Computation of the nodal stresses by local lissage

Abstract:

One presents a local method of computation of nodal stresses starting from the stresses to Gauss points. It is used in options `SIGM_ELNO` and `SIEF_ELNO`.

This method is summarized with calculating the stresses at the tops of an element by multiplying the stresses with Gauss points by a matrix of lissage, constant for each type of element.

For the isoparametric elements of degree 2, the nodal stresses mediums are obtained by average of the values of the stresses at the 2 tops of the edge.

This method of lissage has two advantages:

- the nodal stresses obtained have an order of accuracy moreover than by direct computation with the nodes,
- the method is inexpensive in TEMPS CPU.

This method was generalized:

- with computations of the strains (option `EPSI_ELNO`) and local variables (option `VARI_ELNO`) with the nodes in mechanics,
- the computation of the flux (option `FLUX_ELNO`) to the nodes in thermal.

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This method lean on the observation [bib1] that there exist points where the computation of the stresses, starting from displacements in a primal formulation in displacements, is more precise.

In the isoparametric case of finite elements of order 2 (SEG3 in 1D, QUAD8 and QUAD9 in 2D, HEXA20 in 3D), one shows that Gauss points formula of squaring at 2^n points (n : dimension of space) are such as one can hope, without that being formally shown, for the computation of $\boldsymbol{\sigma}$ the same order of accuracy as for the computation of the field of displacement $\mathbf{u}$.

The idea of the method is to calculate for each element the nodal stresses $\hat{\boldsymbol{\sigma}}$ from $\boldsymbol{\sigma}^{\mathbf{k}}$ with Gauss points, these last being calculated on each element by the formula:

$$\boldsymbol{\sigma}^{\mathbf{k}}=\mathbf{D}\,\mathbf{B}^{\mathbf{k}}\,\mathbf{u}=\mathbf{D}\sum_{i=1}^{NNO}\mathbf{B}_{\mathbf{i}}^{\mathbf{k}}\mathbf{U}_{\mathbf{i}}$$

where:

$\mathbf{D}$ is the matrix of elasticity,
$\mathbf{B}^{\mathbf{k}}$ is the matrix connecting the strains to displacements at the Gauss point k ,
$\mathbf{U}_{\mathbf{i}}$ are nodal displacements (NNO nodes)

2 local Method of minimization by least squares

Generally, one wishes to approximate, within the meaning of the least squares, the spatial distribution of the stresses by $\boldsymbol{\sigma}(x)$ a polynomial function:

$$\hat{\boldsymbol{\sigma}}(x)=\sum_{i=0,\ldots,p} \mathbf{a_i} P^i(x)$$

The problem amounts finding the coefficients $\mathbf{a_i}$ which minimize the functional calculus:

$$\boldsymbol{\chi}=\int\int(\boldsymbol{\sigma}-\hat{\boldsymbol{\sigma}})^2 dx\, dy$$

The values of the function $\boldsymbol{\sigma}$ are known here only with Gauss points: $\boldsymbol{\sigma}^{\mathbf{k}}=\boldsymbol{\sigma}(x_k)$

The minimum will be reached if and only if:

$$\frac{\partial \boldsymbol{\chi}}{\partial \mathbf{a_i}}=0 \qquad \forall\, i=0,\ldots,p$$

In the frame of the finite element method in displacement, one chooses the following function of lissage:

$$\hat{\boldsymbol{\sigma}}(x)=\sum_{i=1}^{n} N_i(x)\hat{\boldsymbol{\sigma}}_{\mathbf{i}}$$

where:

N_i is the shape function associated with the node i on the finite element considered,
$\hat{\boldsymbol{\sigma}}_{\mathbf{i}}$ is the value of the sought stress to i the node,
n the number of nodes retained for the lissage.

One must thus solve the system:

$$\frac{\partial \boldsymbol{\chi}}{\partial \hat{\boldsymbol{\sigma}}_{\mathbf{i}}}=0 \qquad \forall\, i=1,\ldots,n \qquad \text{éq 2-1}$$

One can choose between two methods of local lissage: continuous lissage or discrete lissage.

