Dualisation of the boundary conditions

Summarized:

One explains the principle of **the Lagrange multipliers to solve the linear systems under stresses closely connected** resulting from the imposition of the boundary conditions of the kinematical type. The more definite stiffness matrix obtained not being positive, certain algorithms of resolution thus become unusable. One thus seeks a technique to be able to continue to use the algorithm of factorization LDL^T **without permutation and elimination**. The technique suggested is that of **the "Lagrange doubles"** (used in the Castem2000 code). It is shown that this technique is effective. One gives some indications on the conditioning of the matrixes obtained by this technique.

The problem of the search of **the eigen modes** of **the constrained systems** is then examined. It is shown that a possible solution is to add the boundary conditions dualized to the matrix of "stiffness" and not to touch with the matrix of "mass".

Contents

1 Introduction

We are interested in this document to the dualisation of the boundary conditions (known as kinematics). Two problems distinct from linear algebra are examined:

- the resolution of the linear systems: paragraphs 2,3,4,5,6,
- the search of the eigen modes: paragraph 7.

2 Dualisation of the kinematical boundary conditions, principle of the Lagrange multipliers

In *Code_Aster* (as in the other codes of finite elements), one is brought to solve many linear systems.

Often such a system can be regarded as the algebraical expression of a problem of minimization of a positive quadratic functional calculus $J(u)$ where u belongs to $\mathbb{R}^n$ or n is the number of nodal unknowns while being constrained by a certain number of relations closely connected $C_i(u)-d_i=0$ (boundary conditions of the Dirichlet type).

It is with this problem of minimization under stresses closely connected that one is interested here. In the continuation of the document, one will take as example the case (and the vocabulary) of the linear static mechanics. One will speak about stiffness matrix, vector displacement,… but the technique suggested remains valid for the problems of thermal evolution, into linear or nonlinear.

That is to say the discretized problem:

$$\text{Pb1}:\left\{\min_{\mathbf{u}\in V\subset\mathbb{R}^n} J(\mathbf{u})\right.$$

where:

- $J(\mathbf{u})$ is a quadratic form (total Potential energy)

$$J(\mathbf{u})=\frac{1}{2}(\mathbf{Au},\mathbf{u})-(\mathbf{b},\mathbf{u})$$

  $\mathbf{A}$ is a positive symmetric matrix $((\mathbf{Au},\mathbf{u})\geq 0\ \forall\,\mathbf{u})$ but not inevitably definite ($\mathbf{Au}=0$ is possible for $\mathbf{u}\neq 0$)

- V is the space of kinematically admissible displacements
  (it is under space closely connected of $\mathbb{R}^n$).

This discrete problem is solved numerically while expressing that the “derivative” of $J(\mathbf{u})$ in V is null. One is then brought back to solve a linear system of equations.
The problem is “to derive” $J(\mathbf{u})$ in $V\subset\mathbb{R}^n$.

In general, for practical reasons, the statement of $J(\mathbf{u})$ is calculated in the base of all nodal displacements (without taking account of the stresses): $\mathbf{u}$ belongs then to $\mathbb{R}^n$ where n is the nombre total of nodal unknowns.

If V is under vector space of $\mathbb{R}^n$ generated by $(n-p)$ basic nodal displacements, the derivative of $J(\mathbf{u})$ in V is done very simply: it is enough "and to forget" in A the matrixes b lines and columns correspondings with the d.o.f. removed $(u_i=0)$.

If the constrained degrees of freedom are not put at zero but not assigned to a given value: $u_i=d_i$, it is necessary to modify $\mathbf{b}$.

Finally if the stresses "mix" the d.o.f. between them (linear relations between unknowns) it is necessary to modify $\mathbf{A}$ and $\mathbf{b}$.

The principle of the Lagrange multipliers makes it possible to solve the problem without touching with the matrixes $\mathbf{A}$ and $\mathbf{b}$. The price to be paid is an increase amongst unknowns in the system to be solved.

Instead of solving the problem within the space of V dimension $n-p$, one solves it in space $\mathbb{R}^{n+p}$, the noted additional unknowns λ_i being called Lagrange multipliers.

Principle and justification

Let us take again the preceding problem by clarifying space V

Problem 1:

$$\begin{cases} \min\limits_{\mathbf{u}\in V} J(\mathbf{u}) \\ J(\mathbf{u})=\frac{1}{2}(\mathbf{Au},\mathbf{u})-(\mathbf{b},\mathbf{u}) \\ V=\{\mathbf{u}\in\mathbb{R}^n / C_i(\mathbf{u})=d_i, \forall i=1,p\} \end{cases}$$

C_i Are linear forms on $\mathbb{R}^n$, are d_i to them constant data. One supposes moreover than the p forms C_i are independent between them: the dimension of the space generated by C_i the east p .

One can show (cf [Appendix 1]) that this problem is equivalent to the following:

Problem 2:

$$\begin{cases} \text{trouver } \mathbf{u}\in\mathbf{V} \text{ où} \\ \mathbf{V}=\{\mathbf{u}\in\mathbb{R}^n / C_i(\mathbf{u})=d_i, \forall i=1,p\} \\ \text{tel que } (\mathbf{Au}-\mathbf{b})(\mathbf{v})=0\ \forall\ v\in\mathbf{V_0} \text{ où} \\ \mathbf{V_0}=\{\mathbf{v}\in\mathbb{R}^n \text{ tel que } C_i(\mathbf{v})=0\ \forall\ i=1,p\} \end{cases}$$

Let us rewrite problem 2 differently:

Problem 3:

to find $\mathbf{u} \in \mathbb{R}^n$ such as

$$\forall i \quad \mathbf{C_i}\mathbf{u}-\mathbf{d_i}=0 \qquad \textbf{éq 2-1}$$

$$\forall \mathbf{v} \in V_0, \mathbf{C_i}\mathbf{v}=0 \qquad \textbf{éq 2-2}$$

$$\forall \mathbf{v} \in \mathbf{V_0}, (\mathbf{Au}-\mathbf{b})\mathbf{v}=0 \qquad \textbf{éq 2-3}$$

the equations [éq 2-2] and [éq 2-3] show (by identifying $\mathbb{R}^n$ and its dual) that:

$(\forall i)\mathbf{C_i}$	is orthogonal with $\mathbf{V_0}$
$(\mathbf{Au}-\mathbf{b})$	is orthogonal with $\mathbf{V_0}$

$\mathbf{V_0}$ is under vector space of $\mathbb{R}^n$ orthogonal with $\{\mathbf{C_i}(i=1,p)\}$ ($\mathbf{V_0}$ is of dimension $(n-p)$ because p the conditions $\mathbf{C_i}$ are supposed to be independent).
Since the decomposition of $\mathbb{R}^n$ all in all direct of 2 pennies orthogonal spaces is single, one from of deduced that $(\mathbf{Au}-\mathbf{b})$ belongs to the vector space generated by C_i.

There thus exists a family of scalars λ_i called Lagrange multipliers such as:

$$(\mathbf{Au}-\mathbf{b})+\sum_i \lambda_i \mathbf{C_i}=0$$

This equality is true in $\mathbb{R}^n$

problem 3 becomes then:

Problem 4:

$$\left\{\begin{array}{l} \text{Trouver } \mathbf{u} \in \mathbb{R}^n \ , \ \lambda_i \in \mathbb{R}, (i=1,p) \\ (\forall i=1,p)\mathbf{C_i}\mathbf{u}-\mathbf{d_i}=0 \\ (\mathbf{Au}-\mathbf{b})+\sum_i \lambda_i \mathbf{C_i}=0 \end{array}\right\}$$

The reciprocal one ($Pb4 \Rightarrow Pb3$) is obvious: if there exists λ_i such as:

$$(\mathbf{Au}-\mathbf{b})+\sum_i \lambda_i \mathbf{C_i}=0 \quad \text{then} \quad \forall \mathbf{v} \in V_0 (\mathbf{Au}-\mathbf{b})\mathbf{v}=-\sum_i \lambda_i \mathbf{C_i}.\mathbf{v}=0$$

problem 4 is the sought problem. It will be said that it is the problem with dualized kinematical conditions. Matriciellement one can write it:

$$\begin{gathered}\mathbf{KX}=\mathbf{F}\\ \begin{bmatrix}\mathbf{A} & \mathbf{C}^{\mathrm{T}}\\ \mathbf{C} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{u}\\ \boldsymbol{\lambda}\end{bmatrix}=\begin{bmatrix}\mathbf{b}\\ \mathbf{d}\end{bmatrix}\\ (\mathbf{K})\quad(\mathbf{X})\quad(\mathbf{F})\end{gathered}$$ **éq 2-4**

where:

$$\mathbf{u}\in\mathbb{R}^{n};\lambda\in\mathbb{R}^{p};\mathbf{X}\in\mathbb{R}^{n+p}$$
$$\mathbf{A}\in\mathbf{A}_{n,n};\mathbf{C}\in\mathbf{A}_{p,n};\mathbf{K}\in\mathbf{A}_{n+p,n+p}$$

One realizes that this system can be obtained while seeking to make extreme the functional calculus:

$$L(\mathbf{u},\lambda)=\frac{1}{2}(\mathbf{Au,u})-(\mathbf{b,u})+\lambda(\mathbf{Cu}-\mathbf{d})$$ **éq 2-5**

This functional calculus is called Lagrangian initial problem. The principal interest of this method is to free itself from the stresses: $\mathbf{u}$ and $\boldsymbol{\lambda}$ are sought in $\mathbb{R}^n$ and $\mathbb{R}^p$ ($\mathbf{X}$ in $\mathbb{R}^{n+p}$).

