Finite elements treating the quasi-incompressibility

Summarized:

In certain situations, the structural mechanics behavior of the material imposes that voluminal thermal expansion remains null, in other words that the strain is done with constant volume: isotropic elasticity with Poisson's ratio equal to 0.5 , perfect yieldings in limit analysis…

One proposes here to treat this condition of "incompressibility" or "quasi-incompressibility" by means of a valid formulation as well in the compressible case as in the quasi-incompressible case. For that, one uses a variational formulation at 3 fields where the unknowns are displacement, voluminal strain and the associated multiplier of Lagrange (which would correspond to the pressure in the incompressible case). Two versions of this formulation are proposed: one for the small strains, the other valid one in the presence of large deformations. In the situation of a bi-univocal relation between the pressure and swelling, case of the plasticity of Von Mises, it is possible to come to eliminate the unknown from swelling. There is then a formulation at two fields displacement/pressure.

After some recalls on the difficulties which raise the resolution of the incompressible problems, one describes the mixed finite elements established (in 3D and 2D, plane and axisymmetric into small and large deformations), and one also presents the broad outline of integration in *Code_Aster* (modelizations INCO, INCO_UP, INCO_OSGS, INCO_GD, INCO_LOG and INCO_LUP).

This modelization is necessary to practice the limit analyzes and for modelling elastic behaviors for Poisson's ratios close to 0.5 . It can also be useful in the case of modelizations generating of strong plastic strains and for which the traditional modelizations can be insufficient and generate oscillations of stresses.

Contents

1 Difficulties related to the processing of the incompressibility

In certain situations, the structural mechanics behavior of the material imposes that the strain is done with constant volume. The materials having this property of NON-dilatancy are often qualified "incompressible" materials. We will see that these problems pose two types of difficulties. The first difficulty is related to the writing of the condition of incompressibility, second is related to the numerical problems which this stress generates. These difficulties are found when the material is quasi-incompressible.

One reasons here in small disturbances but the problem remains the same one in the frame of the finished transformations.

1.1 Incompressible" and "quasi-incompressible" behaviors the "

In the frame of the mechanics of the continuums, strain of an isochoric type is characterized by the fact that the gradient of the transformation $\boldsymbol{F}$ is such that $J=\det(\boldsymbol{F})=1$. If one places oneself in the frame of the small disturbances, the preceding condition is reduced to:

$$\mathrm{tr}(\boldsymbol{\varepsilon})=\mathrm{div}\,u=0$$

The tensor $\boldsymbol{\varepsilon}$ is thus only deviatoric: $\boldsymbol{\varepsilon}=\boldsymbol{\varepsilon}^{\boldsymbol{D}}$.

It results from it that in the case of isotropic materials, the invariant $\mathrm{tr}(\boldsymbol{\varepsilon})$ (or $\det(\boldsymbol{F})$) does not intervene in the statement of the density of free energy φ ; thus in the case of incompressible elasticity in HP, one has simply:

$$\varphi(\boldsymbol{\varepsilon})=\mu\,\boldsymbol{\varepsilon}^{\boldsymbol{D}}:\boldsymbol{\varepsilon}^{\boldsymbol{D}}$$

This density makes it possible to express only the deviatoric part of the tensor of the stresses:

$$\boldsymbol{\sigma}^{\boldsymbol{D}}=2\mu\,\boldsymbol{\varepsilon}^{\boldsymbol{D}}$$

In fact, the stress is defined in a constant close p which is opposite average pressure:

$$\boldsymbol{\sigma}=2\mu\boldsymbol{\varepsilon}^{\boldsymbol{D}}+p\boldsymbol{I} \quad (1.1\text{-}1)$$

Note::

Incompressible isotropic elasticity is of course a borderline case of isotropic elasticity with a Poisson's ratio $\nu=\frac{E}{2\nu}-1$ *tending towards* 0.5 *.*

There is not that the elastic materials whose Poisson's ratio is equal or slightly lower than 0.5 *which utilizes the condition of incompressibility. Thus, it intervenes also in the case of plastic rigid material* $\left(\frac{\partial\Phi}{\partial\,\mathrm{tr}\,\boldsymbol{\sigma}}=0\right)$. *Indeed, one has in this case:*

$$\dot{\boldsymbol{\varepsilon}}=\dot{\lambda}\frac{\partial\Phi}{\partial\boldsymbol{\sigma}}\;;\;\dot{\lambda}\geq 0\;;\;\Phi\leq 0\;;\;\dot{\lambda}\,\Phi=0$$

What leads to the condition of incompressibility $\mathrm{tr}(\dot{\boldsymbol{\varepsilon}})=0$ *.*

In addition, in the case of elastoplasticity, when plastic strains become largely higher than the elastic strain, one finds oneself in an almost incompressible case with $\mathrm{tr}(\boldsymbol{\varepsilon})\simeq 0$ *.*

Lastly, the materials checking a behavior model of the type NORTON-HOFF (model used for computations of limit analysis [R7.07.01]) show also the characteristic of incompressibility:

$$\boldsymbol{\varepsilon}(\boldsymbol{v})=\alpha(\sigma_{\mathrm{eq}})^{n-1}\boldsymbol{\sigma}^{\boldsymbol{D}}\,\text{avec}\;\; n\geq 1\;\;\text{et}\;\;\alpha>0$$

where $\sigma_{\mathrm{eq}}=\sqrt{\frac{3}{2}\boldsymbol{\sigma}^{\boldsymbol{D}}:\boldsymbol{\sigma}^{\boldsymbol{D}}}$ *is the equivalent stress of Von Mises.*

1.2 Some possible numerical solutions

If one wants to treat the condition of incompressibility exactly, we saw it, the stress is not completely determined from the strain (cf [éq 1.1-1]). It is thus necessary to use a mixed formulation, i.e. to introduce (at least) another unknown of the problem which will make it possible to determine the tensor of the stresses completely. Several alternatives are possible, simplest consisting in imposing the condition of incompressibility using a multiplier of Lagrange which is then the pressure p.

