External forces of pressure in large displacements

Summarized:

A loading of pressure in large displacements is a following loading. By employing elements of skin, one is brought to calculate, on the one hand, a second member to which computation is close to that in small displacements, and on the other hand, a term of additional stiffness which is not, in general, symmetric.

Contents

1 Introduction

the taking into account of loadings of type pressure (key word PRES_REP in command AFFE_CHAR_MECA [U4.44.01]) raise a certain number of difficulties in the absence of the assumption of small displacements. Indeed, unlike dead loads evoked in [R5.03.01], the pressure depends on displacements since it is about a force whose direction is normal with the field; one speaks then about follower forces, activated by the key word TYPE_CHARGE=' SUIV' in command STAT_NON_LINE [U4.51.03]. Nevertheless, the choice of the present configuration like reference configuration (Lagrangian updated) carried to simple statements - with the help of some notions of differential geometry - work of the forces of pressure and its variation first compared to displacement, the latter being an asymmetric bilinear form.

2 Continuous writing of the problem

2.1 Elements of kinematics in great transformations

One considers a solid Ω subjected to large deformations (see figure 2-1). Either $\boldsymbol{F}$ the tensor gradient of the transformation $\boldsymbol{\phi}$ making pass the initial configuration Ω_0 with the deformed present configuration Ω_t. One notes $\boldsymbol{X}$ the position of a point in Ω_0 and $\boldsymbol{x}$ the position of this same point after strain in Ω_t. $\boldsymbol{u}$ then displacement enters the two configurations. One thus has:

$$\boldsymbol{x}=\boldsymbol{X}+\boldsymbol{u} \quad (1)$$

the tensor gradient of the transformation is written:

$$\boldsymbol{F}=\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}}=\boldsymbol{I}+\nabla\times\boldsymbol{u} \quad 2)2$$

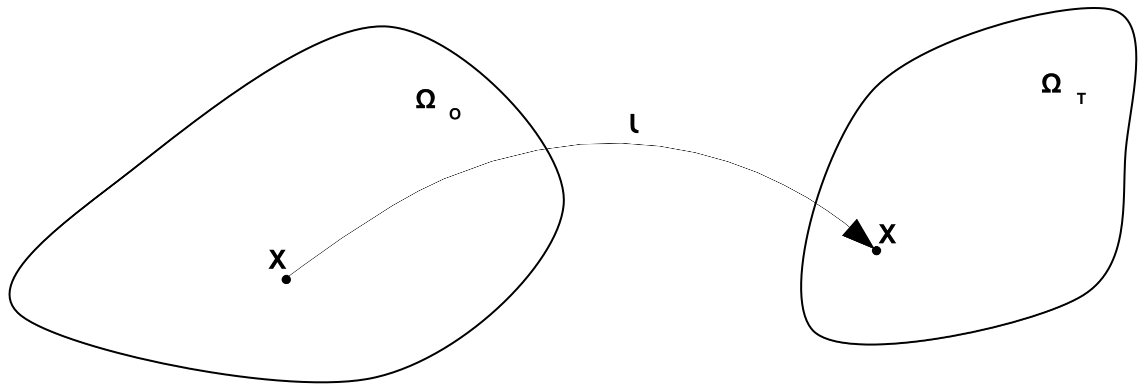

Figure 2-1: Solid in great transformations

2.2 Virtual wor of the external forces of pressure

One considers a normal P pressure on the surface in the reference configuration. This pressure is written p in the present configuration.