The coefficients λ_i are called coefficients of Lagrange of the problem (one will say sometimes “Lagrange”).

3 Disadvantages of this dualisation

One sees according to the statement of Lagrangian that the matrix $\mathbf{K}$ is not positive any more (what was the case of $\mathbf{A}$). Indeed:

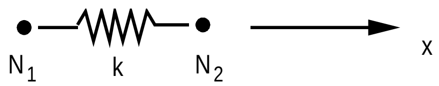
$$\exists \mathbf{u_0} / \mathbf{C}\,\mathbf{u_0} \neq 0 \Rightarrow \exists \lambda_0 / \mathrm{L}(\mathbf{u_0}, \lambda_0) \ldots = \frac{1}{2}(\mathbf{A}\mathbf{u_0}, \mathbf{u_0}) + \lambda_0 \mathbf{C}\mathbf{u_0} < 0$$

The loss of the positivity of the matrix $\mathbf{K}$ involves that the resolution of the system $\mathbf{KX}=\mathbf{F}$ cannot be done any more in general by the classical algorithms of gradient or, by the factorization of Cholesky. The algorithm of factorization LDL^T without permutation of the lines and columns is not guaranteed any more either: it is the latter algorithm which one wants to be able to continue to use.

Let us illustrate the problem on the following example:

Example 1:

a come out from stiffness k connects 2 nodes N_1 and N_2



2 unknowns: u_1 u_2 ; 2 modal forces: f_1 , f_2 $(\mathrm{n}=2)$

$$A=\begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$$

1 stress: $\alpha u_1 + \beta u_2 = \gamma$ $(\mathrm{p}=1)$

The dualized problem is written: $\mathbf{KX}=\mathbf{F}$

with:

$$\mathbf{K}=\begin{bmatrix} k & -k & \alpha \\ -k & k & \beta \\ \alpha & \beta & 0 \end{bmatrix} \qquad \mathbf{X}=\begin{bmatrix} u_1 \\ u_2 \\ \lambda \end{bmatrix} \qquad \mathbf{F}=\begin{bmatrix} f_1 \\ f_2 \\ \gamma \end{bmatrix}$$

Recall of the requirement and sufficient so that the algorithm LDL^T-SP (without permutation) functions:

Let us note $\mathbf{K_i}$ under matrix of $\mathbf{K}$ formed by I first lines and columns of $\mathbf{K}$.
(If $\mathbf{K}$ is of order n $\mathbf{K_n}=\mathbf{K}$, $\mathbf{K_1}=[\mathrm{k}_{11}]$).
There will be no null pivot in the algorithm LDL^T-SP if and only if ($\mathbf{K_i}$ invertible for i is very understood enters 1 and n).
This condition will be noted: `cond1`

the matrix $\mathbf{K}$ above is written with like classification of the unknowns, the order of the components of $\mathbf{X}$:

- $\mathbf{X}=(u_1, u_2, \lambda)$

$$\mathbf{K}=\begin{bmatrix} k & -k & \alpha \\ -k & k & \beta \\ \alpha & \beta & 0 \end{bmatrix}$$

$\mathbf{K_2}=\begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$ do not check the condition `cond1`.

On example 1 let us test new classifications:

- $\mathbf{X}=(\lambda, u_1, u_2)$

$$\mathbf{K}=\begin{bmatrix} 0 & \alpha & \beta \\ \alpha & k & -k \\ \beta & -k & k \end{bmatrix}$$

$\mathbf{K_1}$ do not check the condition `cond1`.

- $\mathbf{X}=(u_1, \lambda, u_2)$

$$\mathbf{K}=\begin{bmatrix} k & \alpha & -k \\ \alpha & 0 & \beta \\ -k & \beta & k \end{bmatrix} \qquad \begin{array}{l} \det \mathbf{K_1}=k \\ \det \mathbf{K_2}=\alpha^2 \\ \det \mathbf{K_3}=-k(\alpha+\beta)^2 \end{array}$$

- k is supposed strictly positive (stiffness of spring)

- $\mathbf{K_3}$ is invertible only if $\alpha+\beta\neq 0$. The case $\alpha+\beta=0$ corresponds indeed to “a bad” physical blocking: the condition: $u_1-u_2=\text{cte}$ do not block “rigid body motions” for $\mathbf{A}$ (without energy).

  So that the total problem has a single solution, it is necessary indeed that the conditions $\mathbf{C_i}\,\mathbf{u}=\mathbf{d_i}$ generate a space of acceptable displacements which does not contain any rigid body motions of $\mathbf{A}$.

  With the notations of [§1] one will write:

$$\ker \mathbf{A}\cap \mathbf{V_0}=\{0\}$$

  One will suppose in the continuation of the document that this condition is checked. It is - with - to say that the stresses $\mathbf{C_i}$ block at least rigid body motions of $\mathbf{A}$ (they can be more numerous of course). When this condition is checked and that the conditions $\mathbf{C_i}$ are independent between them, it will be said that the problem is physically well posed.

- $\mathbf{K_2}$ is invertible only if $\alpha \neq 0$

  It is thus seen that classification (u_1, λ, u_2) checks the condition `cond1` if $\alpha \neq 0$.

  If blocking $\alpha u_1 + \beta u_2 = \gamma$ is reduced to:

$$u_1 = \delta \quad (\alpha, \beta) = (1{,}0) \rightarrow \text{cela marche}$$
$$u_2 = \delta \quad (\alpha, \beta) = (0{,}1) \rightarrow \text{cela ne marche pas}$$

The symmetry of the problem shows that to be able to deal with the problem $(\alpha, \beta) = (0{,}1)$ it is necessary to number $X = (u_2, \lambda, u_1)$.

From this very simple example, one can draw some general conclusions (all negative):

- if one numbers all λ_i after u_i , if $\mathbf{A}$ is singular, the condition `cond1` is not checked for $\mathbf{K_i} = \mathbf{A}$,
- that is to say a condition $\mathbf{C\,u} = \mathbf{d}$ and λ the associated multiplier of Lagrange. The equation $\mathbf{Cu} = \mathbf{d}$ in general does not utilize all the unknowns $\mathbf{u_i}$: the equation constrained certain degrees of freedom. If λ is numbered before the d.o.f. that it constrained the condition `cond1` will not be checked. Indeed let us look under matrix $\mathbf{K_j}$ where j is the number of the equation giving λ .

$$\mathbf{K_j} = \begin{bmatrix} \mathbf{K_{j-1}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

- That is to say a physical structure which, by bad luck, “is retained by its last d.o.f. physics” i.e. such as if one does not block this d.o.f., the matrix is singular, and such as if one blocks it the matrix is invertible. The use of a degree of freedom of Lagrange λ for this blocking is impossible. Indeed, if one numbers λ before the last physical degree of freedom, there will be a null pivot on the level of λ , and if one numbers it after (thus in all last degree of freedom), the matrix $\mathbf{K_{n-1}}$ will not be invertible since blocking is not taken yet into account. One will see with [§4] that the technique of the “double lagrange” makes it possible to solve this problem.

To finish this paragraph we can pass the following remark: If $\mathbf{K}$ is invertible, it is known that there exists a classification of the unknowns making it possible to factorize $\mathbf{K}$ par. LDL^T This classification is for example that resulting from the algorithm LDL^T with permutation (maximum pivot for example). But this renumbering relates to only the lines of the matrix; there is thus loss of the symmetry of $\mathbf{K}$. It is enough to consider the following example:

Example 2:

$$\left.\begin{matrix} \mathbf{A} = [0] \\ C = [1] \end{matrix}\right\} \quad \mathbf{K} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \mathbf{X} = \begin{bmatrix} \mathbf{u} \\ \lambda \end{bmatrix}$$

$\mathbf{K}$ is invertible, but there does not exist any common permutation of the lines and columns $\mathbf{K}$ of allowing a resolution par. LDL^T

All these remarks show that the dualisation proposed in this paragraph does not make it possible to use $LDL^T - SP$.

4 Principle of the “doubles Lagrange”

the method suggested here is that implemented in code CASTEM 2000 (communication personal of Th. CHARAS and P. VERPEAUX). An intuitive presentation could be as follows about it:

It is seen that the dualized problem [éq 2-4] has null terms on the diagonal: those corresponding to the degrees of freedom of Lagrange. This property is also noticed on the Lagrangian one [éq 2-5]: there are no quadratic terms in λ .

This nullity of the diagonal terms prevents certain permutations of lines and columns: one can place a Lagrange before the physical degrees of freedom only it constrained.

The idea is then to break up each coefficient of Lagrange λ into 2 equal parts λ^1 and λ^2 . The equation $\mathbf{Cu}=\mathbf{d}$ is then replaced by:

$$\begin{aligned}\mathbf{Cu}-\alpha\left(\lambda^1-\lambda^2\right)=\mathbf{d}\\ \mathbf{Cu}+\alpha\left(\lambda^1-\lambda^2\right)=\mathbf{d}\end{aligned}$$

where α is a non-zero constant.

