# Hexahedral element at a point of integration, stabilized by the method "Assumed Strain"

**Summarized:**

The hexahedral element with 8 standard nodes under-integrated into 1 point of integration null introduces parasitic modes associated with an energy (modes of sand glass) and can lead to a singularity of the total stiffness matrix for certain boundary conditions. The deficiency of the row of the stiffness matrix, due to under-integration, must thus be filled by adding with the elementary stiffness a matrix known as of stabilization. It is the object of method ASM (Assumed Strain Method) developed here.

The main feature of this method is that the operator discretized gradient $B$ necessarily does not derive from the field of displacement and the classical relations connecting the strain to displacement. Indeed, this method ASM consists in projecting the operator gradient discretized on a suitable subspace in order to avoid the various types of blocking.

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# 1 Introduction

Into computations finite elements, the recourse to the methods of under-integration makes it possible to reduce times computations to a significant degree, which explains their success. The other purpose of these methods is of the finite elements to draw aside the various blockings met in the numeric work implementation.

However, this under-integration does not have only advantages: it unfortunately introduces parasitic modes associated with an energy null, which lead to modes of sand glass, which will deform the mesh in an unrealistic way and end up exploding the solution. This is due to a deficiency of the row of the stiffness matrix due to under-integration. One cures it by adding the elementary stiffness a matrix known as of stabilization. The core of the new stiffness obtained by this layer must be reduced to the only modes corresponding to motions of rigid bodies.

These last years, certain authors developed various elements based on technique ASM (Assumed Strain Method). The main feature of this method is that the operator discretized gradient $B$ necessarily does not derive from the field of displacement and the classical relations connecting the strain to displacement. Indeed, this method ASM consists in projecting the operator gradient discretized on a suitable subspace in order to avoid the various types of blocking. This technique was largely used recently and led to several stabilized finite elements of type quadrangles to 4 nodes or hexahedrons with 8 nodes [1], [2], [3].

It is the element hexahedron with 8 nodes under-integrated into 1 point of integration and stabilized by method ASM, which had in Belytschko and Bindeman [2], which we describe in this document.

# 2 Formulation of element HEXA8 at a point of integration

## 2.1 Field of displacement and operator gradient discretized

In the hexahedral element, the spatial coordinates $x_i$ are connected to the nodal coordinates $x_{iI}$ by means of the isoparametric shape functions $N_I$ by the formulas:

$$x_i = x_{iI} N_I(\xi, \eta, \zeta) = \sum_{I=1}^{8} N_I(\xi, \eta, \zeta) x_{iI}$$ **éq 1.1-1**

One will use in the continuation summation convention for the repeated indices. The indices into tiny $i$ vary from 1 to 3 and represent the spatial directions. The indices in capital letter $I$ vary from 1 to 8 and correspond to the nodes of the element.

The same shape functions are used to define the field of displacement of the element $u_i$ according to nodal displacements $u_{iI}$ :

$$u_i = u_{iI} N_I(\xi, \eta, \zeta)$$

Since the same shape functions apply to the coordinates and displacements, their material derivative is cancelled and the velocity field can be given by:

$$v_i = v_{iI} N_I(\xi, \eta, \zeta)$$ **éq 1.1-2**

the preceding interpolation of the velocity field will make it possible to define strain rate and to write the relations connecting the strains at the nodal speeds. The gradient $v_{i,j}$ of the velocity field is written:

$$v_{i,j} = v_{iI} N_{I,j}$$

By convention, a comma preceding an index into tiny represents a differentiation compared to the spatial coordinates. Tensor strain rate $D_{ij}$ is given then by the symmetric part of the gradient velocity:

$$D_{ij}=\frac{1}{2}(v_{i,j}+v_{j,i})$$

One gives oneself trilinear isoparametric shape functions $N_I(\xi,\eta,\zeta)$ definite by:

$$N_I(\xi,\eta,\zeta)=\frac{1}{8}(1+\xi_I\xi)(1+\eta_I\eta)(1+\zeta_I\zeta)$$

$\xi,\eta,\zeta\in[-1,1], I=1,\dots,8$ with $\xi_I,\eta_I,\zeta_I$ being worth 1 or – 1éq **1.1-3**

