Shell elements: modelizations DKT, DST, DKTG and Summarized

Q4g:

These modelizations of shell elements are intended for computations in small strains and small displacements of curved or plane thin structures. In fact plane elements do not take into account the geometrical curvature of structures, contrary with the shell elements which are curved: it results from it from parasitic bendings which can be reduced by means of more elements in order to be able to approach the curved geometries correctly. The formulation is thus simplified by it and the reduced number of degrees of freedom. These elements are famous as being among most precise for the computation of displacements and the modal analysis.

For each one of these various modelizations several finite elements are available, according to meshes:

1) modelization `DKT` comprises the finite elements triangular (`DKT`) and quadrangular (`DKQ`);
2) modelization `DST` comprises the finite elements triangular (`DST`) and quadrangular (`DSQ`);
3) modelization `DKTG` comprises the finite elements triangular (`DKTG`) and quadrangular (`DKQG`);
4) modelization `Q4G` does not comprise that the quadrangular finite element (`Q4G`)

Note:

In the document [R3.07.09], one presents modelization `Q4GG`. This modelization comprises the quadrangular finite elements (`Q4G`) of which theoretical description is made in this triangular document and the elements (`T3G`).

Contents

1 Introduction

the shell elements and plates are particularly used for modelling thin structures where the relationship between dimensions (characteristic thickness/length) is with more than 1/10. They thus intervene particularly in fields like the civil engineer, the interns of heart REFERENCE MARK, the vibratory analysis ..... One limits oneself to the frame of small displacements and the small strains.

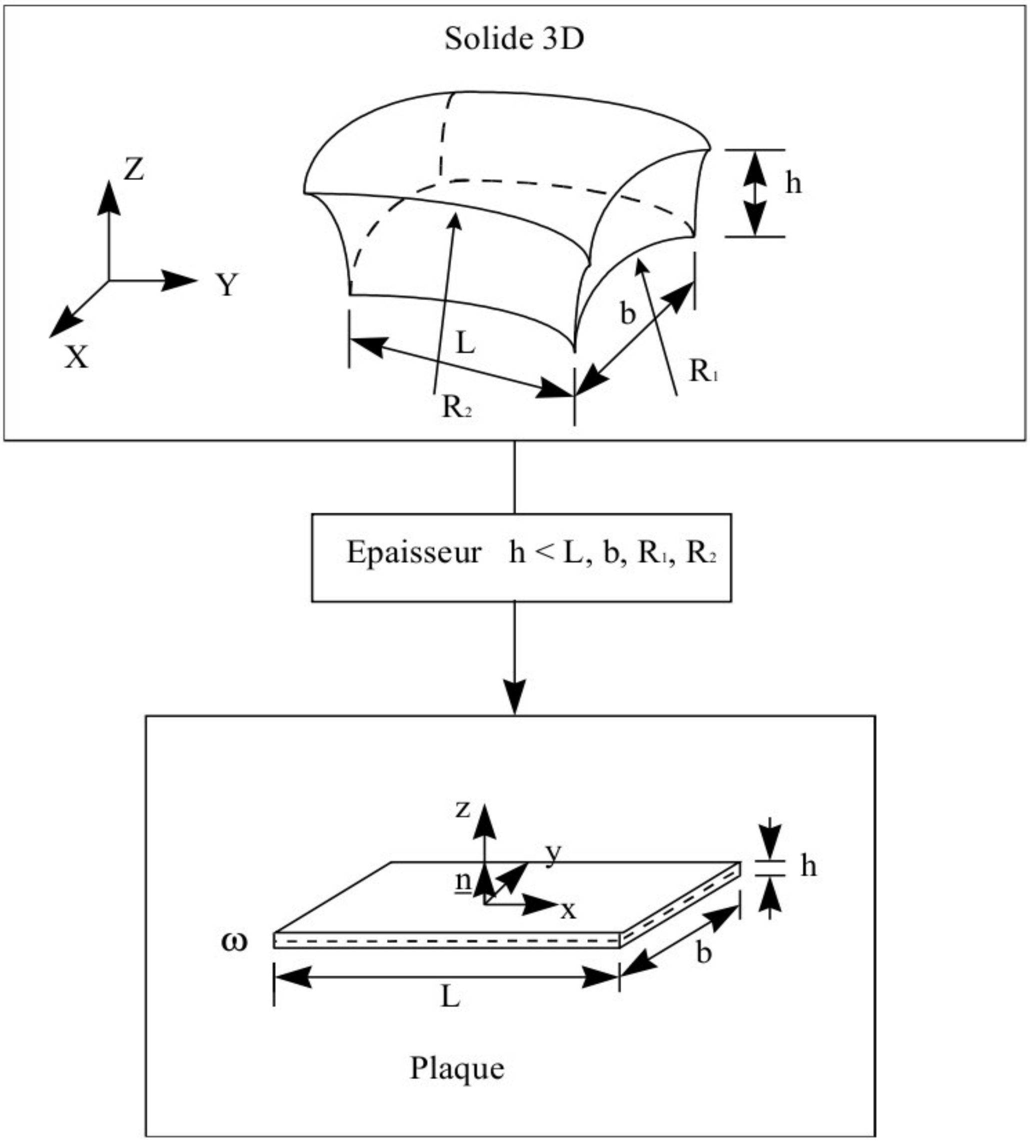
Contrary to the shell elements, the plane shell elements do not make it possible to take into account the geometrical curvature of structure to be represented and induce parasitic bendings. It is thus necessary to use a large number of these elements in order to approach the geometry of structure correctly, and this, more especially as it is curved. On the other hand, one gains in simplicity of formulation and the number of degrees of freedom is reduced. In addition, the formulations “Discrete Shear” (`DST`, `DSQ` and `Q4g`) or “Discrete Kirchhoff” (`DKT`, `DKTG` and `DKQ`, `DKQG`) of the kinematics, with or without transverse distortion respectively, allow good results in terms of displacements and modal analysis.

The way in which these elements in Code_Aster *are established* as certain receipts of use are given to [§5] present note.

2 Geometry

2.1 formulation of the elements plates [bib1]

For the shell elements one defines a surface of reference, or mean surface, planes (plane $x\,y$ for example) and a thickness $h(x, y)$. This thickness must be small compared to other dimensions (extensions, radii of curvature) of structure to modelling. [Figure 2.1-a] below our matter illustrates.



Appear 2.1-a

One attaches to mean surface ω a local orthonormal reference $Oxyz$ associated with the tangent plane of structure different from the total reference $OXYZ$. The position of the points of the plate is given by the Cartesian coordinates (x, y) of mean surface and rise z compared to this surface.

2.1.1 Intrinsic reference

By taking preceding the Oxyz local coordinate system with for origin the first top of the element and for axis OX the side uniting tops 1 and 2, one defines the reference known as intrinsic.

2.2 Theory of the plates

These elements are based on the theory of the plates in small displacements and small strains.

2.2.1 Kinematics

the cross-sections which are the sections perpendicular to mean surface remain right; the material points located on a norm at not deformed mean surface remain on a line in the deformed configuration. It results from this approach that **the fields of displacement vary linearly in the thickness of the plate**. If one indicates by u, v, w displacements of a following $q(x, y, z)$ point x, y and z, one has the kinematics of Hencky-Mindlin thus:

$$\begin{pmatrix} u_x(x,y,z) \\ u_y(x,y,z) \\ u_z(x,y,z) \end{pmatrix} = \begin{pmatrix} u(x,y) \\ v(x,y) \\ w(x,y) \end{pmatrix} + z \begin{pmatrix} \theta_y(x,y) \\ -\theta_x(x,y) \\ 0 \end{pmatrix} = \begin{pmatrix} u(x,y) \\ v(x,y) \\ w(x,y) \end{pmatrix} + z \begin{pmatrix} \beta_x(x,y) \\ \beta_y(x,y) \\ 0 \end{pmatrix}$$

where u, v, w are displacements of the mean surface and θ_x the θ_y rotations of this surface compared to the two axes x and y respectively. One prefers to introduce two rotations $\beta_x(x,y)=\theta_y(x,y), \beta_y(x,y)=-\theta_x(x,y)$.

The three-dimensional strains in any point, with the kinematics introduced previously, are thus given by:

$$\begin{aligned} \varepsilon_{xx} &= e_{xx} + z\kappa_{xx} \\ \varepsilon_{yy} &= e_{yy} + z\kappa_{yy} \\ 2\varepsilon_{xy} &= \gamma_{xy} = 2e_{xy} + 2z\kappa_{xy} \\ 2\varepsilon_{xz} &= \gamma_x \\ 2\varepsilon_{yz} &= \gamma_y \end{aligned}$$

where e_{xx}, e_{yy} and e_{xy} are the membrane strains of mean surface, γ_x and the γ_y strains associated with the transverse shears, and the $\kappa_{xx}, \kappa_{yy}, \kappa_{xy}$ strains of bending (or variations of curvature) of mean surface, which are written:

$$e_{xx}=\frac{\partial u}{\partial x}$$
$$e_{yy}=\frac{\partial v}{\partial y}$$
$$2e_{xy}=\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}$$
$$\kappa_{xx}=\frac{\partial \beta_x}{\partial x}$$
$$\kappa_{yy}=\frac{\partial \beta_y}{\partial y}$$
$$2\kappa_{xy}=\frac{\partial \beta_x}{\partial y}+\frac{\partial \beta_y}{\partial x}$$
$$\gamma_x=\beta_x+\frac{\partial w}{\partial x}$$
$$\gamma_y=\beta_y+\frac{\partial w}{\partial y}$$

Note:

In the theories of plate the introduction of β_x and β_y makes it possible to symmetrize the formulations of the strains and, we will see it thereafter, the balance equations. In the theories of shell one uses rather θ_x and θ_y the associated couples M_x and M_y compared to x and y.

2.2.2 Constitutive law

the behavior of the plates is a behavior 3D in "plane stresses". **The transverse stress** σ_{zz} because of being null regarded as negligible compared to the other components of the tensor of the stresses (assumption of the plane stresses). The most general constitutive law is written then as follows:

$$\begin{pmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\\ \sigma_{xz}\\ \sigma_{yz}\end{pmatrix}=\mathbf{C}(e,\alpha)\begin{pmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\\ \gamma_x\\ \gamma_y\end{pmatrix}=\mathbf{Ce}+z\,\mathbf{C}\kappa+\mathbf{C}\gamma \text{ with } \boldsymbol{e}=\begin{pmatrix}e_{xx}\\ e_{yy}\\ 2e_{xy}\\ 0\\ 0\end{pmatrix},\ \kappa=\begin{pmatrix}\kappa_{xx}\\ \kappa_{yy}\\ 2\kappa_{xy}\\ 0\\ 0\end{pmatrix} \text{ et } \gamma=\begin{pmatrix}0\\ 0\\ 0\\ \gamma_x\\ \gamma_y\end{pmatrix}.$$

where $\mathbf{C}(e,\alpha)$ is the local tangent stiffness matrix combining plane stresses and transverse distortion and α represents all the local variables when the behavior is nonlinear.

For behaviors where the distortions are uncoupled from the strains of membrane and bending, $\mathbf{C}(e,\alpha)$ puts itself in the form:

$$\mathbf{C}=\begin{pmatrix}\mathbf{H} & 0\\ 0 & \mathbf{H}_{\gamma}\end{pmatrix}$$

where $\mathbf{H}(e,\alpha)$ is a matrix 3×3 and $\mathbf{H}_{\gamma}(e,\alpha)$ a matrix 2×2. One will remain in the frame of this assumption.

For an isotropic homogeneous linear behavior elastic, one has as follows:

$$\mathbf{C}=\frac{E}{1-\nu^2}\begin{pmatrix}1 & \nu & 0 & 0 & 0\\ \nu & 1 & 0 & 0 & 0\\ 0 & 0 & \frac{1-\nu}{2} & 0 & 0\\ 0 & 0 & 0 & \frac{k(1-\nu)}{2} & 0\\ 0 & 0 & 0 & 0 & \frac{k(1-\nu)}{2}\end{pmatrix}$$

where k is factor of transverse correction of shears whose meaning is given to the following paragraph.

Note:

One does not describe the variation of the thickness nor that of the transverse strain e_{zz} which one can however calculate by means of the preceding assumption of plane stresses. In addition no restriction is made on the type of behavior that one can represent.

2.2.3 Taken into account of the transverse shears [bib2]

the taking into account of the transverse shears depends on factors of correction determined a priori by energy equivalences with models 3D, so that the stiffness in transverse shears of the model of plate is nearest possible to that defined by the theory of three-dimensional elasticity. Two theories including the strain due to the shears exist and are presented in [bib2].

2.2.3.1 The theory known as of Hencky

This theory as that of Coils-Kirchhoff which results from this immediately rests on the kinematics presented to the §2.2.1. The behavior model is usual and the factor of correction of shears is worth $k=1$.

Note:

The model of Coils-Kirchhoff (DKT (G) and DKQ (G)): When one does not take into account the transverse distortions γ_x and γ_y in the theory of Hencky, the model obtained is that of Coils-Kirchhoff. Two rotations of mean surface are then related to displacements of mean surface by the following relation:

$$\beta_x=-\frac{\partial w}{\partial x}$$
$$\beta_y=-\frac{\partial w}{\partial y}$$

2.2.3.2 The theory known as of Reissner (DST, DSQ and Q4g)

the second theory, known as of Reissner, is developed starting from the stresses. The variation of the membrane stresses (σ_{xx} , σ_{yy} and σ_{xy}) is supposed to be linear in the thickness as in the case of the theory of Hencky where that results from the linearity of the variation of the strains of membrane with the thickness. However, whereas one supposes, in the theory of Hencky, the constant distortion in the thickness and thus the shearing stresses, which violates the boundary conditions $\sigma_{xz}=\sigma_{yz}=0$ on the sides higher and lower of the plate because of constitutive law stated than the §2.2.2., one uses in the frame of the theory of Reissner the balance equations to deduce the variation from it from the shearing stresses in the thickness of the plate, by in particular observing the equilibrium conditions on the sides higher and lower of plate. The internal energy of the model obtained after resolution of the balance equations in 3D, for bending only, with the variation of the plane stresses according to z , reveals, for an elastic material, a relation between the resulting forces and the rotations and the deflection averages. It is in this relation that the factor of correction of shears of instead of $k=5/6$ 1 in the relation appears which binds the shears to the distortion for a homogeneous and isotropic plate. The determination of the factors of correction of shears for orthotropic plates or stratified plates is left in appendix.

2.2.3.3 Equivalence of the approaches Hencky-Coils-Kirchhoff and Reissner

If one compares the slopes of mean surface β_x, β_y to the averages of the slopes in the thickness of the plate and the deflection w with the average deflection, the only difference between the theory of Hencky and that of Reissner is the coefficient of transverse correction of shears of $5/6$ instead of 1. This difference is due to the fact that the starting assumptions are of different nature and especially that the selected variables are not the same ones. Indeed, the deflection on mean surface is not equal to the average of the deflections on the thickness of the plate. It is thus normal that behavior models which utilize different variables are not identical.

The fact of having to solve on the level finite elements of the problems in displacements rather than of the problems in stresses by interpolation of displacements leads us to use the equivalent approach in displacements of the problem of Reissner formulated in stresses.

2.2.3.4 Remarks

Because of preceding equivalence one presents here only the model in displacement for all the elements. In the facts elements DKT, DKTG, DKQG and DKQ are based on the theory of Hencky-Coils-Kirchhoff and the elements DST, DSQ and Q4G are based on the theory of Reissner.

The determination of the factors of correction rests in the frame of another theory, that of Mindlin, on equivalences of eigenfrequency associated with the mode with vibration by transverse shears. One obtains then $k=\pi^2/12$, value very close to $5/6$ for elements DST, DSQ and Q4G in the isotropic case.

In the frame of plasticity the problem of the choice of the coefficient of correction of the transverse shears arises because the equivalent approach in displacements of the problem of Reissner formulated in stresses utilizes the non-linearity of the behavior. One cannot thus deduce some, as it is the case for elastic materials a value of the coefficient of correction of the transverse shears. Plasticity is thus not developed for these elements.

3 Principle of the virtual wors

3.1 Work of strain

the general statement of the work of strain 3D for a plate is worth:

$$W_{\text{def}}=\int_S \int_{-h/2}^{h/2} (\varepsilon_{xx}\sigma_{xx}+\varepsilon_{yy}\sigma_{yy}+\gamma_{xy}\sigma_{xy}+\gamma_x\sigma_{xz}+\gamma_y\sigma_{yz})\,dV$$

where S is mean surface and the position in the thickness of the plate varies between $-h/2$ and $+h/2$.

3.1.1 Statement of the forces resulting

By adopting the kinematics from the §2.2.1, one identifies the work of the internal forces:

$$W_{\text{def}}=\int_S (e_{xx}N_{xx}+e_{yy}N_{yy}+2e_{xy}N_{xy}+\kappa_{xx}M_{xx}+\kappa_{yy}M_{yy}+2\kappa_{xy}M_{xy}+\gamma_x T_x+\gamma_y T_y)\,dS$$ where:

$$N=\begin{pmatrix}N_{xx}\\N_{yy}\\N_{xy}\end{pmatrix}=\int_{-h/2}^{+h/2}\begin{pmatrix}\sigma_{xx}\\\sigma_{yy}\\\sigma_{xy}\end{pmatrix}dz \;\; ; \;\; M=\begin{pmatrix}M_{xx}\\M_{yy}\\M_{xy}\end{pmatrix}=\int_{-h/2}^{+h/2}\begin{pmatrix}\sigma_{xx}\\\sigma_{yy}\\\sigma_{xy}\end{pmatrix}z\,dz \;\; ; \;\; T=\begin{pmatrix}T_x\\T_y\end{pmatrix}=\int_{-h/2}^{+h/2}\begin{pmatrix}\sigma_{xz}\\\sigma_{yz}\end{pmatrix}dz \; .$$

N_{xx}, N_{yy}, N_{xy} are the forces resulting from membrane (in N/m);
M_{xx}, M_{yy}, M_{xy} are the forces resulting from bending or moments (in N);
T_x, T_y are the forces resulting from shears or shears (in N/m).

3.1.2 Relation forces resulting-strains

the statement of the work of strain is also written:

$$W_{\text{def}}=\int_s\int_{-h/2}^{h/2}[\varepsilon\,\mathbf{C}(e,\alpha)\varepsilon]dV=\int_s\int_{-h/2}^{h/2}[\mathbf{eCe}+z\,\mathbf{eC}\kappa+z\kappa\,\mathbf{Ce}+z^2\kappa\,\mathbf{C}\kappa+\gamma\,\mathbf{C}\gamma]dV$$

where $\mathbf{C}(e,\alpha)$ is the local matrix of behavior.

By means of the statement obtained for W_{def} in the preceding paragraph one and the finds the relation following between the resulting forces strains:

$$\begin{aligned}\mathbf{N}&=\mathbf{H}_{m}\mathbf{e}+\mathbf{H}_{\mathbf{mf}}\kappa\\ \mathbf{M}&=\mathbf{H}_{\mathbf{mf}}\mathbf{e}+\mathbf{H}_{f}\kappa\\ \mathbf{T}&=\mathbf{H}_{\mathbf{ct}}\gamma\end{aligned}\quad \text{with}\quad \mathbf{H}_{m}=\int_{-h/2}^{+h/2}\mathbf{H}\,dz\,,\ \mathbf{H}_{\mathbf{mf}}=\int_{-h/2}^{+h/2}\mathbf{H}\,z dz\,,\ \mathbf{H}_{f}=\int_{-h/2}^{+h/2}\mathbf{H}\,z^{2}dz$$

$$\text{where: } H_{\mathbf{ct}}=\begin{pmatrix}G_{11} & 0\\ 0 & G_{22}\end{pmatrix}\quad \mathbf{e}=\begin{pmatrix}e_{xx}\\ e_{yy}\\ 2e_{xy}\end{pmatrix},\ \kappa=\begin{pmatrix}\kappa_{xx}\\ \kappa_{yy}\\ 2\kappa_{xy}\end{pmatrix},\ \gamma=\begin{pmatrix}\gamma_{x}\\ \gamma_{y}\end{pmatrix}$$

The matrixes $\mathbf{H}_{m},\mathbf{H}_{f}\,\text{et}\,\mathbf{H}_{ct}$ are the stiffness matrixes out of membrane, bending and transverse shears, respectively. The matrix $\mathbf{H}_{mf}$ is a stiffness matrix of coupling between the membrane and bending.

For an isotropic homogeneous elastic behavior of plate these matrixes have as a statement:

$$\mathbf{H}_{m}=\frac{Eh}{1-\nu^{2}}\begin{pmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \dfrac{1-\nu}{2}\end{pmatrix},\ \mathbf{H}_{f}=\frac{Eh^{3}}{12(1-\nu^{2})}\begin{pmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \dfrac{1-\nu}{2}\end{pmatrix},\ \mathbf{H}_{\mathbf{ct}}=\frac{kEh}{2(1+\nu)}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},$$

and $\mathbf{H}_{mf}=0$ because there is material symmetry compared to the plane $z=0$.

For an orthotropic material, the behavior is given in appendix.