Let us show the equivalence of the old problem and the new one:

Problem 1 : “simple Lagrange”

to find $\begin{cases}\mathbf{u}\in\mathbb{R}^n\\ \lambda\in\mathbb{R}^p\end{cases}$ such as (S) : $\begin{cases}\mathbf{Au}+\mathbf{C}^{\mathbf{T}}\lambda=\mathbf{b}\\ \mathbf{Cu}=\mathbf{d}\end{cases}$

$$(S)\Leftrightarrow\begin{cases}\lambda^1=\lambda^2\\ \lambda=\lambda^1+\lambda^2\\ \mathbf{Au}+\mathbf{C}^{\mathbf{T}}\lambda=\mathbf{b}\\ \mathbf{Cu}=\mathbf{d}\end{cases}\Leftrightarrow\begin{cases}\mathbf{Au}+\mathbf{C}^{\mathbf{T}}\lambda^1+\mathbf{C}^{\mathbf{T}}\lambda^2=\mathbf{b}\\ \mathbf{Cu}-\alpha\lambda^1+\alpha\lambda^2=\mathbf{d}\\ \mathbf{Cu}+\alpha\lambda^1-\alpha\lambda^2=\mathbf{d}\end{cases}$$

α is a constant $\neq 0$

From where the problem equivalent to the precedent:

Problem 2 : “double Lagrange”

to find $\begin{cases}\mathbf{u}\in\mathbb{R}^n\\ \lambda^1,\lambda^2\in\mathbb{R}^p\times\mathbb{R}^p\end{cases}$ such as $(\mathrm{S}'):\begin{cases}\mathbf{Au}+\mathbf{C}^{\mathrm{T}}\lambda^1+\mathbf{C}^{\mathrm{T}}\lambda^2=\mathbf{b}\\ \mathbf{Cu}-\alpha\lambda^1+\alpha\lambda^2=\mathbf{d}\\ \mathbf{Cu}+\alpha\lambda^1-\alpha\lambda^2=\mathbf{d}\end{cases}$

the new problem can be written:

$$\mathbf{K'}\,\mathbf{X'}=\mathbf{F'}$$

with:

$$\begin{cases}\mathbf{X}'=(\mathbf{u},\lambda_1,\lambda_2)\\ \mathbf{F'}=(\mathbf{b},\mathbf{d},\mathbf{d})\end{cases}$$

$$\mathbf{K'}=\begin{bmatrix}\mathbf{A} & \mathbf{C}^{\mathrm{T}} & \mathbf{C}^{\mathrm{T}}\\ \mathbf{C} & -\alpha\mathbf{l} & \alpha\mathbf{l}\\ \mathbf{C} & \alpha\mathbf{l} & -\alpha\mathbf{l}\end{bmatrix}$$

The problem corresponds to make extreme the functional calculus:

$$L'(\mathbf{u},\lambda^1,\lambda^2)=\frac{1}{2}(\mathbf{Au},\mathbf{u})-(\mathbf{b},\mathbf{u})+(\lambda^1,\mathbf{Cu}-\mathbf{d})$$
$$+(\lambda^2,\mathbf{Cu}-\mathbf{d})-\frac{\alpha}{2}(\lambda^1-\lambda^2,\lambda^1-\lambda^2)$$

One can show (cf Annexes 2) that if one observes a certain rule of classification of the unknowns, and by choosing the constant $\alpha>0$, the matrix $\mathbf{K'}$ checks the condition `cond1`.

This rule is the following one:

That is to say a relation of blocking $\mathbf{Cu}-\mathbf{d}=0$, it corresponds to him 2 Lagrange multipliers λ^1 and λ^2 . This relation utilizes a certain number of physical degrees of freedom.

Regulate R_0 :

For each relation of blocking, it is necessary to place λ^1 before the first constrained physical degree of freedom and λ^2 after the last constrained physical degree of freedom.

To decrease the occupation memory of the matrix $\mathbf{K}$, it is necessary to seek to minimize the bandwidth. It is what one does in *Code_Aster* in “framing” the relations “with nearest”: λ^1 is placed right before the first constrained degree of freedom, λ^2 is placed just after last the d.o.f. constrained.

Illustration:

That is to say a problem with 4 physical degrees of freedom: u_1, u_2, u_3, u_4 .
This system is subjected to 2 conditions:

$$\begin{cases} a_{11} u_1 + a_{13} u_3 = b_1 \\ a_{22} u_2 + a_{24} u_4 = b_2 \end{cases}$$

let us call λ_1^1, λ_1^2 the 2 degrees of freedom of Lagrange associated with the 1st condition and λ_2^1, λ_2^2 those associated with the 2nd condition.

By supposing that the physical degrees of freedom were numbered in the order: u_1, u_2, u_3, u_4 , the total classification of the degrees of freedom retained by *Aster* is then:

$$\lambda_1^1, u_1, \lambda_2^1, u_{2,} u_{3,} \lambda_1^2, u_4, \lambda_2^2$$

λ_1^1 and λ_1^2 frame "with more close" the constrained degrees of freedom (u_1 and u_3)
λ_2^1 and λ_1^2 frame "with more close" the constrained degrees of freedom (u_2 and u_4)

the technique of the "Lagrange doubles" associated with the rule R_0 thus allows to solve any linear system posed physically well with the algorithm of LDL^T without permutation. The demonstration supposes nevertheless that **the matrix $\mathbf{A}$ is symmetric and positive** (not inevitably definite).

Note:

> *Assumptions: $\mathbf{A}$ symmetric and $\mathbf{A}$ positive are necessary to use LDL^T (or LU) without permutation as the two following counterexamples show it:*

- $\mathbf{A_1} = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{bmatrix}$ is symmetric but nonpositive,
- $\mathbf{A_2} = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{-1} & \mathbf{1} \end{bmatrix}$ is positive but asymmetric.

5 Additional advantage

One will show in this paragraph that the technique of the “Lagrange doubles” can make it possible economically to solve a series of problems which would differ only by their kinematical boundary conditions (for example a variable contact zone).

Note:

This possibility is not currently used in the code.

That is to say a system with stresses $\mathbf{KX}=\mathbf{F}$:
Let us write this system while emphasizing a particular stress (this computation remains obviously valid when there are several stresses) $\mathbf{Cu}-\mathbf{d}=0$. To simplify the writing, one chooses $\alpha=1$.

That is to say

- λ^1 the first degree of freedom of Lagrange associated with the stress
- λ^2 the second degree of freedom of Lagrange associated with the stress
- $\mathbf{U}=\mathbf{X}-\{\lambda^1,\lambda^2\}$
- $\tilde{\mathbf{K}}$ = matrix $\mathbf{K}$ projected on $\mathbf{U}$; $\mathbf{b}$ = vector $\mathbf{F}$ project on $\mathbf{U}$

the system is written with these variables:

$$\begin{bmatrix}\tilde{\mathbf{K}} & \mathbf{C}^{\mathbf{T}} & \mathbf{C}^{\mathbf{T}}\\ \mathbf{C} & -1 & 1\\ \mathbf{C} & 1 & -1\end{bmatrix}\begin{bmatrix}\mathbf{U}\\ \lambda^1\\ \lambda^2\end{bmatrix}=\begin{bmatrix}\mathbf{b}\\ \mathbf{d}\\ \mathbf{d}\end{bmatrix}$$

Let us change the coefficient (λ^2,λ^2) : $-1\rightarrow 3$ and let us write the new system:

$$(S)\begin{cases}\tilde{\mathbf{K}}\,\mathbf{U}+(\lambda^1+\lambda^2)\,\mathbf{C}^{\mathbf{T}}=\mathbf{b}\\ \mathbf{CU}-\lambda^1+\lambda^2=\mathbf{d}\\ \mathbf{CU}+\lambda^1+3\lambda^2=\mathbf{d}\end{cases}\Leftrightarrow\quad\begin{matrix}(\lambda^1+\lambda^2)=0 & \textbf{éq 5-1}\\ \mathbf{CU}-\lambda^1+\lambda^2=\mathbf{d} & \textbf{éq 5-2}\\ \tilde{\mathbf{K}}\,\mathbf{U}=\mathbf{b} & \textbf{éq 5-3}\end{matrix}$$

This last system is decoupled: one can solve [éq 5-3] to obtain $\mathbf{U}$ then to calculate λ^1 and λ^2 .

It is noticed that the resolution of [éq 5-3] corresponds to the initial problem without the stress $\mathbf{Cu}-\mathbf{d}=0$. The values of λ^1 and λ^2 then do not have any more the same physical meaning. In other words, the system (S) has the same solution in $\mathbf{U}$ as the subsystem $\tilde{\mathbf{K}}\,\mathbf{U}=\mathbf{b}$; the 2 additional unknowns λ^1 and λ^2 do not disturb the solution in $\mathbf{U}$. The total system can appear of size higher $(+2)$ than what is necessary, but by means of computer, that can be very convenient

Indeed, imagine now that we know in advance that certain kinematic relations are likely to be slackened. Let us number λ^2 the associates with these relations at the end of the system. We can then triangulate partially once and for all the system while stopping before these d.o.f. The part of the triangulated matrix is most important in volume: every physical degrees of freedom and all them λ^1. When a practical problem arises, i.e. when one knows the list of the active linear relations, it is enough to update the last lines of the matrix ($-\alpha$ if the relation is active, $+3\alpha$ if it is not it). One can then finish the triangulation and solve the problem economically.