These shape functions transform a unit cube in space $(\xi,\eta,\zeta)$ into an unspecified hexahedron in space $(x_1,x_2,x_3)=(x,y,z)$. While combining equations **1.1-1, 1.1-2, 1.1-3**, one manages to develop the velocity field as the sum of a constant term, linear terms in $x_i$, and terms utilizing functions $h_\alpha$ :

$$v_i=a_{0i}+a_{1i}x_1+a_{2i}x_2+a_{3i}x_3+c_{1i}h_1+c_{2i}h_2+c_{3i}h_3+c_{4i}h_4$$ I = 1,2,3éq **1.1-4**

$$h_1=\eta\zeta \qquad h_2=\xi\zeta \qquad h_3=\eta\xi \qquad h_4=\xi\eta\zeta$$

Indeed, equation **1.1-1** makes it possible to only write $\xi,\eta,\zeta$ in function $x_i$ , $h_\alpha$ and of a constant parameter. By then injecting these last relations in equation **1.1-2**, one finds the statements **1.1-4** required.

While evaluating equation **1.1-4** with the nodes of the element, one arrives at the 3 systems of 8 equations following:

$$\dot{d}_i=a_{0i}s+a_{1i}x_1+a_{2i}x_2+a_{3i}x_3+c_{1i}h_1+c_{2i}h_2+c_{3i}h_3+c_{4i}h_4 \qquad i=1,2,3$$ **éq 1.1-5**

In the preceding equation and the continuation, the bold characters indicate tensors of order at least 1. Thus, the vectors $\dot{\mathbf{d}}_i$ and $x_i$ represent, respectively, the nodal velocities and the coordinated and are given by:

$$\dot{d}_i^t=(v_{i1},v_{i2},v_{i3},\dots,v_{i8})$$
$$x_i^t=(x_{i1},x_{i2},x_{i3},\dots,x_{i8})$$

The vectors $s$ and $h_\alpha(\alpha=1,\dots,4)$ are given as for them by

$$s^t=(1,1,1,1,1,1,1,1)$$
$$h_1^t=(1,1,-1,-1,-1,-1,1,1)$$
$$h_2^t=(1,-1,-1,1,1,-1,1,1,-1)$$
$$h_3^t=(1,-1,1,-1,1,-1,1,-1)$$
$$h_4^t=(-1,1,-1,1,1,-1,1,-1)$$

arriving at an advantageous writing of the operator discretized gradient $\mathbf{B}$, one introduces the 3 vectors $b_i$ defined by: $b_i^T=N_{,i}(0)=\frac{\partial N}{\partial x_i}|_{\xi=\eta=\zeta=0} \qquad i=1,2,3$ **éq 1.1-6** where $N$ represents $(N_1, N_2, \ldots, N_8)$. These vectors $b_i$ represent the simplest shape of the operator discretized gradient under-integrated introduced by Hallquist and which is based on the evaluating of derivatives of the isoparametric shape functions at the origin of the reference frame $(\xi, \eta, \zeta)$. They can be explicitly given. Moreover, one can check the following conditions of orthogonality:

$$b_i^T . h_\alpha = 0 \qquad b_i^T . s = 0 \qquad b_i^T . x_j = \delta_{ij}$$

$$h_\alpha^T . s = 0 \quad , \quad h_\alpha^T . h_\beta = 8 \delta_{\alpha\beta} \qquad \textbf{éq 1.1-7}$$

$$i = 1 \, à \, 3 \quad , \quad \alpha, \beta = 1 \, à \, 4$$

where $\delta$ the Kronecker symbol indicates.

A this stage, one can determine the constant unknowns who intervene in the writing of the velocity field (**éq 1.1-4)** by multiplying scalairement the equation (**éq 1.1-5)** by $\mathbf{b_j^T}$ and $\mathbf{h_\alpha^T}$, respectively, and by means of the relations of orthogonality above. One obtains:

$$a_{ij} = b_j^T . \dot{d}_i \quad , \quad \boldsymbol{c_{\alpha i}} = \boldsymbol{\gamma_\alpha^T} . \dot{\boldsymbol{d}}_i \qquad \text{with} \qquad \gamma_\alpha = \frac{1}{8}\left[ h_\alpha - \sum_{j=1}^{3} (h_\alpha^T . x_j) b_j \right]$$

the velocity field puts itself finally in the following very practical form:

$$v_i = a_{0i} + (x_1 b_1^T + x_2 b_2^T + x_3 b_3^T + h_1 \gamma_1^T + h_2 \gamma_2^T + h_3 \gamma_3^T + h_4 \gamma_4^T) . \dot{d}_i \quad i = 1,2,3 \qquad \textbf{éq 1.1-8}$$