3.1.3 Elastic internal energy of plate

Taking into account the preceding remarks, the elastic internal energy of the plate is more usually expressed for this kind of geometry in the following way:

$$\Phi_{\text{int}}=\frac{1}{2}\int_{S}[\,\mathbf{e}(\mathbf{H}_{m}\mathbf{e}+\mathbf{H}_{\mathbf{mf}}\kappa)+\kappa(\mathbf{H}_{\mathbf{mf}}\mathbf{e}+\mathbf{H}_{f}\kappa)+\gamma\mathbf{H}_{\mathbf{ct}}\gamma\,]dS$$

that one can break up in the following way:

$$\Phi_{\text{membrane}}=\frac{1}{2}\int_{S}\mathbf{e}\,\mathbf{H}_{m}\mathbf{e}\,dS \qquad \text{energy of membrane}$$

$$\Phi_{\text{flexion}}=\frac{1}{2}\int_{S}\kappa\,\mathbf{H}_{m}\kappa\,dS \qquad \text{energy of bending}$$

$$\Phi_{\text{cisaillement}}=\frac{1}{2}\int_{S}\gamma\,\mathbf{H}_{ct}\gamma\,dS \qquad \text{energy of shears}$$

$$\Phi_{\text{couplage}}=\frac{1}{2}\int_{S}(\mathbf{e}\,\mathbf{H}_{\mathbf{mf}}\kappa+\kappa\,\mathbf{H}_{\mathbf{mf}}\mathbf{e})\,dS \qquad \text{energy of membrane-flexure coupling}$$

3.1.4 Remarks

the relations flexible $\mathbf{H}_m$ $\mathbf{H}_f$, $\mathbf{H}_{mf}$ with $\mathbf{H}$ and $\mathbf{H}_{ct}$ with $\mathbf{H}_\gamma$ are valid whatever the constitutive law: elastic, with unelastic strains (thermoelasticity, plasticity,….).

For a plate made up of N orthotropic layers in elasticity, the matrixes $\mathbf{H}_m$ $\mathbf{H}_f$, $\mathbf{H}_{mf}$ and $\mathbf{H}_{ct}$ are written:

$$\mathbf{H}_m=\sum_{i=1}^{N} h_i \mathbf{H}_i , \mathbf{H}_{mf}=\sum_{i=1}^{N} h_i \eta_i \mathbf{H}_i , \mathbf{H}_f=\sum_{i=1}^{N} \frac{1}{3}(z_{i+1}^3-z_i^3)\mathbf{H}_i , \mathbf{H}_{ct}=\sum_{i=1}^{N} h_i \mathbf{H}_{\gamma i}$$

where: $h_i=z_{i+1}-z_i , \eta_i=\frac{1}{2}\left(z_{i+1}+z_i\right)$ and $\mathbf{H}_i, \mathbf{H}_{\gamma i}$ the matrixes and $\mathbf{H}$ for $\mathbf{H}_\gamma$ the layer represent i .

The homogenization for multi-layer shells can lead to stiffness matrixes of membrane and bending nonproportional of the type:

$$\mathbf{H}_m=\begin{pmatrix} C_{1111} & C_{1122} & 0 \\ C_{1122} & C_{2222} & 0 \\ 0 & 0 & C_{1212} \end{pmatrix}, \mathbf{H}_f=\begin{pmatrix} D_{1111} & D_{1122} & 0 \\ D_{1122} & D_{2222} & 0 \\ 0 & 0 & D_{1212} \end{pmatrix}, \mathbf{H}_{ct}=\begin{pmatrix} G_{11} & 0 \\ 0 & G_{22} \end{pmatrix}$$

for which one cannot find equivalent values of the Young modulus and thickness allowing to find the classical statements of the stiffness, cf [bib7].

3.2 Work of the forces and couples external

work of the forces and couples being exerted on the plate is expressed in the following way:

$$W_{\text{ext}}=\int_S \int_{-h/2}^{+h/2} \mathbf{F}_v . \mathbf{U}\, dV+\int_S \mathbf{F}_s . \mathbf{U}\, dS+\int_C \int_{-h/2}^{+h/2} \mathbf{F}_c . \mathbf{U}\, dz\, ds$$

where $\mathbf{F}_v , \mathbf{F}_s , \mathbf{F}_c$ are the voluminal, surface forces and of contour being exerted on the plate, respectively. C is the part of the contour of the plate to which the forces of contour $\mathbf{F}_c$ are applied. With the kinematics of the §2.2.1, one determines as follows:

$$\begin{aligned} W_{\text{ext}} &=\int_S (f_x u+f_y v+f_z w+c_x \theta_x+c_y \theta_y) dS+\int_C (\phi_x u+\phi_y v+\phi_z w+\chi_x \theta_x+\chi_y \theta_y) ds \\ &=\int_S (f_x u+f_y v+f_z w+c_y \beta_x-c_x \beta_y) dS+\int_C (\phi_x u+\phi_y v+\phi_z w+\chi_y \beta_x-\chi_x \beta_y) ds \end{aligned}$$

- where are present on the plate:

f_x , f_y , f_z : surface forces acting according to x , y and z

$f_i=\int_{-h/2}^{+h/2} \mathbf{F}_v . \mathbf{e}_i dz+\mathbf{F}_s . \mathbf{e}_i$ where $\mathbf{e}_x$ and $\mathbf{e}_y$ are the basic vectors of the tangent plane and $\mathbf{e}_z$ their normal vector.

c_x , c_y : surface couples acting around the axes x and y .

$c_i=\int_{-h/2}^{+h/2} (z\, \mathbf{e}_z \wedge \mathbf{F}_v) . \mathbf{e}_i dz+(\pm\frac{h}{2} \mathbf{e}_z \wedge \mathbf{F}_s) . \mathbf{e}_i$ where $\mathbf{e}_x , \mathbf{e}_y , \mathbf{e}_z$ are the basic vectors previously definite.

- and where are present on the contour of the plate:

  ϕ_x, ϕ_y, ϕ_z : linear forces acting according to x, y and z

  $\phi_i = \int_{-h/2}^{+h/2} \mathbf{F}_c . \mathbf{e}_i dz$ where $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ are the basic vectors previously definite.

  χ_x, χ_y : linear couples around the axes x and y .

  $\chi_i = \int_{-h/2}^{+h/2} (z\,\mathbf{e}_z \wedge \mathbf{F}_c) . \mathbf{e}_i\, dz$ where $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ are the basic vectors previously definite.

Note:

The moments compared to z are null.

3.3 Principle of virtual work

It is written in the following way: $\delta W_{\text{ext}} = \delta W_{\text{def}}$ for all kinematically admissible displacements and rotations virtual.

3.3.1 Kinematics of Hencky

With this kinematics, it results from them after integration by parts of work of strain the balance equations static of the following plates:

- For the forces: $$\begin{aligned} &N_{xx,x} + N_{xy,y} + f_x = 0, \\ &N_{yy,y} + N_{xy,x} + f_y = 0, \\ &T_{x,x} + T_{y,y} + f_z = 0. \end{aligned}$$
- For the couples: $$\begin{aligned} &M_{xx,x} + M_{xy,y} - T_x + c_y = 0, \\ &M_{yy,y} + M_{xy,x} - T_y - c_x = 0. \end{aligned}$$

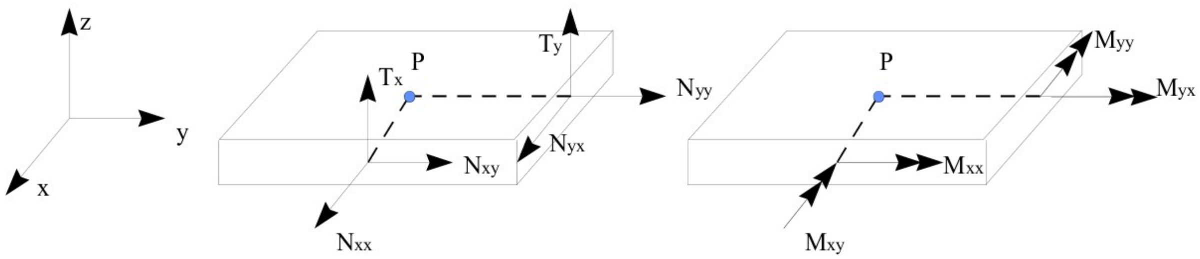
as well as the following boundary conditions on the contour C of S :

$$\begin{aligned} &N_{xx} n_x + N_{xy} n_y = \phi_x \\ &N_{yy} n_y + N_{xy} n_x = \phi_y \\ &T_x n_x + T_y n_y = \phi_z, \\ &M_{xx} n_x + M_{xy} n_y = \chi_y \\ &M_{yy} n_y + M_{xy} n_x = -\chi_x \end{aligned} \quad \text{where} \quad \begin{aligned} &u = \bar{u} \\ &v = \bar{v} \\ &w = \bar{w} \\ &\beta_x = \bar{\theta}_y \\ &\beta_y = -\bar{\theta}_x \end{aligned}$$

where n_x and n_y are the cosine directors of the norm with C directed towards the outside of the plate.

and $\bar{u}$ the trace of on u C indicates .

the physical interpretation of these forces (N , T and M) starting from the preceding equations is given Ci - below:



Appear 3.3.1-a: Resulting forces for a shell element

Note:

N_{xx}, N_{yy} the tractive efforts and N_{xy} the plane shears represent. M_{xx} and M_{yy} the couples of bending and M_{xy} the torque represent. T_x and T_y are the shearing forces transverse.

3.3.2 Kinematics of Coils-Kirchhoff

One recalls that in the frame of this kinematics, one has the following relation binding derivative of the deflection to rotations: $\beta_x=-\frac{\partial w}{\partial x}$, $\beta_y=-\frac{\partial w}{\partial y}$. After a double integration by parts of the work of strain, one obtains the following balance equations static:

- For the forces of membrane: $N_{xx,x}+N_{xy,y}+f_x=0,\quad N_{yy,y}+N_{xy,x}+f_y=0,$
- For the transverse shears and bending stresses:

$$\begin{aligned}&M_{xx,xx}+2\mathrm{M}_{xy,xy}+M_{yy,yy}+f_z+c_{y,x}-c_{x,y}=0,\\&M_{xx,x}+M_{xy,y}-T_x+c_y=0,\\&M_{yy,y}+M_{xy,x}-T_y-c_x=0.\end{aligned}$$

as well as the boundary conditions on the contour C and at the angular points O of the contour C of S :

$$\begin{aligned}&N_{xx}n_x+N_{xy}n_y=\phi_x,\\&N_{yy}n_y+N_{xy}n_x=\phi_y,\\&T_n+M_{ns,s}=\phi_z-\chi_{n,s},\\&M_{nn}=\chi_s,\\&M_{ns}(O+)-M_{ns}(O-)=-[\chi_n(O+)-\chi_n(O-)].\end{aligned}\qquad \text{where}\qquad \begin{aligned}&u=\bar{u}\\&v=\bar{v}\\&w=\bar{w}\\&\beta_n=-\bar{w}_{,n}=\bar{\theta}_s\end{aligned}$$

with
$$\begin{aligned}&T_n=T_xn_x+T_yn_y,\\&M_{nn}=M_{xx}n_x^2+2\mathrm{M}_{xy}n_xn_y+M_{yy}n_y^2,\\&M_{ns}=-M_{xx}n_xn_y+M_{xy}(n_x^2-n_y^2)+M_{yy}n_xn_y.\end{aligned}$$
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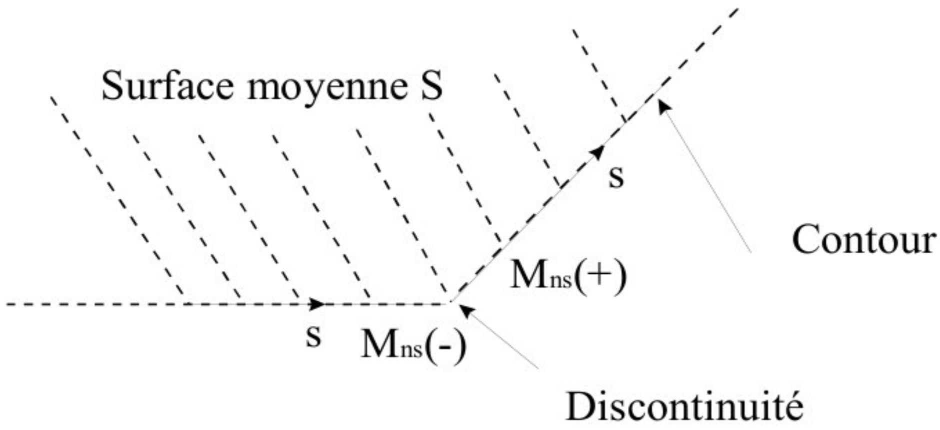


Appear 3.3.2-a: Boundary condition with angular points for a shell element

Note:

The kinematics of Coils-Kirchhoff implies that on the contour of the plate the transverse shearing force is related to the twisting moment. It is noted that the order of the balance equations of bending is higher than with the kinematics of Hencky. Thus, to choose the kinematics of Coils - Kirchhoff amounts increasing the degree of the interpolation functions because one needs a larger regularity for the terms of deflection compared to the terms of membrane because of presence of second derivative of the deflection in the statement of the work of the strains. No shell element of the Code_Aster uses this kinematics. One can thus have differences between the results got with the elements of the Code_Aster and of the analytical results got by means of the kinematics of Coils-Kirchhoff for structures with angular contours.

3.3.3 Principal boundary conditions met [bib1]

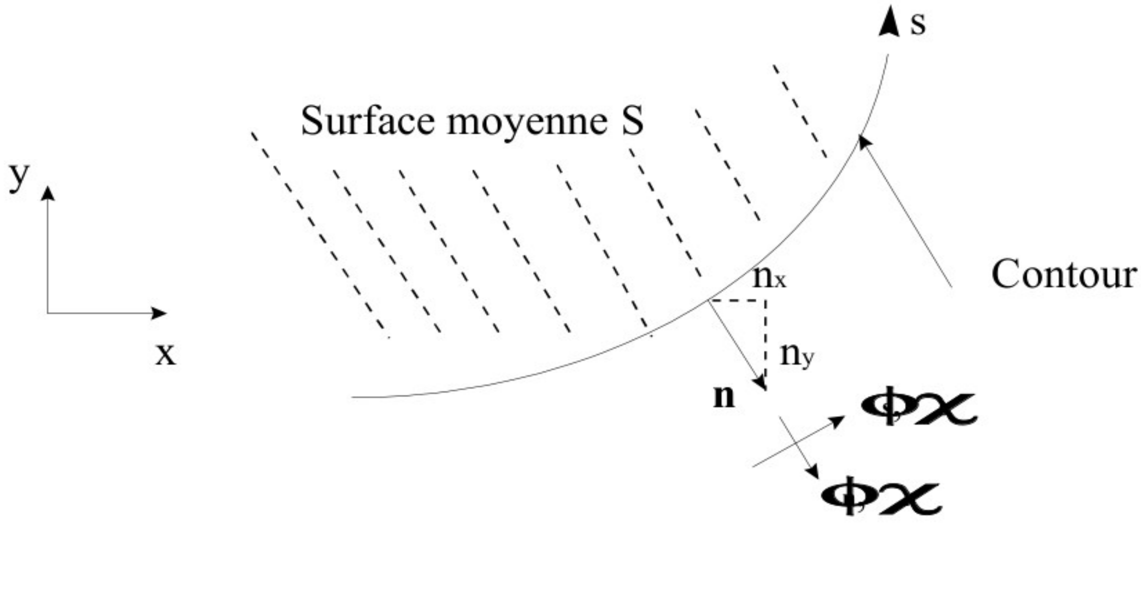


Figure 3.3.3-a: Boundary condition for a shell element

the boundary conditions frequently met are gathered in the table which follows. They are given for the kinematics of Hencky in the reference defined by S and the norm external with the plate:

Fixed support	Support simple	free Edge	Symmetry compared to an axis s	Antisymetry compared to an axis s
$\bar{u}=0$, $\bar{v}=0$, $\bar{w}=0$, $\bar{\theta}_s=0$, $\bar{\theta}_n=0$.	$\bar{u}_n=0$, $\bar{w}=0$, $\bar{\theta}_n=0$.		$\bar{u}_n=0$, $\bar{\theta}_s=0$.	$\bar{u}_s=0$, $\bar{w}=0$, $\bar{\theta}_n=0$.
	$\phi_s=0$, $\chi_s=0$.	$\phi_s=0$, $\phi_n=0$, $\phi_z=0$, $\chi_s=0$, $\chi_n=0$	$\phi_s=0$, $\phi_z=0$, $\chi_n=0$.	$\phi_n=0$, $\chi_s=0$.

with:

$$
\begin{aligned}
& u_n = u n_x + v n_y \, ; u_s = - u n_y + v n_x \, , \\
& \theta_n = \theta_x n_x + \theta_y n_y \, ; \theta_s = -\theta_x n_y + \theta_y n_x \, , \\
& \phi_n = \phi_x n_x^2 + 2\,\phi_{xy} n_x n_y + \phi_y n_y^2 \, , \\
& \phi_s = -\phi_x n_x n_y + \phi_{xy} (n_x^2 - n_y^2) + \phi_y n_x n_y \, , \\
& \chi_n = \chi_x n_x^2 + 2\,\chi_{xy} n_x n_y + \chi_y n_y^2 \, , \\
& \chi_s = -\chi_x n_x n_y + \chi_{xy} (n_x^2 - n_y^2) + \chi_y n_x n_y \, .
\end{aligned}
$$

Note: one numerical

$$
\begin{aligned}
& \beta_s = -\theta_n \, , \\
& \beta_n = \theta_s \, .
\end{aligned}
$$

4 A. Discrétisation of the formulation variational resulting from the principle of virtual work

4.1 Introduction

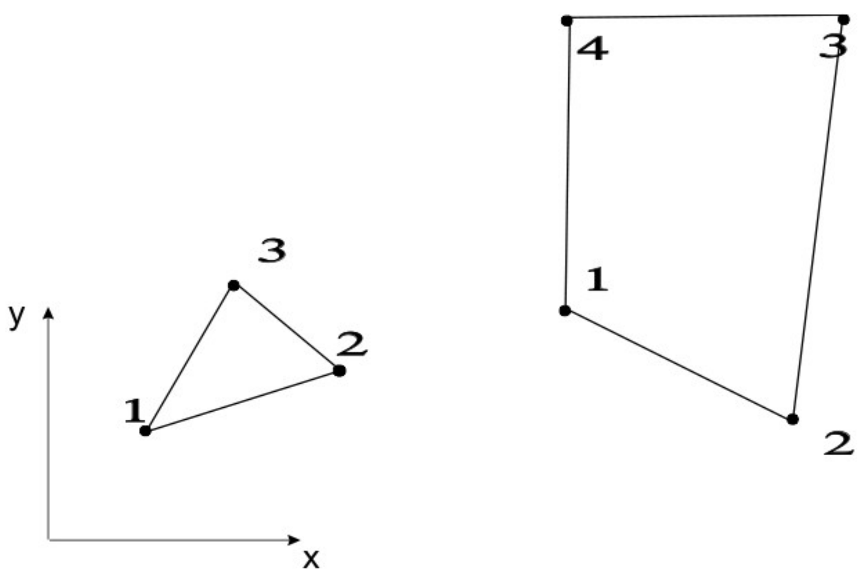
By exploiting the constitutive law, the virtual wor of the internal forces is written (with $H_{\mathbf{mf}}=0$ until the §4.4, which does not remove anything with the generality following results, but allows to reduce the notations):

$$\delta W_{\text{int}}=\int_S (\delta \mathbf{e} \mathbf{H}_{\boldsymbol{m}} \mathbf{e}+\delta \kappa \mathbf{H}_{\boldsymbol{f}} \kappa+\delta \gamma \mathbf{H}_{\boldsymbol{ct}} \gamma) dS$$

with: $\mathbf{e}=\begin{pmatrix} u_{,x} \\ v_{,y} \\ u_{,y}+v_{,x} \end{pmatrix}, \kappa=\begin{pmatrix} \beta_{x,x} \\ \beta_{y,y} \\ \beta_{x,y}+\beta_{y,x} \end{pmatrix}, \gamma=\begin{pmatrix} w_{,x}+\beta_x \\ w_{,y}+\beta_y \end{pmatrix}$.

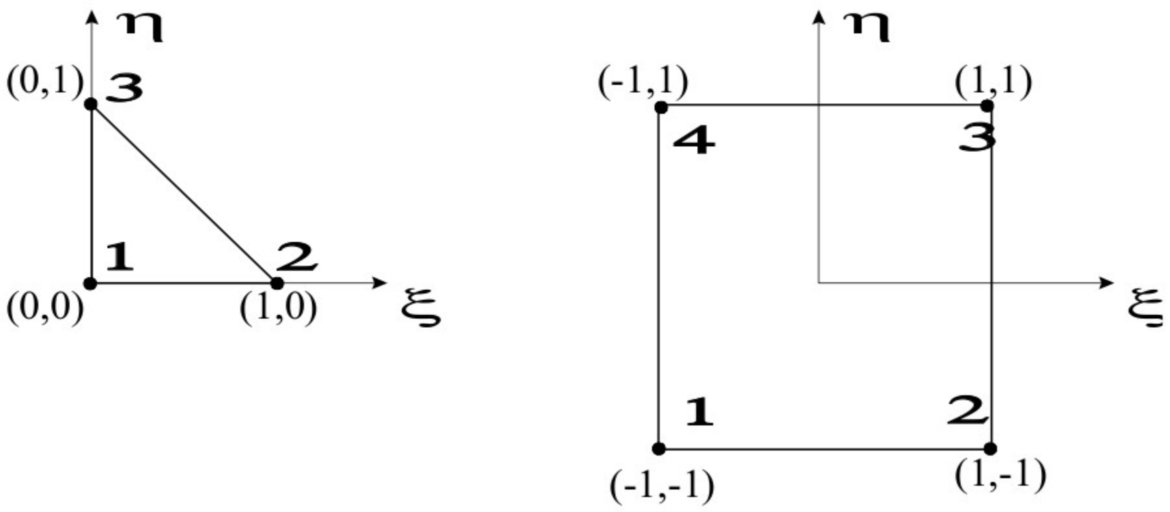
It results from it that the shell elements are elements with five degrees of freedom per node. These degrees of freedom are displacements in the plane of the element u and v , except plane w and the two rotations β_x and β_y .