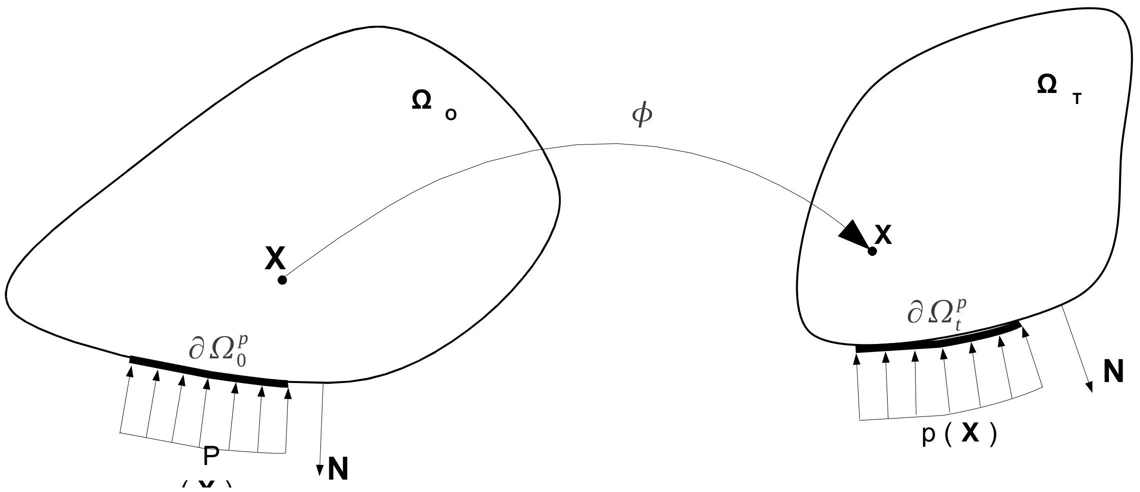

Figure 2-2: Reference configuration and present configuration

In the present configuration, the virtual wor of the external forces of pressure W^p is written simply (see figure 2-2):

$$W^p(\boldsymbol{u}).\delta\boldsymbol{v}=\int_{\partial\Omega_t^p(\boldsymbol{u})} -p.\boldsymbol{n}.\delta\boldsymbol{v}.ds \tag{3}$$

Moreover, one supposes henceforth that the value of the pressure does not depend explicitly on displacement but only on the material point of application:

$$p(\boldsymbol{x})=P(\boldsymbol{\phi}(\boldsymbol{X})) \tag{4}$$

with dimensions follower of the force comes from the dependence of *the norm* to displacement. In this case, one can then express the virtual wor of the forces of pressure in the reference configuration (change of variable in the integral):

$$W^p(\boldsymbol{u}).\ \delta\boldsymbol{v}=\int_{\partial\Omega_0^p} -P\det(\boldsymbol{F})\left[\boldsymbol{F}^{-T}.\boldsymbol{N}\right].\ \delta\boldsymbol{v}(\ \varphi(\boldsymbol{X})).dS \tag{5}$$

On the practical level, one will use the formula (3) to compute: the work of the forces of pressure. However, the formula (5) is best adapted to a derivative compared to the displacement, for which one will see the need in the following paragraph.

2.3 Variation of the virtual wor of the external forces of pressure

In the optics of a resolution of the problem of equilibrium of structure by a method of Newton, one is brought to express the variation of the virtual wor of the external forces of pressure compared to displacement, in a way similar to what was made for the virtual wor of the internal forces in [R5.03.01]. The field of integration being fixed in the statement (5), the derivative under the sign nap is licit, (cf 9):

$$\frac{\partial W^p(\boldsymbol{u})}{\partial\boldsymbol{u}}.\ \delta\boldsymbol{u}.\ \delta\boldsymbol{v}=\int_{\partial\Omega_0^p} -P.\frac{\partial}{\partial\boldsymbol{u}}\left[\det(\boldsymbol{F}).\boldsymbol{F}^{-T}\right].\ \delta\boldsymbol{u}.\boldsymbol{N}.\ \delta\boldsymbol{v}.dS \tag{6}$$

We decide to choose like reference configuration the present configuration, for which $\boldsymbol{F}=\boldsymbol{I}$. This choice led to a simple statement of derivative of the term between hooks:

$$\frac{\partial}{\partial\boldsymbol{u}}\left[\det(\boldsymbol{F}).\boldsymbol{F}^{-T}\right].\ \delta\boldsymbol{u}=\mathrm{div}\ (\delta\boldsymbol{u}).\boldsymbol{I}-\nabla^T\times\delta\boldsymbol{u} \tag{7}$$

Finally, the variation of the virtual wor of the external forces of pressure is written in the present configuration:

$$\frac{\partial W^p(\boldsymbol{u})}{\partial\boldsymbol{u}}.\delta\boldsymbol{u}.\delta\boldsymbol{v}=\int_{\partial\Omega_t^p(\boldsymbol{u})} -p.\frac{\partial}{\partial\boldsymbol{u}}\left[\mathrm{div}\ (\delta\boldsymbol{u}).\boldsymbol{I}-\nabla^T\times\delta\boldsymbol{u}\right].\boldsymbol{n}.\delta\boldsymbol{v}.ds \tag{8}$$

In the statement (8) remains a difficulty. Indeed, one expects to obtain a primarily surface quantity whereas the intégrande reveals terms of normal derivative on the surface. In other words, it is necessary to know the statement of virtual displacements not only on surface of the field but also inside this one (in a vicinity of surface to be able to express normal derivatives). This disadvantage is not pain-killer since in *Code_Aster*, to compute: the elementary terms due to the surface forces, one employs elements of skin for which a normal variation does not have a meaning.