3 Methods of local lissage (ref. [bib2] and [bib3])

3.1 continuous local Lissage

This kind of lissage led to solve the system [éq 2-1] with the functional calculus defined on the finite element running:

$$\chi=\int_e (\boldsymbol{\sigma}-\hat{\boldsymbol{\sigma}})^2=\int_e \left(\boldsymbol{\sigma}-\sum_{i=1}^{n} N_i \hat{\boldsymbol{\sigma}}_{\mathbf{i}}\right)^2$$

Minimization leads to $M^e \hat{\boldsymbol{\sigma}}=F^e$

with:

$$M_{ij}^e=\int_e N_i N_j\, dx\, dy=\sum_{k=1}^{npg} \bar{N}_i(\xi_k) N_j(\xi_k)(det\,\mathbf{J})_k\, \omega_k$$

$$F_i^e=\int_e N_i \boldsymbol{\sigma}\, dx\, dy=\sum_{k=1}^{npg} \bar{N}_i(\xi_k) \boldsymbol{\sigma}_{\mathbf{k}} (det\,\mathbf{J})_k \omega_k$$

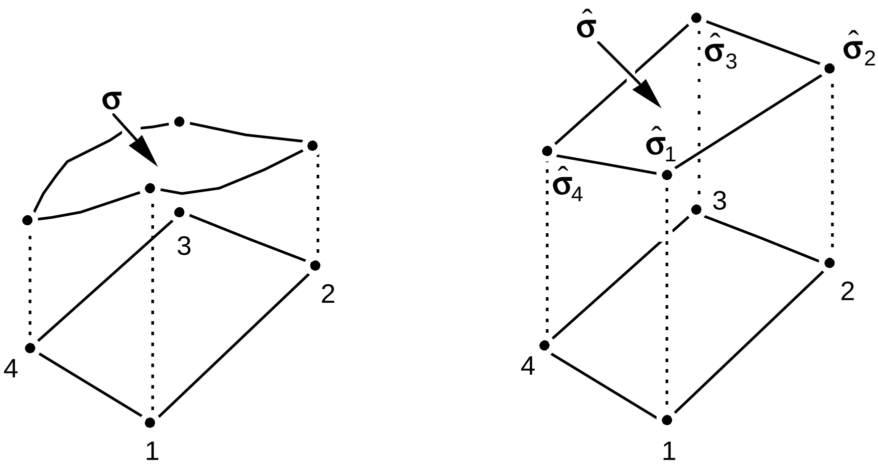
where ξ_k are Gauss points in the element of reference

$(det\,\mathbf{J})_k$ the jacobian of the geometrical transformation between the element of reference and the element running to the point ξ_k .

ω_k : the weight associated with point: ξ_k

ξ_k the stress with point: ξ_k

$\bar{N}_i(\xi_k)$ the value of the shape function in the element of reference to the point ξ_k



calcul direct des contraintes contraintes lissées

Note:

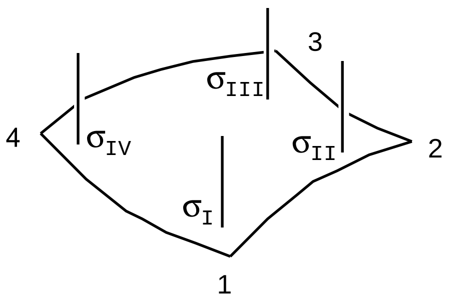
> *If spaces of interpolation of $\boldsymbol{\sigma}$ and of $\hat{\boldsymbol{\sigma}}$ are the same ones, one A. $\boldsymbol{\sigma}=\hat{\boldsymbol{\sigma}}$ In practice, one retains for space of $\hat{\boldsymbol{\sigma}}$ a space smaller than that where is defined $\boldsymbol{\sigma}$ by the finite element.*
> *One sees the restrain between the approximation with Gauss points of $\boldsymbol{\sigma}$ where $\boldsymbol{\sigma}$ thus converges better and this process of lissage whose justification is on the contrary continuous.*
> *The way in which $\boldsymbol{\sigma}$ is calculated with Gauss points does not intervene. Generalization with the nonlinear problems is thus obvious, although it cannot concern the same justification.*

This method is however not adopted because it requires a resolution of system linear for each computation of $\hat{\boldsymbol{\sigma}}$.