Note:

> *If one chooses a procedure of penalization, one is reduced to the quasi-incompressible case and thus to the difficulties evoked below.*

One can also, in particular in the case of linear elasticity, to choose to make the material slightly compressible. In this way, the stress is entirely defined starting from displacement and the use of a mixed formulation is not essential any more. On the other hand, the resolution of these problems with the conventional finite elements in displacement, raises numerical difficulties. Indeed, the kinematical stress that a strain with constant volume represents is very strong, even too strong if the degrees of freedom of the element are not important enough. Thus, the triangle with 3 nodes can present phenomena of blocking, i.e. the "mesh" cannot become deformed. In a less extreme way, most classical elements, in particular linear, behaves in an abnormally rigid way. New elements must thus be used in order to "slacken" the system. These elements can lean on various types of formulation:

- only in mixed
- displacement: displacements/forced, displacements/pressures, strains/forced, voluminal displacements/pressures/thermal expansions,…

In all the cases, if one does not take there keeps, one can have numerical difficulties. Several tracks are used to facilitate the strain of the elements:

- to use under-integration makes it possible to improve the results but it presents a disadvantage: it can lead to the appearance of parasitic modes or hourglass. To solve this problem, one can is to enrich the stiffness matrix thanks to matrixes by stabilization which come to neutralize the hourglass modes, that is to say to use methods of projection which consist in projecting in a smaller space the condition of incompressibility in order to eliminate the phenomena of blocking. Most known is the method B-Bar [bib1],
- enhance the element using additional degrees of freedom: one speaks then about methods with increased strains, modes incompatible,… [bib2]

1.3 Option selected and frames of application

We chose here to choose a formulation which covers theincompressible one as well (until the incompressible one) that the compressible one. For that, the term $\mathrm{tr}(\boldsymbol{\varepsilon})$ is treated like an independent variable. With the multiplier of Lagrange associated, that led to a formulation with 3 or 2 fields. A version large deformations was also developed on the same principle. In this case, the variable independent related to the condition of incompressibility is not any more $\mathrm{tr}(\boldsymbol{\varepsilon})$ but $J=\det(\boldsymbol{F})$.

The advantage of the formulation at 3 fields compared to the version at 2 fields east qu `it makes it possible to use in a transparent way all the elastoplastic constitutive laws available in Aster (not need to separate the deviatoric part and the spherical part of the tensor of the stresses). It is thus not restricted with the elasticity or the elastoplasticity of Von Mises. On the other hand, it introduces a large number of additional degrees of freedom. Some is the formulation selected, one will not be able to treat the case where the Poisson's ratio is strictly equal to 0.5, because one uses for the computation of the elastic stress the term $\frac{E\nu}{(1+\nu)(1-2\nu)}\mathrm{tr}(\boldsymbol{\varepsilon})$, whose denominator is null when $\nu=0.5$.

Consequently, formulations `INCO` **must be used** :

- to deal with the problems of limit analysis for which one supposes that flow is done with constant volume [R7.07.01],
- for dealing with elastic problems of which the Poisson's ratio is higher than 0.45.

They can also be used:

- to deal with the problems where plastic strains are important, which generates oscillations on the level of the stresses (example: in the case of computations on notched samples). Of course, this formulation being more expensive than the formulation in classical displacement, it is to be held for the case which poses problem and where one is interested in the values of the stresses (one can initially try to use under-integrated quadratic elements which improve already the solution).

2 Mixed variational formulation of the problem

2.1 Formulation in the frame of the small strains

Is a solid Ω subjected to:

- a field of displacement imposed $\boldsymbol{u}=\boldsymbol{u}_0$ on Γ_u
- a stress field imposed $\boldsymbol{t}=\boldsymbol{\sigma}.\boldsymbol{n}=\boldsymbol{t}_0$ on Γ_t
- a voluminal field of force $\boldsymbol{f}$ on Ω

In the classical case of the finite elements in displacement (modelization `3D` or `D_PLAN` or `AXIS` in *Code_Aster*), when the problem derives from an energy, the solved problem is the following:

to find $\boldsymbol{u}\in V$ with $\boldsymbol{\sigma}$ checking the behavior model, which minimizes potential energy:

$$\Pi(u)=\frac{1}{2}\int_{\Omega}\boldsymbol{\sigma}:\boldsymbol{\varepsilon}\,d\Omega-\int_{\Omega}\boldsymbol{f}.\boldsymbol{u}\,d\Omega-\int_{\Gamma_t}\boldsymbol{t}.\boldsymbol{u}\,d\Gamma$$

As we explained to [§1], this formulation is not appropriate when one seeks to approach the incompressible solution, i.e. of the condition $\mathrm{div}(\boldsymbol{u})=0$ or $\mathrm{tr}(\boldsymbol{\varepsilon})=0$. To circumvent this difficulty, a solution is to separately treat the spherical part of the tensor of the strains (the part which poses numerical problems) and its deviatoric part. One will thus have:

$$\boldsymbol{\varepsilon}(\boldsymbol{u},g)=\boldsymbol{\varepsilon}^{\boldsymbol{D}}(\boldsymbol{u})+\frac{g}{3}\boldsymbol{I} \text{ where } \boldsymbol{\varepsilon}^{\boldsymbol{D}}(\boldsymbol{u})=\boldsymbol{\varepsilon}(\boldsymbol{u})-\frac{1}{3}tr(\boldsymbol{\varepsilon}(\boldsymbol{u}))\boldsymbol{I} \text{ and } g=tr(\boldsymbol{\varepsilon}(\boldsymbol{u})) \tag{2.1-1}$$

the preceding problem is thus brought back to the resolution of a problem to 2 variables, $\boldsymbol{u}$ and g, under the stress $g=tr(\boldsymbol{\varepsilon}(\boldsymbol{u}))$. It can be brought back to the resolution of an unconstrained problem by introducing a multiplier of Lagrange p ; he is written:

to find $\boldsymbol{u}\in V$, p and g solutions of the problem of POINT-saddles for the Lagrangian one:

$$\mathscr{L}(\boldsymbol{u},p,g)=\int_{\Omega}\left[\boldsymbol{\sigma}:\left(\boldsymbol{\varepsilon}^{\boldsymbol{D}}(\boldsymbol{u})+\frac{g}{3}\boldsymbol{I}\right)+p(\mathrm{div}(\boldsymbol{u})-g)\right]d\Omega-\int_{\Omega}\boldsymbol{f}.\boldsymbol{u}\,d\Omega-\int_{\Gamma_t}\boldsymbol{t}.\boldsymbol{u}\,d\Gamma \tag{2.1-2}$$