2.4 Adoption of a curvilinear parameter setting of surface

to cure the problem mentioned previously, it is necessary to seek to only express the relation (8) using surface quantities. For that, one resorts to elements of differential geometry, 9, which one adopts the notations (in particular, one adopts the summation convention of the repeated indices where the Greek indices take the values 1 and 2 while the Latin indices take the values 1 with 3).

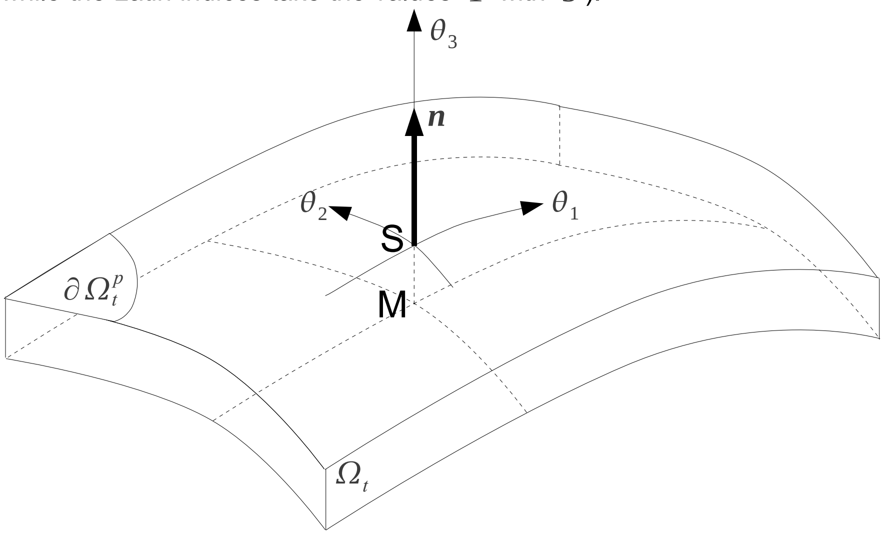

Figure 2-3: Curvilinear parameter setting of the vicinity of the surface subjected to the pressure

Is (θ^1, θ^2) an acceptable parameter setting of surface. To describe volume made up of a vicinity of this surface, one associates a third variable to him θ^3 , which measures the progression according to the unit norm $\boldsymbol{n}$ in (θ^1, θ^2) . One has thus (see figure 2-3):

$$\boldsymbol{OM}(\theta^1, \theta^2, \theta^3) = \boldsymbol{OS}(\theta^1, \theta^2) + \theta^3 . \boldsymbol{n}(\theta^1, \theta^2) \quad (9)$$

With this choice of parameter setting, the natural base covariante $(\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3)$ is written:

$$\boldsymbol{g}_i = \frac{\partial \boldsymbol{OM}}{\partial \theta^i} \quad (10)$$

While the metric tensor $\boldsymbol{g}$ is worth:

$$g_{ij} = \boldsymbol{g}_i . \boldsymbol{g}_j = \begin{bmatrix} g_{11} & g_{12} & 0 \\ g_{21} & g_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (11)$$

In this curvilinear parameter setting, the intégrande (8) has as a statement:

$$-p . g_{ij} . n^i . \left[\delta u^k|_k . \delta v^j - \delta u^j|_k . \delta v^k \right] \quad (12)$$

This term is simplified considerably. Indeed, one can already note that when $j=k$, the term between hook is null. Moreover, in the adopted curvilinear system, the components contravariantes of $\boldsymbol{n}$ are:

$$n^1 = 0, n^2 = 0, n^3 = 1 \quad (13)$$

Lastly, by taking account of the particular form of $\mathbf{g}$ (i.e. $g_{13}=0$, $g_{23}=0$ and $g_{33}=1$), the variation of work is written simply:

$$\frac{\partial W^p(\boldsymbol{u})}{\partial \boldsymbol{u}} . \delta \boldsymbol{u} . \delta \boldsymbol{v} = \int_{\partial \Omega_t^p(\boldsymbol{u})} -p . \left[\delta u^\alpha|_\alpha . \delta v^3 - \delta u^3|_\alpha . \delta v^\alpha \right] . ds \quad (14)$$