By differentiating the formula above compared to $x_j$, one obtains the gradient velocity:

$$v_{i,j} = (b_j^T + \sum_{\alpha=1}^{4} h_{\alpha,j} \gamma_\alpha^T) . \dot{d}_i = (b_j^T + h_{\alpha,j} \gamma_\alpha^T) . \dot{d}_i$$

The operator discretized gradient connecting tensor strain rate to the vector nodal velocities by:

$$\nabla_s v = B . \dot{d}$$

where:

$$\nabla_s \boldsymbol{v} = \begin{bmatrix} v_{x,x} \\ v_{y,y} \\ v_{z,z} \\ v_{x,y} + v_{y,x} \\ v_{x,z} + v_{z,x} \\ v_{y,x} + v_{z,y} \end{bmatrix}, \qquad \dot{\boldsymbol{d}} = \begin{bmatrix} \dot{d}_x \\ \dot{d}_y \\ \dot{d}_z \end{bmatrix}$$

takes the practical matric shape then:

$$B=\begin{bmatrix} \boldsymbol{b}_x^T+\boldsymbol{h}_{\alpha,x}\boldsymbol{\gamma}_\alpha^T & 0 & 0 \\ 0 & \boldsymbol{b}_y^T+\boldsymbol{h}_{\alpha,y}\boldsymbol{\gamma}_\alpha^T & 0 \\ 0 & 0 & \boldsymbol{b}_z^T+\boldsymbol{h}_{\alpha,z}\boldsymbol{\gamma}_\alpha^T \\ \boldsymbol{b}_y^T+\boldsymbol{h}_{\alpha,y}\boldsymbol{\gamma}_\alpha^T & \boldsymbol{b}_x^T+\boldsymbol{h}_{\alpha,x}\boldsymbol{\gamma}_\alpha^T & 0 \\ \boldsymbol{b}_z^T+\boldsymbol{h}_{\alpha,z}\boldsymbol{\gamma}_\alpha^T & 0 & \boldsymbol{b}_x^T+\boldsymbol{h}_{\alpha,x}\boldsymbol{\gamma}_\alpha^T \\ 0 & \boldsymbol{b}_z^T+\boldsymbol{h}_{\alpha,z}\boldsymbol{\gamma}_\alpha^T & \boldsymbol{b}_y^T+\boldsymbol{h}_{\alpha,y}\boldsymbol{\gamma}_\alpha^T \end{bmatrix}$$ **éq the 1.1-9**

vectors $\gamma_\alpha$ which intervene in $\boldsymbol{B}$ check the following conditions of orthogonality:

$$\boldsymbol{\gamma}_\alpha^T.\boldsymbol{x}_j=0 \qquad , \qquad \boldsymbol{\gamma}_\alpha^T\boldsymbol{h}_\beta=\delta_{\alpha\beta}$$

An element based on this formulation is convergent when it is evaluated exactly. However, the evaluating of this operator $\boldsymbol{B}$ of each point of integration makes this element too expensive for the practical applications, and the simplified shape of this element is essential.

## 2.2 Variational formulation of the problem

the extension of the weak form of the variational principle of Hu-Washizu to the case of the mechanics of nonlinear solids is written for a simple element $\Omega_e$

$$\delta\pi(v,\bar{\dot{\boldsymbol{\epsilon}}},\bar{\boldsymbol{\sigma}})=\int_{\Omega_e}\delta\bar{\dot{\boldsymbol{\epsilon}}}^T.\ \boldsymbol{\sigma}\ d\Omega+\delta\int_{\Omega_e}\bar{\boldsymbol{\sigma}}^T\ .\ \left(\nabla_s v-\bar{\dot{\boldsymbol{\epsilon}}}\right)\ d\Omega-\delta\dot{d}^T\ .\ f^{ext}=0$$ **éq 1.2-1**

where $\delta$ a variation represents, $v$ the velocity field, $\bar{\dot{\boldsymbol{\epsilon}}}$ strain rate applied, $\bar{\sigma}$ the applied stress, $\sigma$ the stress evaluated by the constitutive law, $\dot{d}$ the nodal velocities, $f^{ext}$ the external nodal forces and $\nabla_s v$ the symmetric part of the gradient of the velocity field.

