 9
many nodes tops : 4

shape functions and derived first of the quadrangle with 9 nodes:

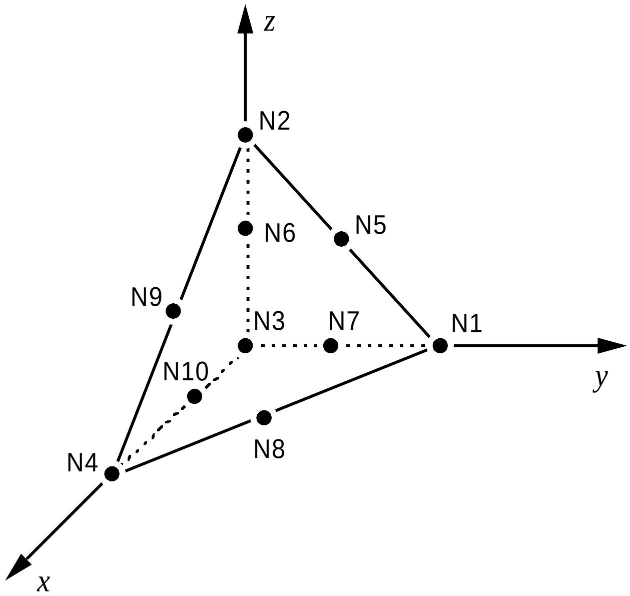
$\{N\}$	$\{\partial N/\partial\xi\}$	$\{\partial N/\partial\eta\}$
$\xi\eta(\xi-1)(\eta-1)/4$	$(2\xi-1)\eta(\eta-1)/4$	$\xi(\xi-1)(2\eta-1)/4$
$\xi\eta(\xi+1)(\eta-1)/4$	$(2\xi+1)\eta(\eta-1)/4$	$\xi(\xi+1)(2\eta-1)/4$
$\xi\eta(\xi+1)(\eta+1)/4$	$(2\xi+1)\eta(\eta+1)/4$	$\xi(\xi+1)(2\eta+1)/4$
$\xi\eta(\xi-1)(\eta+1)/4$	$(2\xi-1)\eta(\eta+1)/4$	$\xi(\xi-1)(2\eta+1)/4$
$(1-\xi^2)\eta(\eta-1)/2$	$-\xi\eta(\eta-1)$	$(1-\xi^2)(2\eta-1)/2$
$\xi(\xi+1)(1-\eta^2)/2$	$(2\xi+1)(1-\eta^2)/2$	$-\xi\eta(\xi+1)$
$(1-\xi^2)\eta(\eta+1)/2$	$-\xi\eta(\eta+1)$	$(1-\xi^2)(2\eta+1)/2$
$\xi(\xi-1)(1-\eta^2)/2$	$(2\xi-1)(1-\eta^2)/2$	$-\xi\eta(\xi-1)$
$(1-\xi^2)(1-\eta^2)$	$-2\xi(1-\eta^2)$	$-2\eta(1-\xi^2)$

second derivative of the quadrangle with 9 nodes:

$\{\partial^2 N/\partial\xi^2\}$	$\{\partial^2 N/\partial\xi\,\partial\eta$	$\{\partial^2 N/\partial\eta^2\}$
$\eta(\eta-1)/2$	$(\xi-1/2)(\eta-1/2)$	$\xi(\xi-1)/2$
$\eta(\eta-1)/2$	$(\xi+1/2)(\eta-1/2)$	$\xi(\xi+1)/2$
$\eta(\eta+1)/2$	$(\xi+1/2)(\eta+1/2)$	$\xi(\xi+1)/2$
$\eta(\eta+1)/2$	$(\xi-1/2)(\eta+1/2)$	$\xi(\xi-1)/2$
$-\eta(\eta-1)$	$-\xi(2\eta-1)$	$1-\xi^2$
$1-\eta^2$	$-\eta(2\xi+1)$	$-\xi(\xi+1$
$-\eta(\eta+1)$	$-\xi(2\eta+1)$	$1-\xi^2$
$1-\eta^2$	$-\eta(2\xi-1)$	$-\xi(\xi-1)$
$-2(1-\eta^2)$	$4\xi\eta$	$-2(1-\xi^2)$

4 Voluminal elements

4.1 Tetrahedrons: `ELREFE TE4`, `T10`



Coordinated of the nodes:

	x	y	z
$N1$	0.	1.	0.
$N2$	0.	0.	1.
$N3$	0.	0.	0.
$N4$	1.	0.	0.
$N5$	0.	0.5.0.5	
$N6$	0.	0.	0.5
$N7$	0.	0.5	0.
$N8$	0.5.0.5		0.
$N9$	0.5	0.	0.5.0.5
$N10$		0.	0.