This problem can be solved, by writing the conditions of optimality:

$$\begin{cases}\dfrac{\partial\mathscr{L}}{\partial\boldsymbol{u}}=\displaystyle\int_{\Omega}(\boldsymbol{\sigma}^{\boldsymbol{D}}+p\boldsymbol{I}):\delta\boldsymbol{\varepsilon}\,d\Omega-\int_{\Omega}\boldsymbol{f}.\delta\boldsymbol{u}\,d\Omega-\int_{\Gamma_t}\boldsymbol{t}.\delta\boldsymbol{u}\,d\Gamma=0\\ \dfrac{\partial\mathscr{L}}{\partial p}=\displaystyle\int_{\Omega}(\mathrm{div}(\boldsymbol{u})-g)\delta p\,d\Omega=0\\ \dfrac{\partial\mathscr{L}}{\partial g}=\displaystyle\int_{\Omega}\left(\frac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})-p\right)\delta g\,d\Omega=0\end{cases} \tag{2.1-3}$$

Note:

the first equation corresponds to the balance equation, the second equation translates the kinematic relation binding g *to* $\boldsymbol{u}$ *, the third equation gives the statement of the multiplier* *of Lagrange p, when the problem does not derive from an energy, one can directly use the system of equations [éq 2.1-3].*

If there exists a bi-univocal relation between the pressure and swelling such as for example for an elastoplastic material with a plasticity criterion of the type von Mises, it is possible to clarify swelling and thus to remove the third equation of the system [éq 2.1-3]. One then obtains the system of two equations to two unknowns which follows:

$$\begin{cases} \int_{\Omega} (\boldsymbol{\sigma}^{\boldsymbol{D}} + p\,\boldsymbol{I}) : \delta\boldsymbol{\varepsilon}\, d\Omega - \int_{\Omega} \boldsymbol{f} . \delta\boldsymbol{u}\, d\Omega - \int_{\Gamma_t} \boldsymbol{t} . \delta\boldsymbol{u}\, d\Gamma = 0 \\ \int_{\Omega} (\mathrm{div}(\boldsymbol{u}) - \frac{p}{\kappa})\, \delta p\, d\Omega = 0 \end{cases} \quad (2.1\text{-}4)$$

Where κ is the modulus of compressibility.

2.2 Formulation in large deformations

As for the small strains, it is possible to propose a variational formulation valid for the large deformations. The principle is identical, but one is based in this case on the decomposition of the tensor gradient of the transformation $\boldsymbol{F} = \boldsymbol{I} + \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}}$ suggested by Flory [bib3]:

$$\boldsymbol{F} = \boldsymbol{F}^s \bar{\boldsymbol{F}} \text{ with } \boldsymbol{F}^s = J^{1/3} \boldsymbol{I} \text{ et } \bar{\boldsymbol{F}} = J^{-1/3} \boldsymbol{F} \text{ et } J = \det(\boldsymbol{F})$$

the idea there still, is to enrich the kinematics by the means of a variable of swelling g, a priori independent of displacements, and slightly related to the variation of volume by a weak relation:

$$B(J) \underset{faible}{\approx} B \circ A(g) = B(A(g))$$

Several relations were tested:

$$\begin{cases} J = 1 + g \\ J^2 = 1 + g \\ \ln(J) = g \\ J = \exp(g) \end{cases}$$

For certain simulations, small differences were observed. For the elements `INCO_GD`, it is finally the linear relation which was established in the code: therefore, in version 9, $B(J) = J$ and $A(g) = 1 + g$. For the elements `INCO_LOG`, it is the relation in logarithm which was retained.

Nevertheless, this choice not being inevitably final, one proposes to write the problem in the general case. An enriched deformation gradient is thus introduced:

$$\tilde{\boldsymbol{F}} = \left(\frac{A(g)}{J}\right)^{\frac{1}{3}} \boldsymbol{F} \quad (2.2\text{-}1)$$

the weak formulation of the problem leans on the search of the point saddles the Lagrangian one $\mathscr{L}$, in which the multiplier of Lagrange p and a third field g, independent of both others, ensuring in a weak way that the relation enters J and g is checked:

$$\mathscr{L}(\boldsymbol{u}, g, p) = \int_{\Omega_0} \psi(\tilde{\boldsymbol{F}})\, d\Omega_0 - W_{ext}(\boldsymbol{u}) + \int_{\Omega_0} p\left[B(J) - B \circ A(g)\right] d\Omega_0 \quad (2.2\text{-}2)$$

where W_{ext} represents the potential of the external forces and $\psi(\tilde{\boldsymbol{F}})$ strain energy.
This problem can be solved as in small strains by writing the conditions of optimality. The variation of Lagrangian is written:

$$\delta\mathcal{L}=\int_{\Omega_0}\left[\boldsymbol{P}:\delta\tilde{\boldsymbol{F}}+p\left(\frac{\partial B(J)}{\partial J}J\frac{\delta J}{J}-\frac{\partial B\circ A(g)}{\partial g}\delta g\right)+\delta p\left(B(J)-B\circ A(g)\right)\right]d\Omega_0-\delta W_{ext}(\boldsymbol{u}) \quad (2.2\text{-}3)$$

with $\boldsymbol{P}$ the first tensor of the Piola-Kirchhoff stresses:

By injecting the variation of the transformation enriched and the statement by the stress of Kirchhoff $\boldsymbol{\tau}=\boldsymbol{P}\,\tilde{\boldsymbol{F}}^{T}$ [1], one obtains the following form for the variation of the Lagrangian one:

$$\begin{aligned}\delta\mathcal{L} &= \int_{\Omega_0}\left(\boldsymbol{\tau}^d+p\frac{\partial B(J)}{\partial J}J\boldsymbol{I}\right):\delta L\,d\Omega_0\\ &+ \int_{\Omega_0}\left(\frac{\mathrm{tr}(\boldsymbol{\tau})}{3}\frac{\partial A(g)/\partial g}{A(g)}-p\frac{\partial B\circ A(g)}{\partial g}\right)\delta g\,d\Omega_0+\int_{\Omega_0}\left(B(J)-B\circ A(g)\right)\delta p\,d\Omega_0\\ &- \delta W_{ext}(\boldsymbol{u})\end{aligned} \quad (2.2\text{-}4)$$

where one introduced the eulerian gradient of displacement ($\boldsymbol{x}$ the vector position at the end of the increment represents):

$$\delta L=\frac{\partial\delta\boldsymbol{u}}{\partial\boldsymbol{x}}=\delta\boldsymbol{F}.\boldsymbol{F}^{-1}$$