The formulation "Assumed strain" retained in the continuation to build the element is based on a simplified form of the variational principle of Hu-Washizu which leans on the fact that the applied stress is selected orthogonal with the difference between the symmetric part of the gradient applied velocity and strain rate. Thus the second term of equation 1.2-1 is eliminated and one obtains:

$$\delta\pi(\bar{\dot{\boldsymbol{\varepsilon}}})=\int_{\Omega_e}\delta\bar{\dot{\boldsymbol{\varepsilon}}}^T.\ \sigma\ d\Omega-\delta\dot{d}^T\ .\ f^{ext}=0$$ **éq 1.2-2**

Pennies this form, the variational principle is independent of the interpolation of the stress, since the applied stress does not intervene any more and thus need does not have to be defined. The discretized equations thus require the only interpolation velocity $v$ and strain rate applied $\bar{\dot{\boldsymbol{\varepsilon}}}$ in the element. If $N_I$ indicates the isoparametric shape functions of the element, one a:

$\boldsymbol{v}(x,t)=\sum_{I=1}^{m}N_I(x)\dot{d}_I(t)$ where $m$ the number of nodes indicates. One from of deduced:

$\nabla_s\boldsymbol{v}(x,t)=\boldsymbol{B}(x).\dot{\boldsymbol{d}}(t)$

Strain rate applied $\bar{\dot{\varepsilon}}$ is defined as for him by: $\bar{\dot{\varepsilon}}(x,t)=\bar{B}\ (x).\ \dot{d}\ (t)$ **éq 1.2-3**

By replacing equation **1.2-3** in variational principle **1.2-2**, one obtains:

$$\delta\dot{d}^T\ .\ \int_{\Omega_e}\bar{B}^T\ .\ \sigma\ d\Omega-\delta\dot{d}^T\ .\ f^{ext}=0$$

As $\delta \dot{d}^T$ can be arbitrarily selected, the preceding equation leads to:

$$f^{\text{int}} = f^{ext}$$

with $f^{\text{int}} = \int_{\Omega_e} \bar{B}^T \,.\, \sigma(\dot{\bar{\boldsymbol{\varepsilon}}})\, d\Omega$

In the equation above, it is well specified that the stress $\sigma$ is calculated by the applied constitutive law starting from strain rate $\dot{\bar{\boldsymbol{\varepsilon}}}$ . For the nonlinear problems, $\sigma$ can also be a function of the integral of strain rate applied and other local variables. The formulation thus obtained is valid for problems including the two types of non-linearities: geometrical and material.

In the case of linear problems, one a:

$$\boldsymbol{\sigma} = \boldsymbol{C} \,.\, \bar{\boldsymbol{\varepsilon}} = \boldsymbol{C} \,.\, \bar{\boldsymbol{B}} \,.\, \boldsymbol{d}$$

the internal forces of the element are written:

$$\boldsymbol{f}^{\text{int}} = \boldsymbol{K}_e . \boldsymbol{d} \qquad \text{with} \qquad \boldsymbol{K}_e = \int_{\Omega_e} \bar{\boldsymbol{B}}^T . \boldsymbol{C} \,.\, \bar{\boldsymbol{B}}\, d\Omega$$

In a standard approach in displacement, strain rate applied is identified with the symmetric part of the gradient velocity, which amounts replacing $\bar{\boldsymbol{B}}$ by $\boldsymbol{B}$ in the preceding statements. One obtains:

$$\boldsymbol{K}_e = \int_{\Omega_e} \boldsymbol{B}^T . \boldsymbol{C} \,.\, \boldsymbol{B}\, d\Omega$$

## 2.3 Modes of "hourglass" associated with an energy null

the matric writing **éq 1.1-9** of the operator discretized gradient will make it possible to understand the origin of the modes of hourglass, or modes of sand glass. As one will see it, these kinematical modes are due to under-integration and are associated with an energy null whereas they induce a non-zero strain. This anomaly is explained by the difference that induced under-integration, between the core of the operator of stiffness discretized and the core of the continuous operator of stiffness.
Let us notice initially that the operator under-integrated discretized gradient (i.e associated with only one point of integration located at the center of the element) is reduced to: éq

$$\boldsymbol{B} = \begin{bmatrix} b_x^T & 0 & 0 \\ 0 & b_y^T & 0 \\ 0 & 0 & b_z^T \\ b_y^T & b_x^T & 0 \\ b_z^T & 0 & b_x^T \\ 0 & b_z^T & b_y^T \end{bmatrix}$$

