# Voluminal finite elements of shells

**Summarized:**

With an aim of supplementing the library of plane finite elements of plate [R3.07.03] currently available in *Code_Aster* (DKT, DST, Q4G…), one proposes to introduce two finite elements of voluminal or three-dimensional shell [bib1]. This modelization COQUE_3D [U3.12.03] makes it possible to carry out structural analyzes shell of an unspecified form with a better approximation of the geometry and kinematics.

One will limit oneself to the frame of the linear kinematics. One thus remains in small displacements and small strains. No restriction is made on the type of behavior in plane stresses.

The two elements which are introduced are the quadratic element quadrangle Hétérosis with 9 nodes and its triangular equivalent with 7 nodes. The formulation of the continuous problem is done in Cartesian coordinates, which makes it possible to avoid explicit computations of the curvatures. These two elements have as a corresponding the linear element of shell with 3 nodes presented in the document [R3.07.02].

These two new elements are validated on existing benchmarks of plate, and on three new benchmarks of shell developed in the documentation of validation and whose principal conclusions are presented briefly here.

This note also presents in appendix how to take into account the anisotropy of the materials.

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# Contents



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# 1 Introduction

One introduces into *Code_Aster* two voluminal finite elements of shell with transverse shears (the quadrange with 9 nodes MEC3QU9H and the triangle with 7 nodes MEC3TR7H) in structural analysis shell of an unspecified form. To represent this kind of structures, one used until now with *Code_Aster* of the shell elements with plane facets which induced parasitic bendings and too restrictive shells of revolution on the type of structure [R3.07.02]. The development was carried out for isotropic materials with linear kinematics. They can thus be used only in the frame of small displacements and small strains. This formulation can be wide with the anisotropic materials [Appendix 1] and with the nonlinear kinematics [R3.07.05].

For the resolution of chained thermomechanical problems, one must use before the finite elements of thermal shell with 7 and 9 nodes described in [R3.11.01].

One develops hereafter the mechanical continuous problem by describing the kinematics of shell of the type Hencky-Mindlin-Naghdi (assumption of the cross-sections or plane) supplemented by a transverse distortion and the thermoelastoplastic constitutive law. Thanks to a parameter of penalization one can pass from a theory with shears to a theory without shears. One presents then the selected finite elements which are isoparametric quadratic elements making it possible to have a fine representation of a curved geometry and good estimates of the stresses. The interpolation and the integration method are also described.

One validates finally the development on some cases of test.

The nonlinear kinematics of these shells is treated in documentation of reference [R3.07.05].

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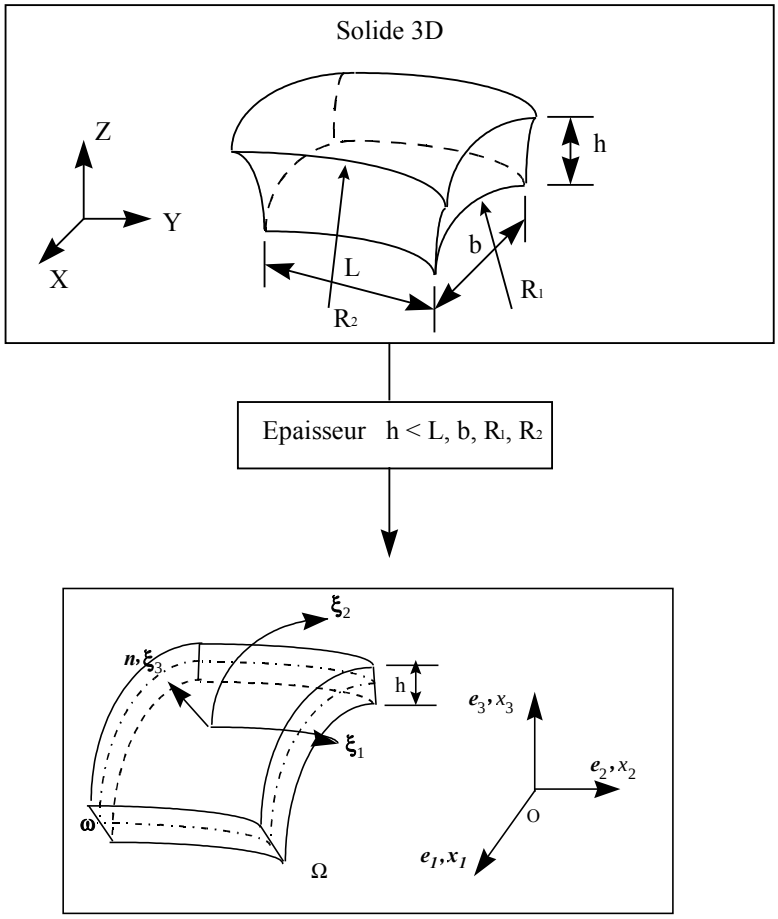
# 2 Geometry

## 2.1 formulation of the shell

For the shell elements voluminal $\Omega$ one defines a surface of reference $\omega$, or mean surface, left (of curvilinear coordinates $\xi_1\xi_2$ for example) and a thickness $h(\xi_1\xi_2)$ measured according to the norm on mean surface. This thickness must be small compared to other dimensions (extensions, radii of curvature) of structure to modelling. Figure [Figure 2.1-a] Ci - below illustrates our matter.




**Appear 2.1-the**

position of the points of the shell is given by the curvilinear coordinates $(\xi_1\xi_2)$ of mean surface $\omega$ and rise $\xi_3$ compared to this surface. $(O, e_k)$ is the total cartesian coordinate system, associated axes $(x_k)$.

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### 2.1.1 Geometrical description of mean surface

**Bases natural local and bases Cartesian local**

Is $P$ an unspecified point of the mean surface of reference $\omega$, one a:

$$\mathbf{OP}=x_k^0\left(\xi_1,\xi_2\right)e_k$$

One defines the vectors $a_\alpha$ of the natural local base of the tangent plane in $P$ $\omega$, attached to $P$ by:

$$a_\alpha=\frac{\partial \mathbf{OP}}{\partial \xi_\alpha}=\mathbf{OP}_{,\boldsymbol{\alpha}}$$

and one defines the unit norm $n$ by:

$$n=\frac{a_1\wedge a_2}{\|a_1\wedge a_2\|}$$

$\xi_3$ is the variable of position in the thickness associated with $n$.

$\left(a_1,a_2,a_3\right)$ constitute the natural base attached to $P$.

The coordinate system curvilinear $\left(\xi_1\xi_2\right)$ not being inevitably orthogonal, the base $\left(a_\alpha\right)$ is thus not inevitably orthogonal (and even less orthonormal). One thus defines an orthonormal local base $t_k$ as follows:

$$t_1=\frac{a_1}{\|a_1\|},t_2=n\wedge t_1 t_3=n$$

and one notes $\left(s_1,s_2\right)$ the coordinate system associated with $\left(t_1,t_2\right)$.

**Intrinsic reference**

the local base $t_k$ will be used to define the intrinsic reference of an element shell by choosing for point $P$ the first top of the element and for vector $a_1$ a vector coinciding with the projection on the first side on the tangent level in the first top.

**Computation of the tensor of curvature**

the tensor of curvature is related to the variation of the norm on $\omega$. It is defined by its mixed components:

$$n_{,\beta}=-C_\beta^\gamma a_\gamma$$

or by its components covariantes: $C_{\alpha\beta}=-a_\alpha.n_{,\beta}=n.a_{\alpha,\beta}$. This tensor is symmetric since $a_{\alpha,\beta}=a_{\beta,\alpha}$. Its trace $\mathrm{tr}\,C_{\alpha\beta}$ is the average curvature and its determinant the gaussian curvature.

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### 2.1.2 Description of the geometry of the shell

Is $Q$ an unspecified point of $\Omega$, volume of the shell of thickness $h$ considered constant, one a:

$$\mathbf{OQ}+\mathbf{OP}+\mathbf{PQ}=\mathbf{OP}+\frac{\xi_3}{2}h\boldsymbol{n}$$

where $\xi_3\in[-1,1]$.

$\left(\xi_1,\xi_2,\frac{\xi_3}{2}h\right)$ constitute a coordinate system curvilinear of $\Omega$.

One can also write $\mathbf{OQ}$ according to his components $(x_k)$ in the global database $(e_k)$ :

$$\mathbf{OQ}=x_k e_k$$

**Base natural local, bases orthonormée local and tensor metric**

As for $P$, one defines the natural base of space 3D $(g_k)$ attached to $Q$ by:

$$g_\alpha=\frac{\partial\mathbf{OQ}}{\partial\xi_\alpha}=a_\alpha+\xi_3\frac{h}{2}n_{,\alpha},\ g_3=\frac{g_1\wedge g_2}{\|g_1\wedge g_2\|}=\boldsymbol{n}$$

As $(g_k)$ is not inevitably orthogonal, one defines an orthonormal local base $(T_k)$ as follows:

$$T_1=\frac{g_1}{\|g_1\|},\ T_2=n\wedge T_1,\ T_3=n$$

and one notes $(x_k)$ the coordinate system associated with $(T_k)$.

One will call $(T_k)$ the local orthonormal base, and the $(x_k)$ coordinated in this local orthonormal base.

By definition, one a:

$$T_k=\frac{\partial\mathbf{OQ}}{\partial\tilde{x}_k}=\frac{\partial x_j}{\partial\tilde{x}_k}e_j=T_k^j e_j$$

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with $\frac{\partial x_j}{\partial \tilde{x}_k} = T_k^j$ the components of $(T_k)$ in the global database $(e_j)$. (They are also the components of the transition matrix of $(T_k)$ with $(e_j)$ since the transition matrix is orthogonal. Thus if $T_k = T_k^j e_j$ one also has $e_k = T_j^k T_j$ ).

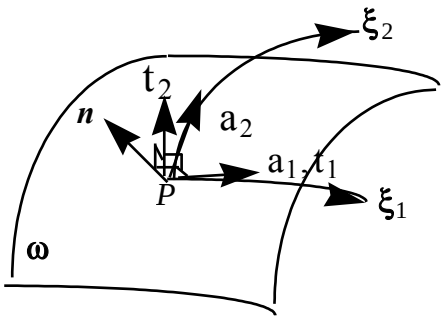
One defines the metric tensor $G$ associated with $Q$ by his components deduced from the scalar products from the vectors from the local orthonormal base:

$$G_{ij} = T_i . T_j$$

This tensor $G$ is worth the identity $\mathbf{Id}$ .

### 2.1.3 Notice

the figures [Figure 2.1.3-a] and [Figure 2.1.3-b] geometrical magnitudes mentioned illustrate Ci - above.




**Appear 2.1.3-a**

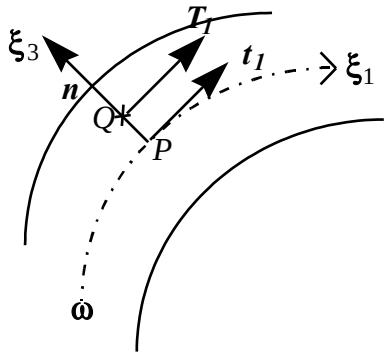



**Figure 2.1.3-b**

It should be noted that the two local orthonormal bases, that associated with mean surface $(t_k)$ and the other with the volume of the shell $(T_k)$ are confused only when the curvature is null. In this case the shell elements are comparable to shell elements

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## 2.2 Theory of the plates and shells

These elements are based on the theory of the plates and the shells according to which:

### 2.2.1 Kinematical

#### 2.2.1.1 Field of displacement

the cross-sections which are the sections perpendicular to mean surface remain right; the material points located on a norm at not deformed mean surface remain on a line in the deformed configuration. It results from this approach that **the fields of displacement vary linearly in the thickness of the shell**.