In short, the system to be solved is the following:

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\boldsymbol{u}}&=\int_{\Omega_0}\delta L:\left(\boldsymbol{\tau}^d+p\frac{\partial B(J)}{\partial J}J\boldsymbol{I}\right)d\Omega_0-\delta W_{ext}=0\\ \frac{\partial\mathcal{L}}{\partial g}&=\int_{\Omega_0}\delta g\left(\frac{\mathrm{tr}(\tau)}{3}\frac{\partial A(g)/\partial g}{A(g)}-p\frac{\partial B\circ A(g)}{\partial g}\right)d\Omega_0=0\\ \frac{\partial\mathcal{L}}{\partial p}&=\int_{\Omega_0}\delta p\left(B(J)-B\circ A(g)\right)d\Omega_0=0\end{aligned} \quad (2.2\text{-}5)$$

Note: *:*

The stress of Kirchhoff resulting from the constitutive law, is thus written $\boldsymbol{\tau}=\boldsymbol{\tau}^{D}+p\frac{\partial B(J)}{\partial J}J\boldsymbol{I}$

1 acts there of a choice and one could just as easily have transported $\boldsymbol{P}$ by the compatible strain $\boldsymbol{F}$ instead of the enriched strain $\tilde{\boldsymbol{F}}$. But the latter has the advantage of preserving the symmetry of the formulation in the elastic case. Moreover, it lends itself better to the architecture of a computer code in which the integration of the constitutive laws is well differentiated from computation from the terms specific to the finite elements (the routines of the behavior do not need to know the existence of two strain measurements) but only the enriched strain).

With regard to obtaining the tangent matrix, she of course asks a little more computation than in small strains, and has the characteristic not to be symmetric in the general case. She is in the code in the following form:

$$\boldsymbol{K}=\begin{bmatrix} K_{UU} & K_{UG} & K_{UP} \\ K_{GU} & K_{GG} & K_{GP} \\ K_{PU} & K_{PG} & K_{PP} \end{bmatrix}$$

Computations are not here detailed. The reader will be able to refer to the reading of [8].

Note:

This formulation makes it possible to regularize with lower costs, the models of ductile damage where the variable of damage is directly connected to the variation of volume.
Indeed, to control the localization of the damage and the strain, the idea is to penalize the strong gradients of damage. As in this formulation at 3 fields, local swelling is treated like a nodal variable, its gradient is easily accessible numerically (subject to an at least linear interpolation).

In the spirit of the formulations with second displacement gradient ([9], [10]), it is enriched by a quadratic term in gradient by swelling. The variation of Lagrangian is written then:

$$\begin{aligned}\delta\mathcal{L} &= \int_{\Omega_0}\left(\boldsymbol{\tau}^d+p\frac{\partial B(J)}{\partial J}J\,\boldsymbol{I}\right):\delta L+\left(\frac{\mathrm{tr}(\boldsymbol{\tau})}{3}\frac{\partial A(g)/\partial g}{A(g)}-p\frac{\partial B\circ A(g)}{\partial g}\right)\delta g\,d\Omega_0 \\ &+ \int_{\Omega_0}\left[\left(B(J)-B\circ A(g)\right)\delta p+c\nabla g\cdot\nabla\delta g\right]d\Omega_0-\delta W_{ext}(\boldsymbol{u})\end{aligned} \quad (2.2\text{-}6)$$

c is a parameter to be determined and homogeneous with a force. This parameter introduces to some extent an internal length of coupling between the points materials. The term added here is isotropic: it is considered that the length interns to introduce is identical in all the directions. For the application to the ductile damage of steels, this assumption seems completely admissible. This formulation is usable for the model of Rousselier, [R5.03.07], with the help of the definition of key word `C_CARA` under operand `NON_LOCAL` of `DEFI_MATERIAU` (see test Code_Aster ssnp122a)

obtaining the formulation at two fields in large deformations follows the same principle as With regard to obtaining the tangent matrix, she asks of course a little more computation than in small strains, and has the characteristic not to be symmetric in the general case. It is in the code in the following form:

3 Discretization by mixed finite elements

3.1 Choice of the discretization

When a mixed formulation is used, it is necessary to discretize at the same time the space of displacements, the multiplier of Lagrange p and “swelling” g . The experience gained on the mixed elements, in particular 2 fields for the incompressible elements, makes it possible to know that the discretization of these fields cannot be unspecified, under penalty of obtaining phenomena of oscillations (in particular on the level of the pressures) or phenomena of blocking (elements not being able to become deformed or too rigid). Thus it is necessary to have a sufficiently significant number of Gauss points of pressure to check the condition of incompressibility almost everywhere and a number of Gauss points of pressure sufficiently low to have more degrees of freedom to calculate than stresses to be checked. One of the requirements to get satisfactory results is the checking by the finite element considered of condition LBB (LADYJENSKAIA, BREZZI, BABUSKA). One can find in [bib5] and [bib6] of the examples of elements satisfying condition LBB.

Here the problem is a little different when the formulation 3 fields east selected.

In the actual position, the discretizations used are not the same ones in the version HP and the version large deformations.