Elements DKT, DKTG and DST are triangular isoparametric elements. Elements DKQ, DKQG, DSQ and $\mathrm{Q4}\gamma$ are quadrilateral isoparametric elements. They are represented below:



Appear 4.1-a: Real elements

the elements of reference are presented below:



Appear 4.1-b: Elements of reference triangle and quadrangle

One defines the reduced reference of the element as the reference (ξ,η) of the element of reference. The local coordinate system of the element, in its plane (x,y) is defined by the user. The direction $X1$ of this local coordinate system is the projection of a direction of reference $\underline{d}$ as regards the element. This direction of reference $\underline{d}$ is chosen by the user who defines it by two nautical angles in the total reference. The norm N with the plane of the element ($12\wedge 13$ for a triangle numbered 123 and $12\wedge 14$ a quadrangle numbered 1234) fixes the second direction. The cross product of the two vectors previously definite $\mathrm{Y1}=\mathrm{N}\wedge\mathrm{X1}$ makes it possible to define the local trihedron in which will be expressed the generalized forces representing the stress state. The user will have to take care that the selected reference axis is not found parallel with the norm of certain shell elements. By default, the direction of reference $\underline{d}$ is the axis X of the total reference of definition of the mesh.

The essential difference between elements DKT, DKQ, DKTG, DKQG on the one hand and DST, DSQ, $\mathrm{Q4}\gamma$ on the other hand comes owing to the fact that for the first the transverse distortion is null, that is to say still $\gamma=0$. The difference between $\mathrm{Q4}\gamma$ and the elements DST and DSQ comes from a choice different of interpolation for the representation of the transverse shears.

4.2 Discretization of the field of displacement

If one discretizes the fields of displacement in the usual way for isoparametric elements i.e.:

$$u=\sum_{i=1}^{N} N_i u_i,\; v=\sum_{i=1}^{N} N_i v_i,\; w=\sum_{i=1}^{N} N_i w_i,\; \beta_x=\sum_{i=1}^{N} N_i \beta_{xi},\; \beta_y=\sum_{i=1}^{N} N_i \beta_{yi},$$

and that one introduces this discretization into the variational formulation of the §4.1 it results from it a blocking in transverse shears analyzed in [bib1] which returns the solution in bending controlled by the effects of transverse shears, and not by bending, when the thickness of the plate becomes small compared to its characteristic dimension.

To cure this disadvantage the variational form presented in introduction is slightly modified so that:

$$\delta W_{\mathrm{int}}=\int_S (\delta\,\mathbf{e}\mathbf{H}_{\boldsymbol{m}}\mathbf{e}+\delta\kappa\,\mathbf{H}_{\boldsymbol{f}}\kappa+\delta\bar{\gamma}\,\mathbf{H}_{\mathbf{ct}}\bar{\gamma})dS=\int_S (\delta\,\mathbf{e}\mathbf{H}_{\boldsymbol{m}}\mathbf{e}+\delta\kappa\,\mathbf{H}_{\boldsymbol{f}}\kappa+\delta\,\mathbf{T}\mathbf{H}_{\mathbf{ct}}^{-1}\mathbf{T})\,dS$$

where $\bar{\gamma}$ are strains of substitution checking $\bar{\gamma}=\gamma$ in a weak way (integral on the sides of the element) and such as $\mathbf{T}=\mathbf{H}_{\mathbf{ct}}\bar{\gamma}$. One checks thus that on the sides ij of the element $\int_i^j(\bar{\gamma}_s-\gamma_s)ds=0$ with $\gamma_s=w_{,s}+\beta_s$.

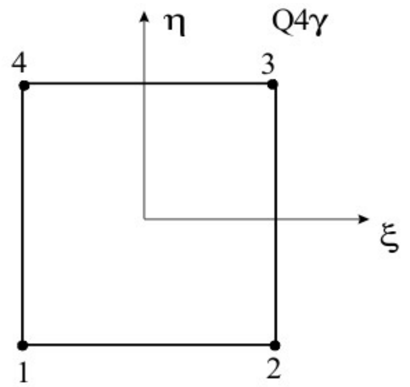
Two approaches are then possible; in the first, that of the element $\mathrm{Q4}\gamma$, one uses the bilinear discretization of the fields of displacement and the fact that $\bar{\gamma}$ is constant on the sides of the element. The relations on the sides ij then make it possible to express the values of $\bar{\gamma}$ on the sides according to the degrees of freedom of bending. In the second approach, which is that of the elements of type DKT, DKTG and DST, one uses the weak formulation of the preceding paragraph which makes it possible to bind bending to the shearing forces to deduce the interpolation from it from the terms of bending.

4.2.1 Q4gamma It

approaches rests on the linear discretization of the fields of displacement presented above:

$$u=\sum_{i=1}^{N}N_i u_i\,,\ v=\sum_{i=1}^{N}N_i v_i\,,\ w=\sum_{i=1}^{N}N_i w_i\,,\ \beta_x=\sum_{i=1}^{N}N_i\beta_{xi}\,,\ \beta_y=\sum_{i=1}^{N}N_i\beta_{yi}\,,$$

where the functions N_i are given below.

	N_i ($i=1,n$)
	$i=1$ with 4 $N_1(\xi,\eta)=\frac{1}{4}(1-\xi)(1-\eta)$ $N_2(\xi,\eta)=\frac{1}{4}(1+\xi)(1-\eta)$ $N_3(\xi,\eta)=\frac{1}{4}(1+\xi)(1+\eta)$ $N_4(\xi,\eta)=\frac{1}{4}(1-\xi)(1+\eta)$

Functions N_i for the elements $\mathrm{Q4}\gamma$

Note:

> *One also notes* $N_i(\xi,\eta)=\frac{1}{4}(1+\xi_i\xi)(1+\eta_i\eta)$ *with* $(\xi_1,\xi_2,\xi_3,\xi_4)=(-1,1,1,-1)$ *and* $(\eta_1,\eta_2,\eta_3,\eta_4)=(-1,-1,1,1)$.

4.2.2 Approaches DKT, DKQ, DKTG, DKQG, DST, DSQ

Like $T_x = M_{xx,x} + M_{xy,y}$ et $T_y = M_{yy,y} + M_{xy,x}$ and $\mathbf{M} = \mathbf{H}_f \kappa$ one from of deduced that $\bar{\gamma}$ is defined according to second derivative of β_x and β_y via two balance equations internal and of the constitutive law in bending. The discretization retained for β_x and β_y , such as β_s is quadratic on the sides and β_n linear, then utilizes of the incomplete quadratic shape functions in the form:

$$\beta_x = \sum_{k=1}^{N} N_k \beta_{xk} + \sum_{k=N+1}^{2N} P_{xk} \alpha_k ,\ \beta_y = \sum_{k=1}^{N} N_k \beta_{yk} + \sum_{k=N+1}^{2N} P_{yk} \alpha_k \text{ with } P_{xk} = P_k C_k \text{ et } P_{yk} = P_k S_k$$

where C_k and S_k are the cosine and directing sines on the side ij to which the node belongs k defined by: $C_k = x_{ji} / L_k = (x_j - x_i)/L_k ; S_k = y_{ij} / L_k = (y_j - y_i)/L_k ; L_k = \left(x_{ji}^2 + y_{ji}^2\right)^{1/2}$.

Note:

To introduce the preceding discretization amounts adding like degrees of freedom with the element of rotations α_k *in the middle of the sides* k *of the element. Indeed, rotations* β_s *and* β_n *such as:*

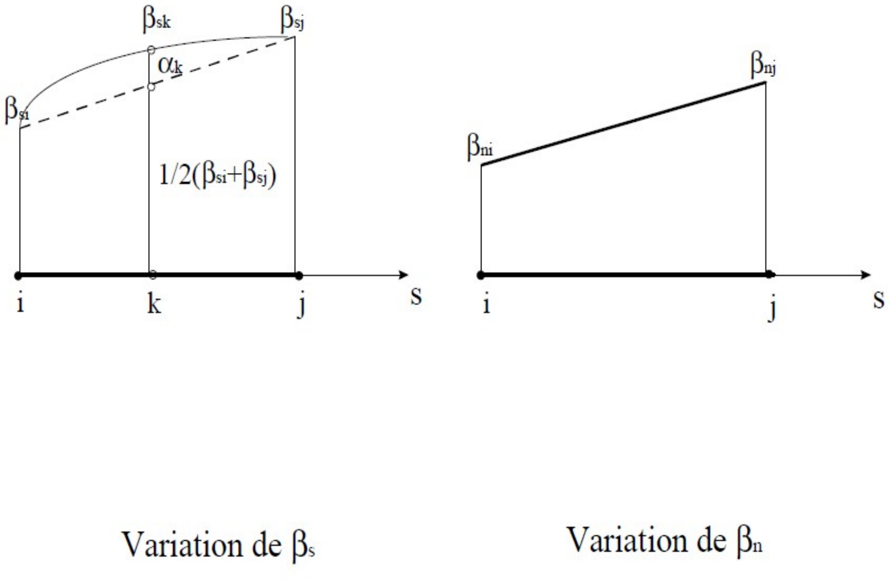
$$\begin{pmatrix} \beta_s \\ \beta_n \end{pmatrix} = \begin{pmatrix} C & S \\ S & -C \end{pmatrix} \begin{pmatrix} \beta_x \\ \beta_y \end{pmatrix}$$

are quadratic for β_s *and linear for* β_n *with:*

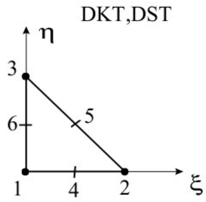
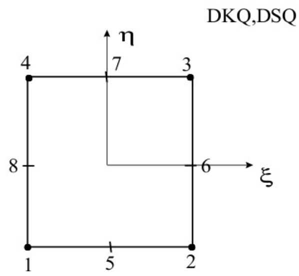
$\beta_s = (1-s')\beta_{si} + s'\beta_{sj} + 4s'(1-s')\alpha_k ; \beta_n = (1-s')\beta_{ni} + s'\beta_{nj}$ *where* $0 \le s' = s/L_k \le 1$.

One observes thus that: $\beta_{sk} = \beta_s(s' = \frac{1}{2}) = \frac{1}{2}(\beta_{si} + \beta_{sj}) + \alpha_k$.

It is the relation $\int_i^j (\bar{\gamma}_s - \gamma_s) ds = 0$ with $\gamma_s = w_{,s} + \beta_s$ which will make it possible to eliminate the additional degrees of freedom and to express them according to displacements and of nodal rotations.



Appears 4.2.2-a: Variations of β_s and β_n

	N_i $(i=1,n)$	P_i $(i=n+1,2n)$
	$i=1$ with 3 $N_1(x,\eta)=\lambda=1-\xi-\eta$ $N_2(x,\eta)=\xi$ $N_3(x,\eta)=\eta$	$i=4\quad 6$ $P_4(\xi,\eta)=4\,\xi\,\lambda$ $P_5(\xi,\eta)=4\,\xi\,\eta$ $P_6(\xi,\eta)=4\,\eta\,\lambda$
	$i=1$ with 4 $N_1(\xi,\eta)=\frac{1}{4}(1-\xi)(1-\eta)$ $N_2(\xi,\eta)=\frac{1}{4}(1+\xi)(1-\eta)$ $N_3(\xi,\eta)=\frac{1}{4}(1+\xi)(1+\eta)$ $N_4(\xi,\eta)=\frac{1}{4}(1-\xi)(1+\eta)$	$i=5\quad 8$ $P_5(\xi,\eta)=\frac{1}{2}(1-\xi^2)(1-\eta)$ $P_6(\xi,\eta)=\frac{1}{2}(1-\eta^2)(1+\xi)$ $P_7(\xi,\eta)=\frac{1}{2}(1-\xi^2)(1+\eta)$ $P_8(\xi,\eta)=\frac{1}{2}(1-\eta^2)(1+\xi)$

Functions N_i and P_i for elements DKT, DST, DKTG, DKQG, DKQ, DSQ

4.3 Discretization of the strain field

the jacobian matrix $\mathbf{J}(\xi,\eta)$ are:

$$\mathbf{J}=\begin{pmatrix} x_{,\xi} & y_{,\xi} \\ x_{,\eta} & y_{,\eta} \end{pmatrix}=\begin{pmatrix} \sum_{i=1}^{N} N_{i,\xi}x_i & \sum_{i=1}^{N} N_{i,\xi}y_i \\ \sum_{i=1}^{N} N_{i,\eta}x_i & \sum_{i=1}^{N} N_{i,\eta}y_i \end{pmatrix}=\begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}.$$

Moreover:

$$\begin{pmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{pmatrix}=j\begin{pmatrix} \dfrac{\partial}{\partial \xi} \\ \dfrac{\partial}{\partial \eta} \end{pmatrix} \text{avec } \mathbf{j}=\begin{pmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{pmatrix}=\mathbf{J}^{-1}=\frac{1}{J}\begin{pmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{pmatrix} \text{où } J=\det\mathbf{J}=J_{11}J_{22}-J_{12}J_{21}$$

It is pointed out that the field of displacement is discretized by:

$$\begin{pmatrix} u \\ v \end{pmatrix}=\sum_{k=1}^{N} N_k(\xi,\eta)\begin{pmatrix} u^k \\ v^k \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} w \\ \beta_x \\ \beta_y \end{pmatrix}=\sum_{k=1}^{N} N_k(\xi,\eta)\begin{pmatrix} w_k \\ \beta_{xk} \\ \beta_{yk} \end{pmatrix}+[\sum_{k=N+1}^{2N}\begin{pmatrix} 0 \\ P_{xk}(\xi,\eta) \\ P_{yk}(\xi,\eta) \end{pmatrix}\alpha_k],$$

the term between hooks being present for the elements of type DKT, DKTG, DST, but not for the elements $Q4\gamma$.

Note: in *Code_Aster*, the jacobian matrix makes it possible to pass from the element of reference to the local coordinate system of the element (and not the total reference) because it is easier to work in this reference.

4.3.1 Discretization of the membrane strain field:

$$e_{xx}=u_{,x}=\sum_{k=1}^{N} N_{k,x}(\xi,\eta)u^k=\sum_{k=1}^{N}(j_{11}N_{k,\xi}+j_{12}N_{k,\eta})u^k,$$
$$e_{yy}=v_{,y}=\sum_{k=1}^{N} N_{k,y}(\xi,\eta)v^k=\sum_{k=1}^{N}(j_{21}N_{k,\xi}+j_{22}N_{k,\eta})v^k,$$
$$2e_{xy}=u_{,x}+v_{,y}=\sum_{k=1}^{N} N_{k,y}(\xi,\eta)u^k+N_{k,x}(\xi,\eta)v^k$$
$$=\sum_{k=1}^{N}(j_{21}N_{k,\xi}+j_{22}N_{k,\eta})u^k+(j_{11}N_{k,\xi}+j_{12}N_{k,\eta})v^k$$

Maybe in matric form:

$\begin{pmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{pmatrix}=\sum_{k=1}^{N}\mathbf{B_{mk}}\mathbf{U_k}$ where $\mathbf{U}_{\boldsymbol{k}}=\begin{pmatrix} u_k \\ v_k \end{pmatrix}$ is the membrane field of displacement to the node k

and:

$$B_{\mathbf{mk}}=\begin{pmatrix} j_{11}N_{k,\xi}+j_{12}N_{k,\eta} & 0 \\ 0 & j_{21}N_{k,\xi}+j_{22}N_{k,\eta} \\ j_{21}N_{k,\xi}+j_{22}N_{k,\eta} & j_{11}N_{k,\xi}+j_{12}N_{k,\eta} \end{pmatrix}$$

The transition matrix of the membrane strains at the field of displacement $U_m=\begin{pmatrix} u_1 \\ v_1 \\ ... \\ u_N \\ v_N \end{pmatrix}$ in the plane of the element is written as follows: $\mathbf{B}_{\boldsymbol{m}[\mathbf{3\times 2N}]}=\left(\mathbf{B}_{\mathbf{m1}}\ldots\mathbf{B}_{\mathbf{mN}}\right)$.

4.3.2 Discretization of the transverse distortion

4.3.2.1 For the elements Q4g

One linearly discretizes the constant $\bar{\gamma}$ field by side so that:

$$\bar{\gamma}=\boldsymbol{j}\,\bar{\gamma}^{ref}$$
$$\bar{\gamma}^{ref}=\begin{pmatrix} \bar{\gamma}_\xi \\ \bar{\gamma}_\eta \end{pmatrix}=\begin{pmatrix} \frac{1-\eta}{2}\gamma_\xi^{12}+\frac{1+\eta}{2}\gamma_\xi^{34} \\ \frac{1-\xi}{2}\gamma_\eta^{23}+\frac{1-\xi}{2}\gamma_\eta^{41} \end{pmatrix}$$

where $\bar{\gamma}^{ref}$ is the transverse field of distortion in the element of reference.

By means of then the relations (cf 4.2): $\begin{matrix}\int_{-1}^{+1}(\bar{\gamma}_{\xi}-(w_{,\xi}+\beta_{\xi}))\,d\xi=0\,;\\ \int_{-1}^{+1}(\bar{\gamma}_{\eta}-(w_{,\eta}+\beta_{\eta}))\,d\eta=0\end{matrix}$ for $\begin{matrix}\xi=\pm 1\\ \eta=\pm 1\end{matrix}$,

one establishes that: $\begin{matrix}\gamma_{\xi}^{ij}=\frac{1}{2}(w_j-w_i+\beta_{\xi i}+\beta_{\xi j});\\ \gamma_{\eta}^{kp}=\frac{1}{2}(w_p-w_k+\beta_{\eta p}+\beta_{\eta k});\end{matrix}$ for $(ij)\in(12{,}34)$ and $(kp)\in(23{,}41)$.

By deferring the two results above in the statement of $\bar{\gamma}^{loc}$, one from of deduced that:

$$\bar{\gamma}^{ref}=\begin{pmatrix}\bar{\gamma}_{\xi}\\ \bar{\gamma}_{\eta}\end{pmatrix}=\begin{pmatrix}\mathbf{B}_{\xi}^{ref}\,u_{\xi}^{ref}\\ \mathbf{B}_{\eta}^{ref}\,u_{\eta}^{ref}\end{pmatrix}=\mathbf{B}^{ref}\,u^{ref}\quad\text{where}\quad u^{ref}=\begin{vmatrix}w_1\\ \beta_{\xi 1}\\ \beta_{\eta 1}\\ \vdots\\ w_N\\ \beta_{\xi N}\\ \beta_{\eta N}\end{vmatrix}\quad\text{and}\quad \mathbf{B}^{ref}=(\mathbf{B}_1,\ldots,\mathbf{B}_N)\quad\text{with}$$

$$\mathbf{B}_k=\begin{pmatrix}N_{k,\xi} & \xi_k N_{k,\xi} & 0\\ N_{k,\eta} & 0 & \eta_k N_{k,\eta}\end{pmatrix}\quad N=4\quad k\in[1,N]\quad \xi_k,\ \eta_k \text{ are defined to 4.2.1.}$$

It is now necessary to express the rotations given here in the element of reference according to rotations in the local coordinate system.

Like $\begin{pmatrix}\beta_{\xi k}\\ \beta_{\eta k}\end{pmatrix}=\boldsymbol{J}_k\begin{pmatrix}\beta_{xk}\\ \beta_{yk}\end{pmatrix}=\begin{pmatrix}J_{11k} & J_{12k}\\ J_{21k} & J_{22k}\end{pmatrix}\begin{pmatrix}\beta_{xk}\\ \beta_{yk}\end{pmatrix}$ one from of deduced that $\bar{\gamma}^{ref}=\mathbf{B}_{loc}u_{loc}$ where

$$u_{loc}=\begin{vmatrix}w_1\\ \beta_{x1}\\ \beta_{y1}\\ \vdots\\ w_N\\ \beta_{xN}\\ \beta_{yN}\end{vmatrix}\quad\text{and}\quad \mathbf{B}_{loc}=(\mathbf{B}_{loc\,1},\ldots,\mathbf{B}_{loc\,N})\quad\text{with}\quad \mathbf{B}_{loc\,k}=\begin{pmatrix}N_{k,\xi} & \xi_k N_{k,\xi}J_{11k} & \xi_k N_{k,\xi}J_{12k}\\ N_{k,\eta} & \eta_k N_{k,\eta}J_{21k} & \eta_k N_{k,\eta}J_{22k}\end{pmatrix}.$$

It will be noticed that the jacobian matrix $\boldsymbol{J}_k$ is expressed in each point of the element.

Finally: $\bar{\gamma}=\begin{pmatrix}\bar{\gamma}_x\\ \bar{\gamma}_y\end{pmatrix}=\begin{pmatrix}j_{11} & j_{12}\\ j_{21} & j_{22}\end{pmatrix}\bar{\gamma}^{ref}=\mathbf{B}_c u_{loc}$ with $\mathbf{B}_{c[2\times 3N]}=\mathbf{j}\mathbf{B}_{loc}$.