If one notes $Q'$, the position of $Q$ after strain, one a:

$$\mathbf{OQ'}=\mathbf{OQ}+\mathbf{QQ'}=\mathbf{OQ}+U(Q)$$

where the field of displacement chosen, corresponding to the kinematics of Hencky-Mindlin, is written:

$$U(Q)=u(P)+\frac{\xi_3}{2}h\beta(P) \text{ with } \beta(P).n=0¿$$

where $u(P)$ and $\beta(P)$ are respectively the vector displacement and the vector rotation of $P$, projection of $Q$ on the mean surface of the shell. The fact that $\beta(P).n=0$ indicates that one does not take into account in this kinematics rotations of the shell around his norm.

**Notation:**

One notes $\tilde{}$ the quantities expressed in the local Cartesian bases $(t_k)$ or $(T_k)$ for the points $P$ and $Q$ respectively. It results from it that:

- the vector three-dimensional displacement $U$ can be written $U=\tilde{U}_k T_k$ or $U=U_k e_k$, where it is expressed respectively in its local orthonormal base or the total Cartesian base,
- the vector displacement of mean surface $u$ can be written $u=\tilde{u}_k t_k$ or $u=u_k e_k$ according to whether it is expressed in its local orthonormal base or the total Cartesian base,
- the vector rotation of mean surface is written $b=\tilde{\beta}_\alpha t_\alpha$ in its local orthornormée base. $\beta$ being the rotation of the norm $\mathbf{n}$ (at mean surface), one also writes $\beta=\theta\wedge n$ with $\theta$, vector rotation of mean surface, such as $\theta=\tilde{\theta}_\alpha t_\alpha$. The equivalence of the two formulations shows that $\tilde{\beta}_1=\tilde{\theta}_2, \tilde{\beta}_2=-\tilde{\theta}_1$.

#### 2.2.1.2 Statement of the three-dimensional strains

the strain tensor is calculated in the local orthonormal Cartesian base $(T_k)$. It is defined like the half-difference of the metric tensors associated with the local orthonormal bases after and before strain. The metric tensor associated with this base in the NON-deformed state is simply the identity $\mathbf{Id}$, while the metric tensor of the deformed state is $\overline{G}_{ij}=T_i'.T'_j$ with $T'_k=\frac{\partial \mathbf{OQ'}}{\partial \tilde{x}_k}$.

The components of the strain tensor in $(T_k)$ are thus given by:

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$$\tilde{\varepsilon}_{\alpha\beta}=\frac{1}{2}\left(\frac{\partial \tilde{U}_{\alpha}}{\partial \tilde{x}_{\beta}}+\frac{\partial \tilde{U}_{\beta}}{\partial \tilde{x}_{\alpha}}\right)$$
$$\tilde{\varepsilon}_{\alpha 3}=\frac{1}{2}\left(\frac{\partial \tilde{U}_{\alpha}}{\partial \tilde{x}_{3}}+\frac{\partial \tilde{U}_{3}}{\partial \tilde{x}_{\alpha}}\right)$$

the equations above are linear relations strain-displacements. The variables of displacement are the components $\tilde{U}_k$ .

The components $\tilde{\varepsilon}_{kl}$ of the tensor $\tilde{\varepsilon}$ can be also expressed according to the components in the total reference $\frac{\partial U_p}{\partial x_m}$ . Indeed as in the total reference $\varepsilon=\varepsilon_{\mathrm{ij}}e^i\times e^j=\varepsilon_{\mathrm{ij}}T^i_kT^j_lT^k\times T^l=\tilde{\varepsilon}_{\mathrm{kl}}T^k\times T^l$ one from of thus deduced immediately that: $\tilde{\varepsilon}_{\mathrm{kl}}=\varepsilon_{\mathrm{ij}}T^i_kT^j_l$ . $\left(e^k\right)$ and $\left(T^k\right)$ are the bases contravariantes associated with $\left(e_k\right)$ and $\left(T_k\right)$ respectively such as: $e^i.e_j=\delta_{ij}$ and .formule $T^i.T_j=\delta_{ij}$ As the bases $\left(e_k\right)$ and $\left(T_k\right)$ are orthonormal, their associated bases contravariantes are confused with themselves. Thus in the same way that one had $T_k=T^j_ke_j$ one finds $T^k=T^j_ke^j$ .
If formula $T=T^i_ke_i\times T^k$ then $T\times T:\varepsilon=\tilde{\varepsilon}_{kl}T^k\times T^l=\tilde{\varepsilon}$ . For the continuation one indicates by $\tilde{\varepsilon}$ the form of the tensor of the strains in the local orthonormal reference and by $\varepsilon$ the form of the same tensor in the total reference. The relation of transition of the one with the other is given above in term of tensors.

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**Note:**

*The terms $T_k^j$ contain the terms of curvature of the shell $\Omega$.*

*One notes in the relations strain-displacements that the component $\tilde{\varepsilon}_{33}$ is not determined by the kinematics. This is to be associated with the assumption of nullity of the transverse normal stresses $\tilde{\sigma}_{33}=0$ justified by the behavior of the shells.*

*In the literature (see for example [bib3]), the modelization of the shells by the approach based on the curvilinear components $\tilde{u}_k$ of displacement reveals explicitly the quantities of curvature on the level of the form of the strain tensor [bib5]. Like, in general, the geometry of the shell is not known explicitly, one must thus numerically determine the geometrical characteristics which are the vectors $a_\alpha, g_\alpha$ ,... and the curvatures $C_{\alpha\beta}$ . With the finite element method it is necessary to derive the shape functions twice (see page 20 of [bib5] and [R3.07.02]) to compute: them $C_{\alpha\beta}$ . This can make their computation vague according to the selected family of the shape functions. The made mistake depends on these last (linear, quadratic, cubic polynomials….) and becomes independent of the refinement of the mesh. A formulation utilizing the derivative first of the shape functions (computation of slopes) does not present this disadvantage. Thus the consequent error with computations of the terms of curvature in a formulation based on the curvilinear approach does not decrease with the refinement of the mesh whereas for the formulation described above it becomes small by increasing the number of finite elements. Within sight of the preceding observations, the approach known as curvilinear was not followed.*

### 2.2.2 Constitutive law

the behavior of the shells is a behavior 3D in “plane stresses”. It binds the components of the stresses and the strains, in the form of vectors, in the local orthonormal base. The transverse stress $\tilde{\sigma}_{33}$ because of being null regarded as negligible compared to the other components of the tensor of the stresses (assumption of the plane stresses). The most general constitutive law is written then as follows:

$$\begin{pmatrix}\tilde{\sigma}_{11}\\ \tilde{\sigma}_{22}\\ \tilde{\sigma}_{12}\\ \tilde{\sigma}_{13}\\ \tilde{\sigma}_{23}\end{pmatrix}=\tilde{C}(\varepsilon,\mu)\begin{pmatrix}\tilde{\varepsilon}_{11}-\tilde{\varepsilon}_{11}^{\text{th}}\\ \tilde{\varepsilon}_{22}-\tilde{\varepsilon}_{22}^{\text{th}}\\ \tilde{\gamma}_{12}\\ \tilde{\gamma}_{1}\\ \tilde{\gamma}_{2}\end{pmatrix}$$

where $\tilde{C}(\varepsilon,\mu)$ is the local matrix of behavior in plane stresses and $\mu$ represents all the local variables when the behavior is nonlinear.

For behaviors where the transverse distortions are uncoupled from the strains of membrane and bending, $\tilde{C}(\varepsilon,\mu)$ is put in the form:

$$\tilde{C}=\begin{pmatrix}\tilde{H} & 0\\ 0 & \tilde{H}_\gamma\end{pmatrix}$$

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where $\tilde{H}(\varepsilon,\mu)$ is a matrix of behavior of membrane-bending $3\times3$ and $\tilde{H}_{\gamma}(\varepsilon,\mu)$ a matrix of transverse behavior of distortion $2\times2$. The two phenomena being decoupled one can also write the behavior in the form:

$$\begin{pmatrix}\tilde{\sigma}_{mf}\\ \tilde{\sigma}_{g}\end{pmatrix}=\tilde{C}(\varepsilon,\mu)\begin{pmatrix}\tilde{\varepsilon}_{mf}\\ \tilde{\gamma}\end{pmatrix}$$ with:

$$\tilde{\sigma}_{mf}=\begin{pmatrix}\tilde{\sigma}_{11}\\ \tilde{\sigma}_{22}\\ \tilde{\sigma}_{12}\end{pmatrix}=\tilde{H}(\varepsilon,\mu)\begin{pmatrix}\tilde{\varepsilon}_{11}-\tilde{\varepsilon}_{11}^{\mathrm{th}}\\ \tilde{\varepsilon}_{22}-\tilde{\varepsilon}_{22}^{\mathrm{th}}\\ \tilde{\gamma}_{12}\end{pmatrix}=\tilde{H}(\varepsilon,\mu)\tilde{\varepsilon}_{mf}$$ and $$\tilde{\sigma}_{\gamma}=\begin{pmatrix}\tilde{\sigma}_{13}\\ \tilde{\sigma}_{23}\end{pmatrix}=\tilde{H}_{\gamma}(\varepsilon,\mu)\begin{pmatrix}\tilde{\gamma}_{1}\\ \tilde{\gamma}_{2}\end{pmatrix}=\tilde{H}_{\gamma}(\varepsilon,\mu)\tilde{\gamma}$$

One will remain from now on in the frame of this assumption.

For an isotropic homogeneous linear behavior elastic, one has as follows:

$$\tilde{C}=\frac{E}{1-\nu^{2}}\begin{pmatrix}1 & \nu & 0 & 0 & 0\\ \nu & 1 & 0 & 0 & 0\\ 0 & 0 & \frac{1-\nu}{2} & 0 & 0\\ 0 & 0 & 0 & \frac{k(1-\nu)}{2} & 0\\ 0 & 0 & 0 & 0 & \frac{k(1-\nu)}{2}\end{pmatrix}$$

where $k$ is factor of transverse correction of shears whose meaning is given in the documentation of reference of the shell elements [R3.07.03], and [bib4] for more details. This coefficient applies $5/6$ to a theory of the Reissner type and 1 in the frame of the theory of Hencky - Mindlin. Lastly, if one chooses $k$ very large, one brings back oneself to a theory of the type Coils-Kirchhoff. One neutralizes the transverse distortion by penalization of associated energy while taking $k=10^{6}h/R$ ( $h$ being the thickness of the shell and $R$ its average radius of curvature).

Always in the isotropic case, the two only non-zero components of $\tilde{\varepsilon}^{th}$ are $\tilde{\varepsilon}_{ii}^{th}$ for $i=1,2$, such as:

$$\tilde{\varepsilon}_{ii}^{th}=\alpha(T-T^{réf})$$

where $\alpha$ is the thermal coefficient of thermal expansion and $T-T^{réf}$ the difference in presumedly known temperature.

**Note:**

*One does not describe the variation of the thickness nor that of the transverse strain $\tilde{\varepsilon}_{33}$ which one can however calculate by means of the preceding assumption of plane stresses. In addition no restriction is made on the type of behavior in plane stresses which one can represent.*
*Same way that $T\times T:\varepsilon=\tilde{\varepsilon}$ one can deduce formula $(T\times T)_{mf}:\varepsilon=T_{mf}:\varepsilon=\tilde{\varepsilon}_{mf}$ formula $(T\times T)_{\gamma}:\varepsilon=T_{\gamma}:\varepsilon=\tilde{\varepsilon}_{\gamma}$, which makes it possible to find $\tilde{\varepsilon}_{mf}$ and $\tilde{\varepsilon}_{\gamma}$ starting from the tensor of the strains in the total reference.*

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# 3 Principle of the virtual wors

## 3.1 Work of strain

In 3D the statement of the work of strain is written:

$$W_{\text{def}}=\int_S \int_{-h/2}^{h/2} (\tilde{\varepsilon}_{ij}\tilde{\sigma}_{ij})\,dV=\int_S \int_{-h/2}^{h/2} (\tilde{\varepsilon}_{ij}\tilde{C}_{ijkl}\tilde{\varepsilon}_{kl})\,dV=\int_S \int_{-h/2}^{h/2} (\varepsilon_{rs}P_i^r P_k^s \tilde{C}_{ijkl} P_k^p P_l^q \varepsilon_{pq})\,dV$$
$$=\int_S \int_{-h/2}^{h/2} (\varepsilon_{rs}C_{rspq}\varepsilon_{pq})\,dV=\int_S \int_{-h/2}^{h/2} (\varepsilon_{ij}\sigma_{ij})\,dV$$

One checks that this statement is invariant compared to the bases in which the tensors are expressed. One chooses for the continuation of this document all to express in the local base $(T_k)$ by knowing that one passes from the local tensor of behavior to the total tensor of behavior by the relation $C_{rspq}=P_i^r P_k^s \tilde{C}_{ijkl} P_k^p P_l^q$ .