**1.3-1 Indeed**

, the terms of $h_{\alpha,i}$ the equation éq **1.1-9** are cancelled at the point of integration. $|\xi = \eta = \zeta = 0$ Now let us analyze the core of the stiffness matrix obtained by integration. In the linear case, this elementary matrix is written: where $\boldsymbol{K}_e = V\, \boldsymbol{B}^T . \boldsymbol{C} \,.\, \boldsymbol{B}$ $V$ the volume of the element indicates.
The examination of the core of the under-integrated stiffness returns under investigation from the row of the matrix. $\boldsymbol{B}$ It is thus necessary null to seek the modes from velocity $\dot{\boldsymbol{d}}$ to strain, i.e. checking: éq

$$\nabla_s \boldsymbol{v} = \boldsymbol{B} \,.\, \dot{\boldsymbol{d}} = 0$$

**1.3-2 One**

must find in the core of $K_e$ the modes associated with rigid body motions, that is to say in 3D, 3 translations and 3 rotations.
The core of the continuous operator is thus of dimension 6 and is reduced to the 6 vectors:

$$\begin{pmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{pmatrix}\begin{pmatrix} y & z & 0 \\ -x & 0 & z \\ 0 & -x & -y \end{pmatrix}$$

The first 3 vectors column check 1.3 **- 2 because of** , $\boldsymbol{b_i^T}.\boldsymbol{s}=0$ the 3 last because of. $b_i^T.x_j=\delta_{ij}$

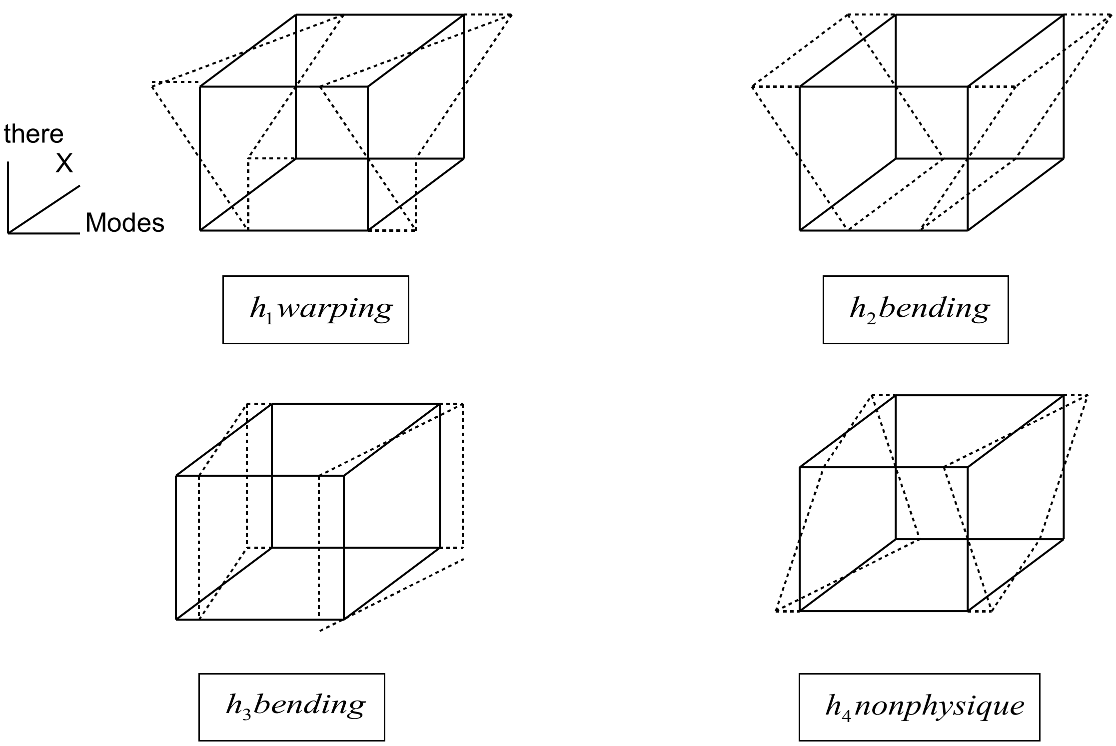
But

in addition to the preceding rigid modes, the operator discretized gradient given into 1.3 **- 1 cancels** the 12 other following vectors: because

$$\begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_1 & 0 \\ 0 & 0 & h_1 \end{pmatrix}\begin{pmatrix} h_2 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_2 \end{pmatrix}\begin{pmatrix} h_3 & 0 & 0 \\ 0 & h_3 & 0 \\ 0 & 0 & h_3 \end{pmatrix}\begin{pmatrix} h_4 & 0 & 0 \\ 0 & h_4 & 0 \\ 0 & 0 & h_4 \end{pmatrix}$$

$b_i^T.h_\alpha=0$

the vectors columns above are called modes of sand glass or modes of hourglass.
The 4 modes according to are $Ox$ below illustrated: Z




**of sand glass according to according to $Ox$ Belytschko and Bindeman**

the operator discretized gradient given in 1.3 **- 1 and** that integrated exactly given into 1.1 **- 9 calculate** the gradients of the linear fields correctly and give identical results when they are applied to the 6 modes of rigid body. On the other hand, the exact formula of properly $B$ calculates the other modes of $h_\alpha$ strain, whereas the under-integrated form gives erroneous results.