4.3.2.2 For the elements of type DKT, DST, DKTG

With regard to the transverse distortions one deduces from $T_x = M_{xx,x} + M_{xy,y}$ et $T_y = M_{yy,y} + M_{xy,x}$ with $\mathbf{M} = \mathbf{H}_f \kappa$ that $\mathbf{T} = \bar{\mathbf{H}}_f \beta_{,xx}$ where:

$\beta^T_{,xx} = \begin{pmatrix} \beta_{x,xx} & \beta_{x,yy} & \beta_{x,xy} & \beta_{y,xx} & \beta_{y,yy} & \beta_{y,xy} \end{pmatrix}$ and $\bar{\mathbf{H}}_f = \begin{pmatrix} H_{11} & H_{33} & 2H_{13} & H_{13} & H_{23} & H_{12}+H_{33} \\ H_{13} & H_{23} & H_{12}+H_{33} & H_{33} & H_{22} & 2H_{23} \end{pmatrix}$ where are $\mathbf{H_{ij}}$ to them the terms (i,j) of $\mathbf{H_f}$.

$$\beta_{x,xx} = \sum_{k=1}^{N} N_{k,xx}(\xi,\eta)\beta_{xk} + \sum_{k=N+1}^{2N} P_{xk,xx}(\xi,\eta)\alpha_k = \sum_{k=1}^{N} N_{k,xx}(\xi,\eta)\beta_{xk} + \sum_{k=N+1}^{N} (j_{11}^2 P_{xk,\xi\xi} + 2j_{11} j_{12} P_{xk,\xi\eta} + j_{12}^2 P_{xk,\eta\eta})\alpha_k,$$

$$\beta_{x,yy} = \sum_{k=1}^{N} N_{k,yy}(\xi,\eta)\beta_{xk} + \sum_{k=N+1}^{2N} P_{xk,yy}(\xi,\eta)\alpha_k = \sum_{k=1}^{N} N_{k,yy}(\xi,\eta)\beta_{xk} + \sum_{k=N+1}^{N} (j_{21}^2 P_{xk,\xi\xi} + 2j_{21} j_{22} P_{xk,\xi\eta} + j_{22}^2 P_{xk,\eta\eta})\alpha_k,$$

$$\beta_{x,xy} = \sum_{k=1}^{N} N_{k,xy}(\xi,\eta)\beta_{xk} + \sum_{k=N+1}^{2N} P_{xk,xy}(\xi,\eta)\alpha_k$$

$$= \sum_{k=1}^{N} N_{k,xy}(\xi,\eta)\beta_{xk} + \sum_{k=N+1}^{N} (j_{11} j_{21} P_{xk,\xi\xi} + [j_{11} j_{22} + j_{12} j_{21}] P_{xk,\xi\eta} + j_{12} j_{22} P_{xk,\eta\eta})\alpha_k,$$

$$\beta_{y,xx} = \sum_{k=1}^{N} N_{k,xx}(\xi,\eta)\beta_{yk} + \sum_{k=N+1}^{2N} P_{yk,xx}(\xi,\eta)\alpha_k = \sum_{k=1}^{N} N_{k,xx}(\xi,\eta)\beta_{yk} + \sum_{k=N+1}^{N} (j_{11}^2 P_{yk,\xi\xi} + 2j_{11} j_{12} P_{yk,\xi\eta} + j_{12}^2 P_{yk,\eta\eta})\alpha_k$$

$$\beta_{y,yy} = \sum_{k=1}^{N} N_{k,yy}(\xi,\eta)\beta_{yk} + \sum_{k=N+1}^{2N} P_{yk,yy}(\xi,\eta)\alpha_k = \sum_{k=1}^{N} N_{k,yy}(\xi,\eta)\beta_{yk} + \sum_{k=N+1}^{N} (j_{21}^2 P_{yk,\xi\xi} + 2j_{21} j_{22} P_{yk,\xi\eta} + j_{22}^2 P_{yk,\eta\eta})\alpha_k,$$

$$\beta_{y,xy} = \sum_{k=1}^{N} N_{k,xy}(\xi,\eta)\beta_{yk} + \sum_{k=N+1}^{2N} P_{yk,xy}(\xi,\eta)\alpha_k$$

$$= \sum_{k=1}^{N} N_{k,xy}(\xi,\eta)\beta_{yk} + \sum_{k=N+1}^{N} (j_{11} j_{21} P_{yk,\xi\xi} + [j_{11} j_{22} + j_{12} j_{21}] P_{yk,\xi\eta} + j_{12} j_{22} P_{yk,\eta\eta})\alpha_k$$

where P_{xk} , P_{yk} and α_k are defined into 4.2.2

is still in matric form:

$$\mathbf{T}=\bar{\mathbf{H}}_f\begin{pmatrix}\beta_{x,xx}\\ \beta_{x,yy}\\ \beta_{x,xy}\\ \beta_{y,xx}\\ \beta_{y,yy}\\ \beta_{y,xy}\end{pmatrix}=$$

$$\bar{\mathbf{H}}_f\sum_{k=1}^{N}\begin{pmatrix}0 & j_{11}^2N_{k,\xi\xi}+2j_{11}j_{12}N_{k,\xi\eta}+j_{12}^2N_{k,\eta\eta} & 0\\ 0 & j_{21}^2N_{k,\xi\xi}+2j_{21}j_{22}N_{k,\xi\eta}+j_{22}^2N_{k,\eta\eta} & 0\\ 0 & j_{11}j_{21}N_{k,\xi\xi}+[j_{11}j_{22}+j_{12}j_{21}]N_{k,\xi\eta}+j_{12}j_{22}N_{k,\eta\eta} & 0\\ 0 & 0 & j_{11}^2N_{k,\xi\xi}+2j_{11}j_{12}N_{k,\xi\eta}+j_{12}^2N_{k,\eta\eta}\\ 0 & 0 & j_{21}^2N_{k,\xi\xi}+2j_{21}j_{22}N_{k,\xi\eta}+j_{22}^2N_{k,\eta\eta}\\ 0 & 0 & j_{11}j_{21}N_{k,\xi\xi}+[j_{11}j_{22}+j_{12}j_{21}]N_{k,\xi\eta}+j_{12}j_{22}N_{k,\eta\eta}\end{pmatrix}\begin{pmatrix}w_k\\ \beta_{xk}\\ \beta_{yk}\end{pmatrix}$$

$$+\bar{\mathbf{H}}_f\sum_{k=1}^{2N}\alpha_k\begin{pmatrix}C_k(j_{11}^2P_{k,\xi\xi}+2j_{11}j_{12}P_{k,\xi\eta}+j_{12}^2P_{k,\eta\eta})\\ C_k(j_{21}^2P_{k,\xi\xi}+2j_{21}j_{22}P_{k,\xi\eta}+j_{22}^2P_{k,\eta\eta})\\ C_k(j_{11}j_{22}P_{k,\xi\xi}+[j_{11}j_{22}+j_{12}j_{21}]P_{k,\xi\eta}+j_{12}j_{22}P_{k,\eta\eta})\\ S_k(j_{11}^2P_{k,\xi\xi}+2j_{11}j_{12}P_{k,\xi\eta}+j_{12}^2P_{k,\eta\eta})\\ S_k(j_{21}^2P_{k,\xi\xi}+2j_{21}j_{22}P_{k,\xi\eta}+j_{22}^2P_{k,\eta\eta})\\ S_k(j_{11}j_{22}P_{k,\xi\xi}+[j_{11}j_{22}+j_{12}j_{21}]P_{k,\xi\eta}+j_{12}j_{22}P_{k,\eta\eta})\end{pmatrix}=$$

$$\bar{\mathbf{H}}_f\sum_{k=1}^{N}\mathbf{P}_{f\beta k}\begin{pmatrix}w_k\\ \beta_{xk}\\ \beta_{yk}\end{pmatrix}+\sum_{k=N+1}^{2N}\bar{\mathbf{H}}_f\mathbf{T_2}\begin{pmatrix}C_kP_{k,\xi\xi}\\ C_kP_{k,\eta\eta}\\ C_kP_{k,\xi\eta}\\ S_kP_{k,\xi\xi}\\ S_kP_{k,\eta\eta}\\ S_kP_{k,\xi\eta}\end{pmatrix}\alpha_k=\bar{\mathbf{H}}_f\sum_{k=1}^{N}\mathbf{P}_{f\beta k}\mathbf{U}_{f\beta k}+\bar{\mathbf{H}}_f\mathbf{T_2}\sum_{k=N+1}^{2N}\mathbf{T}_{ck}\alpha_k=\bar{\mathbf{H}}_f\mathbf{P}_{f\beta}\mathbf{U}_{f\beta}+\bar{\mathbf{H}}_f\mathbf{T_2}\mathbf{T}_{\alpha}\alpha$$

where $\mathbf{T}_{\alpha}=(\mathbf{T}_{c(N+1)}\ldots\mathbf{T}_{c2N})$ and $\mathbf{T_2}=\begin{pmatrix}\mathbf{t_2} & 0\\ 0 & \mathbf{t_2}\end{pmatrix}$ with $\mathbf{t_2}=\begin{pmatrix}j_{11}^2 & j_{12}^2 & 2j_{11}j_{12}\\ j_{21}^2 & j_{22}^2 & 2j_{21}j_{22}\\ j_{11}j_{21} & j_{12}j_{22} & j_{11}j_{22}+j_{12}j_{21}\end{pmatrix}$.

We then use the relation $\int_i^j(\bar{\gamma}_s-\gamma_s)ds=0$ with $\gamma_s=w_{,s}+\beta_s$ for each side ij of the element which makes it possible to obtain them α_k since she is still written:

$$w_j-w_i+\frac{L_k}{2}(C_k\beta_{xi}+S_k\beta_{yi}+C_k\beta_{xj}+S_k\beta_{yj})+\frac{2}{3}L_k\alpha_k=L_k\bar{\gamma}_{sk}$$ where:

$$\bar{\gamma}_{sk}=(C_k\quad S_k)\bar{\gamma}=(C_k\quad S_k)\mathbf{H}_{ct}^{-1}\mathbf{T}=(C_k\quad S_k)\mathbf{H}_{ct}^{-1}[\bar{\mathbf{B}}_{c\beta}\mathbf{U}_{f\beta}+\bar{\mathbf{B}}_{c\alpha}\alpha]$$

where C_k, S_k and L_k are defined into 4.2.2.

Note:

The terms $\bar{\mathbf{B}}_{c\alpha}$ and $\bar{\mathbf{B}}_{c\beta}$ correspond to the integration of the term $\bar{\gamma}_s$ on each side ij of the element. One evaluates the integral by means of two points of gauss of X-coordinates $\pm 1/\sqrt{3}$ and weight $1/2$ in the element of reference $[-1,+1]$. Thus the term $\bar{\mathbf{B}}_{c\alpha}$ and $\bar{\mathbf{B}}_{c\beta}$ can be written:

$$\bar{\mathbf{B}}_{c\alpha}=\frac{1}{2}\left[\bar{H}_f T_2(PG_1)T_\alpha(PG_1)+\bar{H}_f T_2(PG_2)T_\alpha(PG_2)\right]$$ *and*

$$\bar{\mathbf{B}}_{c\beta}=\frac{1}{2}\left[\bar{H}_f P_{f\beta}(PG_1)+\bar{H}_f P_{f\beta}(PG_2)\right] .$$

The relation above is still written in matric form: $\mathbf{A}_\alpha \alpha=\mathbf{A}_w \mathbf{U}_{f\beta}$

with: $$\mathbf{A}_\alpha=\frac{2}{3}\begin{pmatrix} L_{N+1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & L_{2N} \end{pmatrix}-\begin{pmatrix} L_{N+1}C_{N+1} & L_{N+1}S_{N+1} \\ \vdots & \vdots \\ L_{2N}C_{2N} & L_{2N}S_{2N} \end{pmatrix}\mathbf{H}_{ct}^{-1}\bar{\mathbf{B}}_{c\alpha}$$

and:

$$A_w=-\frac{1}{2}\begin{pmatrix} -2 & L_{N+1}C_{N+1} & L_{N+1}S_{N+1} & 2 & L_{N+1}C_{N+1} & L_{N+1}S_{N+1} & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & L_{k+1}C_{k+1} & L_{k+1}S_{k+1} & 2 & L_{k+1}C_{k+1} & L_{k+1}S_{k+1} \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & L_{2N-1}C_{2N-1} & L_{2N-1}S_{2N-1} \\ 2 & L_{2N}C_{2N} & L_{2N}S_{2N} & 0 & 0 & \cdots & \cdots & 0 & 0 \\ \cdots\ 0 & 0 & 0 & & & & & & \\ \cdots\ 0 & 0 & 0 & & & & & & \\ \cdots\ 2 & L_{2N-1}C_{2N-1} & L_{2N-1}S_{2N-1} & & & & & & \\ \cdots\ -2 & L_{2N}C_{2N} & L_{2N}S_{2N} & & & & & & \end{pmatrix}$$

$$+\begin{pmatrix} L_{N+1}C_{N+1} & L_{N+1}S_{N+1} \\ \vdots & \vdots \\ L_{2N}C_{2N} & L_{2N}S_{2N} \end{pmatrix}\mathbf{H}_{ct}^{-1}\bar{\mathbf{B}}_{c\beta}$$

Thus $\alpha=\mathbf{A}_\beta \mathbf{U}_{f\beta}\,\text{avec}\,\mathbf{A}_\beta=\mathbf{A}_\alpha^{-1}\mathbf{A}_w$, which implies $\mathbf{T}=[\bar{\mathbf{B}}_{c\beta}+\bar{\mathbf{B}}_{c\alpha}\mathbf{A}_\beta]\mathbf{U}_{f\beta}$.

Note:

For the elements DST, this statement is simplified a little since $\bar{\mathbf{B}}_{c\beta}=0$ because of linearity of the shape functions N_k $(k=1,2,3)$.
This statement is simpler for elements DKT, DKTG and DKQ since they are without transverse distortion, i.e. $\bar{\gamma}$ = 0, which implies $\mathbf{A}_\alpha=\begin{pmatrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and

¿

$$A_w = -\frac{3}{4}\begin{pmatrix} -2/L_{N+1} & C_{N+1} & S_{N+1} & 2/L_{N+1} & C_{N+1} & S_{N+1} & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & -2/L_{k+1} & C_{k+1} & S_{k+1} & 2/L_{k+1} & C_{k+1} & S_{k+1} & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & -2/L_{2N-1} & C_{2N-1} & S_{2N-1} & \cdots \\ 2/L_{2N} & C_{2N} & S_{2N} & 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots \\ \cdots\ 0 & 0 & 0 \\ \cdots\ 0 & 0 & 0 \\ \cdots\ 2/L_{2N-1} & C_{2N-1} & S_{2N-1} \\ \cdots\ -2/L_{2N} & C_{2N} & S_{2N} \end{pmatrix}$$

It is also noticed that for elements DKT, DKTG the statement of the forces sharp is calculated from the equilibrium and not starting from the behavior (on the basis of the behavior one would find a value zero shears what would not make it possible to carry out the equilibrium!). It results from it according to the §3.1.1 from the non-zero shearing stresses transverse in the thickness from the plate that one is in formulation DKT or DST.

4.3.3 Discretization of the strain field of bending:

4.3.3.1 For the Q4g elements

the relation binding the strains of bending to the field of displacement of bending is written:

$$k_{xx}=\beta_{x,x}=j_{11}\beta_{x,\xi}+j_{12}\beta_{x,\eta}=j_{11}\sum_{k=1}^{N}N_{k,\xi}\beta_{xk}+j_{12}\sum_{k=1}^{N}N_{k,\eta}\beta_{xk},$$

$$\kappa_{yy}=\beta_{y,y}=j_{21}\beta_{y,\xi}+j_{22}\beta_{y,\eta}=j_{21}\sum_{k=1}^{N}N_{k,\xi}\beta_{yk}+j_{22}\sum_{k=1}^{N}N_{k,\eta}\beta_{yk},$$

$$2\kappa_{xy}=\beta_{y,x}+\beta_{x,y}=j_{11}\beta_{y,\xi}+j_{12}\beta_{y,\eta}+j_{21}\beta_{x,\xi}+j_{22}\beta_{x,\eta}=j_{21}\sum_{k=1}^{N}N_{k,\xi}\beta_{xk}+j_{22}\sum_{k=1}^{N}N_{k,\eta}\beta_{xk}$$

$$+j_{11}\sum_{k=1}^{N}N_{k,\xi}\beta_{yk}+j_{12}\sum_{k=1}^{N}N_{k,\eta}\beta_{yk}.$$

That is to say still in matric form:

$\begin{pmatrix}\kappa_{xx}\\ \kappa_{yy}\\ 2\kappa_{xy}\end{pmatrix}=\sum_{k=1}^{N}\mathbf{B}_{fk}\mathbf{U}_{fk}$ where $\mathbf{U}_{fk}=\begin{pmatrix}w_k\\ \beta_{xk}\\ \beta_{yk}\end{pmatrix}$ the field of displacement of bending to the node represents k,

with:

$$\mathbf{B}_{fk}=\begin{pmatrix}0 & j_{11}N_{k,\xi}+j_{12}N_{k,\eta} & 0\\ 0 & 0 & j_{21}N_{k,\xi}+j_{22}N_{k,\eta}\\ 0 & j_{21}N_{k,\xi}+j_{22}N_{k,\eta} & j_{11}N_{k,\xi}+j_{12}N_{k,\eta}\end{pmatrix}.$$

The transition matrix of the field of displacement of bending $\mathbf{U}_f=\begin{pmatrix}w_1\\ \beta_{x1}\\ \beta_{y1}\\ \vdots\\ w_N\\ \beta_{xN}\\ \beta_{yN}\end{pmatrix}$ to the strains of bending is written then: $\mathbf{B}_{f[3\times 3n]}=(\mathbf{B}_{f1},\cdots,\mathbf{B}_{fN})$.

4.3.3.2 For the elements of type DKT, DKTG, DST:

The relation binding the strains of bending to the field of displacement of bending is written:

$$\kappa_{xx}=\beta_{x,x}=j_{11}\beta_{x,\xi}+j_{12}\beta_{x,\eta}=j_{11}(\sum_{k=1}^{N}N_{k,\xi}\beta_{xk}+\sum_{k=N+1}^{2N}P_{xk,\xi}\alpha_k)+j_{12}(\sum_{k=1}^{N}N_{k,\eta}\beta_{xk}+\sum_{k=N+1}^{2N}P_{xk,\eta}\alpha_k),$$
$$\kappa_{yy}=\beta_{y,y}=j_{21}\beta_{y,\xi}+j_{22}\beta_{y,\eta}=j_{21}(\sum_{k=1}^{N}N_{k,\xi}\beta_{yk}+\sum_{k=N+1}^{2N}P_{yk,\xi}\alpha_k)+j_{22}(\sum_{k=1}^{N}N_{k,\eta}\beta_{yk}+\sum_{k=N+1}^{2N}P_{yk,\eta}\alpha_k),$$
$$2\kappa_{xy}=\beta_{y,x}+\beta_{x,y}=j_{11}\beta_{y,\xi}+j_{12}\beta_{y,\eta}+j_{21}\beta_{x,\xi}+j_{22}\beta_{x,\eta}=$$
$$j_{21}(\sum_{k=1}^{N}N_{k,\xi}\beta_{xk}+\sum_{k=N+1}^{2N}P_{xk,\xi}\alpha_k)+j_{22}(\sum_{k=1}^{N}N_{k,\eta}\beta_{xk}+\sum_{k=N+1}^{2N}P_{xk,\eta}\alpha_k)+j_{11}(\sum_{k=1}^{N}N_{k,\xi}\beta_{yk}+\sum_{k=N+1}^{2N}P_{yk,\xi}\alpha_k)$$
$$+j_{12}(\sum_{k=1}^{N}N_{k,\eta}\beta_{yk}+\sum_{k=N+1}^{2N}P_{yk,\eta}\alpha_k).$$

For elements DKT, DKTG, DKQ:

In matric form the preceding relation is also written by introducing the relation $\alpha=\mathbf{A}_{\beta}\mathbf{U}_{f\beta}$:

$$\begin{pmatrix}\kappa_{xx}\\ \kappa_{yy}\\ 2\kappa_{xy}\end{pmatrix}=\begin{pmatrix}j_{11}\mathbf{B}_{\beta x\xi}+j_{12}\mathbf{B}_{\beta x\eta}\\ j_{21}\mathbf{B}_{\beta y\xi}+j_{22}\mathbf{B}_{\beta y\eta}\\ j_{11}\mathbf{B}_{\beta y\xi}+j_{12}\mathbf{B}_{\beta y\eta}+j_{21}\mathbf{B}_{\beta x\xi}+j_{22}\mathbf{B}_{\beta x\eta}\end{pmatrix}\mathbf{U}_f=\mathbf{B}_{f[3\times 3N]}\mathbf{U}_f \quad \text{where} \quad \mathbf{U}_f=\begin{vmatrix}w_1\\ \beta_{x1}\\ \beta_{y1}\\ \vdots\\ w_N\\ \beta_{xN}\\ \beta_{yN}\end{vmatrix}$$ the field

of displacement in bending for the element represents with:

$$B_{\beta x\xi}=(\frac{6P_{N+1,\xi}C_{N+1}}{4L_{N+1}}-\frac{6P_{2N,\xi}C_{2N}}{4L_{2N}},N_{1,x}-\frac{3}{4}(P_{N+1,\xi}C_{N+1}^2+P_{2N,\xi}C_{2N}^2),$$
$$-\frac{3}{4}(P_{N+1,\xi}C_{N+1}S_{N+1}+P_{2N,\xi}C_{2N}S_{2N}),L,$$
$$\frac{6P_{N+k,\xi}C_{N+k}}{4L_{N+k}}-\frac{6P_{N+k-1,\xi}C_{N+k-1}}{4L_{N+k-1}},N_{k,\xi}-\frac{3}{4}(P_{N+k,\xi}C_{N+k}^2+P_{N+k-1,x}C_{N+k-1}^2),$$
$$-\frac{3}{4}(P_{N+k,\xi}C_{N+k}S_{N+k}+P_{N+k-1,\xi}C_{N+k-1}S_{N+k-1}),L$$
$$(k=2,..,N))$$

$$\mathbf{B}_{\beta x\eta}=(\frac{6\mathrm{P}_{N+1,\eta}C_{N+1}}{4\mathrm{L}_{N+1}}-\frac{6\mathrm{P}_{2\mathrm{N},\eta}C_{2\mathrm{N}}}{4\mathrm{L}_{2\mathrm{N}}},N_{1,\eta}-\frac{3}{4}(P_{N+1,\eta}C_{N+1}^2+P_{2\mathrm{N},\eta}C_{2\mathrm{N}}^2),$$
$$-\frac{3}{4}(P_{N+1,\eta}C_{N+1}S_{N+1}+P_{2\mathrm{N},\eta}C_{2\mathrm{N}}S_{2\mathrm{N}}),\cdots,$$
$$\frac{6\mathrm{P}_{N+k,\eta}C_{N+k}}{4\mathrm{L}_{N+k}}-\frac{6\mathrm{P}_{N+k-1,\eta}C_{N+k-1}}{4\mathrm{L}_{N+k-1}},N_{k,\eta}-\frac{3}{4}(P_{N+k,\eta}C_{N+k}^2+P_{N+k-1,h}C_{N+k-1}^2),$$
$$-\frac{3}{4}(P_{N+k,\eta}C_{N+k}S_{N+k}+P_{N+k-1,\eta}C_{N+k-1}S_{N+k-1}),\cdots$$
$$(k=2,..,N))$$

$$\mathbf{B}_{\beta y\xi}=(\frac{6\mathrm{P}_{N+1,\xi}S_{N+1}}{4\mathrm{L}_{N+1}}-\frac{6\mathrm{P}_{2\mathrm{N},\xi}S_{2\mathrm{N}}}{4\mathrm{L}_{2\mathrm{N}}},-\frac{3}{4}(P_{N+1,\xi}C_{N+1}S_{N+1}+P_{2\mathrm{N},\xi}C_{2\mathrm{N}}S_{2\mathrm{N}}),$$
$$N_{1,\xi}-\frac{3}{4}(P_{N+1,\xi}S_{N+1}^2+P_{2\mathrm{N},\xi}S_{2\mathrm{N}}^2),\cdots,$$
$$\frac{6\mathrm{P}_{N+k,\xi}S_{N+k}}{4\mathrm{L}_{N+k}}-\frac{6\mathrm{P}_{N+k-1,\xi}S_{N+k-1}}{4\mathrm{L}_{N+k-1}},-\frac{3}{4}(P_{N+k,\xi}C_{N+k}S_{N+k}+P_{N+k-1,\xi}C_{N+k-1}S_{N+k-1}),$$
$$N_{k,\xi}-\frac{3}{4}(P_{N+k,\xi}S_{N+k}^2+P_{N+k-1,\xi}S_{N+k-1}^2),\cdots$$
$$(k=2,..,N))$$

$$\mathbf{B}_{\beta y\eta}=(\frac{6\mathrm{P}_{N+1,\eta}S_{N+1}}{4\mathrm{L}_{N+1}}-\frac{6\mathrm{P}_{2\mathrm{N},\eta}S_{2\mathrm{N}}}{4\mathrm{L}_{2\mathrm{N}}},-\frac{3}{4}(P_{N+1,\eta}C_{N+1}S_{N+1}+P_{2\mathrm{N},\eta}C_{2\mathrm{N}}S_{2\mathrm{N}}),$$
$$N_{1,\eta}-\frac{3}{4}(P_{N+1,\eta}S_{N+1}^2+P_{2\mathrm{N},\eta}S_{2\mathrm{N}}^2),\cdots,$$
$$\frac{6\mathrm{P}_{N+k,\eta}S_{N+k}}{4\mathrm{L}_{N+k}}-\frac{6\mathrm{P}_{N+k-1,\eta}S_{N+k-1}}{4\mathrm{L}_{N+k-1}},-\frac{3}{4}(P_{N+k,\eta}C_{N+k}S_{N+k}+P_{N+k-1,\eta}C_{N+k-1}S_{N+k-1}),$$
$$N_{k,\eta}-\frac{3}{4}(P_{N+k,\eta}S_{N+k}^2+P_{N+k-1,\eta}S_{N+k-1}^2),\cdots$$
$$(k=2,..,N))$$

For elements DST, DSQ:

The relation binding the strains of bending to the field of displacement in bending is also written in matric form:

$\begin{pmatrix}\kappa_{xx}\\ \kappa_{yy}\\ 2\kappa_{xy}\end{pmatrix}=\sum_{k=1}^{N}\mathbf{B}_{f\beta k}\mathbf{U}_{f\beta k}+\sum_{k=N+1}^{2\mathrm{N}}\mathbf{B}_{f\alpha k}\mathbf{U}_{f\alpha k}$ where $\mathbf{U}_{f\beta k}=\begin{pmatrix}w_k\\ \beta_{xk}\\ \beta_{yk}\end{pmatrix}$ and $\mathbf{U}_{f\alpha k}=\alpha_k$ represent the field of displacement of bending to the node K, so that:

$$\mathbf{B}_{f\beta k}=\begin{pmatrix} 0 & j_{11}N_{k,\xi}+j_{12}N_{k,\eta} & 0 \\ 0 & 0 & j_{21}N_{k,\xi}+j_{22}N_{k,\eta} \\ 0 & j_{21}N_{k,\xi}+j_{22}N_{k,\eta} & j_{11}N_{k,\xi}+j_{12}N_{k,\eta} \end{pmatrix}$$ and

$$\mathbf{B}_{f\alpha k}=\begin{pmatrix} j_{11}P_{xk,\xi}+j_{12}P_{xk,\eta} \\ j_{21}P_{yk,\xi}+j_{22}P_{yk,\eta} \\ j_{11}P_{yk,\xi}+j_{12}P_{yk,\eta}+j_{21}P_{xk,\xi}+j_{22}P_{xk,\eta} \end{pmatrix}.$$

The transition matrix of the field of displacement of bending $\mathbf{U}_f=(\mathbf{U}_{f\beta},\alpha)$ with $\mathbf{U}_{f\beta}=\begin{pmatrix} w_1 \\ \beta_{x1} \\ \beta_{y1} \\ \vdots \\ w_N \\ \beta_{xN} \\ \beta_{yN} \end{pmatrix}$ and

$\alpha=\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{pmatrix}$ to the strains of bending is written then:

$$\mathbf{B}_{f[3\times 4N]}=(\mathbf{B}_{f\beta 1},\cdots,\mathbf{B}_{f\beta N},\mathbf{B}_{f\alpha(N+1)},\cdots,\mathbf{B}_{f\alpha 2N})=(\mathbf{B}_{f\beta[3\times 3N]},\mathbf{B}_{f\alpha[3\times N]}).$$

4.4 Stiffness matrix

the principle of the virtual wors is written in the following way: $\delta W_{\text{ext}}=\delta W_{\text{int}}$ that is to say still in elasticity $\delta\mathbf{U}^T\mathbf{K}\mathbf{U}=\mathbf{F}\delta\mathbf{U}$ in matric form where $\mathbf{K}$ is the stiffness matrix coming from the assembly in the total reference of all the elementary stiffness matrixes.

4.4.1 Elemental stiffness matrix for the Q4g elements

$$\delta W_{\text{int}}^e=\int_e[\delta e(H_m e+H_{mf}\kappa)+\delta\kappa(H_{mf}e+H_f\kappa)+\delta\bar{\gamma}H_{ct}\bar{\gamma}]dS=$$
$$\int_e(\delta U_m^T B_m^T H_m B_m U_m+\delta U_m^T B_m^T H_{mf} B_f U_f+\delta U_f^T B_f^T H_{mf} B_m U_m+\delta U_f^T B_f^T H_f B_f U_f$$
$$+\delta U_f^T B_c^T H_{ct} B_c U_f)dS=$$
$$\delta U_m^T(\int_e B_m^T H_m B_m dS)U_m+\delta U_f^T(\int_e B_f^T H_f B_f dS)U_f+dU_m^T(\int_e B_m^T H_{mf} B_f dS)U_f$$
$$+\delta U_f^T(\int_e B_f^T H_{mf} B_m dS)U_m$$
$$+\delta U_f^T(\int_e B_c^T H_{ct} B_c dS)U_f=\delta U_m^T K_m U_m+\delta U_f^T K_f U_f+\delta U_m^T K_{mf} U_f+\delta U_f^T K_{fm} U_m+\delta U_f^T K_c U_f$$

with $K_{mf}=K_{fm}^{T}$.

This is still written: $\delta W_{\text{int}}^{e}=\left(\delta U_{m},\delta U_{f}\right)K\begin{pmatrix}U_{m}\\U_{f}\end{pmatrix}$ where

$K_{[5N\times 5N]}=\begin{pmatrix}K_{m[2N\times 2N]} & K_{mf[2N\times 3N]}\\K_{mf[3N\times 2N]}^{T} & K_{f[3N\times 3N]}+K_{c[3N\times 3N]}\end{pmatrix}$ is the stiffness matrix of the element.

4.4.2 Elemental stiffness matrix for elements DKT, DKTG, DKQ

Since the relation $\bar{\gamma}=0$ is satisfied, one can write:

$$\delta W_{\text{int}}^{e}=\int_{e}\delta e(H_{m}e+H_{mf}k)+\delta\kappa(H_{mf}e+H_{f}k)\,dS=$$
$$\int_{e}(\delta U_{m}^{T}B_{m}^{T}H_{m}B_{m}U_{m}+\delta U_{m}^{T}B_{m}^{T}H_{mf}B_{f}U_{f}+\delta U_{f}^{T}B_{f}^{T}H_{mf}B_{m}U_{m}+\delta U_{f}^{T}B_{f}^{T}H_{f}B_{f}U_{f})\,dS=$$
$$\delta U_{m}^{T}(\int_{e}B_{m}^{T}H_{m}B_{m}dS)U_{m}+\delta U_{f}^{T}(\int_{e}B_{f}^{T}H_{f}B_{f}dS)U_{f}+\delta U_{m}^{T}(\int_{e}B_{m}^{T}H_{mf}B_{f}dS)U_{f}$$
$$+\delta U_{f}^{T}(\int_{e}B_{f}^{T}H_{mf}B_{m}dS)U_{m}=\delta U_{m}^{T}K_{m}U_{m}+\delta U_{f}^{T}K_{f}U_{f}+\delta U_{m}^{T}K_{mf}U_{f}+\delta U_{f}^{T}K_{fm}U_{m}$$

with $K_{mf}=K_{fm}^{T}$.

This is still written: $\delta W_{\text{int}}^{e}=\left(\delta U_{m},\delta U_{f}\right)K\begin{pmatrix}U_{m}\\U_{f}\end{pmatrix}$

where $K_{[5N\times 5N]}=\begin{pmatrix}K_{m[2N\times 2N]} & K_{mf[2N\times 3N]}\\K_{mf[3N\times 2N]}^{T} & K_{f[3N\times 3N]}\end{pmatrix}$ is the stiffness matrix of the element.

4.4.3 Elemental stiffness matrix for elements DST, DSQ

$$\delta W_{\text{int}}^{e}=\int_{e}\delta e(H_{m}e+H_{mf}\kappa)+\delta\kappa(H_{mf}e+H_{f}\kappa)+\delta\,\mathrm{TH}_{ct}^{-1}T\,dS=$$
$$\int_{e}(\delta U_{m}^{T}B_{m}^{T}H_{m}B_{m}U_{m}+\delta U_{m}^{T}B_{m}^{T}H_{mf}B_{f}U_{f}+\delta U_{f}^{T}B_{f}^{T}H_{mf}B_{m}U_{m}+\delta U_{f}^{T}B_{f}^{T}H_{f}B_{f}U_{f}$$
$$+\delta U_{f\beta^{T}}B_{c\beta}^{T}H_{ct}^{-1}B_{c\beta}U_{f\beta}+\delta U_{f\beta^{T}}B_{c\beta}^{T}H_{ct}^{-1}B_{c\alpha}\alpha+\delta\alpha^{T}B_{c\alpha}^{T}H_{ct}^{-1}B_{c\beta}U_{f\beta}+\delta\alpha^{T}B_{c\alpha}^{T}H_{ct}^{-1}B_{c\alpha}\alpha)\,dS=$$
$$\delta U_{m}^{T}(\int_{e}B_{m}^{T}H_{m}B_{m}dS)U_{m}+\delta U_{f}^{T}(\int_{e}B_{f}^{T}H_{f}B_{f}dS)U_{f}+\delta U_{m}^{T}(\int_{e}B_{m}^{T}H_{mf}B_{f}dS)U_{f}+\delta U_{f}^{T}(\int_{e}B_{f}^{T}H_{mf}B_{m}dS)U_{m}$$
$$+\delta U_{f\beta}^{T}(\int_{e}B_{f\beta}^{T}H_{ct}^{-1}B_{c\beta}dS)U_{f\beta}+\delta U_{f\beta}^{T}(\int_{e}B_{f\beta}^{T}H_{ct}^{-1}B_{c\alpha}dS)\alpha+\delta\alpha^{T}(\int_{e}B_{c\alpha}^{T}H_{ct}^{-1}B_{c\beta}dS)U_{f\beta}+\delta\alpha^{T}(\int_{e}B_{c\alpha}^{T}H_{ct}^{-1}B_{c\alpha}dS)\alpha=$$
$$\delta U_{m}^{T}K_{m}U_{m}+dU_{f}^{T}K_{f}U_{f}+\delta U_{m}^{T}K_{mf}U_{f}+\delta U_{f}^{T}K_{fm}U_{m}+\delta U_{f\beta}^{T}K_{\beta\beta}U_{f\beta}+\delta U_{f\beta}^{T}K_{\beta\alpha}\alpha+\delta\alpha^{T}K_{\alpha\beta}U_{f\beta}+\delta\alpha^{T}K_{c\alpha}\alpha$$

One also knows that $\mathbf{U}_f=(\mathbf{U}_{f\beta},\alpha)$ from where it results that:

$$K_f=\begin{pmatrix} K_{f11} & K_{f12} \\ K_{f12}^T & K_{22} \end{pmatrix} \text{ with: } \begin{array}{l} K_{f11}=\int_s B_{f\beta}^T H_f B_{f\beta} dS ; \\ K_{f12}=\int_s B_{f\beta}^T H_f B_{f\alpha} dS ; \\ K_{f22}=\int_s B_{f\alpha}^T H_f B_{f\alpha} dS . \end{array}$$

$$K_{mf}=\begin{pmatrix} K_{mf11} & K_{mf12} \end{pmatrix} \text{ with: } \begin{array}{l} K_{mf11}=\int_s B_m^T H_{mf} B_{f\beta} dS ; \\ K_{mf12}=\int_s B_m^T H_{mf} B_{f\alpha} dS . \end{array}$$

$$K_{fm}=K_{mf}^T .$$

Using the fact that $\alpha=A_\beta U_{f\beta}$ one from of deduced that:

$$\delta W_{\text{int}}=\delta U_m^T K_m U_m+\delta U_{f\beta}^T K_f^{'} U_{f\beta}+\delta U_m^T K_{mf}^{'} U_{f\beta}+\delta U_{f\beta}^T K_{fm}^{'} U_m \text{ where:}$$

$$\begin{array}{l} K_f^{'}=K_{f11}+K_{\beta\beta}+A_\beta^T(K_{f22}+K_{c\alpha})A_\beta+(K_{f12}+K_{\beta\alpha})A_\beta+A_\beta^T(K_{f12}^T+K_{\beta\alpha}^T) \\ K_{mf}^{'}=K_{mf11}+K_{mf12}A_\beta \end{array} .$$

This is still written: $\delta W_{\text{int}}^e=(\delta U_m,\delta U_{f\beta})K\begin{pmatrix} U_m \\ U_{f\beta} \end{pmatrix}$ where $K_{[5N\times 5N]}=\begin{pmatrix} K_{m[2N\times 2N]} & K'_{mf[2N\times 3N]} \\ K'^T_{mf[3N\times 2N]} & K'_{f[3N\times 3N]} \end{pmatrix}$ is the elemental stiffness matrix for a shell element.

4.4.4 Assembly of the elementary matrixes

the principle of virtual work for all the elements is written:

$\delta W_{\text{int}}=\sum_{e=1}^{\text{nb}\,elem} \delta W_{\text{int}}^e=\delta \mathbf{U}^T \mathbf{K}\mathbf{U}$ where $\mathbf{U}$ is all the degrees of freedom of discretized structure and K comes from the assembly of the elementary matrixes.

4.4.4.1 Degrees of freedom

the process of assembly of the elementary matrixes implies that all the degrees of freedom are expressed in the total reference. In the total reference, the degrees of freedom are three displacements compared to the three axes of the total cartesian coordinate system and the three rotations compared to these three axes. One thus uses transition matrixes of the local coordinate system to the total reference for each element. However it was seen previously that the degrees of freedom of the shell elements are two displacements in the plan of the plate, displacement except plane and two rotations. These rotations not being exactly rotations compared to the axes of the plate since $\beta_x(x,y)=\theta_y(x,y), \beta_y(x,y)=-\theta_x(x,y)$ it is necessary to take account of it with the level of the assembly to reveal the good degrees of freedom θ_{xi},θ_{yi} .

4.4.4.2 Fictitious rotations

rotation compared to the normal with the plate is regarded as not being a degree of freedom. To ensure compatibility between the transition of the local coordinate system the total reference, one thus adds a local additional degree of freedom of rotation to the plate which is that corresponding to rotation compared to the normal with the plane of the element. This implies an expansion of the blocks of dimension $(5,5)$ of the local stiffness matrix into cubes blocks of dimension $(6,6)$ by adding one line and a column corresponding to this rotation. These additional lines and these columns are a priori null. One then carries out the transition of the local stiffness matrix extended to the global stiffness matrix.

In the preceding transformation, one was satisfied to add rotations compared to the norms with the plane of the elements without modifying strain energy. The contribution to the energy brought by these additional degrees of freedom is indeed null and no stiffness is associated for them.

The global stiffness matrix thus obtained presents the risk however to be noninvertible. To avoid this nuisance it is allowed to allot a small stiffness to these additional degrees of freedom on the level of the widened local stiffness matrix. Practically, one chooses it between 10-6 and 10-3 times the diagonal minor term of the stiffness matrix of local bending. The user can choose this multiplicative coefficient `COEF_RIGI_DRZ` itself in `AFFE_CARA_ELEM`; by default it is worth 10-5.

4.5 Mass matrix

the terms of the mass matrix are obtained after discretization of the following variational formulation:

$$\delta W_{mass}^{ac}=\int_{-h/2}^{+h/2}\int_S \rho\,\ddot{\mathbf{u}}\,\delta\,\mathbf{u}\,dzdS=\int_S \rho_m(\ddot{u}\,\delta u+\ddot{v}\,\delta v+\ddot{w}\,\delta w)+\rho_{mf}(\ddot{u}\,\delta\beta_x+\ddot{v}\,\delta\beta_y+\ddot{\beta}_x\,\delta u+\ddot{\beta}_y\,\delta v)$$
$$+\rho_f(\ddot{\beta}_x\,\delta\beta_x+\ddot{\beta}_y\,\delta\beta_y)dS$$

with $\rho_m=\int_{-h/2}^{+h/2}\rho\,dz$, $\rho_{mf}=\int_{-h/2}^{+h/2}\rho\,zdz$, et $\rho_f=\int_{-h/2}^{+h/2}\rho z^2\mathrm{dz}$.

Note:

If the plate is homogeneous or symmetric compared to $z=0$ then $\rho_{mf}=0$. One considers in the continuation of the talk that it is always the case.

4.5.1 Elementary mass matrix classical

4.5.1.1 Q4g Element

the discretization of displacement for this isoparametric element is:

$$\mathbf{u}=\sum_{k=1}^{N} N_k\begin{pmatrix}u_k\\v_k\\w_k\\\beta_{xk}\\\beta_{yk}\end{pmatrix}\Bigg\}\, k=1,\cdots,N$$

The mass matrix, in the base where the degrees of freedom are gathered according to the directions of translation and rotation, has then as a statement:

$$\mathbf{M}=\begin{pmatrix} \mathbf{M}_m & 0 & 0 & \mathbf{M}_{mf} & 0 \\ 0 & \mathbf{M}_m & 0 & 0 & \mathbf{M}_{mf} \\ 0 & 0 & \mathbf{M}_m & 0 & 0 \\ \mathbf{M}_{mf}^T & 0 & 0 & \mathbf{M}_f & 0 \\ 0 & \mathbf{M}_{mf}^T & 0 & 0 & \mathbf{M}_f \end{pmatrix}$$

with: $\mathbf{M}_m=\int_S \rho_m \mathbf{N}^T \mathbf{N}\, dS$, $\mathbf{M}_{mf}=\int_S \rho_{mf} \mathbf{N}^T \mathbf{N}\, \mathrm{dS}$ et $\mathbf{M}_f=\int_S \rho_f \mathbf{N}^T \mathbf{N}\, dS$ and $\mathbf{N}=(N_1 \cdots N_k)$.