The general statement of the work of strain 3D for the shell element is worth:

$$W_{\text{def}}=\int_S \int_{-h/2}^{h/2} (\tilde{\varepsilon}\tilde{\sigma})\,dV=\int_S \int_{-h/2}^{h/2} (\tilde{\varepsilon}\tilde{C}\tilde{\varepsilon})\,dV=\int_S \int_{-h/2}^{h/2} (\tilde{\varepsilon}_{mf}\tilde{H}\tilde{\varepsilon}_{mf})\,dV+\int_S \int_{-h/2}^{h/2} (\tilde{\varepsilon}_{\gamma}\tilde{H}_{\gamma}\tilde{\varepsilon}_{\gamma})\,dV$$

where $S$ is mean surface and the position in the thickness of the shell varies between $-h/2$ and $+h/2$ . It appears in the statement of the work of strain a contribution of strain out of membrane - bending and a contribution of transverse shearing strain.

### 3.1.1 Elastic internal energy of shell

It is expressed in the following way:

$$\Phi_{\text{int}}=\frac{1}{2}\int_S [\frac{E}{1-\nu^2}(\tilde{\varepsilon}_{11}^2+\tilde{\varepsilon}_{22}^2+2\nu\tilde{\varepsilon}_{11}\tilde{\varepsilon}_{22})+G(\tilde{\gamma}_{12}^2+k(\tilde{\gamma}_1^2+\tilde{\gamma}_2^2))]\,dV$$

where K is the factor of correction in transverse shears defined in paragraph 2 and $G=\frac{E}{2(1+\nu)}$ .

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### 3.1.2 Statement of the resulting forces

One notes:

$$N=\begin{pmatrix}N_{11}\\N_{22}\\N_{12}\end{pmatrix}=\int_{-h/2}^{+h/2}\begin{pmatrix}\tilde{\sigma}_{11}\\\tilde{\sigma}_{22}\\\tilde{\sigma}_{12}\end{pmatrix}dz \;;\; M=\begin{pmatrix}M_{11}\\M_{22}\\M_{12}\end{pmatrix}=\int_{-h/2}^{+h/2}\begin{pmatrix}\tilde{\sigma}_{11}\\\tilde{\sigma}_{22}\\\tilde{\sigma}_{12}\end{pmatrix}z\,dz \;;\; T=\begin{pmatrix}T_1\\T_2\end{pmatrix}=\int_{-h/2}^{+h/2}\begin{pmatrix}\tilde{\sigma}_{13}\\\tilde{\sigma}_{23}\end{pmatrix}dz \;.$$

$N_{11}, N_{22}, N_{12}$ are the generalized forces of membrane (in $N/m$);
$M_{11}, M_{22}, M_{12}$ are the generalized forces of bending or moments (in $N$);
$T_1, T_2$ , are the generalized forces of shears or shears (in $N/m$);

The statement of the resulting forces that one gives here is an approximate statement which does not take account of the curvature of the shell (cf p.316 of [bib3]). The mistake made on these forces is then in $h^2/R$ where $1/R$ is the average curvature. When the shell becomes plane, statements given Ci - above are exact and the meaning of the resulting forces can be found in [R3.07.03]. We will not develop more this aspect in addition documented well in [bib3] because the theory of shell used here does not rest on generalized a strains formulation/resulting forces but on three-dimensional/forced a strains formulation.

## 3.2 Work of the forces and external couples

the work of the forces being exerted on the voluminal shell is expressed in the following way:

$$W_{\text{ext}}=\int_S\int_{-h/2}^{+h/2}F_v.U\,dV+\int_S F_s.U\,dS+\int_C\int_{-h/2}^{+h/2}F_c.U\,dz\,ds$$

where $F_v, F_s, F_c$ are the voluminal, surface forces and of contour being exerted on the shell, respectively. $C$ is the part of the contour of the shell to which the forces of contour $F_c$ are applied.

**a) Loads given in the total reference:**

With the kinematics of [§2.2.1], one determines as follows:

$$W_{\text{ext}}=\int_S(f_iu_i+c_i\beta_i)\,dS+\int_C(\phi_iu_i+\chi_i\beta_i)\,ds=\int_S(f_iu_i+c_i(\tilde{\theta}_2t_{1\text{i}}-\tilde{\theta}_1t_{2\text{i}}))dS$$
$$+\int_C(\phi_iu_i+\chi_i(\tilde{\theta}_2t_{1\text{i}}-\tilde{\theta}_1t_{2\text{i}}))\,ds=\int_S(f_iu_i+c_i(\tilde{\theta}_2t_{1\text{i}}-\tilde{\theta}_1t_{2\text{i}}))dS+\int_C(\phi u+\chi\beta)\,ds$$

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- where are present on the shell:

$f_1, f_2, f_3$ : surface forces acting along the axes of the total cartesian coordinate system

$f_i = \int_{-h/2}^{+h/2} F_v . e_i dz + F_s . e_i$ where $e_i$ are the vectors of the total Cartesian base.

$c_1, c_2, c_3$ : surface couples acting around the axes of the total reference.

$c_i = \int_{-h/2}^{+h/2} zF_v . e_i dz \pm \frac{h}{2} F_s . e_i$ where are $e_i$ to them the vectors of the total Cartesian base.

- and where are present on the contour of the shell:

$\phi_1, \phi_2, \phi_3$ : linear forces acting along the axes of the total cartesian coordinate system.

$\phi_i = \int_{-h/2}^{+h/2} F_c . e_i dz$ where are $e_i$ to them the vectors of the total Cartesian base.

$\chi_1, \chi_2, \chi_3$ : linear couples acting around the axes of the total reference.

$\chi_i = \int_{-h/2}^{+h/2} zF_c . e_i dz$ where are $e_i$ to them the vectors of the total Cartesian base.

**Note:**

*One also and the $\phi$ notes $\chi$ linear distributions of force and moment applied to the contour of the finite element.*

**b) Loads given in the local coordinate system:**

One has then:

$$W_{\text{ext}} = \int_S (\sum_{i=1}^{3} \tilde{f}_\alpha t_{\alpha i} u_i + \tilde{c}_1 \tilde{\beta}_1 + c_2 \tilde{\beta}_2)\, dS + \int_C (\sum_{i=1}^{3} \tilde{\phi}_\alpha t_{\alpha i} u_i + \chi_1 \tilde{\beta}_1 + \chi_2 \tilde{\beta}_2)\, ds =$$
$$\int_S (\sum_{i=1}^{3} \tilde{f}_\alpha t_{\alpha i} u_i + \tilde{c}_1 \tilde{\theta}_2 - \tilde{c}_2 \tilde{\theta}_1) dS + \int_C (\sum_{i=1}^{3} \tilde{\phi}_\alpha t_{\alpha i} u_i + \chi_1 \tilde{\theta}_2 - \chi_2 \tilde{\theta}_1) ds$$

the statements of $\tilde{f}_1, \tilde{f}_2, \tilde{f}_3$ and $\tilde{c}_1, \tilde{c}_2, \tilde{c}_3$ are the analogues of the statements obtained for $f_1, f_2, f_3$ and $c_1, c_2, c_3$ by replacing them $e_i$ by $t_i$ .

**Note:**

*For the couple $c$ , the contribution $\tilde{c}_3$ associated with $n$ is null in theory of shell.*

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## 3.3 Work of the inertia forces

work due to the quantities of acceleration is written:

$$W^{ac}=\int_{\Omega}\rho\,\ddot{OQ}'\cdot OQ'\,dv$$

where $\rho$ is the density.

It is supposed that $\ddot{\mathbf{OQ}}'$, the vector of acceleration of the point $Q'$ is following form:

$$\ddot{\mathbf{OQ}}'=\ddot{U}_k e_k+W\wedge\left[W\wedge x_k^0 e_k\right]$$

where one neglected the Coriolis forces and the correction of metric in the thickness.

One notes $\ddot{U}_k=\dfrac{d^2U_k}{dt^2}$, and $\Omega$ is the uniform instantaneous axis of rotation of the total reference $\left(O,e_k\right)$ (compared to a Galilean reference which has the same origin $O$ as the total reference).

One expresses $\Omega$ in the global database $\left(e_k\right)$ :

$$\Omega=\Omega_k e_k$$

For virtual displacement $\mathbf{OQ}'$, one a:

$$\mathbf{OQ}'=U_k e_k$$

work due to the quantities of acceleration becomes then:

$$W^{ac}=\int_{\Omega}\rho\,U_k e_k\left[\ddot{U}_k e_k+\Omega\wedge\left(\Omega\wedge x_k^0 e_k\right)\right]dv=W^{ac}_{mass}+W^{ac}_{cent}$$

with:

$$W^{ac}_{masse}=\int_{\Omega}\rho\,U_k\ddot{U}_k\,dv$$

and:

$$W^{ac}_{cent}=\int_{\Omega}\rho U_k e_k\left[\Omega\wedge\left(\Omega\wedge x_k^0 e_k\right)\right]dv$$

## 3.4 Principle of virtual work

For a static loading, it is written in the following way: $\delta W_{ext}=\delta W_{def}$ where $W_{ext}$ is the sum of various elementary works, corresponding to the various loadings.

In harmonic dynamics (computation of eigen modes), the principle of the virtual wors gives: $\delta W_{ext}+\delta W^{ac}_{mass}=0$

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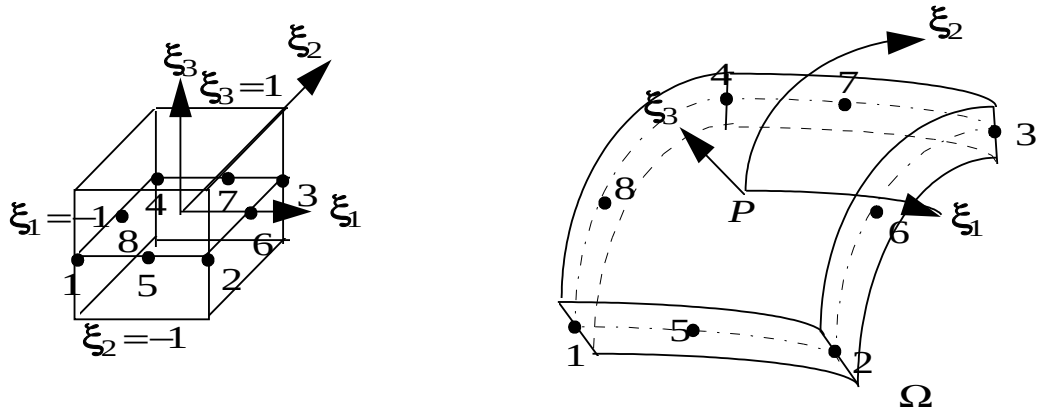
# 4 Numerical discretization of the variational formulation resulting from the principle of virtual work

## 4.1 Introduction

This chapter is devoted to the discretization of the various terms of energy introduced into the preceding chapter. The choice of frame HENCKY-MINDLIN-NAGHDI to describe the kinematics of shell, presented to the paragraph [§2] led to statements of the strains where the derivatives are limited to order 1, contrary to the model of LOVE-KIRCHHOFF. One can thus use a finite element of a restricted nature while ensuring conformity (see p.110 [bib7]).