6 Notice on the conditioning of the system

When one looks at the form of the matrix which one finally will factorize $\mathbf{K'}$ (cf [§3]), one sees that its various submatrices $\mathbf{A}$ $\mathbf{C}$, $\alpha\mathbf{I}$ can be of order of magnitude very different. It is known that in general this situation is not favorable numerically (restricted accuracy of the computers).

It should be noticed that the equations of connection $\mathbf{Cu}-\mathbf{d}=0$ can be multiplied by an arbitrary constant (β) without changing the problem. Moreover, we saw that the matrixes $\alpha\mathbf{I}$ were also arbitrary $(\alpha>0)$. We thus have two parameters allowing "to regulate" the conditioning of the matrix.

We will not make a general demonstration but we are satisfied to examine the most commonplace case which is: a spring, a d.o.f., a connection.

The matrix $\mathbf{K'}$ is written if k is the stiffness of spring:

$$\mathrm{K'}=\begin{bmatrix} k & \beta & \beta \\ \beta & -\alpha & +\alpha \\ \beta & \alpha & -\alpha \end{bmatrix}$$

The conditioning of this matrix is related to the dispersion of its eigenvalues m_i :

Let us calculate the characteristic polynomial of $\mathbf{K'}$:

$$P(\mu)=(\mu+2\alpha)(-\mu^2+k\mu+2\beta^2)$$

$$\Leftrightarrow \begin{aligned} \mu_1 &= -2\alpha && <0 \\ \mu_2 &= \frac{+k+\sqrt{k^2+8\beta^2}}{2} && >0 \\ \mu_3 &= \frac{+k-\sqrt{k^2+8\beta^2}}{2} && <0 \end{aligned}$$

k is the eigenvalue of the nonconstrained system. This eigenvalue is the order of magnitude searched for μ_1, μ_2, μ_3 .

It is noticed that $\mu_1<0, \mu_3<0$ and $\mu_2,>0$ i.e. the 2 eigenvalues added by the coefficients of Lagrange are <0 (it is besides because of that LDL^T-SP is not guaranteed without precautions).

One seeks to obtain eigenvalues of the same order of magnitude:

$$|\mu_1|\simeq|\mu_2|\simeq|\mu_3|$$

$$|\mu_2\,\mu_3|\simeq|\mu_1|^2\Rightarrow 2\beta^2\simeq 4\alpha^2 \qquad \textbf{éq 6-1}$$

So $\beta<<k$ then $\mu_3\simeq 0 \quad \mu_2\simeq k$: it is not result sought.
If $\beta>>k$, then $|\mu_2|\simeq|\mu_3|\simeq\sqrt{2}\,\beta\simeq|\mu_1|$

the three eigenvalues are then in absolute value about β which in front is a very large arbitrary constant k. This solution is not that which one will retain because the value k is in the general case (with a large number of degrees of freedom) of an order of magnitude comparable to the other eigenvalues of the system.

One will choose rather:

$$\beta=\alpha\simeq k\Rightarrow \quad \begin{aligned}\mu_1&\simeq-2\mathrm{k}\\ \mu_2&\simeq 2\mathrm{k}\\ \mu_3&\simeq-k\end{aligned}$$

Practically in *Code_Aster*, one chooses a value of α single for all the system. This value is the average of the extreme values of the diagonal terms associated with the physical degrees of freedom: $\left(\min\left(a_{ii}\right)+\max\left(a_{ii}\right)\right)/2$. Moreover, one takes $\beta=\alpha$.

7 Eigen modes and parameters of Lagrange

7.1 Introduction

This paragraph wants to answer the two following questions:

Q1: Which is the system of values (and vectors) clean **tiny room** to be solved when a mechanical model is subjected to homogeneous linear **kinematical stresses** ?

Q2: Which is the model dualisé (with parameters of Lagrange) equivalent or precedent?

7.2 Mechanical problem to solve

One supposes a mechanical system already discretized by finite elements.

The nodal unknowns are noted $\mathbf{U}=\{u_i\}(i=1,n)$.

Nodal displacements all are not independent: there exist $p(<n)$ homogeneous linear relations between these displacements: $B_j(\mathbf{U})=0(j=1,p)$.
These linear relations are independent between them, i.e. the row of the matrix $\mathbf{B}$ containing the coefficients of p the linear relations is p .

That is to say $\mathbf{K}$ the stiffness matrix of the mechanical system without stresses.
That is to say $\mathbf{M}$ the mass matrix of the mechanical system without stresses.

Which is the system with the eigenvalues to solve find the eigen modes of forced structure?

7.3 Reduced system

Let us notice that if one writes the kinematical linear relations in the form:

$$\mathbf{BU}=0 \quad \textbf{éq 7.3-1}$$

where: $\mathbf{B}$ is a matrix $p\times n$
$\mathbf{U}$ is the vector of the nodal unknowns $\in\mathbb{R}^n$

then:

$$\mathbf{B}\dot{\mathbf{U}}=0 \quad \textbf{éq 7.3-2}$$

and the relation is also valid for the velocities.

Moreover, if $\mathbf{B}$ is of row p, then there exists a square **submatrix** of $\mathbf{B}$ of row p. Let us note $\mathbf{B_1}$ this submatrix.

Then let us make a partition of the unknowns of $\mathbf{U}$ in $\mathbf{U_1}$ and $\mathbf{U_2}$ such as:

$$\mathbf{BU}=0 \quad \Leftrightarrow \quad \begin{bmatrix}\mathbf{B_1} & \mathbf{B_2}\end{bmatrix}\begin{bmatrix}\mathbf{U_1}\\ \mathbf{U_2}\end{bmatrix}=0$$

$$\begin{aligned}&\mathbf{U_1}\in\mathbb{R}^p\\ &\mathbf{U_2}\in\mathbb{R}^{n-p}\\ &\mathbf{B_1}=\text{matrice } p\times p\\ &\mathbf{B_2}=\text{matrice } p\times(n-p)\end{aligned}$$

The linear relations can then be written:

$$\mathbf{B_1}\mathbf{U_1}+\mathbf{B_2}\mathbf{U_2}=0$$

what makes it possible to express the unknowns $\mathbf{U_1}$ according to $\mathbf{U_2}$ since $\mathbf{B_1}$ is invertible.

$$\mathbf{U_1}=-\mathbf{B_1^{-1}}\mathbf{B_2}\mathbf{U_2} \qquad \textbf{éq 7.3-3}$$

reduced Stiffness matrix:

The elastic strain energy of not forced discretized structure is $W_{def}=\frac{1}{2}\mathbf{U^T}\mathbf{KU}$. If one partitionne the matrix $\mathbf{K}$ in the same way that one partitionné $\mathbf{U}$, one obtains:

$$K=\begin{bmatrix}\mathbf{K_1} & \mathbf{K_{12}}\\ \mathbf{K_{12}^T} & \mathbf{K_2}\end{bmatrix}$$

then:

$$2\mathrm{W}_{def}=\mathbf{U_1^T}\mathbf{K_1}\mathbf{U_1}+\mathbf{U_2^T}\mathbf{K_2}\mathbf{U_2}+\mathbf{U_2^T}\mathbf{K_{12}}\mathbf{U_1}+\mathbf{U_1^T}\mathbf{K_{12}^T}\mathbf{U_2}$$

Let us introduce the linear stresses then [éq 7.3-3]:

$$\begin{aligned}2\mathrm{W}_{def}&=\mathbf{U_2^T}\mathbf{B_2^T}\mathbf{B_1^{-T}}\mathbf{K_1}\mathbf{B_1^{-1}}\mathbf{B_2}\mathbf{U_2}+\mathbf{U_2^T}\mathbf{K_2}\mathbf{U_2}\\ &\quad-\mathbf{U_2^T}\mathbf{K_{12}}\mathbf{B_1^{-1}}\mathbf{B_2}\mathbf{U_2}-\mathbf{U_2^T}\mathbf{B_2^T}\mathbf{B_1^{-T}}\mathbf{K_{12}^T}\mathbf{U_2}\\ &=\mathbf{U_2^T}\tilde{\mathbf{K}}_2\mathbf{U_2}\end{aligned}$$

with:

$$\tilde{\mathbf{K}}_2=\mathbf{K}_2+\mathbf{B}_2^{\mathrm{T}}\mathbf{B}_1^{-\mathrm{T}}\mathbf{K}_1\mathbf{B}_1^{-1}\mathbf{B}_2-\mathbf{K}_{12}\mathbf{B}_1^{-1}\mathbf{B}_2-\mathbf{B}_2^{\mathrm{T}}\mathbf{B}_1^{-\mathrm{T}}\mathbf{K}_{12}^{\mathrm{T}} \qquad \textbf{éq 7.3-4}$$

One thus sees that one expressed the strain energy of reduced structure like a bilinear form of $\mathbf{U}_2$. The nodal unknowns of $\mathbf{U}_1$ **were eliminated**. The nodal unknowns U_2 **are not forced any more**.