Shape functions:

Formulate with 4 nodes

$$\begin{cases} w_1(x,y,z)=y \\ w_2(x,y,z)=z \\ w_3(x,y,z)=1-x-y-z \\ w_4(x,y,z)=x \end{cases}$$

Formulates with 10 nodes

$$w_1=y(2y-1) \qquad w_6=4z(1-x-y-z)$$
$$w_2=z(2z-1) \qquad w_7=4y(1-x-y-z)$$
$$w_3=(1-x-y-z)(1-2x-2y-2z) \qquad w_8=4xy$$
$$w_4=x(2x-1) \qquad w_9=4xz$$
$$w_5=4yz \qquad w_{10}=4x(1-x-y-z)$$

Formulates numerical integration:

Formulate to 1 point, of order 1 in x, y, z : (FPG1)

Not	x	y	z	Weight
1	$1/4$	$1/4$	$1/4$	$1/6$

Formula at 4 points, of order 2 in x, y, z : (FPG4)

Not	x	y	z	Weight
1	a	a	a	$1/24$
2	a	a	b	$1/24$
3	a	b	a	$1/24$
4	b	a	a	$1/24$

with: $a=\frac{5-\sqrt{5}}{20}$, $b=\frac{5+3\sqrt{5}}{20}$

Formula at 5 points, of order 3 in x, y, z : (FPG5)

Not	x	y	z	Weight
1	a	a	a	$-2/15$
2	b	b	b	$3/40$
3	b	b	c	$3/40$
4	b	c	b	$3/40$
5	c	b	b	$3/40$

With: $a=0.25 \quad b=\frac{1}{6}$, $c=0.5$

Formulates at 15 points, of order 5 in x, y, z : (FPG15)

Not	x	y	z	Weight
1	a	a	a	$8/405$
2	b_1	b_1	b_1	$\frac{2665-14\sqrt{15}}{226800}$
3	b_1	b_1	c_1	
4	b_1	c_1	b_1	
5	c_1	b_1	b_1	

6	b_2	b_2	b_2	$\frac{2\,665+14\sqrt{15}}{226\,800}$
7	b_2	b_2	c_2	
8	b_2	c_2	b_2	
9	c_2	b_2	b_2	
10	d	d	e	$\frac{5}{567}$
11	d	e	d	
12	e	d	d	
13	d	e	e	
14	e	d	e	
15	e	e	d	

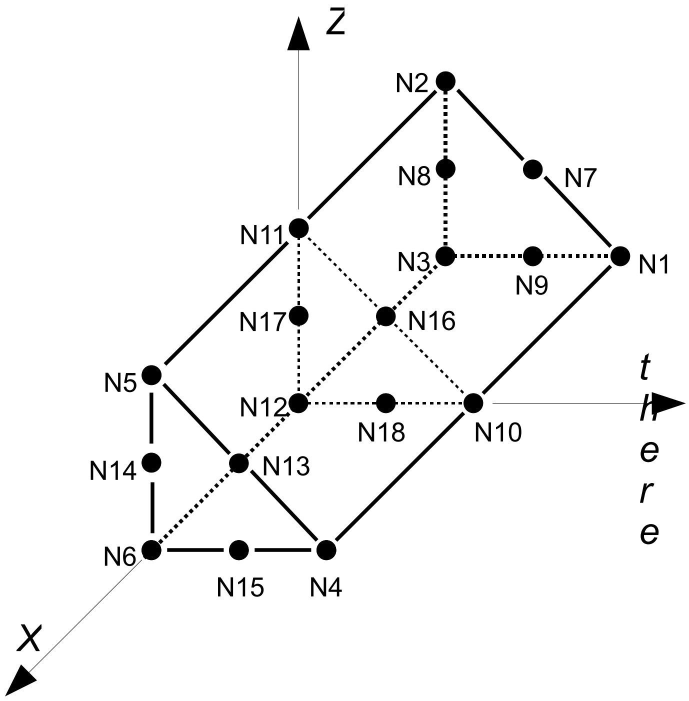
with:

$a=0.25$

$b_1=\frac{7+\sqrt{15}}{34}$ $c_1=\frac{13-3\sqrt{15}}{34}$ $d=\frac{5-\sqrt{15}}{20}$

$b_2=\frac{7-\sqrt{15}}{34}$ $c_2=\frac{13+3\sqrt{15}}{34}$ $e=\frac{5+\sqrt{15}}{20}$