The degrees of freedom are 3 displacements in the total reference and the 2 rotations in local coordinate system.

The selected elements are isoparametric quadrangles or triangles. The quadrangle is represented below. The quadrangles give the best results (see p.202 [bib8]). The best choice consists in taking for these elements of the quadratic interpolation functions (see p.224 [bib8]) in order to correctly model the effects of membrane, bending and shears. According to the results based on many benchmarks of the literature, the best alternative is the quadratic isoparametric quadrangle, which makes it possible to have a fine representation of a curved geometry and good estimates of the stresses. One chooses among the elements with quadratic functions the element hétérosis (Q9H) whose displacements are approached by the interpolation functions of the Sérendip element and the rotations by the functions of the element of Lagrange (cf Annexe3). This choice is justified hereafter.




**Appear 4.1-a: Representations of the isoparametric quadrangle**

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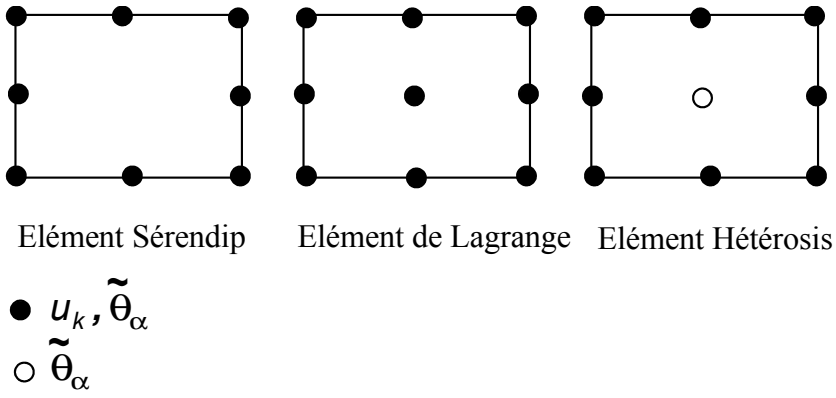
the figure [Figure 4.1-b] summarizes the three families of elements previously named.




**Appear 4.1-b: Families of finite elements for the isoparametric quadrangle**

Of the risks of blocking or locking of membrane or shears appear when the thickness of the shell becomes small compared to its radius of curvature and that the interpolation functions are of a too low nature. To solve them a selective numerical integration is used [bib6]. For certain types of boundary conditions (fixed support) with the Sérendip element locking persists in spite of selective integration. Moreover, for the element of Lagrange, this kind of integration leads to singularities in the stiffness matrix. The element Hétérosis Q9H with selective integration does not encounter the problems mentioned and seems being most powerful for the modelization of the very thin shells (see p.224 [bib8]). It should be noted that this element has a mode of strain without associated energy if it is used only. This mode disappears when one uses more than two elements [bib7].
For the elements triangle, the element Hétérosis T7H is essential for the same reasons but proves definitely less powerful (see paragraph 5 concerning the validation).

One decides to carry out all computations of discretization in the total Cartesian base.

## 4.2 Discretization of the geometrical terms

the punctual coordinates $x_k^0$ $P$ of mean surface $\omega$ are interpolated by the shape functions in the following way:

$$x_k^0 = \sum_{i=1}^{Nb1} N_i^{(1)} x_{ik}^0$$

where the number $Nb1$ and the shape functions $N_i^{(1)}$ depend on the type of element chosen, and $x_{ik}^0$ are the coordinates with the node $i$ of the element.

The vectors covariants $a_\alpha$ (attached to the point $P$) are then given by:

$$a_\alpha = \sum_{i=1}^{Nb1} \frac{\partial N_i^{(1)}}{\partial \xi_\alpha} x_{ik}^0 e_k$$

One avoids the computation of the vectors $T_k$ because the components $T_k^j$ contain the quantities of curvature whose computation is often vague as it was shown in the paragraph [§2.2.1].

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In order to avoid the presence of the terms of curvature, one writes:

$$n=\sum_{i=1}^{Nb1} N_i^{(1)} n_i$$

where $n_i$ is the normal vector with the nodes of the element.

## 4.3 Discretization of the field of displacement

One adopts the following writing for displacement with point: $Q$

$$U=\sum_{i=1}^{Nb1} N_i^{(1)} u_{ik} e_k+\frac{\xi_3}{2}\sum_{i=1}^{Nb2} N_i^{(2)} h_i\left(\tilde{\theta}_{i2} t_{i1}-\tilde{\theta}_{i1} t_{i2}\right)$$

where them $t_\alpha$ are evaluated with the nodes, and where it is observed that the interpolation functions $N_i^{(2)}$ and their number $Nb2$ for rotations $\tilde{\theta}_\alpha$ are a priori different from those used for displacements $u_k$ .

By expressing them $t_{i\alpha}$ according to their components in the total Cartesian base, one obtains:

$$U=\sum_{i=1}^{Nb1} N_i^{(1)} u_{ik} e_k+\frac{\xi_3}{2}\sum_{i=1}^{Nb2} N_i^{(2)} h_i\left(\tilde{\theta}_{i2} t_{i1k}+\tilde{\theta}_{i1} t_{i2k}\right) e_k$$

One calculates then the various elementary terms, in order to obtain the complete discretized formulation. In the continuation one uses Einstein's summation convention, while having with the spirit which the number of interpolations is $Nb1$ for $x_k^0, n, u_k$ , and $Nb2$ for $\tilde{\theta}_\alpha$ $t_\alpha$ .

### 4.3.1 Element Hétérosis Q9H

With this element, the number of interpolations for the geometry $\left(x_k^0, n\right)$ and the displacements $u_k$ is $Nb1=8$ (nodes tops and mediums on the sides), while the number of interpolations for $t_\alpha$ and the rotations $\tilde{\theta}_\alpha$ is $Nb2=9$ (nodes tops and mediums on the sides + barycenter). The total number of degrees of freedom of the element is thus $Nddle=3\times 8+2\times 9=42$ .

The interpolation functions $N_i^{(1)}$ and $N_i^{(2)}$ respectively for the geometry and the displacements, and rotations, can be found for example in [bib2] and are quoted in appendix 2.

The elementary vector of displacement can be put in the following form:

$$\tilde{q}^e=\left(u_{11}, u_{12}, u_{13}, \tilde{\theta}_{11}, \tilde{\theta}_{12}, \ldots, u_{i1}, u_{i2}, u_{i3}, \tilde{\theta}_{i1}, \tilde{\theta}_{i2}, \ldots, \tilde{\theta}_{91}, \tilde{\theta}_{92}\right) i=1,8$$

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### 4.3.2 Element triangle T7H

With this element $Nb1=6$ (nodes tops and mediums on the sides) and $Nb2=7$ (nodes tops and mediums on the sides + barycenter). The total number of degrees of freedom of the element is $Nddle=3\times6+2\times7=32$ .

The 6 interpolation functions $N_i^{(1)}$ which are classical can be found in [bib2] and are quoted in appendix 4. On the other hand the 7 $N_i^{(2)}$ are it much less and their statements are given in Appendix 3.

The elementary vector of displacement can be put in the following form:

$$\tilde{q}^e=\left(u_{I1},u_{I2},u_{I3},\tilde{\theta}_{11},\tilde{\theta}_{12},\ldots,u_{i1},u_{i2},u_{i3},\tilde{\theta}_{i1},\tilde{\theta}_{i2},\ldots,\tilde{\theta}_{71},\tilde{\theta}_{72}\right)i=1,6$$

### 4.3.3 Notices

One notices on the level of the elementary vector $\tilde{q}^e$ the presence of terms associated with the local base and the global database.

## 4.4 Discretization of the strain field

the strain field is expressed like the symmetrized gradient of the field of displacement:

$$\varepsilon=S\nabla U=\frac{1}{2}(\nabla U+\nabla U^T)$$

Like:

$$U(x)=N\left[\xi(x)\right]\tilde{q}^e$$

one thus has:

$$\nabla U=\nabla N(\xi)\frac{\partial\xi}{\partial x}\tilde{q}^e$$

where $N$ gathers the shape functions $N_i^{(1)}$ and $N_i^{(2)}$ the transition matrixes $t_{i\alpha k}$ , $\frac{\partial\xi}{\partial x}$ is the reverse of the jacobian $J$ and $\tilde{q}^e$ is the vector of the degrees of freedom to the nodes (translations $u_k$ and rotations $\tilde{\theta}_\alpha$ ).

Taking into account these relations and of $\tilde{\varepsilon}=T\times T\,\varepsilon$ , one obtains the components of the strain tensor in the local coordinate system:

$$\tilde{\varepsilon}=\tilde{B}\,\tilde{q}^e$$

where $\tilde{B}$ is the interpolation matrix of $\tilde{\varepsilon}$ , such as:

$$\tilde{B}=\boldsymbol{T}\times\boldsymbol{T}\,SJ^{-1}\nabla N\left(\xi\right)$$

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**Note:**

*If the statement of formula* $U(x)=\sum_{i=1}^{Nb1} N_i^{(1)} u_{ik} e_k+\frac{\xi_3}{2}\sum_{i=1}^{Nb2} N_i^{(2)} h_i\left(\tilde{\theta}_{i2} t_{i1k}+\tilde{\theta}_{i1} t_{i2k}\right) e_k=U_t(x)+U_r(x)$ *it is noticed that the terms of membrane are contained in the first part* $U_t(x)$ *of* $U(x)$ *and that the terms of bending are contained in the second part* $U_r(x)$ *of* $U(x)$ *. The terms of transverse shears come from the two contributions. One obtains as follows:* $\tilde{\varepsilon}_m=\tilde{B}_m\tilde{q}^e$ $\tilde{\varepsilon}_f=\tilde{B}_f\tilde{q}^e$ $\tilde{\varepsilon}_\gamma=\tilde{B}_\gamma\tilde{q}^e$ *where*

$\tilde{B}_m=T_{mf}\mathbf{SJ^{-1}}\nabla N_1(x)$

$\tilde{B}_f=T_{mf}\mathbf{SJ^{-1}}\nabla[\xi_3\frac{h}{2}N_2(\xi)]$ *by simple decomposition of the statement* $\tilde{\varepsilon}=\tilde{B}\tilde{q}^e$ *. One calls*

$\tilde{B}_\gamma=T_\gamma\mathbf{SJ^{-1}}\nabla N(\xi)$

*membrane part of the strain projection on the membrane-bending part of the local strain field of the symmetrized gradient of the translations in the total reference. One calls bending part of the strain projection on the membrane-bending part of the local strain field of the symmetrized gradient of rotations in the total reference. One calls transverse distortion projection on the shears part of the local strain field of the symmetrized gradient of total displacement.*

## 4.5 Stiffness matrix

the principle of the virtual wors is written in the following way: $\delta W_{ext}=\delta W_{def}$ that is to say still $\delta U^T \mathrm{KU}=\delta U^T F$ in matric form where $K$ is the stiffness matrix coming from the assembly in the total reference of all the elementary stiffness matrixes. At the elementary level the discretization of the work of strain is written with the preceding notations:

$$\delta W_{def}^{el}=\delta\tilde{q}^{e^t}\int_{-1}^{1}\int_{A_r}\tilde{B}^t\tilde{C}\,\tilde{B}\det J\,d\xi_1 d\xi_2 d\xi_3\tilde{q}^e=\delta\tilde{q}^{e^t}\tilde{K}^e\tilde{q}^e$$

where $A_r$ is the area of reference of the element.