Reduced mass matrix:

Let us adopt the same partition for the mass matrix $\mathbf{M}$. We can write the relation [éq 7.3-2]:

$$\mathbf{B}_1\dot{\mathbf{U}}_1+\mathbf{B}_2\dot{\mathbf{U}}_2=0$$

Same computation as previously then leads us to:

$$2\mathrm{W}_{cin}=\dot{\mathbf{U}}_2^{\mathrm{T}}\tilde{\mathbf{M}}_2\dot{\mathbf{U}}_2$$

with:

$$\tilde{\mathbf{M}}_2=\mathbf{M}_2+\mathbf{B}_2^{\mathrm{T}}\mathbf{B}_1^{-\mathrm{T}}\mathbf{M}_1\mathbf{B}_1^{-1}\mathbf{B}_2-\mathbf{M}_{12}\mathbf{B}_1^{-1}\mathbf{B}_2-\mathbf{B}_2^{\mathrm{T}}\mathbf{B}_1^{-\mathrm{T}}\mathbf{M}_{12}^{\mathrm{T}} \qquad \textbf{éq 7.3-5}$$

Conclusion:

The system to solve find the modes (and the frequencies) clean of structure forced is:

To find $(n-p)(\mathbf{X}_{\mathrm{i}},\,\omega_{\mathrm{i}}^2)\in\mathbb{R}^{n-p}\times\mathbb{R}$ such as $(\tilde{\mathbf{K}}_2-\omega_{\mathrm{i}}^2\tilde{\mathbf{M}}_2)\mathbf{X}_{\mathrm{i}}=0$ with $\tilde{\mathbf{K}}_2$ and $\tilde{\mathbf{M}}_2$ defined by [éq 7.3-4] and [éq 7.3-5].

Application to the blocked degrees of freedom:

In this case:

$$\begin{cases}\mathbf{B}_1=\mathbf{I}\\ \mathbf{B}_2=0\end{cases}$$

from where: $\tilde{\mathbf{K}}_2=\mathbf{K}_2$ and $\tilde{\mathbf{M}}_2=\mathbf{M}_2$

i.e. it is enough “to forget” in $\mathbf{K}$ and the $\mathbf{M}$ lines and columns corresponding to the blocked degrees of freedom.

7.4 Dualisé system

We saw with [§5] that the taking into account of the coefficients of Lagrange (double) in a matrix $\mathbf{A}$ led to the matrix $\mathbf{A}$:

$$\mathbf{A}'=\begin{bmatrix} \mathbf{A} & \beta\mathbf{B}^{\mathrm{T}} & \beta\mathbf{B}^{\mathrm{T}} \\ \beta\mathbf{B} & -\alpha\mathbf{I} & +\alpha\mathbf{I} \\ \beta\mathbf{B} & \alpha\mathbf{I} & -\alpha\mathbf{I} \end{bmatrix}$$

where:

- $\mathbf{B}$ is the matrix of the kinematical conditions: $\mathbf{BU}=0$
- $\alpha\in\mathbb{R}^{+}$ arbitraire $\alpha\neq 0$
- $\beta\in\mathbb{R}$ arbitraire $\beta\neq 0$,

Let us apply the dualisation of the boundary conditions to the matrixes $\mathbf{K}$ and $\mathbf{M}$, by partitionnant the degrees of freedom in $\mathbf{X_1}$, $\mathbf{X_2}$ as with [§7.3]. We obtain the problem with the eigenvalues according to:

$$(S)\left\{\begin{bmatrix} \mathbf{K_1} & \mathbf{K_{12}} & \beta_k\mathbf{B_1^T} & \beta_k\mathbf{B_1^T} \\ \mathbf{K_{12}} & \mathbf{K_2} & \beta_k\mathbf{B_2^T} & \beta_k\mathbf{B_2^T} \\ \beta_k\mathbf{B_1} & \beta_k\mathbf{B_2} & -\alpha_k\mathbf{I} & +\alpha_k\mathbf{I} \\ \beta_k\mathbf{B_1} & \beta_k\mathbf{B_2} & \alpha_k\mathbf{I} & -\alpha_k\mathbf{I} \end{bmatrix} - \omega^2 \begin{bmatrix} \mathbf{M_1} & \mathbf{M_{12}} & \beta_m\mathbf{B_1^T} & \beta_m\mathbf{B_1^T} \\ \mathbf{M_{12}} & \mathbf{M_2} & \beta_m\mathbf{B_2^T} & \beta_m\mathbf{B_2^T} \\ \beta_m\mathbf{B_1} & \beta_m\mathbf{B_2} & -\alpha_m\mathbf{I} & \alpha_m\mathbf{I} \\ \beta_m\mathbf{B_1} & \beta_m\mathbf{B_2} & \alpha_m\mathbf{I} & -\alpha_m\mathbf{I} \end{bmatrix}\right\}\mathbf{X}=0$$

for an own pulsation ω and an eigenvector $\mathbf{X}$ of this system, one can write:

$$\mathbf{\bar{K}_1 X_1}+\mathbf{\bar{K}_{12} X_2}+\beta\,\mathbf{B_1^T}(\lambda_1+\lambda_2)=0 \qquad \textbf{éq 7.4-1}$$

$$\mathbf{\bar{K}_{12} X_1}+\mathbf{\bar{K}_2 X_2}+\beta\,\mathbf{B_2^T}(\lambda_1+\lambda_2)=0 \qquad \textbf{éq 7.4-2}$$

$$\beta(\mathbf{B_1 X_1}+\mathbf{B_2 X_2})-\alpha(\lambda_1-\lambda_2)=0 \qquad \textbf{éq 7.4-3}$$

$$\beta(\mathbf{B_1 X_1}+\mathbf{B_2 X_2})+\alpha(\lambda_1-\lambda_2)=0 \qquad \textbf{éq 7.4-4}$$

with: $\mathbf{\bar{K}_i}=\mathbf{K_i}-\omega^2\mathbf{M_i}$; $\beta=\beta_k-\omega^2\beta_m$; $\alpha=\alpha_k-\omega^2\alpha_m$

The system (S) is of order $(n+2p)$ if $\begin{cases} n & = \text{nombre de ddls physiques} \\ p & = \text{nombre de relations cin�atiques} \end{cases}$.

The characteristic polynomial in ω^2 is **a priori** of degree $n+2p$. Its term moreover high degree is worth: $\prod_{i=1}^{n} (-m_i)\cdot(+\alpha_m)^{2p}$ if m_i is $i^{ème}$ the diagonal term of $\mathbf{M}$.

- it is thus seen **that if** $\alpha_{\mathbf{m}} \neq 0$, the term moreover high degree is $\neq 0$ (because them m_i are >0) and thus the system dualized (S) the most eigenvalues than the reduced system: $(n-p)$. The two systems are thus not equivalent. It is what one notes on the example of [§7.5],

- let us choose $\alpha_{\mathbf{n}} = \beta_{\mathbf{n}} = 0$:

  [éq 7.4-3] and [éq 7.4-4] $\Rightarrow$

$$\begin{cases} \lambda_1 = \lambda_2 \\ \mathbf{X_1} = -\mathbf{B_1^{-1}} \mathbf{B_2} \mathbf{X_2} \end{cases}$$

  [éq 7.4-1] $\Rightarrow$

$$\lambda_1 = \lambda_2 = \frac{-1}{2\beta} \mathbf{B_1^{-T}} \left(-\bar{\mathbf{K}}_1 \mathbf{B_1^{-1}} \mathbf{B_2} + \bar{\mathbf{K}}_{12} \right) \mathbf{X_2}$$

  [éq 7.4-2] $\Rightarrow$

$$-\bar{\mathbf{K}}_{12} \mathbf{B_1^{-1}} \mathbf{B_2} \mathbf{X_2} + \bar{\mathbf{K}}_2 \mathbf{X_2} - \mathbf{B_2^T} \mathbf{B_1^{-T}} \left(-\bar{\mathbf{K}}_1 \mathbf{B_1^{-1}} \mathbf{B_2} + \bar{\mathbf{K}}_{12} \right) \mathbf{X_2} = 0$$

$$\Leftrightarrow (\tilde{\mathbf{K}}_2 - \omega^2 \tilde{\mathbf{M}}_2) \mathbf{X_2} = 0$$

  with:

$$\begin{cases} \tilde{\mathbf{K}}_2 = -\mathbf{K_{12}^T} \mathbf{B_1^{-1}} \mathbf{B_2} + \mathbf{K_2} + \mathbf{B_2^T} \mathbf{B_1^{-T}} \mathbf{K_1} \mathbf{B_1^{-1}} \mathbf{B_2} - \mathbf{B_2^T} \mathbf{B_1^{-T}} \mathbf{K_{12}} \\ \tilde{\mathbf{M}}_2 = -\mathbf{M_{12}^T} \mathbf{B_1^{-1}} \mathbf{B_2} + \mathbf{M_2} + \mathbf{B_2^T} \mathbf{B_1^{-T}} \mathbf{M_1} \mathbf{B_1^{-1}} \mathbf{B_2} - \mathbf{B_2^T} \mathbf{B_1^{-T}} \mathbf{M_{12}} \end{cases}$$

  It is noted that the definitions of $\tilde{\mathbf{K}}_2$ and $\tilde{\mathbf{M}}_2$ are identical to those of the equations [éq7.3 - 4] and [éq 7.3-5].