4.2 Pentahedrons: `ELREFE PE6`, `P15`, `P18`



Coordinated of the nodes:

	x	y	z
$N1$	-1.	1.	0.
$N2$	-1.	0.	1.
$N3$	-1.	0.	0.
$N4$	1.	1.	0.
$N5$	1.	0.	1.
$N6$	1.	0.	0.
$N7$	-1.	0.5.0.5	.
$N8$	-1.	0.	0.5.
$N9$	-1.	0.5	0.
$N10$	0.	1.	0.
$N11$	0.	0.	1.
$N12$	0.	0.	0.
$N13$	1.	0.5.0.5	
$N14$	1.	0.	0.5
$N15$	1.	0.5	0.
$N16$	0.	0.5.0.5	
$N17$	0.	0.	0.5
$N18$	0.	0.5	0.

Shape functions:

Formulate with 6 nodes

$$w_1=\frac{1}{2}y(1-x) \qquad w_4=\frac{1}{2}y(x+1)$$
$$w_2=\frac{1}{2}z(1-x) \qquad w_5=\frac{1}{2}z(x+1)$$
$$w_3=\frac{1}{2}(1-y-z)(1-x) \qquad w_6=\frac{1}{2}(1-y-z)(x+1)$$

Formulates with 15 nodes

$$w_1=y(1-x)(2y-2-x)/2$$
$$w_2=z(1-x)(2z-2-x)/2$$
$$w_3=(x-1)(1-y-z)(x+2y+2z)/2$$
$$w_4=y(1+x)(2y-2+x)/2$$
$$w_5=z(1+x)(2z-2+x)/2$$
$$w_6=(-x-1)(1-y-z)(-x+2y+2z)/2$$
$$w_7=2yz(1-x)$$
$$w_8=2z(1-y-z)(1-x)$$
$$w_9=2y(1-y-z)(1-x)$$
$$w_{10}=y(1-x^2)$$
$$w_{11}=z(1-x^2)$$
$$w_{12}=(1-y-z)(1-x^2)$$
$$w_{13}=2yz(1+x)$$
$$w_{14}=2z(1-y-z)(1+x)$$
$$w_{15}=2y(1-y-z)(1+x)$$

Formulates with 18 nodes

$$w_1=xy(x-1)(2y-1)/2$$
$$w_2=xz(x-1)(2z-1)/2$$
$$w_3=x(x-1)(z+y-1)(2z+2y-1)/2$$
$$w_4=xy(x+1)(2y-1)/2$$
$$w_5=xz(x+1)(2z-1)/2$$
$$w_6=x(x+1)(z+y-1)(2z+2y-1)/2$$
$$w_7=2xyz(x-1)$$
$$w_8=-2xz(x-1)(z+y-1)$$
$$w_9=-2xy(x-1)(z+y-1)$$
$$w_{10}=y(1-x^2)(2y-1)$$
$$w_{11}=z(1-x^2)(2z-1)$$
$$w_{12}=(1-x^2)(z+y-1)(2z+2y-1)$$
$$w_{13}=2xyz(x+1)$$
$$w_{14}=-2xz(x+1)(z+y-1)$$
$$w_{15}=-2xy(x+1)(z+y-1)$$
$$w_{16}=4yz(1-x^2)$$
$$w_{17}=4z(x^2-1)(z+y-1)$$
$$w_{18}=4y(x^2-1)(z+y-1)$$

Formulas of numerical integration at 6 points (order 3 in x , order 2 in y and z) (FPG6)

Not	x	y	z	Weight
1.0.5.0.5	$-1/\sqrt{3}$			$1/6$
2	$-1/\sqrt{3}$	0.	0.5	$1/6$
3.0.5	$-1/\sqrt{3}$		0.	$1/6$
4.0.5.0.5	$1/\sqrt{3}$			$1/6$
5	$1/\sqrt{3}$	0.	0.5	$1/6$
6.0.5	$1/\sqrt{3}$		0.	$1/6$

Formulate numerical integration at 8 points: (FPG8)

2 Gauss points in x (order 3).
4 points of Hammer in y and z (order 3).

Not	x	y	z	Weight
1	$-a$	$1/3$	$1/3$	$-27/96$
2.0.6.0.2	$-a$			$25/96$
3.0.2.0.6	$-a$			$25/96$
4.0.2.0.2	$-a$			$25/96$
5	$+a$	$1/3$	$1/3$	$-27/96$
6.0.6.0.2	$+a$			$25/96$
7.0.2.0.6	$+a$			$25/96$
8.0.2.0.2	$+a$			$25/96$

With $a=0.577350269189626$

Formula of numerical integration at 21 points: (FPG21)