On this statement, one notes that only intervene of the surface operators differential (derivative covariante compared to θ^1 and θ^2 only), which is well the sought-after goal. By introducing the base contravariante $(\boldsymbol{g}^1, \boldsymbol{g}^2, \boldsymbol{g}^3=\boldsymbol{n})$, also called bases dual and which is expressed from the base covariante by:

$$\boldsymbol{g}^i=[\boldsymbol{g}^{-1}]^{ij}.\boldsymbol{g}_j \tag{15}$$

One can free itself of the components curvilinear:

$$\frac{\partial W^p(\boldsymbol{u})}{\partial \boldsymbol{u}}.\ \delta\boldsymbol{u}.\ \delta\boldsymbol{v}=\int_{\partial\Omega_t^p(\boldsymbol{u})} -p.\left[\left(\frac{\partial\ \delta\boldsymbol{u}}{\partial\ \theta^\alpha}.\boldsymbol{g}^\alpha\right).(\delta\boldsymbol{v}.\boldsymbol{n})-\left(\frac{\partial\ \delta\boldsymbol{u}}{\partial\ \theta^\alpha}.\boldsymbol{n}\right).(\delta\boldsymbol{v}.\boldsymbol{g}^\alpha)\right].ds \tag{16}$$

It is henceforth the statement (16) which will be used to compute: the variation of the virtual wor of the forces of pressure.

2.5 Typical case of a structure subjected to an internal or external pressure constant

In the typical case of a constant pressure in a cavity (see figure 2-4), one shows that the forces of pressure derive from a potential Ξ which is not other than the product of the pressure by the volume of the cavity. This result extends to the case from a structure plunged in a fluid with constant pressure.

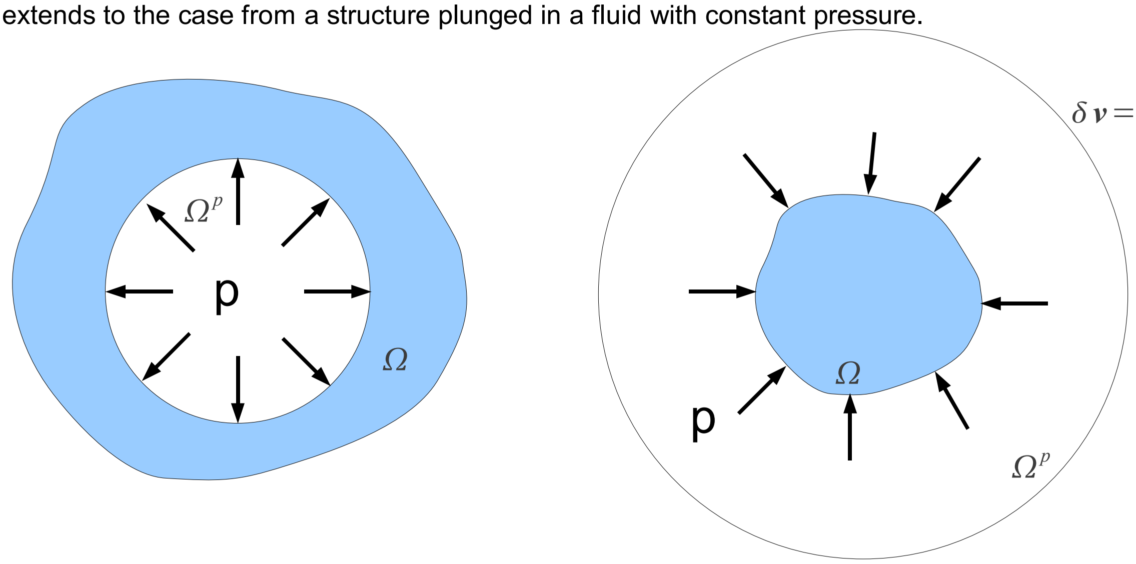

Figure 2-4: Structure under internal or external pressure constant

One writes this potential:

$$\Xi=p.\int_{\partial\Omega_t^p} d\Omega_t=p.\int_{\partial\Omega_0^p}\det\boldsymbol{F}.d\Omega_0 \tag{17}$$

Again, one chooses like reference configuration the present configuration. The variation of Ξ leads then well to the virtual wor of the external forces of pressure:

$$\frac{\partial\Xi}{\partial\boldsymbol{u}}.\delta\boldsymbol{v}=p.\int_{\Omega_t^p}\mathrm{div}(\delta\boldsymbol{v}).d\Omega_t=-\int_{\partial\Omega_t^p} p.\delta\boldsymbol{v}.\boldsymbol{n}.ds=W_p.\delta\boldsymbol{v} \tag{18}$$