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### 4.5.1 Decomposition of the elementary matrixes

This stiffness matrix understands three contributions due to the strains of membrane, bending and transverse distortion. One has as follows: $\tilde{K}^e = \tilde{K}^e_m + \tilde{K}^e_f + \tilde{K}^e_\gamma$ with:

$$\tilde{K}^e_m = \int_{-1}^{1} \int_{A_r} \tilde{B}^t_m H \tilde{B}_m \det J \, d\xi_1 \, d\xi_2 d\xi_3 ;$$

$$\tilde{K}^e_f = \int_{-1}^{1} \int_{A_r} \tilde{B}^t_f H \tilde{B}_f \det J \, d\xi_1 \, d\xi_2 d\xi_3 ;$$

$$\tilde{K}^e_\gamma = \int_{-1}^{1} \int_{A_r} \tilde{B}^t_\gamma H_\gamma \tilde{B}_\gamma \det J \, d\xi_1 \, d\xi_2 d\xi_3 .$$

### 4.5.2 Assembly of the elementary matrixes

the principle of virtual work for all the elements is written:

$\delta W_{def} = \sum_{e=1}^{\text{nb elem}} \delta W^e_{def} = \delta \boldsymbol{U}^T \mathbf{K} \mathbf{U}$ where $U$ is all the degrees of freedom of discretized structure and $K$ comes from the assembly of the elementary matrixes.

#### 4.5.2.1 Degrees of freedom

the process of assembly of the elementary matrixes implies that all the degrees of freedom are expressed in the total reference. In the total reference, the degrees of freedom are three displacements compared to the three axes of the total cartesian coordinate system and the three rotations compared to these three axes. One thus uses, for the degrees of freedom of rotation, of the transition matrixes of the orthonormal local coordinate system $t_\alpha$ to the total reference for each element.

#### 4.5.2.2 Fictitious rotations

rotation compared to the normal with the shell is not a true degree of freedom. To ensure compatibility between the transition of the local coordinate system the total reference, one thus adds a local additional degree of freedom of rotation to the shell which is that corresponding to rotation compared to the normal at the mean surface of the element. This implies an expansion of the blocks of dimension $(5,5)$ of the local stiffness matrix into cubes blocks of dimension $(6,6)$ by adding one line and a column corresponding to this rotation. These additional lines and these columns are a priori null. One then carries out the transition of the local stiffness matrix extended to the global stiffness matrix.
In the preceding transformation, one was satisfied to add rotations compared to the norms on the surface of the elements without modifying strain energy. The contribution to the energy brought by these additional degrees of freedom is indeed null and no stiffness is associated for them.
The global stiffness matrix thus obtained presents the risk however to be noninvertible. To avoid this nuisance it is allowed to allot a small stiffness to these additional degrees of freedom on the level of the widened local stiffness matrix. Practically, one chooses it between 10-6 and 10-3 times the diagonal minor term of the stiffness matrix of local rotation. The user can choose this multiplicative coefficient `COEF_RIGI_DRZ` itself in `AFFE_CARA_ELEM`; by default it is worth 10-5.

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## 4.6 Mass matrix

the virtual wor of the effects of inertia can be expressed in the form:

$$\delta W_{mass}^{ac} = \int_{\Omega} \rho \ddot{U}(Q).\delta U(Q) d\Omega$$

One supposes that the strains and the displacements remain sufficiently small so that the norm at the mean surface of the shell remains unchanged.
With these assumptions, we can write the field of virtual displacement:

$$\delta U(Q)(\xi_1,\xi_2,\xi_3) = \delta u(P)(\xi_1,\xi_2) + \xi_3 \frac{h}{2} \delta\theta(\xi_1,\xi_2) \wedge n(\xi_1,\xi_2)$$

and the field of acceleration:

$$\ddot{U}(Q)(\xi_1,\xi_2,\xi_3) = \ddot{u}(P)(\xi_1,\xi_2) + \xi_3 \frac{h}{2} \ddot{\theta}(\xi_1,\xi_2) \wedge n(\xi_1,\xi_2)$$

In this statement, we neglected the gyroscopic terms.

### 4.6.1 Discretization of displacement for the mass matrix

At the point $Q$ , one takes as interpolation of the field of displacement:

$$\delta U(Q)(\xi_1,\xi_2,\xi_3) = \sum_{I=1}^{Nb1} N_I^1(\xi_1,\xi_2) \begin{pmatrix} \delta u_{I1} \\ \delta u_{I2} \\ \delta u_{I3} \end{pmatrix} - \xi_3 \frac{h}{2} \sum_{I=1}^{Nb2} N_I^2(\xi_1,\xi_2) \begin{bmatrix} 0 & -n_{I3} & n_{I2} \\ n_{I3} & 0 & -n_{I1} \\ -n_{I2} & n_{I1} & 0 \end{bmatrix} \begin{pmatrix} \delta\theta_{I1} \\ \delta\theta_{I2} \\ \delta\theta_{I3} \end{pmatrix}$$

formulate acceleration, the interpolation is written:

$$\ddot{U}(Q)(\xi_1,\xi_2,\xi_3) = \sum_{I=1}^{Nb1} N_I^1(\xi_1,\xi_2) \begin{pmatrix} \ddot{u}_{I1} \\ \ddot{u}_{I2} \\ \ddot{u}_{I3} \end{pmatrix} - \xi_3 \frac{h}{2} \sum_{I=1}^{Nb2} N_I^2(\xi_1,\xi_2) \begin{bmatrix} 0 & -n_{I3} & n_{I2} \\ n_{I3} & 0 & -n_{I1} \\ -n_{I2} & n_{I1} & 0 \end{bmatrix} \begin{pmatrix} \ddot{\theta}_{I1} \\ \ddot{\theta}_{I2} \\ \ddot{\theta}_{I3} \end{pmatrix}$$

We rewrite the two preceding equations in the matric form:

$$\delta U(Q)(\xi_1,\xi_2,\xi_3) = N \delta u^e$$
$$\ddot{U}(Q)(\xi_1,\xi_2,\xi_3) = N \ddot{u}^e$$

where $N$ is the interpolation matrix, whose statement is:

$$\mathbf{N} = \left[ \left[ N_I^1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \xi_3 \frac{h}{2} N_I^2 \begin{pmatrix} 0 & -n_{I3} & n_{I2} \\ n_{I3} & 0 & -n_{I1} \\ -n_{I2} & n_{I1} & 0 \end{pmatrix} \right]_{I=1,Nb1} \quad -\xi_3 \frac{h}{2} N_{Nb2}^2 \begin{pmatrix} 0 & -n_{Nb23} & n_{Nb22} \\ n_{Nb23} & 0 & -n_{Nb21} \\ -n_{Nb22} & n_{Nb21} & 0 \end{pmatrix} \right]$$

the vector $u^e$ is the elementary nodal vector of displacements in the total reference which is put in the following form:

### 4.6.2 Elementary mass matrix

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With the preceding notations, the virtual wor of the effects of inertia is put in the following matric form:

$$\delta W_{mass}^{inertie} = \delta u^{eT} M^e \ddot{u}^e$$

with $M^e$ the coherent mass matrix which can be expressed in the form:

$$M^e = \int_{\Omega e} \rho N^T N \det(J(\xi_3)) d\xi_1 d\xi_2 d\xi_3$$

It is important to note that because of the curvature, a coupling of the terms of translation with those of rotation is possible (indeed, $\det(J(\xi_3))$ is not constant in the thickness).

### 4.6.3 Assembly of the elementary mass matrixes

the assembly of the mass matrixes follows same logic as that of the stiffness matrixes. The degrees of freedom are the same ones and one finds the processing specific to normal rotations on the surface of the shell. Although the coherent mass matrix is built in the total reference, it remains singular compared to the rotation of the norm in each node. We need to supply this matrix on the basis of the variational form:

$$\delta W_n^{inertie} = \sum_{I=1}^{Nb2} m_e \delta\theta_I (n_I \times n_I) \ddot{\theta}_I$$

where $m_e$ is selected constant by element and calculated according to the formula:

$$m_e = C m_{max}$$

$m_{\max}$ being the major term due to rotations (in the local coordinate system of the element) on the diagonal of the matrix $M^e$. It is thus to note that with this intention it was necessary to bring back the contribution of the rotations initially expressed in the total reference of the element, in the local coordinate system of the element by change of reference.

For modal computations utilizing at the same time the computation of the stiffness matrix and that of the mass matrix, it is necessary to take a mass on the degree of normal rotation on the surface of the shell being worth $C$ time the minor term diagonal of the mass matrix for the terms of rotation in the local coordinate system, where $C$ is worth between 10-6 and 10-3. One chooses to confuse the values of this coefficient with those of the `COEF_RIGI_DRZ` for the equivalent operation on the stiffness matrix. By default $C$ is thus worth $10^{-5}$. That makes it possible to inhibit, during a modal analysis, the modes being able to appear on the additional degree of freedom of rotation around the norm on the surface of the shell.

## 4.7 Numerical integration for surface

### 4.7.1 elasticity Integration

For the element Hétérosis Q9H the bending part of the stiffness matrix is integrated classically with 9 Gauss points while the parts membrane and shears are obtained by integration reduced with 4 Gauss points.

For element T7H, by analogy with Q9H, the stiffness matrix is obtained with 7 points of integration of Hammer for the bending part and 3 points of integration of Hammer for the parts shears and membrane.

| Cordonnées of the points | Weight $\omega_t$ |
|---|---|
| $\xi_1 = 1/3 ; \eta_1 = 1/3$ | $9/80$ |

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| | |
|---|---|
| $\xi_2=a\,;\eta_2=a$<br>$a=\frac{6+\sqrt{15}}{21}$ | $A=\frac{155+\sqrt{15}}{2400}$ |
| $\xi_3=1-2\mathrm{a}\,;\eta_3=a$ | $A$ |
| $\xi_4=a\,;\eta_4=1-2\mathrm{a}$ | $A$ |
| $\xi_5=b\,;\eta_5=b$<br>$b=4/7-a$ | $31/240-A$ |
| $\xi_6=1-2\mathrm{b}\,;\eta_6=b$ | $31/240-A$ |
| $\xi_7=b\,;\eta_7=1-2\mathrm{b}$ | $31/240-A$ |

$$\int_0^1\int_0^{1-\xi} y(\xi,\eta)\,d\eta\,d\xi = \sum_{i=1}^{n}\omega_i\,y(\xi_i,\eta_i)$$

**normal numerical Formulas of integration on triangle T7H (Hammer)**

| **Cordonnées of the points** | **Weight** $\omega_t$ |
|---|---|
| $\xi_1=-a\,;\eta_1=-a$<br>$a=-0.774596669241483$ | $25/81$ |
| $\xi_2=0.\,;\eta_2=-a$ | $40/81$ |
| $\xi_3=a\,;\eta_3=-a$ | $25/81$ |
| $\xi_4=a\,;\eta_4=0$ | $40/81$ |
| $\xi_5=a\,;\eta_5=a$ | $25/81$ |
| $\xi_6=0\,;\eta_6=a$ | $40/81$ |
| $\xi_7=-a\,;\eta_7=a$ | $25/81$ |
| $\xi_8=-a\,;\eta_8=0$ | $40/81$ |
| $\xi_9=0\,;\eta_9=0$ | $64/81$ |

$$\int_{-1}^{1}\int_{-1}^{1} y(\xi,\eta)\,d\eta\,d\xi = \sum_{i=1}^{n}\omega_i\,y(\xi_i,\eta_i)$$

**normal numerical Formulas of integration $3\times3$ on quadrangle Q9H (Gauss)**

One notices that the order of Gauss points of the preceding formula is not the same as for the isoparametric elements. The first 8 points are described here while turning in the direct meaning.

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The principle of reduced integration consists in evaluating the shear strains and of membrane at the points of reduced integration and extrapolating them at the points of classical integration. This amounts supposing that these strains are bilinear on element Q9H and linear on the T7H. The shape functions chosen to make this extrapolation are the bilinear of the quadrangle with 4 nodes for Q9H and linear classical shape functions of the triangle with 3 nodes for the T7H being worth 1 at the points of reduced integration.