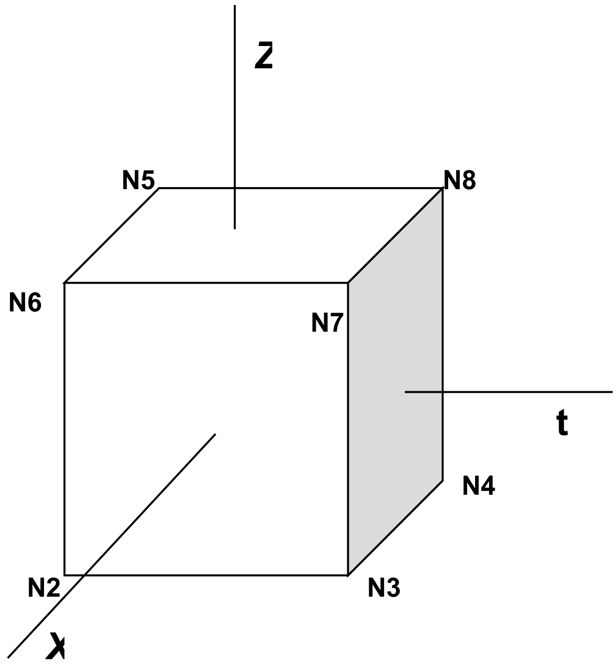
3 Gauss points in x (order 5).
7 points of Hammer in y and z (order 5 in y and z).

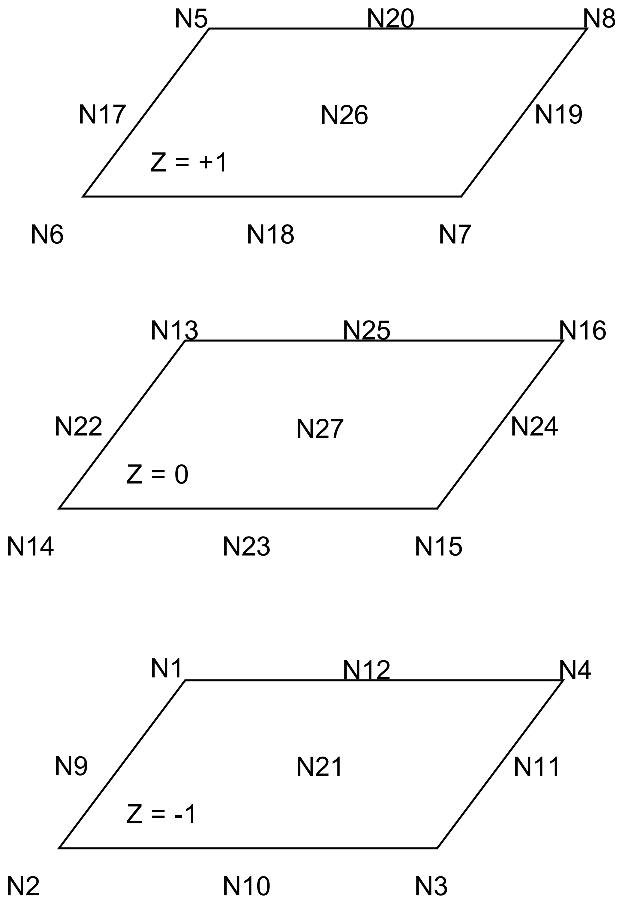
Not	x	y	z	Weight
1	$-\alpha$	$1/3$	$1/3$	$c_1\frac{9}{80}$
2 3 4	$-\alpha$ $-\alpha$ $-\alpha$	a $1-2a$ a	a a $1-2a$	$c_1\left(\frac{155+\sqrt{15}}{2400}\right)$
5 6 7	$-\alpha$ $-\alpha$ $-\alpha$	b $1-2b$ b	b b $1-2b$	$c_1\left(\frac{155-\sqrt{15}}{2400}\right)$
8	0.	$1/3$	$1/3$	$c_2\frac{9}{80}$
9 10 11	0. 0. 0.	a $1-2a$ a	a a $1-2a$	$c_2\left(\frac{155+\sqrt{15}}{2400}\right)$
12 13 14	0. 0. 0.	b $1-2b$ b	b b $1-2b$	$c_2\left(\frac{155-\sqrt{15}}{2400}\right)$
15	α	$1/3$	$1/3$	$c_1\frac{9}{80}$
16 17 18	α α α	b $1-2a$ a	a a $1-2a$	$c_1\left(\frac{155+\sqrt{15}}{2400}\right)$
19 20 21	α α α	b $1-2b$ b	b b $1-2b$	$c_1\left(\frac{155-\sqrt{15}}{2400}\right)$

with:

$\alpha=\sqrt{\frac{3}{5}}$ $c_1=\frac{5}{9}$ $c_2=\frac{8}{9}$ $a=\frac{6+\sqrt{15}}{21}$ $b=\frac{6-\sqrt{15}}{21}$

4.3 Hexahedrons: `ELREFE HE8 , H20 , H27`





Coordinated of the nodes:

	x	y	z
N1	-1.	-1.	-1.
N2	1.	-1.	-1.
N3	1.	1.	-1.
N4	-1.	1.	-1.
N5	-1.	-1.	1.
N6	1.	-1.	1.
N7	1.	1.	1.
N8	-1.	1.	1.
N9	0.	-1.	-1.
N10	1.	0.	-1.
N11	0.	1.	-1.
N12	-1.	0.	-1.
N13	-1.	-1.	0.
N14	1.	-1.	0.
N15	1.	1.	0.
N16	-1.	1.	0.
N17	0.	-1.	1.
N18	1.	0.	1.
N19	0.	1.	1.
N20	-1.	0.	1.
N21	0.	0.	-1.
N22	0.	-1.	0.
N23	1.	0.	0.
N24	0.	1.	0.
N25	-1.	0.	0.
N26	0.	0.	1.
N27	0.	0.	0.

Shape functions:

Formulate with 8 nodes

$$w_1=\frac{1}{8}(1-x)(1-y)(1-z) \qquad w_5=\frac{1}{8}(1-x)(1-y)(1+z)$$
$$w_2=\frac{1}{8}(1+x)(1-y)(1-z) \qquad w_6=\frac{1}{8}(1+x)(1-y)(1+z)$$
$$w_3=\frac{1}{8}(1+x)(1+y)(1-z) \qquad w_7=\frac{1}{8}(1+x)(1+y)(1+z)$$
$$w_4=\frac{1}{8}(1-x)(1+y)(1-z) \qquad w_8=\frac{1}{8}(1-x)(1+y)(1+z)$$

Formulates with 20 nodes

$$w_1=\frac{1}{8}(1-x)(1-y)(1-z)(-2-x-y-z) \qquad w_{11}=\frac{1}{4}(1-x^2)(1+y)(1-z)$$
$$w_2=\frac{1}{8}(1+x)(1-y)(1-z)(-2+x-y-z) \qquad w_{12}=\frac{1}{4}(1-y^2)(1-x)(1-z)$$
$$w_3=\frac{1}{8}(1+x)(1+y)(1-z)(-2+x+y-z) \qquad w_{13}=\frac{1}{4}(1-z^2)(1-x)(1-y)$$
$$w_4=\frac{1}{8}(1-x)(1+y)(1-z)(-2-x+y-z) \qquad w_{14}=\frac{1}{4}(1-z^2)(1+x)(1-y)$$
$$w_5=\frac{1}{8}(1-x)(1-y)(1+z)(-2-x-y+z) \qquad w_{15}=\frac{1}{4}(1-z^2)(1+x)(1+y)$$
$$w_6=\frac{1}{8}(1+x)(1-y)(1+z)(-2+x-y+z) \qquad w_{16}=\frac{1}{4}(1-z^2)(1-x)(1+y)$$
$$w_7=\frac{1}{8}(1+x)(1+y)(1+z)(-2+x+y+z) \qquad w_{17}=\frac{1}{4}(1-x^2)(1-y)(1+z)$$
$$w_8=\frac{1}{8}(1-x)(1+y)(1+z)(-2-x+y+z) \qquad w_{18}=\frac{1}{4}(1-y^2)(1+x)(1+z)$$
$$w_9=\frac{1}{4}(1-x^2)(1-y)(1-z) \qquad w_{19}=\frac{1}{4}(1-x^2)(1+y)(1+z)$$
$$w_{10}=\frac{1}{4}(1-y^2)(1+x)(1-z) \qquad w_{20}=\frac{1}{4}(1-y^2)(1-x)(1+z)$$