3.1.1 Small strains

For the small strains, we took as a starting point the classical uses of the mixed formulations (e.g. [bib7]), by means of element of type $P2/P1/P1$ for the formulation at 3 fields. In other words, displacement is quadratic, the pressure and swelling is all the two linear ones. The finite elements used for the formulation at 3 fields are thus the following:

in 2D:	u	triangle with 6 nodes	quadrilateral with 8 nodes	
	p, g	triangle with 3 nodes	quadrilateral with 4 nodes	
in 3D:	u	tetrahedron with 10 nodes	hexahedron with 20 nodes	pentahedron with 15 nodes
	p, g	tetrahedron with 4 nodes	hexahedron with 8 nodes	pentahedron with 6 nodes

For each type of element, one uses only one family of Gauss points:

- 3 points for triangles
- 4 points for quadrilaterals
- 4 points for tetrahedrons
- 8 points for hexahedrons
- 21 points for the pentahedrons

For the formulation at two fields, an element of the type $P2/P1$ was introduced. Displacement thus has a quadratic interpolation while the pressure is interpolated linearly. Into the case of the use of a discretization in triangles or linear tetrahedrons, two methods of stabilization were introduced. The first corresponds to the stabilized finite element $P1+/P1$. $+$ Corresponds to the introduction of an additional degree of freedom to the center of the element into the interpolation of displacements. This additional degree is commonly called “bubble”. This method of stabilization functions only on elements simplexes (triangle in 2D and tetrahedron in 3D). It has the advantage of using very few degrees of freedom. The second method of stabilization corresponds to the method Orthogonal Sub-Grid Scale (OSGS) [bib11]. The advantage of this method is to function for all topologies of elements. Its principal disadvantage is to introduce a third unknown (and thus additional degrees of freedom) corresponding to the field of pressure project π on orthogonal space at the fields of displacement.

The finite elements used for the formulation at 2 fields are thus the following:

Interpolation		P1+/P1	P1/P1 OSGS	P2/P1	P1/P1 OSGS	P2/P1	P1/P1 OSGS	P2/P1
in 2D:	u	triangle with 3 nodes	triangle with 3 nodes	triangle with 6 nodes	quadrilateral with 4 nodes	quadrilateral with 8 nodes		
	p	triangle with 3 nodes	triangle with 3 nodes	triangle with 3 nodes	quadrilateral with 4 nodes	quadrilateral with 4 nodes		
	π		triangle with 3 nodes		quadrilateral with 4 nodes			

in 3D:	u	tetrahedron with 4 nodes + bubble	tetrahedron with 4 nodes	tetrahedron with 10 cubic	nodes with 8 cubic	nodes with 20 nodes	pentahedron with 6 nodes	pentahedron with 15 nodes
	p	tetrahedron with 4 nodes	tetrahedron with 4 nodes	tetrahedron with 4 cubic	nodes with 8 cubic	nodes with 8 nodes	pentahedron with 6 nodes	pentahedron with 6 nodes
	π		tetrahedron with 4 cubic		nodes with 8 nodes		pentahedron with 6 nodes	

the families of Gauss points used are the same ones as those of the formulation at 3 fields. It will be noted that for the elements $P1+/P1$, one uses one Gauss point for integration.

3.1.2 Large deformations

From version 11, the choice of the interpolations in large deformations is identical to that of the small strains. The elements are of the type $P2/P1/P1$ for the formulations with 3 fields and $P2/P1$ for the formulation at 2 fields. The finite elements used for the formulation at 3 fields are thus the following:

in 2D:	u	triangle with 6 nodes	quadrilateral with 8 nodes	
	p, g	triangle with 3 nodes	quadrilateral with 4 nodes	
in 3D:	u	tetrahedron with 10 nodes	hexahedron with 20 nodes	pentahedron with 15 nodes
	p, g	tetrahedron with 4 nodes	hexahedron with 8 nodes	pentahedron with 6 nodes

They are the same families of Gauss points as those of the small strains which were used.

3.2 Writing of the discrete problem

One approaches here initially, the writing of the discrete problem in the frame of the formulation at 3 fields. Maybe $\mathbf{u}^e$, p^e and g^e, vectors of the elementary nodal unknowns (respectively displacement, pressure and swelling). If $\mathbf{N}^u$, $\mathbf{N}^p$ and $\mathbf{N}^g$ are the shape functions (respectively interpolations of displacement, pressure and swelling) associated with the finite element considered:

$$\mathbf{u}=\mathbf{N}^u \mathbf{u}^e$$
$$p=\mathbf{N}^p p^e$$
$$g=\mathbf{N}^g g^e$$

3.2.1 Writing in small strains

$\mathbf{B}$ is the classical derivative matrix making it possible to pass from $\mathbf{u}^e$ to $\boldsymbol{\varepsilon}$:

$$\boldsymbol{\varepsilon}=\mathbf{B}\,\mathbf{u}^e$$

In the formulation, one distinguishes e_{dev} and e_{dil} , which leads us to define the operators B_{dev} and B_{dil} such as: $\varepsilon^D=B_{dev}U^e$ and $\frac{\mathrm{tr}\ \varepsilon}{3}=B_{dil}U^e$

forms It discretized equations of the problem at 3 fields [éq 2.1-3] is written:

$$\boldsymbol{F}_u=\int_\Omega \boldsymbol{B}^T(\boldsymbol{\sigma}^D+p\boldsymbol{I})\,d\Omega=\boldsymbol{F}_{ext}$$

$$\boldsymbol{F}_p=\int_\Omega (\boldsymbol{N}^p)^T(\boldsymbol{B}_{dil}\boldsymbol{u}-g)\,d\Omega=0$$

$$\boldsymbol{F}_g=\int_\Omega (\boldsymbol{N}^g)^T\left(\frac{1}{3}\operatorname{tr}(\boldsymbol{\sigma})-p\right)d\Omega=0$$

The tangent matrix of the problem is symmetric and leans on the following terms:

$$\boldsymbol{K}_{uu}=\frac{\partial\boldsymbol{F}_u}{\partial\boldsymbol{u}^e}=\int_\Omega \boldsymbol{B}_{dev}^T\boldsymbol{D}\,\boldsymbol{B}_{dev}\,d\Omega$$

$$\boldsymbol{K}_{up}=\frac{\partial\boldsymbol{F}_u}{\partial p^e}=\int_\Omega \boldsymbol{B}_{dil}^T\boldsymbol{N}^p\,d\Omega$$

$$\boldsymbol{K}_{ug}=\frac{\partial\boldsymbol{F}_u}{\partial g^e}=\frac{1}{3}\int_\Omega \operatorname{tr}(\boldsymbol{B}_{dev}^T\boldsymbol{D})\boldsymbol{N}^g\,d\Omega$$

$$\boldsymbol{K}_{pp}=\frac{\partial\boldsymbol{F}_p}{\partial p^e}=\boldsymbol{0}$$

$$\boldsymbol{K}_{pg}=\frac{\partial\boldsymbol{F}_p}{\partial g^e}=-\int_\Omega (\boldsymbol{N}^p)^T\boldsymbol{N}^g\,d\Omega$$

$$\boldsymbol{K}_{gg}=\frac{\partial\boldsymbol{F}_g}{\partial g^e}=\frac{1}{9}\int_\Omega (\boldsymbol{N}^g)^T\operatorname{tr}(\boldsymbol{D})\boldsymbol{N}^g\,d\Omega$$