Deficiency in the row of the operator under $B$ - integrated (row 6 only instead of 18 for integrated $B$ exactly) results in unrealistic oscillations of the mesh leading to solutions which explode for certain boundary conditions. These

under-integrated elements thus require techniques known as of stabilization. Stabilization

# 3 of the type "Assumed Strain Method" (ASM)

the approach followed to stabilize the under-integrated hexahedral element is that of Belytschko and Bindeman [2]. The method developed to control the modes of hourglass in an under-integrated element is known as Assumed Strain Method. It B implies moreover the construction of the suitable shape of the matrix making it possible to avoid numerical blockings. Formulas

## 3.1 of Hallquist and Flanagan-Belytschko Jusqu

"here, to establish the formulas 1.1 **- 9 of** L" operator B, one used the vectors based $\boldsymbol{b_i}$ on derivative of the shape functions in the beginning (equation 1.1 **- 6).** This writing of is $B$ known as form of Hallquist. One second form is considered now, based on the mean values of derivatives of the shape functions of the element. This form is due to Flanagan and Belytschko and is written: These

$$\hat{\boldsymbol{b}}_i^T=\frac{1}{V}\int_{\Omega_e} \boldsymbol{N}_{,i}(\xi,\eta,\zeta)\,d\,\Omega ¿ \qquad i=1,2,3$$

vectors can be calculated explicitly. They can also be integrated numerically exactly.
The conditions of orthogonality of the §1.2 remain checked excluded the first on the elements nonparallelipipedic: on all the

$$\hat{\boldsymbol{b}}_i^T.\boldsymbol{h}_j\begin{cases}=0 & \text{sur les parallélipipèdes}\\ \neq 0 & \text{sinon}\end{cases}$$

$\hat{\boldsymbol{b}}_i^T.\boldsymbol{h}_4=0$ elements. By

supposing still true this relation of orthogonality, the displacement gradient is written like previously: with

$$\boldsymbol{v}_{i,j}=(\hat{\boldsymbol{b}}_j^T+\boldsymbol{h}_{\alpha,j}\hat{\boldsymbol{\gamma}}_\alpha^T).\dot{\boldsymbol{d}}_i \qquad \text{One } \hat{\boldsymbol{\gamma}}_\alpha=\frac{1}{8}\left[\boldsymbol{h}_\alpha-\sum_{j=1}^{3}(\boldsymbol{h}_\alpha^T.\boldsymbol{x}_j)\hat{\boldsymbol{b}}_j\right]$$

can continue to write: where $\nabla_s v=\hat{B}\cdot\dot{d}$
appears $\hat{B}$ as the sum of a constant term and $\hat{B}_c$ a nonconstant term defined $\hat{B}_n$ by: with

$$\hat{\boldsymbol{B}}=\hat{\boldsymbol{B}}_c+\hat{\boldsymbol{B}}_n$$

$$\hat{\boldsymbol{B}}_c=\begin{bmatrix}\hat{b}_x^T & 0 & 0\\ 0 & \hat{b}_y^T & 0\\ 0 & 0 & \hat{b}_z^T\\ \hat{b}_y^T & \hat{b}_x^T & 0\\ \hat{b}_z^T & 0 & \hat{b}_x^T\\ 0 & \hat{b}_z^T & \hat{b}_y^T\end{bmatrix}\quad \hat{\boldsymbol{B}}_n=\begin{bmatrix}\hat{X}_{1234}^T & 0 & 0\\ 0 & \hat{Y}_{1234}^T & 0\\ 0 & 0 & \hat{Z}_{1234}^T\\ \hat{Y}_{1234}^T & X_{1234}^T & 0\\ \hat{Z}_{1234}^T & 0 & X_{1234}^T\\ 0 & \hat{Z}_{1234}^T & \hat{Y}_{1234}^T\end{bmatrix}$$