Formulates with 27 nodes

$$w_1=\frac{1}{8}x(x-1)y(y-1)z(z-1)$$
$$w_2=\frac{1}{8}x(x+1)\ y(y-1)\ z(z-1)$$
$$w_3=\frac{1}{8}x(x+1)\ y(y+1)\ z(z-1)$$
$$w_4=\frac{1}{8}x(x-1)\ y(y+1)\ z(z-1)$$
$$w_5=\frac{1}{8}x(x-1)\ y(y-1)\ z(z+1)$$
$$w_6=\frac{1}{8}x(x+1)\ y(y-1)\ z(z+1)$$
$$w_7=\frac{1}{8}x(x+1)\ y(y+1)\ z(z+1)$$
$$w_8=\frac{1}{8}x(x-1)\ y(y+1)\ z(z+1)$$
$$w_9=\frac{1}{4}(1-x^2)\ y(y-1)\ z(z-1)$$
$$w_{10}=\frac{1}{4}x(x+1)(1-y^2)\ z(z-1)$$
$$w_{11}=\frac{1}{4}(1-x^2)\ y(y+1)\ z(z-1)$$
$$w_{12}=\frac{1}{4}x(x-1)(1-y^2)\ z(z-1)$$
$$w_{13}=\frac{1}{4}x(x-1)\ y(y-1)(1-z^2)$$
$$w_{14}=\frac{1}{4}x(x+1)\ y(y-1)(1-z^2)$$
$$w_{15}=\frac{1}{4}x(x+1)y(y+1)(1-z^2)$$
$$w_{16}=\frac{1}{4}x(x-1)y(y+1)(1-z^2)$$
$$w_{17}=\frac{1}{4}(1-x^2)y(y-1)z(z+1)$$
$$w_{18}=\frac{1}{4}x(x+1)(1-y^2)z(z+1)$$
$$w_{19}=\frac{1}{4}(1-x^2)y(y+1)z(z+1)$$
$$w_{20}=\frac{1}{4}x(x-1)(1-y^2)z(z+1)$$
$$w_{21}=\frac{1}{2}(1-x^2)(1-y^2)z(z-1)$$
$$w_{22}=\frac{1}{2}(1-x^2)y(y-1)(1-z^2)$$
$$w_{23}=\frac{1}{2}x(x+1)(1-y^2)(1-z^2)$$
$$w_{24}=\frac{1}{2}(1-x^2)y(y+1)(1-z^2)$$
$$w_{25}=\frac{1}{2}x(x-1)(1-y^2)(1-z^2)$$
$$w_{26}=\frac{1}{2}(1-x^2)(1-y^2)z(z+1)$$
$$w_{27}=(1-x^2)(1-y^2)(1-z^2)$$

Formulates squaring of Gauss at 2 points in each direction (order 3) (FPG8)

Not	x	y	z	Weights
1	$-1/\sqrt{3}$	$-1/\sqrt{3}$	$-1/\sqrt{3}$	1.
2	$-1/\sqrt{3}$	$-1/\sqrt{3}$	$1/\sqrt{3}$	1.
3	$-1/\sqrt{3}$	$1/\sqrt{3}$	$-1/\sqrt{3}$	1.
4	$-1/\sqrt{3}$	$1/\sqrt{3}$	$1/\sqrt{3}$	1.5
	$1/\sqrt{3}$	$-1/\sqrt{3}$	$-1/\sqrt{3}$	1.
6	$1/\sqrt{3}$	$-1/\sqrt{3}$	$1/\sqrt{3}$	1.
7	$1/\sqrt{3}$	$1/\sqrt{3}$	$-1/\sqrt{3}$	1.
8	$1/\sqrt{3}$	$1/\sqrt{3}$	$1/\sqrt{3}$	1.

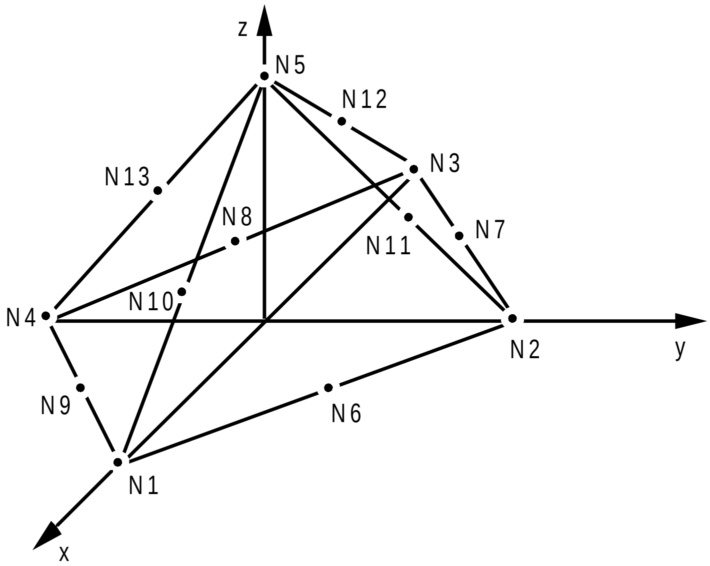
Formulate squaring of Gauss at 3 points in each direction (order 5): (FPG27)