In this case, typical case the variation of the virtual wor is also the second variation of the potential Ξ, i.e. a symmetric bilinear **form** :

$$\frac{\partial W^p(\boldsymbol{u})}{\partial\boldsymbol{u}}.\delta\boldsymbol{u}.\delta\boldsymbol{v}=\frac{\partial^2\Xi(\boldsymbol{u})}{\partial\boldsymbol{u}^2}.\delta\boldsymbol{u}.\delta\boldsymbol{v} \tag{19}$$

3 Discretization

3.1 Introduction into *Code_Aster*

In *Code_Aster*, of the finite elements of skin (surface elements plunged in a three-dimensional space) are employed to discretize real and virtual displacements intervening in surface statements such as (3) and (16). These last make it possible respectively to express the second member vector and the stiffness matrix due to the pressure, whose employment by the algorithm of `STAT_NON_LINE` is specified in [R5.03.01]
One thus develops four options:
1.`RIGI_MECA_PRSU_R` : stiffness matrix for a following pressure like real constant
2.`RIGI_MECA_PRSU_F` : stiffness matrix for a following pressure like real function
3.`CHAR_MECA_PRSU_R` : second member vector for a following pressure like real constant
4.`CHAR_MECA_PRSU_F` : second member vector for a following pressure as real function
These options are developed for the elements `3D`, `D_PLAN` and `AXIS`. One can apply a normal following pressure but not following tangent shears. The pressure can be a real function of time or a real constant.

3.2 Discretization of the terms of differential geometry

the first two vectors of the base covariante $\{\boldsymbol{G}_{\alpha=1,2}\}$ is calculated starting from the displacement and of derivatives of the shape functions $[\boldsymbol{B}_\alpha]$:

$$\{\boldsymbol{G}_\alpha\}=[\boldsymbol{B}_\alpha].\{\boldsymbol{u}\} \quad (20)$$

the norm $\{\boldsymbol{N}\}$ is calculated like the cross product of these the first two vectors $\{\boldsymbol{G}_{\alpha=1,2}\}$:

$$\boldsymbol{n}=\frac{\boldsymbol{g}_1\wedge\boldsymbol{g}_2}{\|\boldsymbol{g}_1\wedge\boldsymbol{g}_2\|} \quad (21)$$

One can also calculate the metric tensor $[\boldsymbol{G}_{\alpha\beta}]$:

$$[\boldsymbol{G}_{\alpha\beta}]=\{\boldsymbol{G}_\alpha\}\langle\boldsymbol{G}_\beta\rangle$$

formulate2
222

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$$J=\det[\boldsymbol{G}_{\alpha\beta}] \quad (23)$$

One can calculate the metric matrix contravariante:

$$[\boldsymbol{G}^{\delta\gamma}]=[\boldsymbol{G}_{\alpha\beta}]^{-1} \quad (24)$$

And finally to extract the base contravariante $\{\boldsymbol{G}^\delta\}$:

$$\{\boldsymbol{G}^\delta\}=[\boldsymbol{G}^{\delta\gamma}]\{\boldsymbol{G}_\alpha\}$$

formulate2
525

3.3) Vector of

the follower forces of the virtual wor of the forces of pressure (3) is in fact identical to that carried out in small displacements, with the help of a preliminary reactualization of the geometry. One leaves the statement of the virtual wors:

$$W^p(\boldsymbol{u}).\delta\boldsymbol{v}=\int_{\partial\Omega_t^p(\boldsymbol{u})}-p.\boldsymbol{n}.\delta\boldsymbol{v}.ds \quad (26)$$

In discretized form:

$$W_p(\boldsymbol{u}).\ \delta\boldsymbol{v}=\langle\delta\boldsymbol{V}\rangle\{F^p(\boldsymbol{u})\} \quad (27)$$

the variation of displacements is written starting from the shape functions:

$$\delta\boldsymbol{v}=\langle\boldsymbol{\varphi}\rangle.\{\delta\boldsymbol{V}\} \quad (28)$$

One discretized all the terms of differential geometry in the preceding paragraph, it but does not remain us any more to discretize the integral by means of a diagram of Gauss with the weights $\omega_{i_{pg}}$:

$$\int_{\partial \Omega_t^p(\boldsymbol{u})} A.\,ds=\sum_{i_{pg}} A_{\xi_{pg}}.\,\omega_{i_{pg}} \quad (29)$$

the diagrams of integration used are summarized in the table below.