4.5.1.2 Elements of type DKT, DST

Like $\begin{pmatrix} w \\ \beta_x \\ \beta_y \end{pmatrix}=\sum_{k=1}^{N} N_k(\xi,\eta)\begin{pmatrix} w_k \\ \beta_{xk} \\ \beta_{yk} \end{pmatrix}+\sum_{k=N+1}^{2N}\begin{pmatrix} 0 \\ P_{xk}(\xi,\eta) \\ P_{yk}(\xi,\eta) \end{pmatrix}\alpha_k$ where $\alpha=\mathbf{A}_\beta \mathbf{U}_{f\beta}$ one from of deduced that:

$$\begin{pmatrix} w \\ \beta_x \\ \beta_y \end{pmatrix}=\sum_{k=1}^{N}\begin{pmatrix} N_k(\xi,\eta) & 0 & 0 \\ N_{kxw}(\xi,\eta) & N_{kxx}(\xi,\eta) & N_{kxy}(\xi,\eta) \\ N_{kyw}(\xi,\eta) & N_{kyx}(\xi,\eta) & N_{kyy}(\xi,\eta) \end{pmatrix}\begin{pmatrix} w_k \\ \beta_{xk} \\ \beta_{yk} \end{pmatrix} .$$

The membrane part of the elementary matrix of mass is the same one as for Q4g with $k=3$ instead of 4 in $\mathbf{N}$. The bending part is composed of the blocks kp (k ième line and p ième column) following:

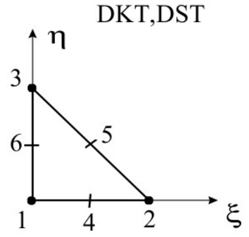
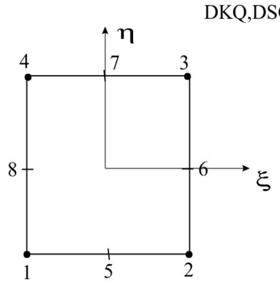
$$\rho_f\begin{pmatrix} N_{kxw}N_{pxw}+N_{kyw}N_{pyw}+\rho_m N_k N_p/\rho_f & N_{kxw}N_{pxx}+N_{kyw}N_{pyx} & N_{kxw}N_{pxy}+N_{kyw}N_{pyy} \\ N_{kxx}N_{pxw}+N_{kyx}N_{pyw} & N_{kxx}N_{pxx}+N_{kyx}N_{pyx} & N_{kxx}N_{pxy}+N_{kyx}N_{pyy} \\ N_{kxy}N_{pxw}+N_{kyy}N_{pyw} & N_{kxy}N_{pxx}+N_{kyy}N_{pyx} & N_{kxy}N_{pxy}+N_{kyy}N_{pyy} \end{pmatrix}$$

4.5.2 Improved elementary mass matrix

As the deflection of a flexbeam can be represented by a linear approximation with difficulty, one can enrich the shape functions for the terms by bending. This approach is used in *Code_Aster* for the elements of the type DKT, DST and Q4g where the shape functions used in the computation of the mass matrix of bending are of order 3. The interpolation for w is written as follows:

$$w=\sum_{k=1}^{N} N_{(k-1)N+1}(\xi,\eta)w_k+N_{(k-1)N+2}(\xi,\eta)w_{,\xi k}+N_{(k-1)N+3}(\xi,\eta)w_{,\eta k}$$

where the shape functions are given for the triangle and the quadrangle in the following table:

	DKT,DST 	DKQ,DSQ,Q4γ 
Interpolation for w	$\lambda=1-\xi-\eta$ $i=1$ with 9 $N_1(\xi,\eta)=3\lambda^2-2\lambda^3+2\xi\eta\lambda$ $N_2(\xi,\eta)=\lambda^2\xi+\xi\eta\lambda/2$ $N_3(\xi,\eta)=\lambda^2\eta+\xi\eta\lambda/2$ $N_4(\xi,\eta)=3\xi^2-2\xi^3+2\xi\eta\lambda$ $N_5(\xi,\eta)=\xi^2(-1+\xi)-\xi\eta\lambda$ $N_6(\xi,\eta)=\xi^2\eta+\xi\eta\lambda/2$ $N_7(\xi,\eta)=3\eta^2-2\eta^3+2\xi\eta\lambda$ $N_8(\xi,\eta)=\eta^2\xi+\xi\eta\lambda/2$ $N_9(\xi,\eta)=\eta^2(-1+\eta)-\xi\eta\lambda$	$i=1 \quad 12$ $N_1(\xi,\eta)=\frac{1}{8}(1-\xi)(1-\eta)(2-\xi^2-\eta^2-\xi-\eta)$ $N_2(\xi,\eta)=\frac{1}{8}(1-\xi)(1-\eta)(1-\xi^2)$ $N_3(\xi,\eta)=\frac{1}{8}(1-\xi)(1-\eta)(1-\eta^2)$ $N_4(\xi,\eta)=\frac{1}{8}(1+\xi)(1-\eta)(2-\xi^2-\eta^2+\xi-\eta)$ $N_5(\xi,\eta)=-\frac{1}{8}(1+\xi)(1-\eta)(1-\xi^2)$ $N_6(\xi,\eta)=\frac{1}{8}(1+\xi)(1-\eta)(1-\eta^2)$ $N_7(\xi,\eta)=\frac{1}{8}(1+\xi)(1+\eta)(2-\xi^2-\eta^2+\xi+\eta)$ $N_8(\xi,\eta)=-\frac{1}{8}(1+\xi)(1+\eta)(1-\xi^2)$ $N_9(\xi,\eta)=-\frac{1}{8}(1+\xi)(1+\eta)(1-\eta^2)$ $N_{10}(\xi,\eta)=\frac{1}{8}(1-\xi)(1+\eta)(2-\xi^2-\eta^2-\xi+\eta)$ $N_{11}(\xi,\eta)=\frac{1}{8}(1-\xi)(1+\eta)(1-\xi^2)$ $N_{12}(\xi,\eta)=\frac{1}{8}(1-\xi)(1+\eta)(1-\eta^2)$

Interpolation functions for the deflection of the elements of type DKT, DST, DKTG and Q4G, in dynamics and modal.

4.5.2.1 Elements of type DKT

It is known that in the approximation of one Coils-Kirchhoff has $\beta_x = -w_{,x}$ and $\beta_y = -w_{,y}$ in any point of the element.

Because of discretization stated above one a:

$$w=\sum_{k=1}^{N} N_{(k-1)N+1}(\xi,\eta)w_k+(J_{11}N_{(k-1)N+2}(\xi,\eta)+J_{21}N_{(k-1)N+3}(\xi,\eta))w_{,xk}+(J_{12}N_{(k-1)N+2}(\xi,\eta)$$
$$+J_{22}N_{(k-1)N+3}(\xi,\eta))w_{,yk}$$

since: $\begin{pmatrix} w_{,\xi k} \\ w_{,\eta k} \end{pmatrix} = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} \begin{pmatrix} w_{,xk} \\ w_{,yk} \end{pmatrix}$.

This is still written:

$$w=\sum_{k=1}^{N} N'_{(k-1)N+1}(\xi,\eta)w_k+N'_{(k-1)N+2}(\xi,\eta)\beta_{xk}+N'_{(k-1)N+3}(\xi,\eta)\beta_{yk}$$

where:
$$\begin{aligned} N'_{(k-1)N+1}(\xi,\eta) &= N_{(k-1)N+1}(\xi,\eta) \\ N'_{(k-1)N+2}(\xi,\eta) &= -J_{11}N_{(k-1)N+2}(\xi,\eta)-J_{21}N_{(k-1)N+3}(\xi,\eta) \\ N'_{(k-1)N+3}(\xi,\eta) &= -J_{12}N_{(k-1)N+2}(\xi,\eta)-J_{22}N_{(k-1)N+3}(\xi,\eta) \end{aligned} .$$

By not taking account of the effects of inertia, the mass matrix has the following form thus:

$$\mathbf{M}=\begin{bmatrix} \mathbf{M}_m & 0 & 0 \\ 0 & \mathbf{M}_m & 0 \\ 0 & 0 & \mathbf{M}_f \end{bmatrix} \text{ where } \mathbf{M}_f=\int_S \rho_m \mathbf{N}' \mathbf{N}' dS .$$

4.5.2.2 Elements of type DST

It is known that for these elements one has $\beta_x = \gamma_x - w_{,x}$ and $\beta_y = \gamma_y - w_{,y}$ where the distortion γ is constant on the element.

Like:

$$w=\sum_{k=1}^{N} N_{(k-1)N+1}(\xi,\eta)w_k+(J_{11}N_{(k-1)N+2}(\xi,\eta)+J_{21}N_{(k-1)N+3}(\xi,\eta))w_{,xk}+(J_{12}N_{(k-1)N+2}(\xi,\eta)$$
$$+J_{22}N_{(k-1)N+3}(\xi,\eta))w_{,yk}$$

one can also write:

$$w=\sum_{k=1}^{N} N'_{(k-1)N+1}(\xi,\eta)w_k+N'_{(k-1)N+2}(\xi,\eta)\beta_{xk}+N'_{(k-1)N+3}(\xi,\eta)\beta_{yk}$$
$$+(J_{11}\bar{\gamma}_x+J_{12}\bar{\gamma}_y)SN_{(k-1)N+2}(\xi,\eta)+(J_{21}\bar{\gamma}_x+J_{22}\bar{\gamma}_y)SN_{(k-1)N+3}(\xi,\eta)$$

where: $$N'_{(k-1)N+1}(\xi,\eta)=N_{(k-1)N+1}(\xi,\eta)$$
$$N'_{(k-1)N+2}(\xi,\eta)=-J_{11}N_{(k-1)N+2}(\xi,\eta)-J_{21}N_{(k-1)N+3}(\xi,\eta) \ ,$$
$$N'_{(k-1)N+3}(\xi,\eta)=-J_{12}N_{(k-1)N+2}(\xi,\eta)-J_{22}N_{(k-1)N+3}(\xi,\eta)$$

$$\sum N_{(k-1)N+1}(\xi,\eta)=\sum_{k=1}^{N} N_{(k-1)N+1}(\xi,\eta)$$
$$\sum N_{(k-1)N+2}(\xi,\eta)=\sum_{k=1}^{N} N_{(k-1)N+2}(\xi,\eta)$$
$$\sum N_{(k-1)N+3}(\xi,\eta)=\sum_{k=1}^{N} N_{(k-1)N+3}(\xi,\eta)$$

and $$\begin{pmatrix}\bar{\gamma}_x\\ \bar{\gamma}_y\end{pmatrix}=\mathbf{H}_{ct}^{-1}[\mathbf{B}_{c\beta}+\mathbf{B}_{c\alpha}\mathbf{A}_{\beta}]\begin{vmatrix}w_1\\ \beta_{x1}\\ \beta_{y1}\\ \vdots\\ w_N\\ \beta_{xN}\\ \beta_{yN}\end{vmatrix}=\mathbf{T}_{\gamma w}\begin{vmatrix}w_1\\ \beta_{x1}\\ \beta_{y1}\\ \vdots\\ w_N\\ \beta_{xN}\\ \beta_{yN}\end{vmatrix}.$$

One obtains the interpolation then for w :

$$w=\sum_{k=1}^{N} N''_{(k-1)N+1}(\xi,\eta)w_k+N''_{(k-1)N+2}(\xi,\eta)\beta_{xk}+N''_{(k-1)N+3}(\xi,\eta)\beta_{yk}$$

where:
$$N''_{(k-1)N+1}(\xi,\eta)=N'_{(k-1)N+1}(\xi,\eta)+$$
$$(J_{11}T_{\gamma w}(1,(k-1)N+1)+J_{12}T_{\gamma w}(2,(k-1)N+1))\sum N_{(j-1)N+2}(\xi,\eta)+$$
$$(J_{21}T_{\gamma w}(1,(k-1)N+1)+J_{22}T_{\gamma w}(2,(k-1)N+1))\sum N_{(j-1)N+3}(\xi,\eta)$$
$$N''_{(k-1)N+2}(\xi,\eta)=N'_{(k-1)N+2}(\xi,\eta)+$$
$$(J_{11}T_{\gamma w}(1,(k-1)N+2)+J_{12}T_{\gamma w}(2,(k-1)N+2))\sum N_{(j-1)N+2}(\xi,\eta)+$$
$$(J_{21}T_{\gamma w}(1,(k-1)N+2)+J_{22}T_{\gamma w}(2,(k-1)N+2))\sum N_{(j-1)N+3}(\xi,\eta)$$
$$N''_{(k-1)N+3}(\xi,\eta)=N'_{(k-1)N+3}(\xi,\eta)+$$
$$(J_{11}T_{\gamma w}(1,(k-1)N+3)+J_{12}T_{\gamma w}(2,(k-1)N+3))\sum N_{(j-1)N+2}(\xi,\eta)+$$
$$(J_{21}T_{\gamma w}(1,(k-1)N+3)+J_{22}T_{\gamma w}(2,(k-1)N+3))\sum N_{(j-1)N+3}(\xi,\eta)$$

By not taking account of the effects of inertia, the mass matrix has the following form thus:

$$\mathbf{M}=\begin{pmatrix}\mathbf{M_m} & 0 & 0\\ 0 & \mathbf{M_m} & 0\\ 0 & 0 & \mathbf{M_f}\end{pmatrix} \text{ where } \mathbf{M}_f=\int_S \rho_m \mathbf{N}'' \mathbf{N}'' dS .$$

4.5.2.3 Elements of the Q4g type

One proceeds in the same way that for the elements of type DST but with:

$$\begin{pmatrix}\bar{\gamma}_x\\ \bar{\gamma}_y\end{pmatrix}=\mathbf{B_c}\begin{pmatrix}w_1\\ \beta_{x1}\\ \beta_{y1}\\ M\\ w_N\\ \beta_{xN}\\ \beta_{yN}\end{pmatrix}$$ where $\mathbf{B_c}$ is the matrix established with the § 4.3.2.1.

4.5.2.4 Notice

One neglects in the form of the elementary mass matrix the terms of inertia of rotation $\int_S \rho_f (\ddot{\beta}_x \delta \beta_x + \ddot{\beta}_y \delta \beta_y) dS$ because the latter are negligible [bib3] compared to the others. Indeed a multiplicative factor of $h^2/12$ the dregs to the other terms and they become negligible for a thickness ratio over characteristic length lower than $1/20$.

4.5.3 Assembly of the elementary mass matrixes

the assembly of the mass matrixes follows same logic as that of the stiffness matrixes. The degrees of freedom are the same ones and one finds the processing specific to normal rotations with the plan of the plate. For modal computations utilizing at the same time the computation of the stiffness matrix and that of the mass matrix, it is necessary to take a stiffness or a mass on the degree of normal rotation to the plan of the plate from 103 to 106 times smaller than the diagonal minor term of the stiffness matrix or of mass for the terms of bending. That makes it possible to inhibit the modes being able to appear on the additional degree of freedom of rotation around the norm with the plan of the plate. By default, one takes a stiffness or a mass on the degree of normal rotation to the plan of the plate 105 times smaller than the diagonal minor term of the stiffness matrix or of mass for the terms of bending

4.5.4 lumped Mass matrix

the use of a lumped mass matrix has two advantages: it is simpler to implement numerically and it allows a better convergence. However the results are less good than with the classical diagram (consistent matrix) for which the error is minimal [bib5]. The lumped mass matrix is recommended in theory only for transient computations using an explicit diagram of temporal integration, which is used almost exclusively in fast dynamics. Contrary to computations with the implicit schemes where with each increment (and each iteration Newton in nonlinear case) one assembles and opposite a matrix, given like a linear combination between the stiffness matrix, the mass matrix and the damping matrix, in explicit computations only the mass matrix is assembled and reversed. Therefore, by means of a diagonal mass matrix the gain in term of the TEMPS CPU of the resolution as well as the gain in term of the storage of the matrixes are enormous compared to the use of a consistent mass matrix.

Code_Aster not being a code specialized in fast dynamics, this advantage of the diagonal matrix is not exploited. The option of the lumped mass matrix is thus only one choice of modelization, allowing possible comparisons other computer codes.

One presents two methods for diagonalizing here a coherent mass matrix. It will be also shown in what the choice between the two methods is conditioned by the choice on the coherent matrix between the classic (§4.5.1) and improved (§4.5.2).

The technique undoubtedly simplest to obtain a lumped matrix is to retain the diagonal value for each degree of freedom as the sum of the elements of line of the coherent matrix. Moreover, it is pointed out that the most important property of a lumped mass matrix is that it makes it possible to represent a motion of the rigid body correctly. It is satisfied by the method of “summation by line”, put in equations in the following way:

$$M_C^\alpha = \sum_\beta M^{\alpha\beta} .$$

Unfortunately the summation of the lines does not guarantee that all the terms M_C^α are positive. The negative terms in particular by means of appear the improved coherent mass matrix (mentioned in §4.5.2). This reason and for most elements in Code_Aster, one chose another approach, which had in Hinton (see [bib5]), where the diagonal terms corresponding to the directions x y z $M_H^{\alpha x}$, $M_H^{\alpha y}$ and $M_H^{\alpha z}$, are calculated like:

$$M_H^{\alpha x} = \frac{\int_V \rho\, dV}{\sum_\beta M_x^{\beta\beta}} M_x^{\alpha\alpha} \qquad M_H^{\alpha y} = \frac{\int_V \rho\, dV}{\sum_\beta M_y^{\beta\beta}} M_y^{\alpha\alpha} \qquad M_H^{\alpha z} = \frac{\int_V \rho\, dV}{\sum_\beta M_z^{\beta\beta}} M_z^{\alpha\alpha} \qquad \text{eq. 2.4-1.}$$

Where in [eq. the 2.4-1] indices α correspond to the numbers of nodes. Although the method of Hinton is generally more robust, it is unsuited to the elements plates and shell, since [eq. 2.4-1] does not make it possible to include the terms of inertia, the terms $M_H^{\alpha \cdot}$ defined in [eq. 2.4-1] having inevitably the units of mass and never of inertia.

Consequently, for the elements treated here one modifies the computation of the consistent mass matrix being used for computation of the lumped mass matrix. One adopts the classical method described in §4.5.1 for the coherent mass matrix, then the approach of “summation by line” for the lumping. The impacted option of *Code_Aster* is `MASS_MECA_EXPLI` and only for elements `DKT` and `DKTG` . For the others one does not have the lumped mass matrix.

4.5.5 Modification of the terms of inertia

the lumped mass matrix described in §4.5.4 is not very effective for a computation in explicit dynamics, where time step of stability is strongly penalized by a bad conditioning of the mass matrix. Terms corresponding to rotations, i.e. the terms of inertia are the principal culprits, since much smaller than the terms of translation, i.e. displacements. For this reason, one proposed in [bib12] a method to modify the problematic terms while avoiding degrading the quality of the solution. Although the old and not completely rigorous approach suggested in [bib12] is largely referenced by the literature of the field and was not really prone to remarkable improvements.