Not	x	y	z	Weight
1	$-\alpha$	$-\alpha$	$-\alpha$	c_1^3
2	$-\alpha$	$-\alpha$	0.	$c_1^2 c_2$
3	$-\alpha$	$-\alpha$	α	c_1^3
4	$-\alpha$	0.5	$-\alpha$	$c_1^2 c_2$
	$-\alpha$	0.	0.	$c_1 c_2^2$
6	$-\alpha$	0.	α	$c_1^2 c_2$
7	$-\alpha$	α	$-\alpha$	c_1^3
8	$-\alpha$	α	0.	$c_1^2 c_2$
9	$-\alpha$	α	α	c_1^3
10	0.	$-\alpha$	$-\alpha$	$c_1^2 c_2$
11	0.	$-\alpha$	0.	$c_1 c_2^2$
12	0.	$-\alpha$	α	$c_1^2 c_2$
13	0.	0.	$-\alpha$	$c_1 c_2^2$
14	0.	0.	0.	c_2^3
15	0.	0.	α	$c_1 c_2^2$
16	0.	α	$-\alpha$	$c_1^2 c_2$
17	0.	α	0.	$c_1 c_2^2$
18	0.	α	α	$c_1^2 c_2$
19	α	$-\alpha$	$-\alpha$	c_1^3
20	α	$-\alpha$	0.	$c_1^2 c_2$
21	α	$-\alpha$	α	c_1^3
22	α	0.	$-\alpha$	$c_1^2 c_2$
23	α	0.	0.	$c_1 c_2^2$
24	α	0.25	α	$c_1^2 c_2$
	α	α	$-\alpha$	c_1^3
26	α	α	0.	$c_1^2 c_2$
27	α	α	α	c_1^3

with:

$\alpha=\sqrt{\frac{3}{5}}$ $c_1=\frac{5}{9}$ $c_2=\frac{8}{9}$

4.4 Pyramids: `ELREFE PY5`, `P13`



the square base is consisted the quadrangle $N_1 N_2 N_3 N_4$ and N_5 is the top of the pyramid.

	x	y	z
N_1	1.	0.	0.
N_2	0.	1.	0.
N_3	– 1.	0.	0.
N_4	0.	– 1.	0.
N_5	0.	0.	1.
N_6	0.5.0.5		0.
N_7	– 0.5	0.5	0.
N_8	– 0.5	– 0.5	0.
N_9	0.5	– 0.5	0.
N_{10}	0.5	0.	0.5
N_{11}	0.	0.5.0.5	
N_{12}	– 0.5	0.	0.5
N_{13}	0.	– 0.5	0.5

Shape functions:

Formulate with 5 nodes

$$w_1=\frac{(-x+y+z-1)(-x-y+z-1)}{4(1-z)}$$
$$w_2=\frac{(-x-y+z-1)(x-y+z-1)}{4(1-z)}$$
$$w_3=\frac{(x+y+z-1)(x-y+z-1)}{4(1-z)}$$
$$w_4=\frac{(x+y+z-1)(-x+y+z-1)}{4(1-z)}$$
$$w_5=z$$

Formulates with 13 nodes

$$w_1=\frac{(-x+y+z-1)(-x-y+z-1)(x-0.5)}{2(1-z)}$$
$$w_2=\frac{(-x-y+z-1)(x-y+z-1)(y-0.5)}{2(1-z)}$$
$$w_3=\frac{(x-y+z-1)(x+y+z-1)(-x-0.5)}{2(1-z)}$$
$$w_4=\frac{(x+y+z-1)(-x+y+z-1)(-y-0.5)}{2(1-z)}$$
$$w_5=2\mathrm{z}(z-0.5)$$
$$w_6=-\frac{(-x+y+z-1)(-x-y+z-1)(x-y+z-1)}{2(1-z)}$$
$$w_7=-\frac{(-x-y+z-1)(x-y+z-1)(x+y+z-1)}{2(1-z)}$$
$$w_8=-\frac{(x-y+z-1)(x+y+z-1)(-x+y+z-1)}{2(1-z)}$$
$$w_9=-\frac{(x+y+z-1)(-x+y+z-1)(-x-y+z-1)}{2(1-z)}$$
$$w_{10}=\frac{z(-x+y+z-1)(-x-y+z-1)}{1-z}$$
$$w_{11}=\frac{z(-x-y+z-1)(x-y+z-1)}{1-z}$$
$$w_{12}=\frac{z(x-y+z-1)(x+y+z-1)}{1-z}$$
$$w_{13}=\frac{z(x+y+z-1)(-x+y+z-1)}{1-z}$$

Formulates numerical integration at 5 points (FPG5):