With regard to the formulation at 2 fields, she results easily from the preceding one. The forms discretized of the equations give us:

$$\boldsymbol{F}_u=\int_\Omega \boldsymbol{B}^T(\boldsymbol{\sigma}^D+p\boldsymbol{I})\,d\Omega=\boldsymbol{F}_{ext}$$

$$\boldsymbol{F}_p=\int_\Omega (\boldsymbol{N}^p)^T\left(\boldsymbol{B}_{dil}\boldsymbol{u}-\frac{\boldsymbol{N}^p}{\kappa}\right)d\Omega=0$$

The tangent matrix of the problem is symmetric and leans on the following terms:

$$\boldsymbol{K}_{uu}=\frac{\partial\boldsymbol{F}_u}{\partial\boldsymbol{u}^e}=\int_\Omega \boldsymbol{B}_{dev}^T\boldsymbol{D}\,\boldsymbol{B}_{dev}\,d\Omega$$

$$\boldsymbol{K}_{up}=\frac{\partial\boldsymbol{F}_u}{\partial p^e}=\int_\Omega \boldsymbol{B}_{dil}^T\boldsymbol{N}^p\,d\Omega$$

$$\boldsymbol{K}_{pp}=\frac{\partial\boldsymbol{F}_p}{\partial p^e}=-\frac{1}{\kappa}\int_\Omega (\boldsymbol{N}^p)^T\boldsymbol{N}^p\,d\Omega$$

3.2.2 Writing in great transformations

the writing being a little tiresome, the reader will be able to refer to the reading of [8] to have more information.

4 Integration in *Code_Aster* of the finite elements incompressible

4.1 general Presentation of the incompressible element in small strains

the finite elements are integrated in *Code_Aster* in 2D plane strains, 2D axisymmetric and 3D. The 3 modelizations are accessible by means of the following options for `AFFE_MODELE` :

- `"3D_INCO"`, `"3D_INCO_UP"` or `"3D_INCO_OSGS"` for 3D and respectively for the formulation with 3, 2 fields and 2 fields stabilized with method OSGS,
- `"D_PLAN_INCO"`, `"D_PLAN_INCO_UP" or "D_PLAN_INCO_OSGS"` for 2D in plane strains and respectively for the formulation with 3, 2 fields and 2 fields stabilized with method OSGS,
- `"AXIS_INCO"`, `"AXIS_INCO_UP"` or `"AXIS_INCO_OSGS"` for the 2D axisymmetric one and respectively for the formulation with 3, 2 fields and 2 fields stabilized with method OSGS.

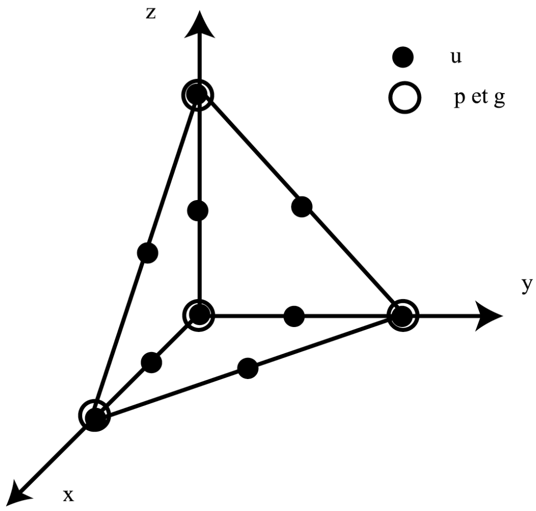
In the catalog of the elements, the incompressible elements can apply to meshes:

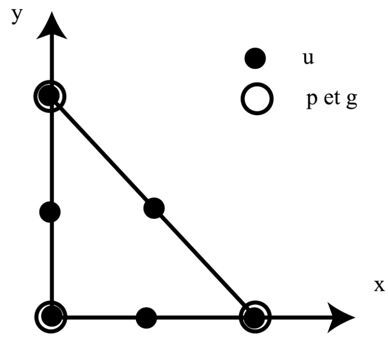
Meshes	Many	Formulation nodes in Many	displacements nodes in pressure or swelling	Many nodes of gradient of pressure project
`TRIA3`	2 fields	3	3	
`TRIA3`	2 fields OSGS	3	3	3
`TRIA6`	2 and 3 fields	6	3	
`QUAD4`	2 fields OSGS	4	4	4
`QUAD8`	2 and 3 fields	8	4	
`HEXA20`	2 and 3 fields	20	8	
`TETRA4`	2 fields	4	4	
`TETRA4`	2 fields OSGS	4	4	4
`TETRA10`	2 and 3 fields	10	4	
`PENTA6`	2 fields OSGS	6	6	6
`PENTA15`	2 and 3 fields	15	6	

In the routines of initializations of the incompressible elements, one defines:

- 1 only family of Gauss points (cf §3.13.1),
- 2 families of shape functions respectively associated with displacements (shape functions with degree 2) and under the terms with pressure and swelling (of degree 1) if one is in formulation 3 fields.

Let us take as example the tetrahedral element with 10 nodes: the degrees of freedom in displacement are carried by all the nodes, on the other hand, only the 4 nodes tops have the degrees of freedom p and g .





Accessible components for the field DEPL are thus

- displacements: DX, DY and DZ in 3D with all the nodes,
- pressure: NEAR for the nodes tops,
- swelling (formulation at 3 fields): GONF for the nodes tops.

4.2 General presentation of the incompressible element in large deformations

the finite elements are integrated in *Code_Aster* in 2D plane strains, 2D axisymmetric and 3D. The 3 modelizations using the formalism of large deformations of SIMO_MIEHE and leaning on a formulation at 3 fields are accessible by means of the following options for AFFE_MODELE :

- "3D_INCO_GD" for 3D,
- "D_PLAN_INCO_GD" for 2D in plane strains,
- "AXIS_INCO_GD" for the 2D axisymmetric one.