For more details on the principle of reduced or selective integration, one can refer to [bib6].

| Cordonnées of the points | Weight $\omega_t$ |
|---|---|
| $\xi_1=1/6 ; \eta_1=1/6$ | $1/6$ |
| $\xi_2=2/3 ; \eta_2=1/6$ | $1/6$ |
| $\xi_3=1/6 ; \eta_3=2/3$ | $1/6$ |

$$\int_0^1 \int_0^{1-\xi} y(\xi,\eta)\, d\eta\, d\xi = \sum_{i=1}^{n} \omega_i\, y(\xi_i,\eta_i)$$

**numerical Formulas of integration reduced on triangle T7H (Hammer)**

For the elements quadrangle a Gauss quadrature $2\times 2$ is used.

| Cordonnées of the points | Weight $\omega_t$ |
|---|---|
| $\xi_1=1/\sqrt{3} ; \eta_1=1/\sqrt{3}$ | $1$ |
| $\xi_2=1/\sqrt{3} ; \eta_2=-1/\sqrt{3}$ | $1$ |
| $\xi_3=-1/\sqrt{3} ; \eta_3=1/\sqrt{3}$ | $1$ |
| $\xi_3=-1/\sqrt{3} ; \eta_3=-1/\sqrt{3}$ | $1$ |

$$\int_{-1}^{1} \int_{-1}^{1} y(\xi,\eta)\, d\eta\, d\xi = \sum_{i=1}^{n} \omega_i\, y(\xi_i,\eta_i)$$

**numerical Formulas of integration reduced $2\times 2$ on quadrangle Q9H (Gauss)**

### 4.7.2 Integration in the thickness

integration in the thickness is made with three points for the two elements.

| Cordonnées of the points | Weight $\omega_t$ |
|---|---|
| $\xi_1=-1$ | $1/3$ |
| $\xi_2=0$ | $4/3$ |
| $\xi_3=+1$ | $1/3$ |

$$\int_{-1}^{1} y(\xi)\, d\xi = \sum_{i=1}^{n} \omega_i\, y(\xi_i)$$

**Formula of numerical integration in the thickness in elasticity**

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## 4.8 Numerical integration for plasticity

the principle of surface integration remains the same one as in elasticity, but the initial thickness is divided into $N$ identical layers of thickness. There are three points of integration per layer. The points of integration are located in higher skin of layer, in the middle of the layer and in lower skin of layer. For $N$ layers, the number of points of integration is of $2\mathrm{N}+1$ . One advises to use from 3 to 5 layers in the thickness for a number of points of integration being worth 7,9 and 11 respectively.

For the stiffness, one calculates for each layer, in plane stresses, the contribution to the stiffness matrixes of membrane, bending and transverse distortion. These contributions are added and assembled to obtain the total tangent stiffness matrix.

For each layer, one calculates the state of the stresses $(\sigma_{11,}\sigma_{22,}\sigma_{12})$ and all the local variables, in the middle of the layer and in skins higher and lower of layer, starting from the local plastic behavior and of the local strain field $(\varepsilon_{11,}\varepsilon_{22,}\varepsilon_{12})$ . The positioning of the points of integration enables us to have the rightest estimates, because not extrapolated, in skins lower and higher of layer, where it is known that the stresses are likely to be maximum. The plastic behavior does not understand for the moment the terms of transverse shears which are treated in an elastic way, because the transverse shears are uncoupled from the membrane behavior in plane stresses.

| Cordonnées of the points | Weights $\omega_t$ |
|---|---|
| $\xi_1=-1$ | 1/3 |
| $\xi_2=0$ | 4/3 |
| $\xi_3=+1$ | 1/3 |
| $\int_{-1}^{1} y(\xi)\,d\xi=$ | $\sum_{i=1}^{n}\omega_i\, y(\xi_i)$ |

**Formula of numerical integration for a layer in the thickness in plasticity**

**Note:**

*One already mentioned with [§2.2.2] that the value of the coefficient of correction in transverse shears for the shell elements and of shell was obtained by identification of elastic complementary energies after resolution of the equilibrium 3D. This method is not usable any more in elastoplasticity and the choice of the coefficient of correction in transverse shears is posed then. The transverse terms of shears are thus not affected by plasticity and are treated elastically, for want of anything better. If one places oneself in theory of Coils-Kirchhoff for a value of this coefficient of $10^6 h/R$ ( $h$ being the thickness of the shell and $R$ its average radius of curvature) the transverse terms of shears become negligible and the approach is more rigorous.*

## 4.9 Discretization of elementary works for the loadings

### 4.9.1 elementary Discretization of the work of the forces and external couples being exerted on mean surface

According to the paragraph [§3.2], one recalls that one has for these forces and couples:

$$\delta W_{ext}=\int_S (f\,\delta u+c\,\delta\beta)\,dS$$

where $S$ is the mean surface of the shell.
For the first term of this statement one has as follows:

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#### 4.9.1.1 Loads given in the total reference

$$\delta W_{ext}=\int_{A_r}[F_k N_i^1 \delta u_{ik}+c_k N_j^2(\delta\tilde{\theta}_{j2}t_{j1k}-\delta\tilde{\theta}_{j1}t_{j2k})]\frac{2}{h}\det J^\circ d\xi_1 d\xi_2$$

with $\det J^\circ=\det J(\xi_3=0)$

#### 4.9.1.2 Loads given in the local coordinate system

$$\delta W_{ext}=\int_{A_r}[F_\alpha N_j^1 t_{j\alpha k} N_i^1 \delta u_{ik}+c_\alpha t_{\alpha k} N_j^2(\delta\tilde{\theta}_{j2}t_{j1k}-\delta\tilde{\theta}_{j1}t_{j2k})]\frac{2}{h}\det J^\circ d\xi_1 d\xi_2$$

### 4.9.2 elementary Discretization of the work of the forces and external couples being exerted on contour

According to the paragraph [§3.2], it is pointed out that one has for these forces and couples:

$$\delta W_{ext}=\int_C(\phi\delta u+\chi\delta\beta)ds$$

where $C$ is the average contour of the shell. $\phi$ and the $\chi$ linear distributions of force and moment applied to the contour of the shell in the total reference.

The discretization gives then: $\delta W_{ext}=\int_C[\phi_k N_i^1 \delta u_{ik}+\chi_k N_j^2(\delta\tilde{\theta}_{j2}t_{j1k}-\delta\tilde{\theta}_{j1}t_{j2k})]ds$

### 4.9.3 Discretization of the term of gravity

One has for this term:

$$\delta W_{\text{pes}}=\int_{\Omega_e}\rho g\delta U(Q)dV=\int_{\Omega_e}\rho g_k\delta U_k(Q)dV=\int_{\Omega_e}\rho g_k[N_i^1\delta u_{ik}+\frac{\xi_3}{2}N_j^2(\delta\tilde{\theta}_{j2}t_{j1k}-\delta\tilde{\theta}_{j1}t_{j2k})]dV$$

Is: $\delta W_{pes}=\int_{\Omega_e}\rho g_k N_i^1\delta u_{ik}dV$ by supposing negligible the second term of statement Ci above.

### 4.9.4 Discretization of the term of pressure

One supposes that the pressure $p$ is applied to the mean surface $\omega$ of the shell. One has then:

$$\delta W_{pres}=\int_{A_r}ep\,n\delta u(P)dS=\int_{A_r}ep(a_1\wedge a_2)\delta u(P)d\xi_1 d\xi_2$$
$$dW_{pres}=\int_{A_r}ep\,n\delta u(P)dS=\int_{A_r}ep(a_1\wedge a_2)\delta u(P)d\xi_1 d\xi_2$$

where $e=\pm 1$ according to whether $p$ is applied in intern skin or external.

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Like $a_\alpha = a_{\alpha k} e_k$ , this is still written: $\delta W_{pres} = \int_{A_r} epN_i^1 \delta u_{ik} \nu_k d\xi_1 d\xi_2$ where

$$\begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} = \begin{pmatrix} J_{12}^\circ J_{23}^\circ - J_{13}^\circ J_{22}^\circ \\ J_{13}^\circ J_{21}^\circ - J_{11}^\circ J_{23}^\circ \\ J_{11}^\circ J_{22}^\circ - J_{12}^\circ J_{21}^\circ \end{pmatrix}, J_{ij}^\circ = J_{ij}\left(\xi_3 = 0\right) .$$

### 4.9.5 Discretization of the terms of inertia centrifuges

One adds with the statement of the field of accelerations of the paragraph [§4.6] the term corresponding to the centrifugal accelerative forces if the total reference $(O, e_k)$ is in uniform rotation $\Omega$ compared to a Galilean reference which has the same origin $O$ as the total reference. The statement of the field of accelerations becomes as follows:

$$\ddot{U}(Q)(\xi_1, \xi_2, \xi_3) = \ddot{u}(P)(\xi_1, \xi_2) + \xi_3 \frac{h}{2} \ddot{\theta}(\xi_1, x_2) \wedge n(\xi_1, \xi_2) + \Omega \wedge [\Omega \wedge \mathbf{OP}]$$

where one neglected the Coriolis forces and the correction of metric in the thickness.

One expresses $\Omega$ in the global database ( $e_k$ ): $\Omega = \Omega_k e_k$ .

By taking again the statement of: $dW^{inertie} = \int_\Omega \rho \ddot{\mathbf{U}}(Q) . \mathbf{dU}(Q) d\Omega$ , one identifies the contribution of the terms of centrifugal inertia: $dW_{cent}^{inertie} = \int_{\Omega_e} \rho \delta u_k e_k [\Omega \wedge (\Omega \wedge x_k^0 e_k)] dV$ by neglecting the terms of rotation in virtual displacement. The terms of mass are unchanged compared to [§4.6].

As one a:

$$\Omega \wedge x_k^0 e_k = \Omega_p e_p \wedge x_k^0 e_k = \Omega_p x_k^0 e_{qpk} e_q$$

where $e_{qpk}$ is the permutation of Lévi-Strauss.

One also writes:

$$\Omega \wedge (\Omega \wedge x_k^0 e_k) = e_{qpk} e_{srq} \Omega_r \Omega_p x_k^0 e_k$$

From where it results from it that:

$$W_{cent}^{inertie} = \int_{-1}^{1} \int_{Ar} \rho \delta u_{is} N_i^{(1)} e_{qpk} e_{srq} \Omega_r \Omega_p x_{\mathrm{jk}}^0 N_j^{(1)} \det J \, dx_1 dx_2 dx_3$$

### 4.9.6 Taken into account of the loadings of thermal thermal expansion

One treats only the case where the thermoelastic characteristics $E$ $\nu$ , $\alpha$ depend only on the average temperature $\bar{T}$ in the thickness. Moreover, the material is thermoelastic isotropic homogeneous in the thickness.

The variational formulation of work due to thermal thermal expansions is written:

The temperature is represented by the model thermal at three fields according to [R3.11.01]:

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$$T\left(\xi_\alpha,\xi_3\right)=T^m\left(\xi_\alpha\right).P_1\left(\xi_3\right)+T^s\left(\xi_\alpha\right).P_2\left(\xi_3\right)+T^i\left(\xi_\alpha\right).P_3\left(\xi_3\right),$$

with: $P_j\left(\xi_3\right)$ : three polynomials of LAGRANGE in the thickness: $]-1,+1[$ :

$$P_1\left(\xi_3\right)=1-\left(\xi_3\right)^2\ ;\ P_2\left(\xi_3\right)=\frac{\xi_3}{2}\left(1+\xi_3\right)\ ;\ P_3\left(\xi_3\right)=-\frac{\xi_3}{2}\left(1-\xi_3\right)\ ;$$

FROM the representation of the temperature above, one obtains:

- the average temperature in the thickness:
$$\bar{T}\left(\xi_\alpha\right)=\frac{1}{2}\int_{-1}^{+1}T\left(\xi_\alpha,\xi_3\right)d\,\xi_3=\frac{1}{6}\left(4\mathrm{T}^m\left(\xi_\alpha\right)+T^s\left(\xi_\alpha\right)+T^i\left(\xi_\alpha\right)\right)\ ;$$
- the average variation in temperature in the thickness:
$$\hat{T}\left(\xi_\alpha\right)=3\int_{-1}^{+1}T\left(\xi_\alpha,\xi_3\right)\xi_3 d\,\xi_3=T^s\left(\xi_\alpha\right)-T^i\left(\xi_\alpha\right)\ ;$$

Thus the temperature can be written in the following way:

$$T\left(\xi_\alpha,\xi_3\right)=\bar{T}\left(\xi_\alpha\right)+\hat{T}\left(\xi_\alpha\right).\xi_3/2+\tilde{T}\left(\xi_\alpha,\xi_3\right)\ \text{such as:}$$
$$\int_{-1}^{+1}\tilde{T}\left(\xi_\alpha,\xi_3\right)=0\,;\int_{-1}^{+1}\xi_3\tilde{T}\left(\xi_\alpha,\xi_3\right)=0\ .$$

If the temperature is indeed closely connected in the thickness one has $\tilde{T}=0$ .