This $\hat{X}_{1234}^T=\sum_{\alpha=1}^{4} h_{\alpha,x}\hat{\boldsymbol{\gamma}}_{\boldsymbol{\alpha}}^T$ $\hat{Y}_{1234}^T=\sum_{\alpha=1}^{4} h_{\alpha,y}\hat{\boldsymbol{\gamma}}_{\boldsymbol{\alpha}}^T$ $\hat{Z}_{1234}^T=\sum_{\alpha=1}^{4} h_{\alpha,z}\hat{\boldsymbol{\gamma}}_{\boldsymbol{\alpha}}^T$

formulation, even if it is less rigorous than that of Hallquist (since it supposes true all the relations of orthogonality) gives better results in terms of accuracy and convergence. Projection

## 3.2 on a strain field One $\bar{B}$

applies a projection to the operator gradient discretized $\hat{B}$ to deduce an operator having from it $\bar{B}$ certain good properties. The purpose of projection is double: It
•makes it possible, on the one hand, to eliminate voluminal blocking from the finite element in the incompressible case It
•avoids, on the other hand, blocking due under the excessive terms of transverse shears in the problems with dominant bending.

The operator $\hat{B}$ is replaced by an operator such as $\bar{B}$ : Only

$$\begin{cases}\hat{\boldsymbol{B}}=\hat{\boldsymbol{B}}_c+\hat{\boldsymbol{B}}_n\\ \bar{\boldsymbol{B}}=\hat{\boldsymbol{B}}_c+\bar{\boldsymbol{B}}_n\end{cases}$$

the nonconstant part $\hat{\boldsymbol{B}}_n$ is projected, the constant part remains $\hat{\boldsymbol{B}}_c$ unchanged. Two

different projections lead to the 2 following finite elements: Element

•ASQBI (Assumed Strain Quintessential Bending Incompressible)! éq

$$\bar{\boldsymbol{B}}_n=\begin{bmatrix}\hat{X}_{1234}^T & -\bar{\nu}\,\hat{Y}_3^T-\nu\,\hat{Y}_{24}^T & -\bar{\nu}\,\hat{Z}_2^T-\nu\,\hat{Z}_{34}^T\\ -\bar{\nu}\,\hat{X}_3^T-\nu\,\hat{X}_{14}^T & \hat{Y}_{1234}^T & -\bar{\nu}\,\hat{Z}_1^T-\nu\,\hat{Z}_{34}^T\\ -\bar{\nu}\,\hat{X}_2^T-\nu\,\hat{X}_{14}^T & -\bar{\nu}\,\hat{Y}_1^T-\nu\,\hat{Y}_{24}^T & \hat{Z}_{1234}^T\\ \hat{Y}_{12}^T & \hat{X}_{12}^T & 0\\ \hat{Z}_{13}^T & 0 & \hat{X}_{13}^T\\ 0 & \hat{Z}_{23}^T & \hat{Y}_{23}^T\end{bmatrix}$$ **2.2-1 Where**

: , $\bar{\nu}=\frac{\nu}{1-\nu}$ Element $\hat{\boldsymbol{X}}_{14}^T=h_{1,x}\hat{\boldsymbol{\gamma}}_{\boldsymbol{1}}{}^{\boldsymbol{T}}+h_{4,x}\hat{\boldsymbol{\gamma}}_{\boldsymbol{4}}{}^{\boldsymbol{T}}$ $\hat{\boldsymbol{Z}}_2^T=h_{2,z}\hat{\boldsymbol{\gamma}}_{\boldsymbol{2}}{}^{\boldsymbol{T}}$

•ADS (Assumed Deviatoric Strain): éq

$$\bar{B}_n = \begin{bmatrix} \frac{2}{3}\hat{X}^T_{1234} & -\frac{1}{3}\hat{Y}^T_{1234} & -\frac{1}{3}\hat{Z}^T_{1234} \\ -\frac{1}{3}\hat{X}^T_{1234} & \frac{2}{3}\hat{Y}^T_{1234} & -\frac{1}{3}\hat{Z}^T_{1234} \\ -\frac{1}{3}\hat{X}^T_{1234} & -\frac{1}{3}\hat{Y}^T_{1234} & \frac{2}{3}\hat{Z}^T_{1234} \\ \hat{Y}^T_{12} & \hat{X}^T_{12} & 0 \\ \hat{Z}^T_{13} & 0 & \hat{X}^T_{13} \\ 0 & \hat{Z}^T_{23} & \hat{Y}^T_{23} \end{bmatrix}$$ **the 2.2-2**