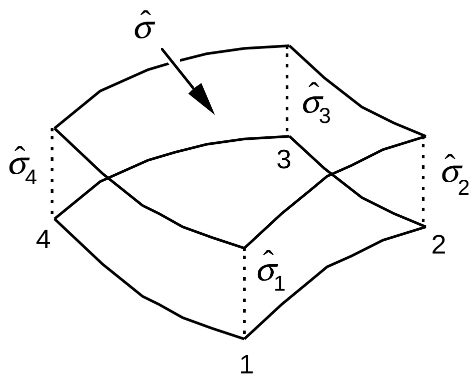
3.2 Discrete local lissage

In this case, the functional calculus $\boldsymbol{\chi}$ is replaced by the summation:

$$\tilde{\boldsymbol{\chi}}=\sum_{k=1}^{npg}(\boldsymbol{\sigma}(\xi_k)-\hat{\boldsymbol{\sigma}}(\xi_k))^2=\sum_{k=1}^{npg}\left(\boldsymbol{\sigma}(\xi_k)-\sum_{i=1}^{n}\hat{\boldsymbol{\sigma}}_{\boldsymbol{i}}N_i(\xi_k)\right)^2$$



contraintes aux points de GAUSS



contraintes lissées

The system to be solved is written there still: $\dfrac{\partial\tilde{\boldsymbol{\chi}}}{\partial\hat{\boldsymbol{\sigma}}_{\mathbf{i}}}=0$ that is to say:

$$\sum_{k=1}^{npg}\sum_{j=1}^{n}N_i(\xi_k)N_j(\xi_k)\hat{\boldsymbol{\sigma}}_{\mathbf{j}}=\sum_{k=1}^{npg}N_i(\xi_k)\boldsymbol{\sigma}(\xi_k)\quad\forall i=1,\dots,n$$

maybe in matric form: $\mathbf{M}\{\hat{\boldsymbol{\sigma}}_{\mathbf{noeud}}\}=\mathbf{P}\{\boldsymbol{\sigma}_{\mathbf{GAUSS}}\}$

The matrixes $\mathbf{M}$ (square $n\times n$) and $\mathbf{P}$ (rectangular $n\times npg$) are then independent of the current element e .

They can thus be calculated once and for all on the element of reference.

Note:

This method is more economic than the preceding one and gives comparable results [bib2],
There still, the way thus $\boldsymbol{\sigma}^{\mathbf{k}}$ is calculated in each Gauss point is indifferent (since the number of Gauss points used for computation of $\boldsymbol{\sigma}$ and that of $\hat{\boldsymbol{\sigma}}$ is the same one). One will be able to thus use this method in nonlinear.

4 Application of the method to computation of the nodal stresses for various elements

the local lissage adopted in *Code_Aster* is the discrete local lissage [§2.2], which makes it possible to avoid the computation of integrals on the element.

On all the elements of continuous medium 2D and 3D, one chose a space of lissage leaning on the shape functions relating to the tops of the element.

The method thus makes it possible to obtain the stresses at the tops. In the case of the elements of order 2, one calculates the nodal stresses mediums by taking the arithmetic mean value of the two tops "framing" the medium node considered.

One gives the transition matrixes hereafter allowing to calculate the nodal stresses tops starting from the stresses with Gauss points. These matrixes can be square or rectangular. Indeed, the transition matrixes $\mathbf{M}^{-1}\mathbf{P}$ are calculated once and for all with the initialization of each type of finite element (in `AFFE_MODELE`). Two types of matrixes exist:

square $\mathbf{M}^{-1}\mathbf{P}$ matrixes, which are to be used when the number of Gauss points used for computation of the stresses to Gauss points $\boldsymbol{\sigma}^{\mathbf{k}}$ is identical to the number of nodes tops,
of the rectangular $\mathbf{M}^{-1}\mathbf{P}$ matrixes, which are to be used when the number of Gauss points of $\boldsymbol{\sigma}^{\mathbf{k}}$ is different (in general higher) with the number of nodes tops.