It is necessary to evaluate the three-dimensional thermal stresses, in each point of integration in the thickness. These stresses of thermal origin withdrawn from the usual mechanical stresses are calculated at the points of integration in the thickness by:

$$\tilde{\sigma}_{\beta\gamma}^{ther}=\frac{\alpha.E}{1-\nu^2}\left(\bar{T}-T^{réf}+\hat{T}.\xi_3/2\right)\delta_{\beta\gamma}$$

### 4.9.7 Assembly

the variational formulation of the work of the external forces for all the elements is written then:

$\delta W_{\text{ext}}=\sum_{e=1}^{\text{nb}\,elem}\delta W_{\text{ext}}^e=\delta U^T F$ where $U$ is all the degrees of freedom of discretized structure and $F$ comes from the assembly of the vectors forces elementary.

As for the stiffness matrixes, the process of assembly of the vectors forces elementary implies that all the degrees of freedom are expressed in the total reference. In the total reference, the degrees of freedom are three displacements compared to the three axes of the total cartesian coordinate system and the three rotations compared to these three axes. One thus uses transition matrixes of the local coordinate system to the total reference for rotations of each element.

**Note:**

> *The external forces can also be defined in the reference user. One then uses a transition matrix of the reference user towards the local coordinate system of the element to have the statement of these forces in the local coordinate system of the element and to deduce the vector from it elementary corresponding room forces For the assembly one passes then from the local coordinate system of the element to the total reference.*

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# 5 Validation

to judge relevance of thick the shell formulation, the few examples of application according to relate to as well the linear static as the computation of eigen modes. Three new cases tests relative to the two finite elements described in the preceding parts were integrated in *Code_Aster*. They come to enrich the benchmarks by the shell elements already present in the environment from *Code_Aster.* Most of these benchmarks were indexed in [bib10].

The three new benchmarks, two in static plus one in dynamics, are classical examples of validation drawn from [bib3]. The reference solutions, analytical or numerical, resulting from [bib3] are compared with the numerical results given by *Code_Aster*. For more information on these benchmarks, one will refer to the documentation of validation indicated in reference.

## 5.1 Case test in linear static

### 5.1.1 Case static test n° 1

the first case test is that of a cylindrical panel subjected to its own weight [V3.03.107].

This test makes it possible to highlight effects of membrane more important than those of bending. It makes it possible to measure the performance of the elements shells compared to elements DKT or DKQ whose interpolation out of membrane is linear.

### 5.1.2 Static case test n° the 2

second cases test is that of a helicoid shell subjected to two concentrated types of loading [V3.03.108].

The helicoid shape of the shell makes it possible to study the geometrical representation of the finite elements. The concentrated loadings can be:

- in the plane: the influence due to the effects of membrane is then not important and the behavior dominating is that due to bending,
- except plane: the effects of membrane affect the behavior of the shell.

## 5.2 Case test in dynamics

This case test is a simplified model of paddle of compressor, which is in fact a cylindrical panel [V2.03.102].

This test highlights the performances of the elements in dynamic behavior by the data of the frequencies and the eigen modes.

The frequencies and eigen modes of the paddle are experimental values which are used of results as reference.

# 6 Thermomechanical sequence

## 6.1 Description

For the resolution of chained thermomechanical problems, one must of the finite elements use for thermal computation thermal shell [R3.11.01] whose field of temperature is recovered like input datum of *Code_Aster* for mechanical computation. It is necessary thus that there is compatibility between the thermal field given by the thermal shells and that recovered by the mechanical shells. This last is defined by the knowledge of 3 fields `TEMP_SUP`, `TEMP_MIL` and `TEMP_INF` given in skins lower, medium and higher of shell.

The table below indicates compatibilities between the shell elements mechanics and of thermal shell.

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| THERMAL modelization | Nets | Finite element | to use with | Mesh | Finite element | MECHANICAL Modelization |
|---|---|---|---|---|---|---|
| COQUE | QUAD9 | THCOQU9 | ////////// | QUAD9 | MEC3QU9H | COQUE_3D |
| COQUE | TRIA7 | THCOTR7 | ////////// | TRIA7 | MEC3TR7H | COQUE_3D |

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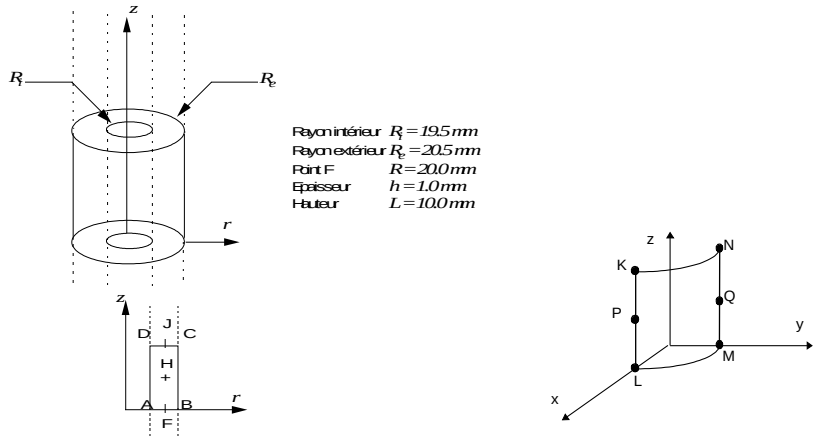
**Note:**

- *The nodes of the thermal shell elements and mechanical shells must correspond. The meshes for the thermal and the mechanics will thus have the same number and the same type of meshes.*
- *The surface thermal shell elements are treated like plane elements by projection of the initial geometry on the level defined by the first 3 tops.*

The thermomechanical sequence is also possible if one knows by experimental measurements the variation of the field of temperature in the thickness of structure or certain parts of structure. In this case one works with a card of temperature defined a priori; the field of temperature is not given any more by three values TEMP_INF, TEMP_MIL and TEMP_SUP of thermal computation obtained by EVOL_THER. It can be much richer and contain an arbitrary number of points of discretization in the thickness of the shell. Operator DEFI_NAPPE allows to create such profiles of temperatures starting from the abundant data by the user. These profiles are affected by the command CREA_CHAMP (cf benchmark HSNS100B). It will be noted that it is not necessary for mechanical computation that the number of points of integration in the thickness is equal to the number of points of discretization of the field of temperature in the thickness. The field of temperature is automatically interpolated at the points of integration in the thickness of the shell elements.

## 6.2 Benchmarks

the benchmarks for the thermomechanical sequence between thermal shell elements and shell elements are the HPLA100C (elements MEC3QU9H) and HPLA100D (elements MEC3TR7H). It is about a heavy thermo-elastic hollow roll in uniform rotation [V7.01.100] subjected to a phenomenon of thermal thermal expansion where the fields of temperature are calculated with THER_LINEAIRE by a steady computation.




Thermal thermal expansion is worth: $T(\rho)-T_{ref}(\rho)=0.5(T_s+T_i)+2.(T_s+T_i)(r-R)/h$

with:

- $T_s=0.5°C, T_i=-0.5°C, T_{ref}=0.°C$
- $T_s=0.1°C, T_i=0.1°C, T_{ref}=0.°C$

One tests the stresses, the forces and bending moments in $L$ and $M$. The results of reference are analytical. One obtains very good performances whatever the type of element considered.

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# 7 Establishment of the shell elements in *Code_Aster*

## 7.1 Description

These elements (of names MEC3TR7H and MEC3QU9H) lean on meshes curved TRIA7 and QUAD9. These elements are not exact with the nodes and it is necessary to net with several elements to get correct results.

## 7.2 Introduced use and developments

These elements are used in the following way:

```
MY = CREA_MAILLAGE (    MESH: MAILINI
                        MODI_MAILLE: (OPTION: "QUAD8_9"
                        TOUT: "YES")…)
```

One calls on a routine MODI_MAILLE of modification of the mesh to pass from the elements quadrangles to 8 nodes to the elements quadrangles to 9 nodes or many elements triangles to 6 nodes to the elements triangles to 7 nodes.

AFFE_MODELE (MODELISATION: "COQUE_3D"…) for the triangle and the quadrangle

One calls on routine INI080 for the position of the points of Hammer and Gauss on the surface of the shell and the corresponding weights.

```
AFFE_CARA_ELEM (COQUE: (THICKNESS: "EP"
              ANGL_REP: ( α , β )
              COEF_RIGI_DRZ: "CTOR")
```

to make postprocessings (forced, generalized forces,…) in a reference chosen by the user who is not the local coordinate system of the element, one defines the direction $X1$ of the reference user like the projection of a direction of reference $\underline{d}$ on the surface $\omega$ of the element. This direction of reference $\underline{d}$ is chosen by the user who defines it by two nautical angles in the total reference. The norm $N$ on the surface of the element fixes the second direction at the point of observation concerned. The cross product of the two vectors previously definite $Y1 = N \wedge X1$ makes it possible to define the local trihedron in which will be expressed the generalized forces representing the stress state. The user will have to take care that the selected reference axis is not found parallel with the norm of certain shell elements. By default, the direction of reference $\underline{d}$ is the axis $X$ of the total reference of definition of the mesh.
Value CTOR corresponds to coefficent that the user can introduce for the processing of the terms of stiffness and mass according to normal rotation on the surface of the shell. This coefficient must be sufficiently small not to disturb the energy assessment of the element and not too small so that the stiffness matrixes and of mass are invertible. A value of 10-5 is put by default.

ELAS: (E: Young NU: $\nu$ ALPHA: $\alpha$ . RHO: $\rho$ .)
For a homogeneous isotropic thermoelastic behavior in the thickness one uses key word ELAS in DEFI_MATERIAU where the coefficients are defined $E$ , Poisson's ratio, Young modulus $\nu$ , $\alpha$ thermal coefficient of thermal expansion and RHO density $\tilde{}$

AFFE_CHAR_MECA (DDL_IMPO: (
DX:. DY:. DZ:. DRX:. DRY:. DRZ:. DDL of shell in the total reference.
FORCE_COQUE: (FX:. FY:. FZ:. MX:. MY:. MZ:. ). They is the surface forces on shell elements. These forces can be given in the total reference or the reference user defined by ANGL_REP.

FORCE_NODALE: (FX:. FY:. FZ:. MX:. MY:. MZ:. ). They is the forces of shell in the total reference.

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## 7.3 Computation in linear elasticity

the stiffness matrix and the mass matrix (respectively options RIGI_MECA and MASS_MECA) are integrated numerically in the TE0401 and TE0406, respectively. The computation account holds owing to the fact that the terms corresponding to the degrees of freedom of rotation of shell are expressed in the local coordinate system of the element. A transition matrix makes it possible to pass from the local degrees of freedom to the total degrees of freedom.

Elementary computations (CALC_CHAMP) currently available correspond to the options:

- EPSI_ELNO and SIGM_ELNO which and the provide the strains nodal stresses in the reference user of the element in lower skin, with semi thickness and in higher skin of shell. One stores these values in the following way: 6 components of strain or stresses,
- EPXX EPYY EPZZ EPXY EPXZ EPYZ or SIXX SIYY SIZZ SIXY SIXZ SIYZ,
- EFGE_ELNO: who gives the forces generalize by element with the nodes starting from displacements: NXX, NYY, NXY, MXX, MYY, MXY, QX, QY.
- SIEF_ELGA: who gives the stresses by element to Gauss points in the local coordinate system of the element starting from displacements: SIXX, SIYY, SIZZ, SIXY, SIXZ, SIYZ.
- EPOT_ELEM: who gives the elastic strain energy of strain per element starting from displacements.
- ECIN_ELEM: who gives kinetic energy by element.