One focuses oneself on the terms due to bending θ_x , θ_y and $w(=u_z)$, the terms corresponding to the membrane being obtained in a way classical and also applied to the elements 2D. The definite M mass matrix §4.5 becomes:

$$M^{\alpha\beta}=m^{\alpha\beta}\begin{pmatrix}\frac{h^3}{12} & 0 & 0\\ 0 & \frac{h^3}{12} & 0\\ 0 & 0 & h\end{pmatrix},$$ éq. 2.5-1

where h is the thickness of the plate and $m^{\alpha\beta}$ definite like:

$$m^{\alpha\beta}=\int_S \rho N^{\alpha} N^{\beta} dS$$.éq . 2.5-2

It is noted that [eq. 2.5-1] and [eq. 2.5-2] are equivalent to [eq. 2.3-1] for the kinematics of plates. In [bib12] one proposes to build the lumped matrix from [éq. 2.5-2] by means of the squaring of Lobatto, whose alternatives are the trapezoidal diagram and the diagram of Simpson, where the points of integration coincide with the nodes. The construction of the mass matrix is done through EF beam, linear and with two nodes, by means of the trapezoidal diagram, leading to:

$$M_0^{\text{pout}}=\frac{1}{2}\rho LA\begin{pmatrix}\frac{I}{A} & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & \frac{I}{A} & 0\\ 0 & 0 & 0 & 1\end{pmatrix},$$ éq. 2.5-3

for which the vector of d.d.l is written like. , $\left(\theta_1 \quad w_1 \quad \theta_2 \quad w_2\right)^T$ and A L are I the area of the section, the length and the main moment of inertia of the element beam, respectively. The use of the matrix [éq. 2.5-3] seeming too restrictive compared to the stability condition, one proposes in [bib12] rather: , where

$$M^{\text{pout}}=\frac{1}{2}\rho LA\begin{pmatrix}\alpha & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & \alpha & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

the parameter α is introduced so that its adjustment can maximize time step stability. According to [bib12] its optimal value would be. By $\alpha=\frac{1}{8}L^2$ directly applying these results by analogy to the plates, one replaces the matrix of [éq. 2.5-1] by: , éq

$$M^{\alpha\beta}=m^{\alpha\beta}\begin{pmatrix}\frac{hA^e}{8} & 0 & 0\\ 0 & \frac{hA^e}{8} & 0\\ 0 & 0 & h\end{pmatrix}.$$ 2.5-4 where

is A^e the area of the element considered. In the transition of the beam to the plate one supposed a certain equivalence between the length of the element beam and the area of the element plates, so that. It $A^e \approx L^2$ is pointed out that the approach suggested in [bib12] is not rigorous from the geometrical point of view and that it is focused on the maximization of the step of stability. In the established version, one makes sure of the desired effect of the modification of [éq. 2.5-1] with [éq. 2.5-4] by means of: , éq

$$M^{\alpha\beta}=m^{\alpha\beta}\begin{pmatrix}\max\left(\frac{h^3}{12},\frac{hA^e}{8}\right) & 0 & 0\\ 0 & \max\left(\frac{h^3}{12},\frac{hA^e}{8}\right) & 0\\ 0 & 0 & h\end{pmatrix}.$$ 2.5-5 because

[éq 2.5-4] is not interesting that for the meshes coarse, a priori more current, while [éq. 2.5-1] becomes favorable for very fine meshes. Numerical integration

4.6 for elasticity For

triangular elements DKT, DKTG and DST the stiffness matrixes are obtained exactly with three points of integration of Hammer Cordonnées

of the points Weight	1/6 ω_l
$\xi_1=1/6\,;\eta_1=1/6$	1/6
$\xi_2=2/3\,;\eta_2=1/6$	1/6
$\xi_3=1/6\,;\eta_3=2/3$	Formulas
$\int_0^1\int_0^{1-\xi} y(\xi,\eta)\,d\eta\,d\xi=$	$\sum_{i=1}^{n}\omega_i\, y(\xi_i,\eta_i)$

of numerical integration on a triangle (Hammer) For

the elements quadrangle a Gauss quadrature 2x2 is used. Cordonnées

of the points Weight	1 1 ω_l
$\xi_1=1/\sqrt{3}\,;\eta_1=1/\sqrt{3}$	1
$\xi_2=1/\sqrt{3}\,;\eta_2=-1/\sqrt{3}$	1
$\xi_3=-1/\sqrt{3}\,;\eta_3=1/\sqrt{3}$	Formulas
$\xi_3=-1/\sqrt{3}\,;\eta_3=-1/\sqrt{3}$	
$\int_0^1\int_0^{1-\xi} y(\xi,\eta)\,d\eta\,d\xi=$	$\sum_{i=1}^{n}\omega_i\, y(\xi_i,\eta_i)$

of numerical integration on a quadrangle (Gauss) Numerical integration

4.7 for plasticity

integration on the surface of the element is supplemented by an integration in the thickness of the behavior since: where

$\mathbf{H}_{\boldsymbol{m}}=\int_{-h/2}^{+h/2}\mathbf{H}\,\mathrm{dz},\mathbf{H}_{\boldsymbol{mf}}=\int_{-h/2}^{+h/2}\mathbf{H}\,\mathrm{zdz},\mathbf{H}_{\boldsymbol{f}}=\int_{-h/2}^{+h/2}\mathbf{H}\,z^2\mathrm{dz}$ is $\mathbf{H}$ the matrix of plastic behavior local.

The initial thickness is divided into identical N layers of thickness and there are three points of integration per layer (except for elements DKTG and DKQG which have only one layer and a point of integration in the layer). The points of integration are located in higher skin of layer, in the middle of the layer and in lower skin of layer. For layers N , the number of points of integration is of. One $2\mathrm{N}+1$ advises to use from 3 to 5 layers in the thickness for a number of points of integration being worth 7,9 and 11 respectively. For

the stiffness, one calculates for each layer, in plane stresses, the contribution to the stiffness matrixes of membrane, bending and membrane-flexure coupling. These contributions are added and assembled to obtain the total tangent stiffness matrix. For each layer, one calculates the state of the stresses and L $(\sigma_{xx},\sigma_{yy},\sigma_{xy})$ "together of the local variables, in the middle of the layer and in skins higher and lower of layer, starting from the local plastic behavior and of the local strain field. $(\varepsilon_{xx},\varepsilon_{yy},\varepsilon_{xy})$ The positioning of the points D" integration enables us to have the rightest estimates, because not extrapolated, in skins lower and higher of layer, where it is known that the stresses are likely to be maximum. Cordonnées

of the points Weights	1/3 ω_l
$\xi_1=-1$	4/3

$\xi_2=0$	1/3
$\xi_3=+1$	Formula
$\int_{-1}^{1} y(\xi)\,d\xi=$	$\sum_{i=1}^{n}\omega_i\, y(\xi_i,\eta_i)$

of numerical integration for a layer in the thickness Note

: One

already mentioned with the §2.2.3 that the value of the coefficient of correction in transverse shears for the elements DST, DSQ and Q4g was obtained by identification of elastic energies complémenatires after resolution of the equilibrium 3D. This method is not usable any more in elastoplasticity and the choice of the coefficient of correction in transverse shears is posed then. Plasticity is thus not developed for these elements. Discretization

4.8 of work external the variational

formulation of external work for the shell elements is written: By taking account of

$$\delta W_{ext}=\int_S (f_x\delta u+f_y\delta v+f_z\delta w+m_x\delta\beta_x+m_y\delta\beta_y)dS+\int_C (f_x\delta u+f_y\delta v+f_z\delta w+m_x\delta\beta_x+m_y\delta\beta_y)ds$$

a linear discretization of displacements, one can write for an element: The variational

$$\begin{aligned}
\delta W^e_{ext}=&\sum_{k=1}^{N}\int_S (f_x N_k(\xi,\eta)du_k+f_y N_k(\xi,\eta)\delta v_k+f_z N_k(\xi,\eta)\delta w_k\\
&+m_x N_k(\xi,\eta)\delta\beta_{xk}+m_y N_k(\xi,\eta)\delta\beta_{yk})dS\\
&+\int_C (\phi_x N_k(\xi,\eta)\delta u_k+\phi_y N_k(\xi,\eta)\delta v_k+\phi_z N_k(\xi,\eta)\delta w_k\\
&+\mu_x N_k(\xi,\eta)\delta\beta_{xk}+\mu_y N_k(\xi,\eta)\delta\beta_{yk})ds\\
=&\sum_{k=1}^{N}(\int_S f_x N_k(\xi,\eta)dS+\int_C \phi_x N_k(\xi,\eta)ds\vdots\;\int_S f_y N_k(\xi,\eta)dS+\int_C \phi_y N_k(\xi,\eta)ds\vdots\\
&\int_S f_z N_k(\xi,\eta)dS+\int_C \phi_z N_k(\xi,\eta)ds\vdots\;\int_S m_x N_k(\xi,\eta)dS+\int_C \mu_x N_k(\xi,\eta)ds\vdots\\
&\int_S m_y N_k(\xi,\eta)dS+\int_C \mu_y N_k(\xi,\eta)ds\vdots)\delta\mathbf{U}^e_k\\
=&\sum_{k=1}^{N}\mathbf{F}^e_k\delta\mathbf{U}^e_k=\mathbf{F}^e\delta\mathbf{U}^e
\end{aligned}$$

formulation of the work of the external forces for all the elements is written then: where

$\delta W_{\mathbf{ext}}=\sum_{e=1}^{\mathrm{nb}elem}\delta W^e_{\mathbf{ext}}=\mathbf{F}\,\delta\mathbf{U}=\delta\mathbf{U}^T\mathbf{F}^T$ is $\mathbf{U}$ all the degrees of freedom of discretized structure and comes $\mathbf{F}$ from the assembly of the vectors forces elementary. As

for the stiffness matrixes, the process of assembly of the vectors forces elementary implies that all the degrees of freedom are expressed in the total reference. In the total reference, the degrees of freedom are three displacements compared to the three axes of the total cartesian coordinate system and the three rotations compared to these three axes. One thus uses transition matrixes of the local coordinate system to the total reference for each element. Note:

The external forces can also be defined in the reference user. One then uses a transition matrix of the reference user towards the local coordinate system of the element to have the statement of these forces in the local coordinate system of the element and to deduce the vector from it elementary corresponding room forces For the assembly one passes then from the local coordinate system of the element to the total reference. Taking

4.9 into account of the thermal loadings Thermo

4.9.1 - elasticity of the plates the temperature

is represented by the model thermal at three fields according to [R3.11.01]: , with

$$T(x_\gamma, x_3) = T^m(x_\gamma).P_1(x_3) + T^s(x_\gamma).P_2(x_3) + T^i(x_\gamma).P_3(x_3)$$

: : $T^m(x_\gamma)$ the temperature on the average average:

$T^s(x_\gamma)$ the temperature on the higher skin:

$T^i(x_\gamma)$ the temperature on the lower skin:

$P_j(x_3)$ three polynomials of LAGRANGE in the thickness: : from $]-h/2,+h/2[$

$$P_1(x_3) = 1-(2x_3/h)^2 \;;\; P_2(x_3) = \frac{x_3}{h}(1+2x_3/h) \;;\; P_3(x_3) = -\frac{x_3}{h}(1-2x_3/h) \;;$$

the representation of the temperature above, one obtains: the average temperature

- in the thickness: ;

$$\bar{T}(x_\gamma) = \frac{1}{h}\int_{-h/2}^{+h/2} T(x_\gamma, x_3)\, dx_3 = \frac{1}{6}\left(4T^m(x_\gamma) + T^s(x_\gamma) + T^i(x_\gamma)\right)$$ the average

- variation in temperature in the thickness: ; Thus

$$\hat{T}(x_\gamma) = \frac{12}{h^2}\int_{-h/2}^{+h/2} T(x_\gamma, x_3)\, x_3 dx_3 = T^s(x_\gamma) - T^i(x_\gamma)$$

the temperature can be written in the following way: such as

$T(x_\gamma, x_3) = \bar{T}(x_\gamma) + \hat{T}(x_\gamma).x_3/h + \tilde{T}(x_\gamma, x_3)$: . If

$$\int_{-h/2}^{h/2} \tilde{T}(x_\gamma, x_3) = 0 \;;\; \int_{-h/2}^{h/2} x_3\,\tilde{T}(x_\gamma, x_3) = 0$$

the temperature is indeed closely connected in the thickness one has. Code_Aster $\tilde{T} = 0$

treats three different thermoelastic situations, where the elastic characteristics thermo -, E ν depend α only on the average temperature in $\bar{T}$ the thickness: the case

- where the material is thermoelastic isotropic homogeneous in the thickness; the case
- where the plate models an orthotropic grid (concrete reinforcing steels); the case
- where the behavior of the plate is deduced from a thermoelastic homogenization, cf [bib7]. For

the shell elements in thermoelasticity, the heating effects are taken into account via generalized forces, membrane and bending. Thus, in the case of a homogeneous plate, knowing the coefficient of thermal expansion, α the generalized thermal forces are defined starting from the plane stresses in the thickness by: Maybe

$$N^{ther}_{\beta\gamma}=\int_{-h/2}^{+h/2} C_{\beta\gamma\eta\zeta}\, e^{ther}_{\eta\zeta}\, dx_3=\int_{-h/2}^{+h/2} \alpha\, C_{\beta\gamma\eta\zeta}\left(T-T^{réf}\right)\delta_{\eta\zeta}\, dx_3$$
$$M^{ther}_{\beta\gamma}=\int_{-h/2}^{+h/2} x_3\, C_{\beta\gamma\eta\zeta}\, e^{ther}_{\eta\zeta}\, dx_3=\int_{-h/2}^{+h/2} \alpha\, x_3\, C_{\beta\gamma\eta\zeta}\left(T-T^{réf}\right)\delta_{\eta\zeta}\, dx_3$$
$$V^{ther}_{\beta}=0$$

in the homogeneous isotropic thermoelastic case in the thickness:

$$N^{ther}_{\beta\gamma}=\alpha . C_{\beta\gamma\eta\zeta} . h . \left(\bar{T}-T^{réf}\right)\delta_{\eta\zeta}=\alpha . \frac{Eh}{1-\nu} . \left(\bar{T}-T^{réf}\right)\delta_{\beta\gamma};$$
$$M^{ther}_{\beta\gamma}=\alpha . C_{\beta\gamma\eta\zeta} . \frac{h^2}{12}\hat{T}=\alpha . \frac{Eh^2}{12(1-\nu)} . \hat{T}\,\delta_{\beta\gamma};\ V^{ther}_{\beta}=0 .$$

The thermal stresses of origin withdrawn from the usual mechanical stresses are calculated in three positions (sup., moy. and inf.) in the thickness: In

$$\sigma^{ther}_{\beta\gamma}=\frac{\alpha . E}{1-\nu}\left(\bar{T}-T^{réf}+\hat{T} . x_3/h\right)\delta_{\beta\gamma}$$

the case deduced from the thermoelastic homogenization, cf [bib7], the generalized thermal forces are defined by the general relation, starting from the “correctors” of membrane, those $c^{\beta\gamma}$ of bending, and $x^{\beta\gamma}$ that of thermal expansion, like $\mathbf{u}^{dil}$ averages on representative ground volume (cell): z In

$$N^{ther}_{\beta\gamma}=\langle\langle C_{\beta\gamma\eta\zeta} . \alpha . \left(\bar{T}-T^{réf}+\hat{T}\left(x_\gamma\right) . z_3/h+\tilde{T}\left(x_\gamma, x_3\right)\right)\delta_{\eta\zeta}\rangle\rangle_Z+\langle\langle e_{ij}\left(c^{\beta\gamma}\right)C_{ijkl} . e_{kl}\left(u^{dil}\right)\rangle\rangle_Z ;$$
$$M^{ther}_{\beta\gamma}=\langle\langle z_3 . C_{\beta\gamma\eta\zeta} . \alpha . \left(\bar{T}-T^{réf}+\hat{T}\left(x_\gamma\right) . z_3/h+\tilde{T}\left(x_\gamma, x_3\right)\right)\delta_{\eta\zeta}\rangle\rangle_Z+\langle\langle e_{ij}\left(x^{\beta\gamma}\right)C_{ijkl} . e_{kl}\left(u^{dil}\right)\rangle\rangle_Z ;$$
$$V^{ther}_{\beta}=0$$

this case when one limits oneself to the orthotropic situations without coupling bending-membrane, one neglects the role of on $\tilde{T}\left(x_\gamma, x_3\right)$ the corrector, and $\mathbf{u}^{dil}$ one thus finds that the thermal forces which appear to the second member have as a statement: One

$$N^{ther}_{\beta\gamma}=\alpha . H^{m}_{\beta\gamma\eta\zeta} . \left(\bar{T}-T^{réf}\right)\delta_{hz} ;$$
$$M^{ther}_{\beta\gamma}=\alpha . H^{f}_{\beta\gamma\eta\zeta} . \hat{T}\,\delta_{\eta\zeta} ;\ V^{ther}_{\beta}=0$$

cannot however go back to the complete three-dimensional stresses: it would be necessary to know the “correctors” within the basic cell having been used with the determination of the coefficients as homogenized behavior. In

the thermoelastoplastic situations, or for the shells (elements of family COQUE_3D), `it` is necessary to evaluate the three-dimensional stresses, of which thermal stresses, in each point of integration in the thickness. Note:

To go back

to the complete three-dimensional stresses is not immediate for the multi-layer shells (stratified) because it is necessary to know layer by layer the stress state; in elasticity, this one results from the strain state and the behavior on the level of each layer. Thermomechanical

4.9.2 sequence For

the resolution of chained thermomechanical problems, one must of the finite elements use for thermal computation thermal shell [R3.11.01] whose field of temperature is recovered like input datum of Code_Aster *for* mechanical computation. It is necessary thus that there is compatibility between the thermal field given by the thermal shells and that recovered by the mechanical plates. This last is defined by the knowledge of 3 fields TEMP_SUP , TEMP _MIL and TEMP_INF given in skins lower, medium and higher of shell. The table

below indicates compatibilities between the shell elements and the shell elements thermal: THERMAL

modelization Nets	**Finite element**	**to use**	**with Mesh**	**Finite element**	**MECHANICAL**	**Modelization COQUE**
QUAD4	THCOQU	4 QUAD4		MEDKQU	4 MEDKQG 4 MEDSQU 4 MEQ4 QU4 DKT	DKTG DST Q4G COQUE
SORTED	3 THCOTR	3 SORTED		3 MEDKTR	3 MEDKTG 3 MEDSTR 3 DKT	DKTG DST Note:

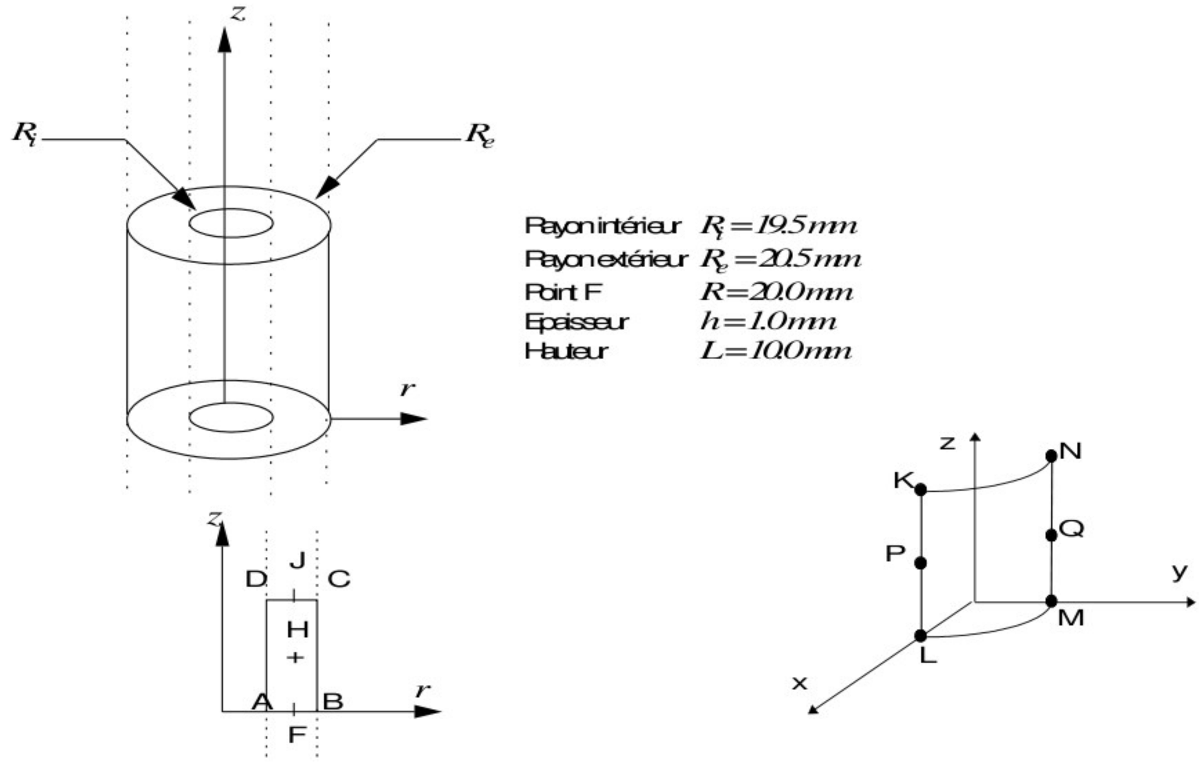
The nodes of the thermal shell elements and mechanical plates must correspond. The meshes will be identical.
The surface thermal shell elements are treated like plane elements by projection of the initial geometry on the level defined by the first 3 tops. The sequence

with definite multi-layer materials via command DEFI _COMPOSITE [U4.23 .03] is not available for time. The thermomechanical

sequence is also possible if one knows by experimental measurements the variation of the field of temperature in the thickness of structure or certain parts of structure. In this case one works with a card of temperature defined a priori; the field of temperature is not given any more by three values TEMP_INF , TEMP _MIL and TEMP_SUP of thermal computation obtained by EVOL_THER . It can be much richer and contain an arbitrary number of points of discretization in the thickness of the shell. Operator DEFI _NAPPE allows to create such profiles of temperatures starting from the data provided by the user. These profiles are affected by the command CREA_CHAMP (cf the hpla100e benchmark). It will be noted that it is not necessary for mechanical computation that the number of points of integration in the thickness is equal to the number of points of discretization of the field of temperature in the thickness. The field of temperature is automatically interpolated at the points of integration in the thickness of the shell elements or of shells by the command CREA_RESU operation PREP_VRC2 . For
the elements DKTG on the other hand , which do not have subpoints in the thickness, one should not use PREP_VRC2 . Three values TEMP_INF , TEMP_MIL and TEMP_SUP are assigned to the same command variables name, directly recoverable in the programs. Benchmarks

4.9.3

the benchmarks for the thermomechanical sequence between thermal shell elements and shell elements are the hpla100e (elements DKT) and hpla100f (elements DKQ). It is about a heavy thermo-elastic hollow roll in uniform rotation [V7.01.100] subjected to a phenomenon of thermal thermal expansion where the fields of temperature are calculated with THER_LINEAIRE `by` a steady computation. Thermal



thermal expansion is worth: with $T(\rho) - T_{réf}(\rho) = 0.5(T_s + T_i) + 2.(T_s + T_i)(r - R)/h$

: One tests

- $T_s = 0.5\,°C$, $T_i = -0.5\,°C$, $T_{réf} = 0.\,°C$
- $T_s = 0.1\,°C$, $T_i = 0.1\,°C$, $T_{réf} = 0.\,°C$

the stresses, the forces and bending moments in and L . M The results of reference are analytical. One obtains very good performances whatever the type of element considered. Establishment

5 of the shell elements in Code_Aster *Description*

5.1 : These

elements (of names MEDKTR 3, MEDSTR 3, MEDKQU 4, MEDSQU 4, MEDKTG3, MEDKQG4 and MEQ4 QU4) lean on meshes SORTED 3 and QUAD4 plane . These elements are not exact with the nodes and it is necessary to net with several elements to get correct results. Introduced