diagrams of projection are detailed in [2]. One will indicate nevertheless the broad outlines of obtaining their operators. $\bar{B}_n$ For

the element ADS, that initially amounts breaking up the operator into $\hat{B}_n$ the sum of a spherical term and a term deviatoric. Then, the spherical part under-is integrated (i.e evaluated at the point, which $\xi=\eta=\zeta=0$ comes down cancelling it). This procedure thus makes it possible to treat the diagonal terms of the strain, therefore voluminal blocking, and one can check in éq 2.2 **- 2 that** the sum of the first three terms in each vector column is null. To avoid blocking in transverse shears, it is necessary to act this time on the nondiagonal terms of the strain, and thus to cancel those responsible for excessive shears. Result of this stage results in the suppression of certain terms in the three last lines of the operator (to compare $\hat{B}_n$ the formulas giving and) $\hat{B}_n$ . For $\bar{B}_n$

element ASQBI, the processing of blocking in shears is the same one as for element ADS. It results from it that the three last lines of the operator are $\bar{B}_n$ identical in the two elements. On the other hand, the processing of voluminal blocking is a little different for element ASQBI (see éq 2.2 **- 1). One** can check however that, when, therefore $\nu \rightarrow \frac{1}{2}$ , $\bar{\nu} \rightarrow 1$ the sum of the first three terms in each vector column of is null $\bar{B}_n$ . Choice

## 3.3 of the finite elements One

compared two finite elements ADS and ASQBI on a certain number of tests. It arises that ASQBI gives better results that ADS in elasticity whereas ADS gives better results in plasticity, its behavior being a little too flexible in the elastic cases. The case test of the cantilever beam in bending [V6.04.196] in is a good illustration. These observations are in good conformity with those of Belytschko and Bindeman [2]. One

thus chose to establish in Aster element ASQBI in elasticity and element ASD in plasticity to profit in each case from the best element. Moreover, one avoids with the user having to choose the finite element. It is pointed out that the only difference between the two elements is the matrix. Integration $\bar{B}_n$

# 4 of the element in Code_Aster *Description*

## 4.1 and use This element

leans on meshes 3D the voluminal HEXA8 . `Modelization`

### 4.1.1 One 3D_SI

assigns the modelization `to meshes` the indicated `HEXA8` . The usual elements of face of the modelization 3D are affected on meshes the QUAD4 . `Material`

### 4.1.2 All

the coefficients material relating to the valid constitutive laws in small strains for the modelizations 3D are usable. Boundary conditions

### 4.1.3 and loading All

the loadings and boundary conditions available on the elements 3D and the elements of face are available. Options

### 4.1.4 of postprocessing All

the options of postprocessing usually available for the modelizations 3D are usable. Computation

### 4.1.5 in linear buckling option

RIGI_ `MECA_GE` being activated in the catalog of the element, it is possible to carry out a classical computation of buckling after assembly of the stiffness matrixes elastic and geometrical. Nonlinear

### 4.1.6 computations All

the behaviors available for the modelization 3D usual are usable, in small strains. With regard to the large deformations, the only option usable is the "approximation PETIT_REAC . `L"` numerical integration

is carried out with a Gauss point, just like in nonlinear material. Establishment

## 4.2 the options

are activated in the catalog meca_ `hexs8.catastrophes. Validation`

## 4.3 the tests

validating this element are, in version 9 of Code_Aster *: SSNV* 196

- : Beam 3D in bending in elasticity and plasticity HSNV125
- G, PERFE01A, SDNV103, SSND105 use only one element to validate nonlinear behaviors, thus allowing savings in time computation (1 only Gauss point) Conclusion

# 5 the element

hexahedron 8 nodes under-integrated into a point of integration and stabilized by method ASM gives correct results on problems where the standard `HEXA8` blocks (bending, shears), as benchmark SSNV196 shows it. Moreover it gives significant gains in time computation. It presents

a disadvantage in static however (the same one as element 2D similar QUAD4 `under` integrated stabilized): when the elements are not parallelepipedic, the results are affected, even false if the elements have angles distant from 90°. The recommended solution is then to resort on these meshes to the standard `HEXA8` (to 8 Gauss points). In dynamics clarifies on the other hand, this disadvantage does not remain. But the current establishment relates to only the static. Bibliography

# 6 T. BELYTSCHKO

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# 7 of the versions of the document: Version

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