Not	x	y	z	Weights
1.0.5		0.	H 1	2/15
2	0.	0.5	h1	2/15
3	– 0.5	0.	h1	2/15
4	0.	– 0.5	h1	2/15
5	0.	0.	H2	2/15

with:

$$h_1 = 0.1531754163448146$$
$$h_2 = 0.6372983346207416$$

Formulate numerical integration at 6 points (FPG6):

Not	x	y	z	Weight
1	has	0.	h_1	p_1
2	0.	a	h_1	p_1
3	$-a$	0.	h_1	p_1
4	0.5	$-a$	h_1	p_1
	0.	0.	h_2	p_2
6	0.	0.	h_3	p_3

with:

p_1 = 0.1024890634400000
p_2 = 0.1100000000000000
p_3 = 0.1467104129066667

a = 0.5702963741068025
h_1 = 0.1666666666666666
h_2 = 0.08063183038464675
h_3 = 0.6098484849057127

Formula of numerical integration at 27 points (FPG27):

Not	x	y	z	Weight
1	0.	0.	$1/2$	a_1
2	$\frac{b_1}{2}(1-z)$	$\frac{b_1}{2}(1-z)$	$1/2$	b_6
3	$-\frac{b_1}{2}(1-z)$	$\frac{b_1}{2}(1-z)$	$1/2$	b_6
4	$-\frac{b_1}{2}(1-z)$	$-\frac{b_1}{2}(1-z)$	$1/2$	b_6
5	$\frac{b_1}{2}(1-z)$	$-\frac{b_1}{2}(1-z)$	$1/2$	b_6
6	0.	0.	$\frac{1-b_1}{2}$	b_6
7	0.	0.	$\frac{1+b_1}{2}$	b_6
8	$c_1(1-z)$	0.	$(1-c_1)/2$	c_8
9	0.	$c_1(1-z)$	$(1-c_1)/2$	c_8
10	$-c_1(1-z)$	0.	$(1-c_1)/2$	c_8
11	0.	$-c_1(1-z)$	$(1-c_1)/2$	c_8
12	$c_1(1-z)$	0.	$(1+c_1)/2$	c_8
13	0.	$c_1(1-z)$	$(1+c_1)/2$	c_8
14	$-c_1(1-z)$	0.	$(1+c_1)/2$	c_8
15	0.	$-c_1(1-z)$	$(1+c_1)/2$	c_8
16	$\frac{d_1}{2}(1-z)$	$\frac{d_1}{2}(1-z)$	$(1-d_1)/2$	d_{12}
17	$-\frac{d_1}{2}(1-z)$	$\frac{d_1}{2}(1-z)$	$(1-d_1)/2$	d_{12}
18	$-\frac{d_1}{2}(1-z)$	$-\frac{d_1}{2}(1-z)$	$(1-d_1)/2$	d_{12}
19	$\frac{d_1}{2}(1-z)$	$-\frac{d_1}{2}(1-z)$	$(1-d_1)/2$	d_{12}
20	$d_1(1-z)$	0.	$1/2$	d_{12}
21	0.	$d_1(1-z)$	$1/2$	d_{12}
22	$-d_1(1-z)$	0.	$1/2$	d_{12}
23	0.	$-d_1(1-z)$	$1/2$	d_{12}
24	$\frac{d_1}{2}(1-z)$	$\frac{d_1}{2}(1-z)$	$(1+d_1)/2$	d_{12}
25	$-\frac{d_1}{2}(1-z)$	$\frac{d_1}{2}(1-z)$	$(1+d_1)/2$	d_{12}
26	$-\frac{d_1}{2}(1-z)$	$-\frac{d_1}{2}(1-z)$	$(1+d_1)/2$	d_{12}

27	$\frac{d_1}{2}(1-z)$	$-\frac{d_1}{2}(1-z)$	$(1+d_1)/2$	d_{12}

with:

a_1 = 0.788073483
b_6 = 0.499369002
b_1 = 0.848418011
c_8 = 0.478508449
c_1 = 0.652816472
d_{12} = 0.032303742
d_1 = 1.106412899

5 Bibliography

1 DHATT G., TOUZOT G.: A presentation of the finite element method 2nd edition. Editor: MALOINE S.A. Year 984

6 History of the versions of the document

Index Doc.	**Version Aster**	**Author (S) or contributor (S), organization**	**Description of the modifications**
E	8.4	J.Pellet, X.Desroches, EDF/ R & D	Version 8 complete.
F	9.2	J.Pellet EDF/R & D /AMA	Correction concerning the HEXA27, cf drives REX 11036
F	9.4	J.Pellet EDF/R & D /AMA	Correction page 21 of the shape function w5 of the HEXA27 (file 12170)