4.1 Square transition matrixes

These matrixes are used in the elements for which the number of Gauss points of the computation of `SIEF_ELGA/SIGM_ELGA` is equal to the number of tops. The option calculates in first the stresses in a number of Gauss points equal to the number of tops. Then the matrixes $\mathbf{M}^{-1}\mathbf{P}$ (given afterwards) are used to compute: the nodal stresses. They is the elements:

in 2D: QUAD4, TRIA6, under-integrated QUAD8,
in 3D: TETRA4, PENTA6, HEXA8, PYRAM5 and HEXA20 under-integrated.

4.1.1 Square transition matrixes for the elements 2D

4.1.1.1 Triangles

$$\mathbf{M}^{-1}\mathbf{P}=\frac{1}{3}\begin{bmatrix}5 & -1 & -1\\ -1 & 5 & -1\\ -1 & -1 & 5\end{bmatrix}$$

4.1.1.2 Quadrangles

$$\mathbf{M}^{-1}\mathbf{P}=\begin{bmatrix} 1+\frac{\sqrt{3}}{2} & -\frac{1}{2} & 1-\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1+\frac{\sqrt{3}}{2} & -\frac{1}{2} & 1-\frac{\sqrt{3}}{2} \\ 1-\frac{\sqrt{3}}{2} & -\frac{1}{2} & 1+\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1-\frac{\sqrt{3}}{2} & -\frac{1}{2} & 1+\frac{\sqrt{3}}{2} \end{bmatrix}$$

4.1.2 square Transition matrixes for the Pentahedral elements

4.1.2.1 3D

$$\mathbf{M}^{-1}\mathbf{P}=\frac{1}{a-b}\begin{bmatrix} a & a & a-1 & a \\ a & a-1 & a & a \\ a-1 & a & a & a \\ a & a & a & a-1 \end{bmatrix}$$

$$\text{avec}\quad a=\frac{5-\sqrt{5}}{20}\quad b=\frac{5+3\sqrt{5}}{20}$$

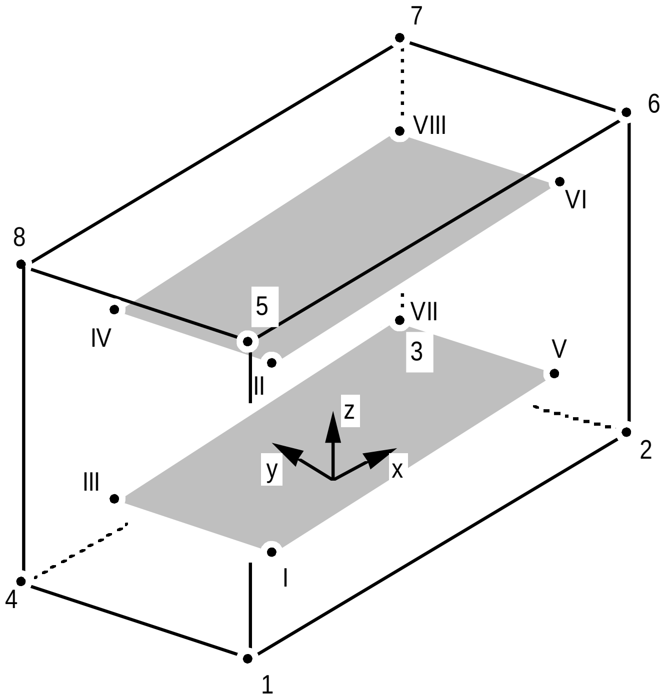
4.1.2.2 Tetrahedrons

$$\mathbf{M}^{-1}\mathbf{P}=\begin{bmatrix} \alpha & -\alpha & \alpha & 1-\alpha & \alpha-1 & 1-\alpha \\ \alpha & \alpha & -\alpha & 1-\alpha & 1-\alpha & \alpha-1 \\ -\alpha & \alpha & \alpha & \alpha-1 & 1-\alpha & 1-\alpha \\ 1-\alpha & \alpha-1 & 1-\alpha & \alpha & -\alpha & \alpha \\ 1-\alpha & 1-\alpha & \alpha-1 & \alpha & \alpha & -\alpha \\ \alpha-1 & 1-\alpha & 1-\alpha & -\alpha & \alpha & \alpha \end{bmatrix}$$