The 3 modelizations using the formalism of large deformations of GDEF_LOG and leaning on a formulation at 3 fields are accessible by means of the following options for AFFE_MODELE :

- "3D_INCO_LOG" for 3D,
- "D_PLAN_INCO_LOG" for 2D in plane strains,
- "AXIS_INCO_LOG" for the 2D axisymmetric one.

The 3 modelizations using the formalism of large deformations of GDEF_LOG and leaning on a formulation at 2 fields are accessible by means of the following options for AFFE_MODELE :

- "3D_INCO_LUP" for 3D,
- "D_PLAN_INCO_LUP" for 2D in plane strains,
- "AXIS_INCO_LUP" for the 2D axisymmetric one.

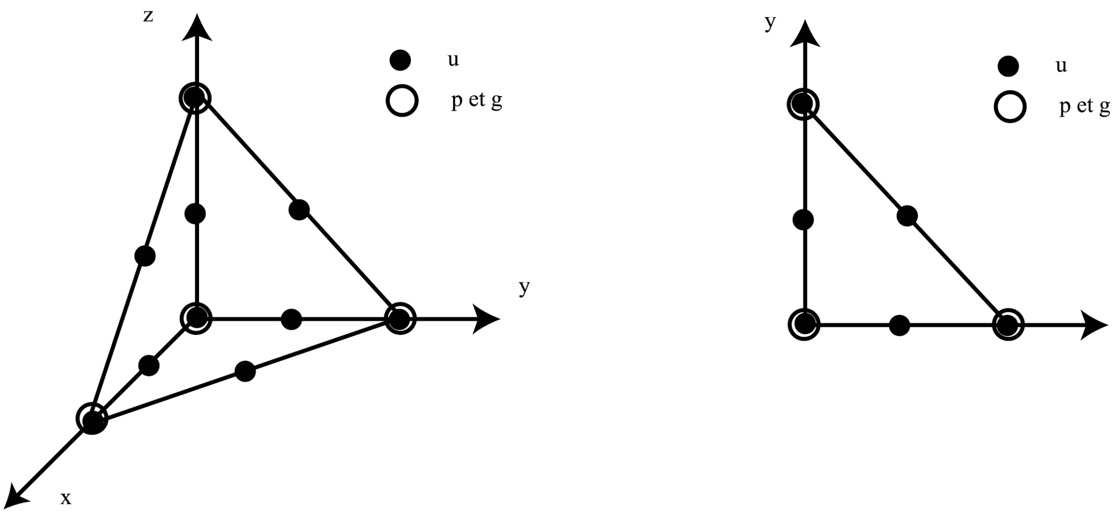
In the catalog of the elements, the incompressible elements can apply to meshes:

Meshes	Many nodes in Many	displacements nodes in pressure (and swelling)
TRIA6	6	3
QUAD8	8	4
HEXA20	20	8
TETRA10	10	4
PENTA15	15	6

In the routines of initialization of the incompressible elements, one define:

- 1 only family of Gauss points (cf §3.13.1),
- 2 families of shape functions respectively associated with displacements and the pressure (shape functions with degree 2) and under the terms with swelling (shape functions of degree 1).

Let us take as example the tetrahedral element with 10 nodes: the degrees of freedom in displacement and pressure are carried by all the nodes, on the other hand, only the 4 nodes tops have the degrees of freedom of swelling.



Accessible components for the field DEPL are thus

- displacements: DX, DY and DZ in 3D with all the nodes,
- pressure: NEAR for the top nodes,
- swelling: GONF for the top nodes.

4.3 Use of the modelization

By choice, modelizations INCO, INCO_UP, INCO_OSGS, INCO_GD, INCO_LOG and INCO_UP are accessible only with STAT_NON_LINE and DYNA_NON_LINE and option COMP_INCR. Under this key word, the version small strains is accessible by means of DEFORMATION=' PETIT', the version large deformations by means of DEFORMATION=' SIMO_MIEHE' or DEFORMATION=' GDEF_LOG'. The behavior models usable are those available respectively into small strains and large deformations SIMO_MIEHE or GDEF_LOG.

It is thus not possible to use the modelizations with the commands:

- MECA_STATIQUE
- CALC_MATR_ELEM/CALC_VECT_ELEM/ASSE_MATRICE/ASSE_VECTEUR/RESOUDRE
- STAT_NON_LINE (COMP_ELAS =…)

Being given the shape of the tangent matrix for the formulations to 3 fields (INCO, INCO_GD and INCO_LOG), it is often necessary to use the MUMPS solver to solve the linear systems.

It is advised to use the convergence criterion by stress of reference RESI_REFE_RELA.

4.4 Formulation of the elementary terms of the second member

the loads can be gravity, of the surface forces distributed, the pressures. The elementary terms are calculated in a classical way for the degrees of freedom of displacement and one zero affects the value for the degrees of freedom of pressure and swelling.

4.5 Computation of the strains and the stresses

In this formulation, it is appropriate to distinguish the stress field resulting from the constitutive law $\boldsymbol{\sigma}_{ldc}$, of the stress field which checks the equilibrium and which is defined by the relation $\boldsymbol{\sigma}=\boldsymbol{\sigma}_{ldc}^{D}+p\boldsymbol{I}$.

In small strains, it is the latter field which is stored in SIEF_ELGA as well as the relation binding the multiplier p and $\boldsymbol{\sigma}_{ldc}$. In short, the components of SIEF_ELGA are:

- SIXX, SIYY, SIZZ, SIXY in 2D like SIXZ and SIYZ in 3D: components of the tensor $\boldsymbol{\sigma}=\boldsymbol{\sigma}_{ldc}^{D}+p\boldsymbol{I}$,
- SIP which is equal to $\left(\frac{1}{3}\mathrm{tr}(\boldsymbol{\sigma}_{ldc})-p\right)$,

In large deformations, SIEF_ELGA contains the stresses of Cauchy $\boldsymbol{\sigma}_{ldc}$, resulting from the constitutive law.

It is also possible to recompute EPSI_ELGA, which is the strain field with the classical meaning.