Geometrical mesh	Diagram of Gauss (see [R3.01.01])
3D	
TRIA3	FPG3
TRIA6	FPG4
QUAD4	FPG4
QUAD8	FPG9
QUAD9	FPG9
D_PLAN	
SEG2	FPG2
SEG3	FPG4
AXIS	
SEG2	FPG2
SEG3	FPG4

In *Code_Aster*, the pressure p given by `AFFE_CHAR_MECA` is localised with the nodes, one thus must by interpolating the pressure p of the nodes $p^{i_{no}}$ towards the Gauss point $p_{\xi_{pg}}$:

$$p_{\xi_{pg}}=\sum_{i_{no}} N_{\xi_{pg}}^{i_{no}}.\,p^{i_{no}} \quad (30)$$

Finally:

$$\{F^p(\boldsymbol{u})\}=-\sum_{i_{pg}} p_{\xi_{pg}}.\,\omega_{i_{pg}}.\,n_{\xi_{pg}}.\,J_{\xi_{pg}}.\{\boldsymbol{\varphi}_{\xi_{pg}}\}$$

formulate3 131

3.4) Matrix of

the follower forces of the variation of the virtual wor of the forces of pressure (16) is worth:

$$\frac{\partial W^p(\boldsymbol{u})}{\partial \boldsymbol{u}}.\,\delta\boldsymbol{u}.\,\delta\boldsymbol{v}=\int_{\partial \Omega_t^p(\boldsymbol{u})} -p.\left[\left(\frac{\partial\,\delta\boldsymbol{u}}{\partial\,\theta^\alpha}.\boldsymbol{g}^\alpha\right).(\delta\boldsymbol{v}.\boldsymbol{n})-\left(\frac{\partial\,\delta\boldsymbol{u}}{\partial\,\theta^\alpha}.\boldsymbol{n}\right).(\delta\boldsymbol{v}.\boldsymbol{g}^\alpha)\right].ds \quad (32)$$

It allows to extract the tangent matrix from the follower forces:

$$\frac{\partial W^p(\boldsymbol{u})}{\partial \boldsymbol{u}}.\,\delta\boldsymbol{u}.\,\delta\boldsymbol{v}=\langle\delta\boldsymbol{V}\rangle[-\boldsymbol{K}^p(\boldsymbol{u})]\{\delta\boldsymbol{U}\} \quad (33)$$

the minus sign comes owing to the fact that the contribution of the matrix is to the first member.

Finally:

$$[\boldsymbol{K}^p(\boldsymbol{u})]=\sum_{i_{pg}} p_{\xi_{pg}}.\,\omega_{i_{pg}}.\,J_{\xi_{pg}}.[\boldsymbol{B}_{\xi_{pg}}]^T.\left(\{\boldsymbol{G}^\delta_{\xi_{pg}}\}.\langle\boldsymbol{n}_{\xi_{pg}}\rangle-\{\boldsymbol{n}_{\xi_{pg}}\}.\langle\boldsymbol{G}^\delta_{\xi_{pg}}\rangle\right).[\boldsymbol{\varphi}_{\xi_{pg}}]$$

formulate3 434

3.5) Choice of

the matrix, the matrix is not **symmetric** (except typical case of a structure subjected to an internal or external pressure constant, cf §2.56). It is noted also in practice, that for strong variations of the geometry (by means of a behavior very-elastic in large deformations like ELAS_HYPER), the fact of symmetrizing this matrix is not a good strategy (failure of convergence). One thus decides to keep this nonsymmetrical matrix, in spite of (light) the overcost induced by the factorization of such a matrix.

4 Bibliography

[1] “Foundations of solid mechanics”, Fung Y.C., Prentice Hall. 1965, pp 31-57.

[2] “Computation of derivative of a quantity compared to a crack tip by the method theta”, Mialon P., EDF - Bulletin of the Management of the Studies and Searches - Series C - n° 3. 1988, pp 1-28.

History of the versions of the document

Version Aster	**Author (S) Organization (S)**	**Description of the modifications**
3	3E.Lorentz	initial Text
11	**M.Abbas**	systematic Use of the nonsymmetrical matrix, harmonization of the notations, notes on the discretization, new figures