$$\alpha=\frac{\sqrt{3}+1}{2}$$

4.1.2.3 Hexahedrons

$$\mathbf{M}^{-1}\mathbf{P}=\begin{bmatrix} a & b & b & c & b & c & c & d \\ b & c & c & d & a & b & b & c \\ c & d & b & c & b & c & a & b \\ b & c & a & b & c & d & b & c \\ b & a & c & b & c & b & d & c \\ c & b & d & c & b & a & c & b \\ d & c & c & b & c & b & b & a \\ c & b & b & a & d & c & c & b \end{bmatrix}\quad a=\frac{5+3\sqrt{3}}{4}\quad b=\frac{1+\sqrt{3}}{4}\quad c=\frac{\sqrt{3}-1}{4}\quad d=\frac{5-3\sqrt{3}}{4}$$



Appears 4.1.2.3 - has: Classification of Gauss points on the hexahedron with 8 nodes

4.2 rectangular $\mathbf{M}^{-1}\mathbf{P}$ Transition matrixes

In nonlinear for some element types (TRIA3, QUAD8 and QUAD9 in 2D, TETRA10, PENTA15 and HEXA20 in 3D), the stresses and the local variables with Gauss points are calculated on a richer family of Gauss points (9 points for the quadrangles, 15 points for the tetrahedrons, 21 points for the pentahedrons, 27 points for the hexahedrons).

The discrete local lissage is then carried out from these fields and transport with the nodes utilizes matrixes different from the preceding ones. They are not square any more, because of dimension (many tops, number of Gauss points). The transition matrixes $\mathbf{M}^{-1}\mathbf{P}$ are not calculated explicitly, in particular $\mathbf{M}$ is reversed by *Code_Aster*.

In the typical case of the triangle with 3 nodes, the fields are supposed to be constant by element (only one Gauss point) and:

$$\mathbf{M}^{-1}\mathbf{P}=\begin{bmatrix}1\\1\\1\end{bmatrix}$$

For example, the computation carried out by option `SIGM_ELNO` is then the following:

If the stresses were calculated on a family having a number of Gauss points higher than many tops (for announced elements Ci - above). $\mathbf{M}^{-1}\mathbf{P}$ is then rectangular, and

$$\hat{\boldsymbol{\sigma}}_{\mathbf{i}}=\sum_{i=1}^{\text{nb sommets}}\;\sum_{k=1}^{\text{nb pts Gauss}}\left(\mathbf{M}^{-1}\mathbf{P}\right)_{ik}\boldsymbol{\sigma}^{\mathbf{k}}$$

If not, if the number of Gauss points is equal to the number of tops, $\mathbf{M}^{-1}\mathbf{P}$ is then square. One calculates $\hat{\boldsymbol{\sigma}}_{\mathbf{i}}=\left(\mathbf{M}^{-1}\mathbf{P}\right)_{ik}\boldsymbol{\sigma}^{\mathbf{k}}$ [§4.1].

5 Other computation options using the same method

the method described previously is used in *Code_Aster* to compute: the strains, the local variables and the flux with the nodes.

The produced fields are `cham_elem` with the nodes.

6 Other methods of lissage of stresses

There exist two other methods of lissage, relating only to the stresses, used by the estimators of Zhu-Zienkiewicz version 1 and 2 [R4.10.01 §3].

The stress fields with the produced nodes are then `cham_no`.

The corresponding computation options are accessible by the command `CALC_ERREUR` [U4.81.06].

7 Bibliography

1) BARLOW J. - Optimal stress hirings in finite element models - International Newspaper for Numerical Methods in Engineering Vol.10 p 243 - 251 (1976).

2) HINTON E., CAMPBELL JJ. - Total Room and smoothing of discontinuous finite element functions using has least public gardens method - International Newspaper for Numerical Methods in Engineering Vol.8 p 461 - 480 (1974).

3) HINTON E., SCOTT F.C., RICKETTS R.E. - Room least public gardens stress smoothing for parabolic isoparametric elements - Int. J. for Num. Meth. in Eng. Flight 9 p 235 - 256 (1975)

8 Description of the versions of the document

Version Aster	Author (S) Organization (S)	Description of the modifications
10.2	*X. DESROCHES (EDF/IMA/T62)*	Small corrections