  It is thus seen that any eigenvector $\mathbf{X}$ of the dualized system is also eigenvector of the reduced system (with the same own pulsation) if one projects it on space U_2.

Reciprocally, any eigenvector $\mathbf{X_2}$ of the reduced problem can be prolonged in an eigenvector of the dualized system $\mathbf{X^T}=\left[\mathbf{X_1^T},\mathbf{X_2^T},\lambda_1^T,\lambda_2^T\right]$.

with:

$$\begin{cases}\mathbf{X_1}=-\mathbf{B_1^{-1}}\mathbf{B_2}\mathbf{X_2}\\ \lambda_1=-\dfrac{1}{2\beta}\mathbf{B_1^{-T}}(-\bar{\mathbf{K}}_1\mathbf{B_1^{-1}}\mathbf{B_2}+\bar{\mathbf{K}}_{12})\mathbf{X_2}\\ \lambda_2=-\dfrac{1}{2\beta}\mathbf{B_1^{-T}}(-\bar{\mathbf{K}}_1\mathbf{B_1^{-1}}\mathbf{B_2}+\bar{\mathbf{K}}_{12})\mathbf{X_2}\end{cases}$$

The two systems are thus equivalent, they have the same eigen modes and the same eigenvalues.

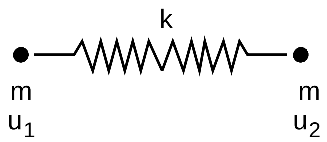
The dualized system, although of size higher than the reduced system, does not have more eigenvalues than the reduced system (the dimension of clean space is the same one).

Conclusion:

The dualized system is equivalent to the reduced system as soon as one chooses $\alpha_m=\beta_m=0$, it is - with - to say if the dualized stiffness matrix is taken but which one does not modify the mass matrix. It is what is made in Aster.

7.5 Example

Is the system:



$$\mathbf{K}=\begin{bmatrix}k & -k\\ -k & k\end{bmatrix}\qquad \mathbf{M}=\begin{bmatrix}m & 0\\ 0 & m\end{bmatrix}$$

the stress is added to him $\alpha\,\mathbf{u_1}+\beta\,\mathbf{u_2}=0(\alpha\neq 0)$; That is to say $\gamma=\dfrac{\beta}{\alpha}$.

The reduced system is then:

- $\mathbf{K_1}=k\,;\mathbf{K_2}=k\,;\mathbf{K_{12}}=-k\,;\mathbf{M_1}=m\,;\mathbf{M_2}=m\,;\mathbf{M_{12}}=0$
- $\mathbf{B_1}=\alpha\,;\mathbf{B_2}=\beta$

$$\Rightarrow\tilde{\mathbf{K}}_2=k(1+\gamma)^2;\tilde{\mathbf{M}}_2=m(1+\gamma^2)$$

$$\Rightarrow\omega^2=\frac{k}{m}\frac{(1+\gamma)^2}{1+\gamma^2};\mathbf{X_2}=1$$

It is noted that the eigenvalue ω^2 depends on the ratio $\gamma=\frac{\beta}{\alpha}$.

If $\gamma=0$,$\omega^2=\frac{k}{m}$

If $\gamma=1,\omega^2=\frac{2k}{m}$

If $g\rightarrow\infty,\omega^2\rightarrow\frac{k}{m}$

Choose $(\alpha=\beta=1)$ to simplify and write the dualized system:

$$\left(\begin{bmatrix} k & -k & \beta_k & \beta_k \\ -k & k & \beta_k & \beta_k \\ \beta_k & \beta_k & -\alpha_k & \alpha_k \\ \beta_k & \beta_k & \alpha_k & -\alpha_k \end{bmatrix} - \lambda \begin{bmatrix} m & 0 & \beta_m & \beta_m \\ 0 & m & \beta_m & \beta_m \\ \beta_m & \beta_m & -\alpha_m & \alpha_m \\ \beta_m & \beta_m & \alpha_m & -\alpha_m \end{bmatrix}\right)\begin{pmatrix} x_1 \\ x_2 \\ \lambda_1 \\ \lambda_2 \end{pmatrix}=0$$

the eigenvalues of this system are:

$$\omega^2=\left\{\frac{\beta_k}{\beta_m},\frac{\beta_k}{\beta_m},\frac{\beta_k}{\beta_m},\frac{2k}{m}\right\}$$

It is noted that one finds the real eigenvalue (the fourth), but that one finds 3 eigenvalues parasitic due to nonthe nullity of α_m and β_m .

If one chooses $\alpha_m=\beta_m=0$, computation shows that the characteristic polynomial is of degree 1 and that its only solution is:

$$\begin{cases} \omega^2=\frac{2k}{m} \\ \mathbf{X}=\{-1,+1\} \end{cases}$$

who is the sought solution.

7.6 Conclusions

- If $\mathbf{K}$ and $\mathbf{M}$ are the stiffness matrixes and of mass of a NON-constrained system.
- If the linear stresses can be written in the form:

$$[\mathbf{B}_1\,\mathbf{B}_2]\begin{bmatrix}\mathbf{U}_1\\\mathbf{U}_2\end{bmatrix}=0 \text{ with } \mathbf{B}_1 \text{ formula}$$

- Then the eigen modes of forced structure are those of the reduced system:

$$\left(\tilde{\mathbf{K}}_2-\omega^2\tilde{\mathbf{M}}_2\right)\mathbf{X}_2=0$$

with:

$$\begin{cases}\tilde{\mathbf{K}}_2=-\mathbf{K}_{12}^{\mathbf{T}}\mathbf{B}_1^{-1}\mathbf{B}_2+\mathbf{K}_2+\mathbf{B}_2^{\mathbf{T}}\mathbf{B}_1^{-\mathbf{T}}\mathbf{K}_1\mathbf{B}_1^{-1}\mathbf{B}_2-\mathbf{B}_2^{\mathbf{T}}\mathbf{B}_1^{-\mathbf{T}}\mathbf{K}_{12}\\\tilde{\mathbf{M}}_2=-\mathbf{M}_{12}^{\mathbf{T}}\mathbf{B}_1^{-1}\mathbf{B}_2+\mathbf{M}_2+\mathbf{B}_2^{\mathbf{T}}\mathbf{B}_1^{-\mathbf{T}}\mathbf{M}_1\mathbf{B}_1^{-1}\mathbf{B}_2-\mathbf{B}_2^{\mathbf{T}}\mathbf{B}_1^{-\mathbf{T}}\mathbf{M}_{12}\end{cases}$$

- The dualized system (double Lagrange) which is written:

$$(\tilde{\tilde{\mathbf{K}}}-\omega^2\tilde{\tilde{\mathbf{M}}})\tilde{\tilde{\mathbf{X}}}=0$$

with:

$$\tilde{\tilde{\mathbf{X}}}^T=\begin{bmatrix}\mathbf{X}_1 & \mathbf{X}_{12} & \lambda_1 & \lambda_2\end{bmatrix}$$

$$\tilde{\tilde{\mathbf{K}}}=\begin{bmatrix}\mathbf{K}_1 & \mathbf{K}_{12} & \mathbf{B}_1^{\mathbf{T}} & \mathbf{B}_1^{\mathbf{T}}\\\mathbf{K}_{12}^{\mathbf{T}} & \mathbf{K}_2 & \mathbf{B}_2^{\mathbf{T}} & \mathbf{B}_2^{\mathbf{T}}\\\mathbf{B}_1 & \mathbf{B}_2 & -\mathbf{I} & \mathbf{I}\\\mathbf{B}_1 & \mathbf{B}_2 & \mathbf{I} & -\mathbf{I}\end{bmatrix}$$

$$\tilde{\tilde{\mathbf{M}}}=\begin{bmatrix}\mathbf{M}_1 & \mathbf{M}_{12} & 0 & 0\\\mathbf{M}_{12}^{\mathbf{T}} & \mathbf{M}_2 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{bmatrix}$$

has the same solutions then as the reduced system.

8 Description of the versions of the document

Version Aster	Author (S) Organization (S)	Description of the modifications
5	***J.PELLET (EDF-R&D/AMA)***	

Annexe 1

Is problem 1

$$\left|\begin{array}{l}\min_{\mathbf{u}\in\mathbf{V}} J(\mathbf{u})\\ J(\mathbf{u})=\frac{1}{2}(\mathbf{Au},\mathbf{u})-(\mathbf{b},\mathbf{u})\\ \mathbf{V}\text{ pennies spaces closely connected } \mathbb{R}^n=\left\{\mathbf{u}\in\mathbb{R}^n \text{ tel que } \mathbf{C_i}\,\mathbf{u}-\mathbf{d_i}=0, \forall\, i=1,p\right\}\\ \mathbf{A}\text{ and } \mathbf{b} \text{ are defined on } \mathbb{R}^n\\ \mathbf{A}\text{ positive symmetric matrix of order } n\,.\end{array}\right.$$

Problem 2

$$\left|\begin{array}{l}\text{To find } \mathbf{u}\in\mathbf{V} \text{ such as: } \left(\left(\mathbf{Au},\mathbf{v_0}\right)-\left(\mathbf{b},\mathbf{v_0}\right)=0 \quad \forall\,\mathbf{v_0}\in\mathbf{V_0}\right)\\ \mathbf{V}=\left\{\mathbf{u}\in\mathbb{R}^n \text{ tel que } \mathbf{C_i}\,\mathbf{u}-\mathbf{d_i}=\mathbf{0},\ \forall\, i=1,p\right\}\\ \mathbf{V_0}=\left\{\mathbf{v_0}\in\mathbb{R}^n \text{ tel que } \mathbf{C_i}\,\mathbf{u}=0,\ \forall\, i=1,p\right\}\end{array}\right.$$

One will show that the two preceding problems are equivalent.
Let us notice first of all that problem 2 is equivalent to problem 2 '.