One can also carry out a computation of Yield-point load with POST_ELEM.

5 Incompressible

5.1 validation elastic Case

test SSLV130 (cf [V3.04.130]) makes it possible to check the validity of the modelization in the case of an incompressible elastic cylinder subjected to an internal pressure. Its equivalent in large deformations also exist: test SSNV112 (cf [V6.04.112]).

5.2 Elastoplastic case

the goal of this example is to illustrate the contribution of modelization `INCO` if plastic strains are important compared to the elastic strain. One studies for that a notched sample into axisymmetric, subjected to an imposed displacement. The geometry and the loading are represented on the figure below. The mesh consists of 548 TRI6.

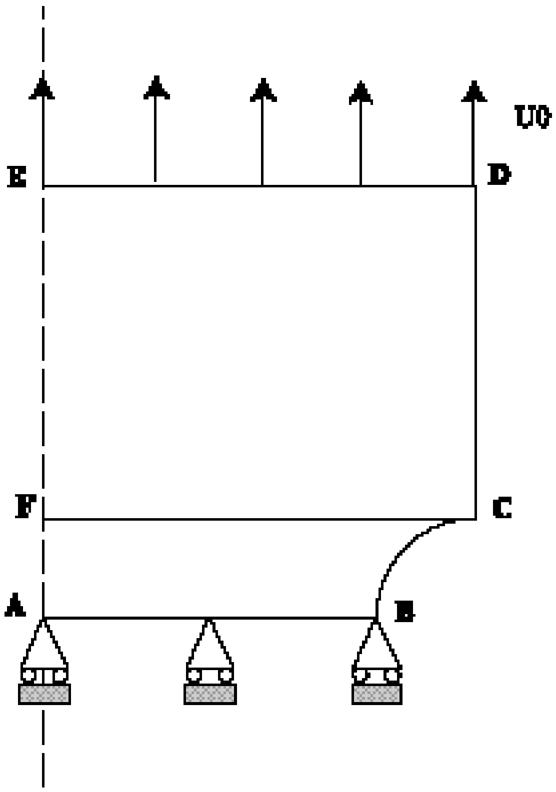


Figure 5.2-a : Geometry and boundary conditions

the behavior of material is of the elastoplastic type with linear isotropic hardening (`VMIS_ISOT_LINE`). The parameters are the following:

- $E=200\,000\,MPa$
- $\nu=0.3$
- $\sigma_y=200\,MPa$
- $E_T=1000\,MPa$

On the figure [Figure 5.2-b], one compares the stress σ_{yy} obtained on the path FC (cf [Figure 5.2-a]) with the classical modelization AXIS and modelization AXIS_INCO.

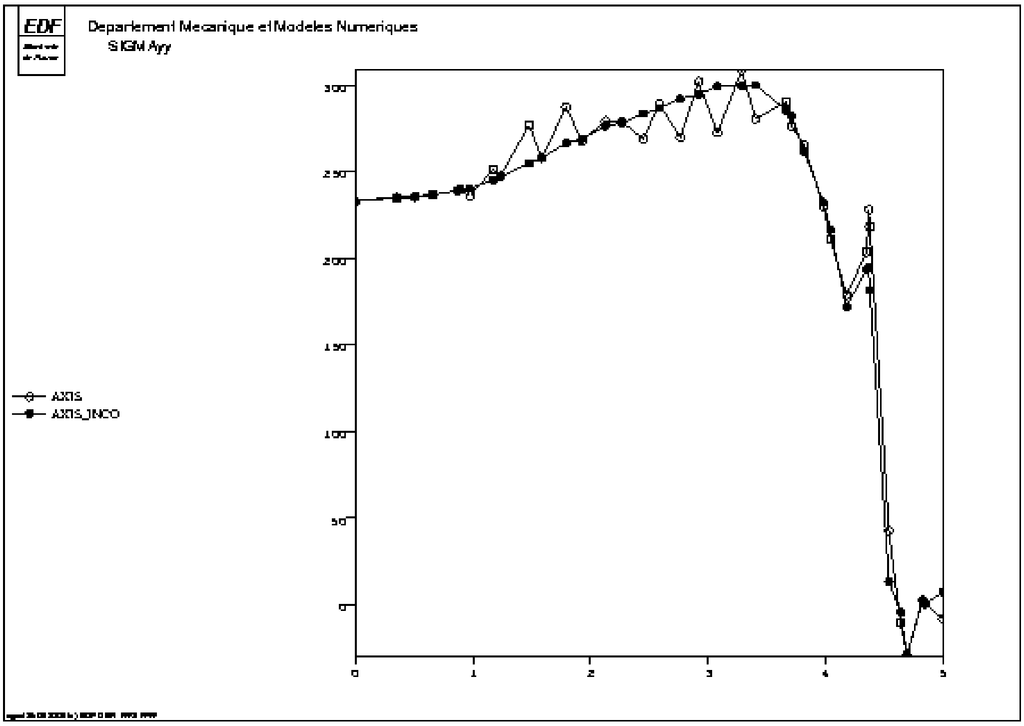


Figure 5.2-b : σ_{yy} along line FC

One sees very clearly that the solution obtained with formulation INCO makes it possible to be freed from the parasitic oscillations.

6 Bibliography

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7 of the versions Version

Aster Author	(S) Organization (S) Description	of the modifications 6.2
S.MICHEL	- PONNELLE, E.LORENTZ EDF R & D AMA initial	Version 6.4
S.MICHEL	- PONNELLE, E.LORENTZ EDF R & D light AMA	Updated for the version 6.4.7.2
S.MICHEL	- PONNELLE, E.LORENTZ EDF R & D AMA Addition	of the formulation in great transformations 7.4
S.MICHEL	- PONNELLE, E.LORENTZ EDF R & D AMA Addition	of the pentahedral elements 9.4
S.MICHEL	- PONNELLE EDF R & D AMA E.LORENTZ EDF R & D SINETICS New	quasi-incompressible elements formulation in great transformations 10.3
S.FAYOLLE	EDF	formulation at 2 fields in small strain 11.2

	R & D AMA New	$U-P$
S.FAYOLLE	EDF R &D AMA New	formulations OSGS and INCO_LOG 11.3
S.FAYOLLE	EDF R & D AMA New	formulations INCO_LUP