5.2 use and developments: These

elements are used in the following way: AFFE_MODELE

- (MODELISATION = "DKT".) for the triangle and the quadrangle of the type DKT AFFE_MODELE
- (MODELISATION = "DST".) for the triangle and the quadrangle of the type DST AFFE_MODELE
- (MODELISATION = "DKTG".) for the triangle and the quadrangle of the type DKTG AFFE_MODELE
- (MODELISATION = "Q4G".) for the quadrangle of the type Q4G One calls

on routine INI079 for the position of the points of Hammer and Gauss on the surface of the plate and the corresponding weights. AFFE_CARA_ELEM

- (COQUE=_F (EPAISSEUR=' EP' ANGL_REP
  = ("" α ") β COEF_RIGI_DRZ
  = "CTOR")

to make postprocessings (forced, generalized forces,…) in a reference chosen by the user who is not the local coordinate system of the element, one gives a direction of reference D defined by two nautical angles in the total reference. The projection of this direction of reference as regards the plate fixes a direction of reference $X1$. The norm with the plane into fixed one second and the cross product of the two vectors previously definite make it possible to define the local trihedron in which will be expressed the generalized forces and the forced. The user will have to take care that the selected reference axis is not found parallel with the norm of certain shell elements of the model. By default this direction of reference is the axis of the total X reference of definition of the mesh. Value

CTOR corresponds to coefficent that the user can introduce for the processing of the terms of stiffness and mass according to normal rotation with the plan of the plate. This coefficient must be sufficiently small not to disturb the energy assessment of the element and not too small so that the stiffness matrixes and of mass are invertible. A value of 10-5 is put by default. ELAS

- =_F (E =young NU = nu ALPHA = alpha . RHO = rho .) For

a homogeneous isotropic thermoelastic behavior in the thickness one uses key word ELAS in DEFI _MATERIAU where one defines the coefficients modulus E Young, . ν Poisson's ratio, thermal λ coefficient of thermal expansion and RHO density˜ ELAS_ORTH

- (_FO) =_F (E_L
  =ygl. E_T =ygt. G_LT =glt. G_TZ =gtz. NU_LT =nult. ALPHA_L
  =alphal. ALPHA_T =alphat.) For

an orthotropic thermoelastic behavior whose axes of orthotropy are, and L T for z isotropy of axis (fibers L in the direction L coated by a matrix, for example) it is necessary to give the seven independent coefficients ygl, modulus Young longitudinal, ygt, transverse modulus Young, glt, shear modulus in the plane, gtz LT, shear modulus in the nult plane TZ, Poisson's ratio in the thermal plane LT and the coefficients of thermal expansion alphal and alphat for longitudinal and transverse thermal thermal expansion, respectively. The orthotropic **behavior available is only associated with the key word DEFI _COMPOSITE which makes it possible to define a multi-layer composite shell. For** only one orthotropic material, one will thus use DEFI _COMPOSITE with only one layer. If one wishes to use ELAS_ORTH with transverse shears, modelization DST necessarily should be employed. If one uses modelizations DKT, or DKTG, the transverse shears are not taken into account. ELAS_COQUE

- (_FO) =F (MEMB_L =C1111. MEMB_LT =C1122. MEMB_T =C2222. MEMB_G_LT =C1212. FLEX_L =D1111. FLEX_LT =D1122. FLEX_T =D2222. FLEX_G_LT =D1212. CISA_L =G11…. CISA_T =G22…. ALPHA =alpha. RHO =rho.) This behavior

was added in DEFI _MATERIAU to take into account stiffness matrixes nonproportional out of membrane and bending, obtained by homogenization of a multi-layer material. The coefficients of the stiffness matrixes are then introduced with the hand by the user into the reference user defined by the key word ANGL_REP . The thickness given in AFFE_CARA_ELEM is only used with the density defined by RHO. alpha is thermal thermal expansion. If one wishes to use ELAS_COQUE with transverse shears, modelization DST necessarily should be employed. If modelization DKT is used, the transverse shears are not taken into account. DEFI

- _COMPOSITE _F (COUCHE = THICKNESS : "EP" MATER = "material" ORIENTATION = (theta)) This key word

makes it possible to define a multi-layer composite shell on the basis of the sub-base towards the roadbase from its characteristics sleep by layer, thickness, type of the material constitutive and directional sense of fibers compared to a reference axis. The type of the constitutive material is produced by the operator DEFI _MATERIAU under key word ELAS_ORTH . theta is the angle of the first direction of orthotropy (longitudinal meaning or meaning of fibers) in the tangent plane with the element compared to the first direction of the reference of reference defined by ANGL_REP . By default theta is null, if not it must be provided in degrees and must be understood enters and $-90°$. AFFE_CHAR_MECA $+90°$

- (DDL_IMPO =_F (DX =. DY =. DZ =. DRX =. DRY =. DRZ =. degree of freedom of plate in the total reference. FORCE_COQUE =_F (FX =. FY =. FZ =. MX =. MY =. MZ =.) It S "acts of the surface forces (membrane and bending) on shell elements. These forces can be given in the total reference or the reference user defined by ANGL_REP . FORCE_NODALE

- =_F (FX =. FY =. FZ =. MX =. MY =. MZ =.) It S " acts of the forces of shell in the total reference. Computation

5.3 in linear elasticity: The stiffness matrix

and the mass matrix (respectively options RIGI_MECA and MASS_MECA) are integrated numerically. It is not checked if the mesh is plane or not. The computation account holds owing to the fact that the terms corresponding to the degrees of freedom of plate are expressed in the local coordinate system of the element. A transition matrix makes it possible to pass from the local degrees of freedom to the total degrees of freedom.

Elementary computations (CALC_CHAMP) currently available correspond to the options: EPSI_ELNO

- and SIGM_ELNO which and the provide the strains nodal stresses in the reference user of the element in lower skin, with semi thickness and in higher skin of plate, the position being specified by the user. One stores these values in the following way: 6 components of strain or stresses: EPXX
- EPYY EPZZ EPXY EPXZ EPYZ or SIXX SIYY SIZZ SIXY SIXZ SIYZ DEGE_ELNO
- : who gives the strains generalized by "element to the nodes starting from displacements in the reference user: EXX, EYY, EXY, KXX, KYY, KXY, GAX, GAY. EFGE_ELNO
- : who gives the forces generalize by element with the nodes starting from displacements: NXX, NYY, NXY, MXX, MYY, MXY, QX, QY. SIEF_ELGA
- : who gives the forces generalize by element with Gauss points starting from displacements: NXX, NYY, NXY, MXX, MYY, MXY, QX, QY. EPOT_ELEM
- : who gives L" elastic strain energy of strain per element starting from displacements. ECIN_ELEM
- : who gives kinetic energy by element. Finally

one calculates also option FORC_NODA of computation of the nodal forces for operator CALC_CHAMP . Plastic

5.4 design the stiffness matrix

is there too integrated numerically. One calls on the computation option STAT_NON_LINE in which one defines in the level of the nonlinear behavior the number of layers to be used for numerical integration. For

modelizations DKT, all the models of plane stresses available in Code_Aster *can* be used. STAT_NON_LINE

```
(…. COMP_INCR
 =_F (RELATION = ' "COQUE_NCOU
=" COUCHES' MANY)….
) For
```

the modelizations DST and Q4G, only linear elasticity is usable. For

the modelization DKTG, the only constitutive laws used are total models (since there is only one point of integration in the thickness), connecting the strains generalized to the generalized stresses. These models are, in version 9.4: GLRC_DM and GLRC_DAMAGE , like their coupling with elastoplastic models out of membrane (KIT_DDI) .

Currently available elementary computations correspond to the options: EPSI_ELNO

- which provides the strains by element to the nodes in the reference user starting from displacements, in lower skin, with semi thickness and in higher skin of plate. SIGM_ELNO
- which makes it possible to obtain the stress field in the thickness by element with the nodes for all the subpoints (all the layers and all the positions: in lower skin, in the medium and in higher skin of layer). EFGE_ELNO
- which makes it possible to obtain the forces generalized by element with the nodes in the reference user. VARI_ELNO
- which calculates the field of local variables and the forced by element with the nodes for all the layers, in the local coordinate system of the element. Conclusion

6

the plane finite elements of plate that we describe here are used in thin structure computations, in small displacements and strains, whose thickness ratio over characteristic length is lower than. As $1/10$ these elements are plane, they do not take into account the curvature of structures, and it is necessary to refine the meshes if this one would be important. It is

elements for which the strains and the forced in the plane of the element vary linearly with the thickness of the plate. Moreover, the distortion associated with the transverse shears is constant in the thickness of the element. Two families of shell elements exist: elements DKT, DKQ (or DKTG, DKQG) for which the transverse distortion are null and the elements DST, DSQ and Q4G for which it remains constant and non-zero in the thickness. One advises to use the second type of elements when the structure to be netted has a thickness ratio over characteristic length understood enters and $1/20$ the $1/10$ first in the remainder of the cases. When the transverse distortion is non-zero, shell elements DST, DSQ and Q4G 3D do not satisfy the equilibrium conditions and the boundary conditions on the transverse nullity of the shearing stresses on the sides higher and lower with plate, compatible with a constant transverse distortion in the thickness of the plate. It results from it thus that on the level of the behavior a coefficient of for $5/6$ a homogeneous plate corrects the usual relation between the stresses and the distortion transverses in order to 3D ensure the equality between energies of shears of the model and the model of plate constant distortion. In this case, the deflection has as w an interpretation average transverse displacement in the thickness of the plate.

The nonlinear behaviors in plane stresses are available for shell elements DKT and DKQ only. Indeed the rigorous taking into account of non-zero constant transverse shears on the thickness and the determination of the correction associated on the shear stiffness compared to a model satisfying the equilibrium conditions and the boundary conditions are not possible and thus return the use of elements DST, DSQ and rigorously impossible Q4G in plasticity. For

the elements of family DKTG, only of the total behaviors (membrane relations moment-curvatures and forces – elongations) are available.

Elements corresponding to the machine elements exist in thermal; the mechanical sequences thermo - are thus available except, for time, the stratified materials. Bibliography

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8 of the versions of the document Index document

Version Aster	Author (S)	Organization (S) Description	of the modifications A5 P. MASSIN
		EDF R & D/ AMA initial Text	B 9.4 X.
DESROCHES		, D.MARKOVIC, EDF R & D AMA Addition of	DKTG, and the lumped mass matrixes: Orthotropic

Annexe 1 plates For an orthotropic

material like that represented below, composed for example of fibers of direction L coated with a matrix, whose axes of orthotropy are L, T and Z with isotropy of axis L, the statement for the matrixes and in $\mathbf{H}$ orthotropic reference $\mathbf{H}_\gamma$ the previously definite one becomes: and with

$$\mathbf{H}_L=\begin{pmatrix} H_{LL} & H_{LT} & 0 \\ H_{LT} & H_{TT} & 0 \\ 0 & 0 & G_{LT} \end{pmatrix} \text{ and } \mathbf{H}_{L\gamma}=\begin{pmatrix} G_{LZ} & 0 \\ 0 & G_{TZ} \end{pmatrix}$$

$$H_{LL}=\frac{E_L}{1-\nu_{LT}\nu_{TL}}\,;H_{TT}=\frac{E_T}{1-\nu_{LT}\nu_{TL}} \quad H_{LT}=\frac{E_T\nu_{LT}}{1-\nu_{LT}\nu_{TL}}=\frac{E_L\nu_{TL}}{1-\nu_{LT}\nu_{TL}}$$

. The knowledge

$$G_{LZ}=\frac{E_L}{2(1+\nu_{LZ})} \quad G_{TZ}=\frac{E_T}{2(1+\nu_{TZ})}$$

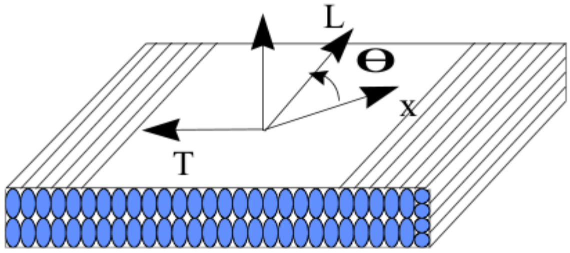
of the five independent coefficients is sufficient $E_L, E_T, G_{LT}, G_{TZ}, \nu_{LT}$ to determine the coefficients of the matrixes and since $\mathbf{H}$ $\mathbf{H}_\gamma$: and. If

$$\nu_{TL}=\frac{E_T\nu_{LT}}{E_L} \text{ one } G_{LZ}=G_{LT}$$

indicates by the angle θ between orthotropic reference and the principal axis of the reference defined by the user by means of ANGL_REP `it is established` that: and with

$$\mathbf{H}=\mathbf{T}_1^T\mathbf{H}_L\mathbf{T}_1 : \ \mathbf{H}_\gamma=\mathbf{T}_2^T\mathbf{H}_{L\gamma}\mathbf{T}_2$$

and where $\mathbf{T}_1=\begin{pmatrix} C^2 & S^2 & CS \\ S^2 & C^2 & -CS \\ -2CS & 2CS & C^2-S^2 \end{pmatrix}$ and $\mathbf{T}_2=\begin{pmatrix} C & S \\ -S & C \end{pmatrix}$ as $C=\cos\theta, S=\sin\theta$ indicated $\theta=(x, L)$ on the figure below. In the case of



forced initial of thermal origin, we have moreover: where and

$$\sigma_{th}=-\mathbf{T}_1^T\mathbf{H}_L\begin{pmatrix} \alpha_L\Delta T \\ \alpha_T\Delta T \\ 0 \end{pmatrix}$$

are α_L α_T the coefficients of thermal expansion thermal in the directions and L the temperature variation T ΔT . : Factors

Annexe 2 of transverse correction of shears for orthotropic or stratified plates the matrix

is defined $\mathbf{H_{ct}}$ so that the surface density of transverse energy of shears obtained in the case of the three-dimensional distribution of the stresses resulting from the resolution of the equilibrium is equal to that of the model of plate based on the assumptions of Reissner, for a behavior in pure bending. One must thus find such as $\mathbf{H_{ct}}$: with and

$$\frac{1}{2}\int_{-h/2}^{+h/2}\tau\,\mathbf{H}_{\gamma}^{-1}\tau=\frac{1}{2}\mathbf{T}\mathbf{H}_{ct}^{-1}\mathbf{T}=\frac{1}{2}\gamma\,\mathbf{H_{ct}}\gamma\ .\ \tau=\begin{pmatrix}\sigma_{xz}\\ \sigma_{yz}\end{pmatrix}\ \text{To obtain}\ \mathbf{T}=\int_{-h/2}^{+h/2}\tau\,dz=\mathbf{H_{ct}}\gamma$$

one uses $\mathbf{H_{ct}}$ the distribution of following τ 3D obtained z from the resolution of the balance equations without external couples: with for

$$\sigma_{xz}=-\int_{-h/2}^{z}(\sigma_{xx,x}+\sigma_{xy,y})\,d\zeta\,;\,s_{yz}=-\int_{-h/2}^{z}(\sigma_{xy,x}+\sigma_{yy,y})\,d\zeta\ .\ \text{If}\ \sigma_{xz}=\sigma_{yz}=0\quad z=\pm h/2$$

there is no coupling membrane bending (symmetry compared to), the stresses $z=0$ in the plane of the element in the case of have $\sigma_{xx},\sigma_{yy},\sigma_{xy}$ as a statement a behavior of pure bending: with. If

$$\sigma=z\,\mathbf{A}(z)\mathbf{M}\ \text{and}\ \mathbf{A}(z)=\mathbf{H}(z)\mathbf{H}_f^{-1}$$

do not depend $\mathbf{H}(z)$ $\mathbf{H}_f$ on and one can x determine y . Indeed $\mathbf{H_{ct}}$: where and

$$\tau(z)=\mathbf{D_1}(z)\mathbf{T}+\mathbf{D_2}(z)\lambda\ \text{like}\ \mathbf{T}=\begin{pmatrix}T_x\\ T_y\end{pmatrix}=\begin{pmatrix}M_{xx,x}+M_{xy,y}\\ M_{xy,x}+M_{yy,y}\end{pmatrix}\quad \lambda=\begin{pmatrix}M_{xx,x}-M_{xy,y}\\ M_{xy,x}-M_{yy,y}\\ M_{yy,x}\\ M_{xx,y}\end{pmatrix}$$

: . It results

$$\mathbf{D_1}=-\int_{-h/2}^{z}\frac{\zeta}{2}\begin{pmatrix}A_{11}+A_{33} & A_{13}+A_{32}\\ A_{31}+A_{23} & A_{22}+A_{33}\end{pmatrix}d\zeta\ \text{from it}$$

$$\mathbf{D_2}=-\int_{-h/2}^{z}\frac{\zeta}{2}\begin{pmatrix}A_{11}-A_{33} & A_{13}-A_{32} & 2A_{12} & 2A_{31}\\ A_{31}-A_{23} & A_{33}-A_{22} & 2A_{32} & 2A_{21}\end{pmatrix}d\zeta$$

that with: $\frac{1}{2}\int_{-h/2}^{+h/2}\tau\,\mathbf{H}_{\gamma}^{-1}\tau=\frac{1}{2}\begin{pmatrix}\mathbf{T}\\ \lambda\end{pmatrix}\begin{pmatrix}C_{\mathbf{11}} & C_{\mathbf{12}}\\ C_{\mathbf{12}}^T & C_{\mathbf{22}}\end{pmatrix}\begin{pmatrix}\mathbf{T}\\ \lambda\end{pmatrix}$ As in addition

$$\mathbf{C_{11}}=\int_{-h/2}^{+h/2}\mathbf{D}_1^T\mathbf{H}_{\gamma}^{-1}\mathbf{D_1}\,dz\,;$$
$$\mathbf{C_{12}}=\int_{-h/2}^{+h/2}\mathbf{D}_1^T\mathbf{H}_{\gamma}^{-1}\mathbf{D_2}\,dz\,;$$
$$\mathbf{C_{22}}=\int_{-h/2}^{+h/2}\mathbf{D}_2^T\mathbf{H}_{\gamma}^{-1}\mathbf{D_2}\,dz$$

one proposes $\frac{1}{2}\int_{-h/2}^{+h/2} \tau \mathbf{H}_{\gamma}^{-1} \tau = \frac{1}{2} \mathbf{T} \mathbf{H}_{ct}^{-1} \mathbf{T}$ to take to satisfy $\mathbf{H}_{ct} = \mathbf{C}_{11}^{-1}$ as well as possible the two equations whatever and. While T comparing λ

thus calculated $\mathbf{H}_{ct}$ with one reveals $\bar{\mathbf{H}}_{ct} = \int_{-h/2}^{+h/2} \mathbf{H}_{\gamma} dz$ the coefficients of correction of following transverse shears: . For $k_1 = H_{ct}^{11} / \bar{H}_{ct}^{11} ; k_{12} = H_{ct}^{12} / \bar{H}_{ct}^{12} ; k_2 = H_{ct}^{22} / \bar{H}_{ct}^{22}$

a homogeneous, isotropic or anisotropic plate, one finds as follows: with. Note: $\mathbf{H}_{ct} = kh \, \mathbf{H}_{\gamma}$ $k = 5/6$

This method

is valid only when the composite plate is symmetric compared to. For $z = 0$ a multi-layer

- material, one establishes that: where: and

$$\mathbf{C}_{11} = \sum_{i=1}^{N} \frac{h_i}{4} \left(\sum_{p=1}^{i-1} h_p \eta_p \mathbf{A}_p^T - \frac{1}{2} z_i^2 \mathbf{A}_i^T \right) \mathbf{H}_{\gamma}^{-1} \left(\sum_{p=1}^{i-1} h_p \eta_p \mathbf{A}_p - \frac{1}{2} z_i^2 \mathbf{A}_i \right) +$$
$$\frac{1}{24} (z_{i+1}^3 - z_i^3) \left[\mathbf{A}_i^T \mathbf{H}_{\gamma}^{-1} \left(\sum_{p=1}^{i-1} h_p \eta_p \mathbf{A}_p - \frac{1}{2} z_i^2 \mathbf{A}_i \right) + \left(\sum_{p=1}^{i-1} h_p \eta_p \mathbf{A}_p^T - \frac{1}{2} z_i^2 \mathbf{A}_i^T \right) \mathbf{H}_{\gamma}^{-1} \mathbf{A}_i \right]$$
$$+ \frac{1}{80} (z_{i+1}^5 - z_i^5) \mathbf{A}_i^T \mathbf{H}_{\gamma}^{-1} \mathbf{A}_i$$

$h_i = z_{i+1} - z_i , \eta_i = \frac{1}{2}(z_{i+1} + z_i)$ $\mathbf{A}_i$ the matrix for the layer represents $\begin{pmatrix} A_{11} + A_{33} & A_{13} + A_{32} \\ A_{31} + A_{23} & A_{22} + A_{33} \end{pmatrix}$. The validity i

- of the choice can be $\mathbf{H}_{ct} = \mathbf{C}_{11}^{-1}$ examined a posteriori when one has an estimate of the solution (fields of displacements and plane stresses, in particular). One can then estimate the difference between the two estimates on energy. A approach of computation in two stages for the multi-layer plates and shells (with diagonal $\mathbf{H}_{ct}$ and two coefficients and) has D k_1 " k_2 developed elsewhere by Noor and Burton [bib11] [bib12]. In the case of
- an isotropic homogeneous plate or anisotropic L" equality between two energies is satisfied in a strict sense since. The choice $\mathbf{D}_2 = 0$ makes above is then valid and no examination a posteriori is necessary.