Problem 2 '

$$\left|\begin{array}{l}\text{To find } \mathbf{u}\in\mathbf{V} \text{ such as: } ((\mathbf{Au},\mathbf{v}-\mathbf{u})-(\mathbf{b},\,\mathbf{v}-\mathbf{u})=0 \quad \forall\,\mathbf{v}\in V)\\ \mathbf{V}=\left\{u\in\mathbb{R}^n \text{ tel que } \mathbf{C_i}\,\mathbf{u}-\mathbf{d_i}=0,\ \forall\, i=1,p\right\}\end{array}\right.$$

There is indeed bijection between the group of $\left\{\mathbf{v}-\mathbf{u},\mathbf{u}\in\mathbf{V},\mathbf{v}\in\mathbf{V}\right\}$ and the group $\mathbf{V_0}$.

Let us show that problem 2 ' is equivalent to problem 1:

That is to say $\mathbf{u}$ solution of 2 '

Then, $\forall\,\mathbf{v}\in\mathbf{V}$

$$\begin{aligned}J(\mathbf{v})-J(\mathbf{u})&=\frac{1}{2}(\mathbf{Av},\mathbf{v})-(\mathbf{b},\mathbf{v})-\frac{1}{2}(\mathbf{Au},\mathbf{u})+(\mathbf{b},\mathbf{u})\\ &=\frac{1}{2}(\mathbf{Av},\mathbf{v})-(\mathbf{Au},\mathbf{v}-\mathbf{u})-\frac{1}{2}(\mathbf{Au},\mathbf{u})\\ &=\frac{1}{2}((\mathbf{Av},\mathbf{v})-2(\mathbf{Au},\mathbf{v})+(\mathbf{Au},\mathbf{u}))=\frac{1}{2}(\mathbf{A}(\mathbf{u}-\mathbf{v},\mathbf{u}-\mathbf{v}))\geq 0\end{aligned}$$ N

Let us calculate derivative first of all of $J(\mathbf{u})$:

$\forall \mathbf{u} \in \mathbf{V}, \forall \mathbf{v_0} \in \mathbf{V_0},$

$$J'(\mathbf{u})\cdot\mathbf{v_0} = \lim_{\varepsilon\to 0}\frac{J(\mathbf{u}+\varepsilon\,\mathbf{v_0})-J(\mathbf{u})}{\varepsilon}$$
$$= \lim_{\varepsilon\to 0}\left[(\mathbf{Au},\mathbf{v_0})+\frac{\varepsilon}{2}(\mathbf{Av_0},\mathbf{v_0})-(\mathbf{b},\mathbf{v_0})\right]=(\mathbf{Au}-\mathbf{b})\,\mathbf{v_0}$$

That is to say $\mathbf{u}$ the solution of Pb1

$\forall\,\mathbf{V}\in\mathbf{V}$, let us pose $\mathbf{v}_0=\mathbf{v}-\mathbf{u}$; $\mathbf{v}_0\in\mathbf{V}_0$.

$$\forall\,\varepsilon \quad \frac{J(u+\varepsilon\,\mathbf{v_0})-J(\mathbf{u})}{\varepsilon}\geq 0 \quad \Rightarrow \quad (J'(\mathbf{u})\cdot\mathbf{v_0}\geq 0 \;\;,\; \forall\,\mathbf{v_0}\in\mathbf{V_0})$$

It is seen that $J'(\mathbf{u})$ who is a linear form on $\mathbf{V_0}$ must be systematically positive. This is possible only if this form is identically null.

One concludes from it that:

$$J'(\mathbf{u})\cdot(\mathbf{v}-\mathbf{u})=(\mathbf{Au}-\mathbf{b})(\mathbf{v}-\mathbf{u})=0 \;\;,\; \forall\,\mathbf{v}\in\mathbf{V} \qquad \text{N}$$

Annexe 2

Definitions, notations

$\mathbf{A}$ is the unconstrained stiffness matrix $(n\ ' n)$ (symmetric and positive)
$\mathbf{C}$ is the matrix of blocking: $\mathbf{Cu}-d=0$ (C matrix $n\ ' p(p<n)$)
$\mathbf{u}$ of the physical degrees of freedom is the vector $\in R^n$
λ^1 of the first degrees of freedom of Lagrange is the vector $\in \mathbb{R}^p$
λ^2 of the second degrees of freedom of Lagrange One is $\in \mathbb{R}^p$

$$\mathbf{x}=(\mathbf{u}, \lambda^1, \lambda^2)\in \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^p$$

the vector notes:

- $\mathbf{U}$ all the physical degrees of freedom,
- Λ^1 all the first degrees of freedom of Lagrange,
- $\mathbf{\Lambda^2}$ all the second degrees of freedom of Lagrange.

$\alpha \in \mathbb{R}^+$

$\mathbf{K}$ stamp of a symmetric $2\ p+n$ nature

$$\mathbf{K}=\begin{bmatrix} \mathbf{A} & \mathbf{C^T} & \mathbf{C^T} \\ \mathbf{C} & -\alpha\mathbf{l} & \alpha\mathbf{l} \\ \mathbf{C} & \alpha\mathbf{l} & -\alpha\mathbf{l} \end{bmatrix}$$

the matrix $\mathbf{K}$ written above corresponds to a certain classification of the unknowns:

$$\mathbf{x}=(\mathbf{u}, \lambda^1, \lambda^2)$$

The genuine matrix $\mathbf{K}$ which one seeks to show that it is factorisable by LDL^T without permutation is not written with this classification. The only rule of classification taken into account is the following one:

Regulate $R0$ **:**
The two degrees of freedom of Lagrange associated with an equation with connection $\mathbf{C_i U}-\mathbf{d_i}=0$ frame the d.o.f. physiques constrained by this equation.

Thereafter, to simplify the writing, one will take $\alpha=1$.

One seeks to show that very under matrix $\mathbf{K_i}$ of $\mathbf{K}$ is invertible.
That is to say under matrix $\mathbf{K_i}$ given. It corresponds to a division of the degrees of freedom: those of row $\le i$, those of row $\ge i$.

We will note:

$\tilde{U}$ the subset of U corresponding to the degrees of freedom of row $\le i$.

$\tilde{\tilde{U}}$ the subset of U corresponding to the degrees of freedom of row $> i$.

L_1 the couples such as formula is $(\lambda_1^1, \lambda_1^2)$ $\text{rang}\ (\lambda_1^1) < \text{rang}\ (\lambda_1^2) \le i$
$L_1^1 = \{\lambda_1^1\}$; $L_1^2 = \{\lambda_1^2\}$

L_3 the couples such as formula is $(\lambda_3^1, \lambda_3^2)$ $i < \text{rang}\ (\lambda_3^1) < \text{rang}\ (\lambda_3^2)$
$L_3^1 = \{\lambda_3^1\}$; $L_3^2 = \{\lambda_3^2\}$

L_2 the couples such as formulates One $(\lambda_2^1, \lambda_2^2)$ is $\text{rang}\ (\lambda_2^1) \le i < \text{rang}\ (\lambda_2^2)$
$L_2^1 = \{\lambda_2^1\}$; $L_2^2 = \{\lambda_2^2\}$

all A. $L = \bigcup_{\substack{i=1,3 \\ j=1,2}} L_i^j$

the matrix C can cut out in 3 parts corresponding to cutting (L_1, L_2, L_3)

$\mathbf{C_1}$
$\mathbf{C_2}$
$\mathbf{C_3}$

Each matrix $\mathbf{C_i}$ can cut out in 2 parts corresponding to cutting $(\tilde{U}, \tilde{\tilde{U}})$

$\tilde{C}_i$	$\tilde{\tilde{C}}_i$

the matrix A can Using cut out in 4 parts corresponding to $(\tilde{U}, \tilde{\tilde{U}})$

$$\mathbf{A} = \begin{bmatrix} \tilde{\mathbf{A}} & \bar{\mathbf{A}} \\ \bar{\mathbf{A}}^{\mathbf{T}} & \tilde{\tilde{\mathbf{A}}} \end{bmatrix}$$

cutting these notations, the problem to be solved is to show that the Ki matrix is invertible.

$$\mathbf{K_i} = \begin{bmatrix} -\mathbf{I} & \mathbf{I} & 0 & \tilde{\mathbf{C}}_1 \\ \mathbf{I} & -\mathbf{I} & 0 & \tilde{\mathbf{C}}_1 \\ 0 & 0 & -\mathbf{I} & \tilde{\mathbf{C}}_2 \\ \tilde{\mathbf{C}}_1^{\mathbf{T}} & \tilde{\mathbf{C}}_1^{\mathbf{T}} & \tilde{\mathbf{C}}_2^{\mathbf{T}} & \tilde{\mathbf{A}} \end{bmatrix}$$

This matrix corresponds to the vector $\mathbf{X_i}=\begin{bmatrix}\lambda_1^1\\ \lambda_1^2\\ \lambda_2^1\\ \tilde{\mathbf{U}}\end{bmatrix}$

It should be shown that: $\mathbf{K_i}.\mathbf{X_i}=0\Rightarrow\mathbf{X_i}=0$

The problem is equivalent to:

Problem 1:

$$(S)=\begin{cases}-\lambda_1^1+\lambda_1^2+\tilde{\mathbf{C}}_\mathbf{1}\tilde{\mathbf{u}}=0\\ \lambda_1^1-\lambda_1^2+\tilde{\mathbf{C}}_\mathbf{1}\tilde{\mathbf{u}}=0\\ -\lambda_2^1+\tilde{\mathbf{C}}_\mathbf{2}\tilde{\mathbf{u}}=0\\ \tilde{\mathbf{C}}_\mathbf{1}^\mathbf{T}\left(\lambda_1^1+\lambda_1^2\right)+\tilde{\mathbf{C}}_\mathbf{2}^\mathbf{T}\lambda_2^1+\tilde{\mathbf{A}}.\tilde{\mathbf{u}}=0\end{cases}\Rightarrow\begin{cases}\tilde{\mathbf{u}}=0\\ \lambda_1^1=\lambda_1^2=0\\ \lambda_2^1=0\end{cases}$$

General case:
It is supposed that $\tilde{\mathrm{U}}\neq\varnothing$; $\mathbf{L_1}\neq\varnothing$; $\mathbf{L_2}\neq\varnothing$

$(S)\Rightarrow$

$$\lambda_1^1=\lambda_1^2 \quad \textbf{éq An2-1}$$

$$\tilde{\mathbf{C}}_\mathbf{1}\tilde{\mathbf{u}}=0 \quad \textbf{éq An2-2}$$

$$\lambda_2^1=\tilde{\mathbf{C}}_\mathbf{2}\tilde{\mathbf{u}} \quad \textbf{éq An2-3}$$

$$2\tilde{\mathbf{C}}_\mathbf{1}^\mathbf{T}\lambda_1^1+\left(\tilde{\mathbf{C}}_\mathbf{2}^\mathbf{T}\tilde{\mathbf{C}}_\mathbf{2}+\tilde{\mathbf{A}}\right)\tilde{\mathbf{u}}=0 \quad \textbf{éq An2-4}$$

Of [éq An2-4], one deduces:

$$2\tilde{\mathbf{u}}^\mathbf{T}\tilde{\mathbf{C}}_\mathbf{1}^\mathbf{T}\mathbf{l}_\mathbf{1}^\mathbf{1}+\tilde{\mathbf{u}}^\mathbf{T}\left(\tilde{\mathbf{C}}_\mathbf{2}^\mathbf{T}\tilde{\mathbf{C}}_\mathbf{2}+\tilde{\mathbf{A}}\right)\tilde{\mathbf{u}}=0$$

From [éq An2-2], one obtains:

$$\tilde{\mathbf{u}^\mathbf{T}}\tilde{\mathbf{C}}_\mathbf{1}^\mathbf{T}=0 \qquad \Rightarrow \qquad \tilde{\mathbf{u}^\mathbf{T}}\left(\tilde{\mathbf{C}}_\mathbf{2}^\mathbf{T}\tilde{\mathbf{C}}_\mathbf{2}+\tilde{\mathbf{A}}\right)\tilde{\mathbf{u}}=0$$
$$\Rightarrow\tilde{\mathbf{u}^\mathbf{T}}\tilde{\mathbf{C}}_\mathbf{2}^\mathbf{T}\tilde{\mathbf{C}}_\mathbf{2}\tilde{\mathbf{u}}+\tilde{\mathbf{u}^\mathbf{T}}\tilde{\mathbf{A}}\tilde{\mathbf{u}}=0$$

However $\tilde{\mathbf{A}}$ is symmetric positive (under matrix of a positive matrix) and $\tilde{\mathbf{C}}_2^{\mathrm{T}}\tilde{\mathbf{C}}_2$ is as a positive symmetric matrix, therefore this sum cannot be null as if the two terms are null.

$$\Rightarrow \begin{cases} \tilde{\mathbf{u}}^{\mathrm{T}}\tilde{\mathbf{C}}_2^{\mathrm{T}}.\tilde{\mathbf{C}}_2\,\tilde{\mathbf{u}}=0 \\ \tilde{\mathbf{u}}^{\mathrm{T}}\tilde{\mathbf{A}}\,\tilde{\mathbf{u}}=0 \end{cases}$$

$$\Rightarrow \tilde{\mathbf{C}}_2\,\tilde{\mathbf{u}}=0 \Rightarrow \lambda_2^1=0 \qquad \textbf{éq An2-5}$$

$\tilde{\mathbf{A}}$ is a positive matrix, one wants to show that:

$$\tilde{\mathbf{u}}^{\mathrm{T}}\tilde{\mathbf{A}}\,\tilde{\mathbf{u}}=0 \Rightarrow \tilde{\mathbf{u}}=0 \qquad \textbf{éq An2-6}$$

It remains us to show that: $\tilde{\mathbf{u}}=0\,\text{et}\,\lambda_1^1=0$

- $\tilde{\mathbf{u}}=0$

Let us prolong $\tilde{\mathbf{u}}$ on $\mathbb{R}^n$ by $\tilde{\tilde{\mathbf{u}}}=0$ $\quad \mathbf{u}=(\tilde{\mathbf{u}}\,,\tilde{\tilde{\mathbf{u}}})$:

$$\mathbf{u}^{\mathrm{T}}\mathbf{A}\mathbf{u}=\tilde{\mathbf{u}}^{\mathrm{T}}\tilde{\mathbf{A}}\,\tilde{\mathbf{u}}=0 \Rightarrow \mathbf{u}\in\ker\mathbf{A}$$

$$\mathbf{C}\mathbf{u}=\begin{bmatrix}\mathbf{C}_1\mathbf{u}\\ \mathbf{C}_2\mathbf{u}\\ \mathbf{C}_3\mathbf{u}\end{bmatrix}=\begin{bmatrix}\tilde{\mathbf{C}}_1\tilde{\mathbf{u}}\\ \tilde{\mathbf{C}}_2\tilde{\mathbf{u}}\\ \tilde{\mathbf{C}}_3\tilde{\mathbf{u}}\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Indeed $\tilde{\mathbf{C}}_3=0$ because if not, there would exist d.o.f. of $\tilde{\mathbf{u}}$ constrained by equations not yet taken into account (of row $>i$) what is contrary with R_0 .

The prolongation $\mathbf{u}$ of $\tilde{\mathbf{u}}$ is thus in the cores of $\mathbf{A}$ and $\mathbf{C}$. One will show that it is then null. Let us take again the problem with “simple Lagrange”.

$$(S_2)=\begin{cases}\mathbf{A}\mathbf{u}+\mathbf{C}^{\mathrm{T}}\lambda=\mathbf{b}\\ \mathbf{C}\mathbf{u}=\mathbf{d}\end{cases}$$

So $\mathbf{u_0}\neq 0$ is such that $\mathbf{A}\mathbf{u_0}=0$ and $\mathbf{C}\mathbf{u_0}=0$.
If u_1 is solution of S2, it is seen that then $\mathbf{u_1}+\mu\,\mathbf{u_0}$ is also solution. What is impossible because we suppose our problem posed physically well.
One concludes from it that $\mathbf{u}=0 \quad \Rightarrow \tilde{\mathbf{u}}=0$.

- $\lambda_1^1=0$

[éq An2-4] gives:

$$\tilde{\mathbf{C}}_1^{\mathbf{T}} \lambda_1^1=0 \qquad \textbf{éq An2-7}$$

In the same way that the rule $\mathbf{R_0}$ imposing $\tilde{\mathbf{C}}_3=0$, one sees that $\tilde{\tilde{\mathbf{C}}}_1=0$.

[éq An2-6] gives:

$$\mathbf{C}_1^{\mathbf{T}} \lambda_1^1=\begin{bmatrix} \tilde{\mathbf{C}}_1^T \lambda_1^1 \\ \tilde{\tilde{\mathbf{C}}}_1^T l_1^1 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}=0 \qquad \textbf{éq An2-8}$$

Let us reason by the absurdity: so $\lambda_1^1 \neq 0$ is such that $\mathbf{C}_1^{\mathbf{T}} \lambda_1^1=0$, it is that there exists a linear combination of the lines of $\mathbf{C_1}$ which is null, which is contradictory with the fact that the lines of $\mathbf{C_1}$ are independent from/to each other (physical problem good posed).

Thus $\lambda_1^1=0$.

Typical case:

When one (or more) of the sets $\tilde{\mathbf{u}}$ L_1 , L_2 is empty, the system (S) is simplified. One can check that the reasoning which one made in the general case, makes it possible to show similar results.