Finally the TE0416 calculates also option FORC_NODA of computation of the nodal forces for operator CALC_CHAMP.

## 7.4 Plastic design

the stiffness matrix is also integrated numerically, by layers, in the TE0414. One calls on the computation option STAT_NON_LINE in which one defines in the level of the nonlinear behavior the number of layers to be used for numerical integration. All the models of plane stresses available in *the Code_Aster* can be used.

```
STAT_NON_LINE (….
 COMP_INCR: (RELATION: ""
COQUE_NCOU: " LAYER MANY")
….)
```

Currently available elementary computations correspond to the options:

- EPSI_ELNO which provides the strains by element to the nodes in the reference user starting from displacements, in lower skin, with semi thickness and in higher skin of shell.
- SIGM_ELNO which makes it possible to obtain the stress field in the thickness by element with the nodes for all the subpoints (all the layers and for all the positions: in lower skin, in the medium or in higher skin of layer). These values are given in the reference user.
- EFGE_ELNO which makes it possible to obtain the forces generalized by element with the nodes in the reference user.
- VARI_ELNO which calculates the field of local variables and the forced by element with the nodes for all the layers, in the local coordinate system of the element.

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# 8 Conclusion

the curved finite elements of shell that we describe here are used in curved thin structure computations whose thickness ratio over characteristic length is lower than 1/10. Two finite elements of voluminal shell leaning on meshes quadrangular and triangular ones were introduced into *Code_Aster*. They were selected with a quite particular aim: to be able to represent a complete curved structure behavior whereas until now one could use only elements with plane facets which induced parasitic bendings and required to refine the meshes.

It is elements for which the strains and the forced in the plane of the element vary linearly with the thickness of the shell. The selected kinematics is a kinematical shell of the Hencky-Mindlin-Naghdi type making it possible to utilize the transverse energy of shears. The distortion associated with the transverse shears is constant in the thickness of the element. The variable correction on the transverse $k$ shear coefficient offers a flexibility in use making it possible to pass from the theory of HENCKY-MINDLIN-NAGHDI for $k=1$, with that of REISSNER for $k=5/6$ and to that of LOVE_KIRCHHOFF (for very mean structures) if one chooses a value of $k$ equal to $10^6\times h/L$, $h$ being the thickness and $L$ a characteristic distance (average radius of curvature, enforcement zone of the loads…). As in this last case, one uses a method of penalization to make small the terms of shears transverse, one can, if one takes a value of $k$ too important, make singular the numerical system. In this case, it is necessary to decrease the value of $k$.

The value by default of $k$ is of $5/6$. It is generally used when the structure to be netted has a thickness ratio over characteristic length understood enters $1/20$ and $1/10$. For lower thickness where the transverse distortion becomes low one can want to use a value of $k=10^6\times h/L$ (to be able to make comparisons with shell elements DKT for example). When the transverse distortion is non-zero, the shell elements 3D do not satisfy the equilibrium conditions and the boundary conditions on the transverse nullity of the shearing stresses on the sides higher and lower with shell, compatible with a constant transverse distortion in the thickness of the shell. It results from it thus that on the level of the behavior a coefficient of 5/6 for a homogeneous shell corrects the usual relation between the stresses and the transverse distortion in order to 3D ensure the equality between energies of shears of the model and the model of shell constant distortion. In this case, the deflection $\tilde{u}_3$ has as an interpretation average transverse displacement in the thickness of the shell and not the displacement of the mean surface of the shell.

For structures of low thickness in order to avoid the phenomena of blocking, one uses under - integration reduced for the parts membrane and shears of the stiffness matrix. The choice on the finite elements was made on the elements quadrangle Hétérosis Q9H and triangle T7H. Indeed, among the finite elements with quadratic interpolation functions, the performance of the element Hétérosis Q9H is known. It is in particular higher than that of the elements Sérendip Q9S or the elements of Lagrange Q9. This performance rests however on the selective integration of the element with reduced integration of the terms of membrane and shears on the one hand, and normal integration of the terms of bending on the other hand. By analogy with Q9H, one took the finite element T7H like triangular element of form. However, as far as possible, one will use the Q9H rather than the T7H which is definitely less powerful.

The nonlinear behaviors in plane stresses are available for these elements. It is announced however that the stresses generated by the transverse distortion are treated elastically, for want of anything better. Indeed the rigorous taking into account of non-zero constant transverse shears on the thickness and the determination of the correction associated on the shear stiffness compared to a model satisfying the equilibrium conditions and the boundary conditions are not possible and thus return the use of these elements, when the transverse shears are non-zero, rigorously impossible in plasticity. Rigorously, for nonlinear behaviors, it would thus be necessary to use these elements in the frame of the theory of Coils-Kirchhoff.

Elements corresponding to the machine elements exist in thermal; the mechanical sequences thermo - are thus available with of the finite elements of thermal shells to 7 and 9 nodes. Extensions of the preceding formulation presented in appendix allow also the taking into account of the anisotropy of the materials and kinematical non-linearity. This second extension is operational in *Code_Aster* and is the object of a documentation of reference [R3.07.05].

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# 10 of the versions of the document Version

| Aster Author | (S) Organization (S) Description | of modifications 5 P. |
|---|---|---|
| Massin | , A.Laulusa EDF R & D /MMN initial | Text 7.4 |
| X.Desroc hes | Updated | : minor modifications Extension |

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# Annexe 1 to the anisotropic materials not programmed One considers

that the shell consists of an orthotropic material, axes of orthotropy associated $\tilde{\tilde{x}}_k$ with the base. $k_k$ The constitutive law in these axes is written: formulate

$$\underset{(6\times1)}{\tilde{\tilde{\varepsilon}}} = \underset{(6\times6)}{\tilde{\tilde{S}}_k} \underset{(6\times1)}{\tilde{\tilde{\sigma}}}$$

is $\tilde{S}$ the matrix of flexibility of the component. Are $k$

and $\tilde{\varepsilon}$ , $\tilde{\sigma}$ the strain tensors and stresses in the axes, one $\tilde{x}_k$ a: formulates

$$\tilde{\sigma} = {}^t Q \tilde{\tilde{\sigma}} Q$$
$$\tilde{\varepsilon} = {}^t Q \tilde{\tilde{\varepsilon}} Q$$

() is $Q = \left[T_1, T_2, T_3\right]_{/k_k}$ $Q_{ij} = T_i . k_j$ the matrix of the cosine directors of in $T_k$ the base. In $k_k$

vectorial form, one a: formulates

$$\tilde{\sigma} = \tilde{T} \tilde{\tilde{\sigma}}$$
$$\tilde{\varepsilon} = \tilde{T} \tilde{\tilde{\varepsilon}}$$

the components of are $\tilde{T}$ defined according to those of. Conversely $Q$

, one a: thus

$$\tilde{\tilde{\sigma}} = \tilde{T}^{-1} \tilde{\sigma}$$
$$\tilde{\tilde{\varepsilon}} = \tilde{T}^{-1} \tilde{\varepsilon}$$

, one obtains: that one

$$\tilde{\varepsilon} = \tilde{T} \tilde{\tilde{S}}_k \tilde{T}^{-1} \tilde{\sigma}$$

writes:

$$\tilde{\varepsilon} = \tilde{S}_k \tilde{\sigma}$$

To be coherent with the assumption of plane stress, one $\tilde{\sigma}_{33} = 0$ writes: formulate

$$\underset{(5\times1)}{\tilde{\varepsilon}_r} = \underset{(5\times5)}{\tilde{S}_{kr}} \underset{(5\times1)}{\tilde{\sigma}_r}$$

the symbol like $r$ tiny room, which gives: formulate

$$\tilde{\sigma}_r = \tilde{C}_k \tilde{\varepsilon}_r , \tilde{C}_k = \tilde{S}_{kr}^{-1}$$

récrit by omitting the symbol, $r$ elastic

$$\tilde{\sigma} = \tilde{C}_k \tilde{\varepsilon}$$

strain energy is $W^{el}$ : formulate

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$$W^{el}=\frac{1}{2}{}^t\tilde{q}^e\int_{-1}^{1}\int_{Ar}{}^t\tilde{B}\tilde{C}_k\tilde{B}\det J\,d\xi_1 d\xi_2 d\xi_3\tilde{q}^e$$

the shell consists of layers $Nc$ , each layer being regarded as a component, then $k$ : formulate

$$W^{el}=\frac{1}{2}{}^t\tilde{q}^e\sum_{k=1}^{Nc}\int_{2e_k^-/h}^{2e_k^+/h}\int_{Ar}{}^t\tilde{B}\tilde{C}_k\tilde{}B\det J\,d\xi_1 d\xi_2 d\xi_3\tilde{}q^e$$

and $e_k^-$ are $e_k^+$ the X-coordinates of the limits lower and higher of the layer of thickness $k$ , with $e_k=e_k^+-e_k^-$ and $e_1^-=-h/2$ . While $e_{Nc}^+=h/2$ posing

: formulate

$$\xi_3=\frac{e_k}{h}\overline{\xi}_3+\frac{e_k^++e_k^-}{h},\overline{\xi}_3\in[-1,1]$$

formulates

$$W^{el}=\frac{1}{2}{}^t\tilde{q}^e\sum_{k=1}^{Nc}\frac{e_k}{h}\int_{-1}^{1}\int_{Ar}{}^t\tilde{B}\tilde{C}_k\tilde{B}\det J\left(\xi_1,\xi_2,\overline{\xi}_3\right)d\xi_1 d\xi_2 d\overline{\xi}_3\tilde{q}^e$$

, for work due to thermal thermal expansions, one $W^{th}$ a: formulates

$$\tilde{\tilde{\varepsilon}}_{th}^k=\left(\alpha_1^k T,\alpha_2^k T,\alpha_3^k T,0,0,0\right)$$

are to them $\alpha_i^k$ the coefficients of thermal expansion thermal of the layer in $k$ the axes of orthotropy (). With $\tilde{\tilde{\xi}}_k$

the relation: formulate

$$\tilde{\varepsilon}_{th}^k=\tilde{T}\tilde{\tilde{\varepsilon}}_{th}^k$$

: formula

$$W^{th}=-{}^t\tilde{q}^e\int_{-1}^{1}\int_{Ar}{}^t\tilde{B}\left(-\tilde{C}_k\tilde{\varepsilon}_{th}^k\right)\det J\,d\xi_1 d\xi_2 d\xi_3$$

: formulate

$$W^{th}={}^t\tilde{q}^e\sum_{h=1}^{Nc}\frac{e_k}{h}\int_{-1}^{1}\int_{Ar}{}^t\tilde{B}\tilde{C}_k\tilde{\varepsilon}_{th}^k\det J\,d\xi_1 d\xi_2 d\overline{\xi}_3$$

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# Annexe 2 for element Q9H These

functions are given on page 174 of [bib8]. A2.1

## Shape functions for the translations

the 8 shape functions of Lagrange incomplete of the element quadrangle Q9H [A2.2-a Figure] for the interpolation of displacements are $u_k$ : with

- $N_i^{(1)}(\xi_1,\xi_2)=\frac{1}{4}(-1+\xi_{1i}\xi_1+\xi_{2i}x_2)(1+\xi_{1i}\xi_1)(1+x_{2i}\xi_2)\,i=1,2,3,4$
- $N_i^{(1)}(\xi_1,\xi_2)=\frac{1}{2}(1-x_1^2)(1+\xi_{2i}\xi_2)\,i=5,7$
- $N_i^{(1)}(\xi_1,\xi_2)=\frac{1}{2}(1-x_2^2)(1+\xi_{1i}\xi_1)\,i=6,8$

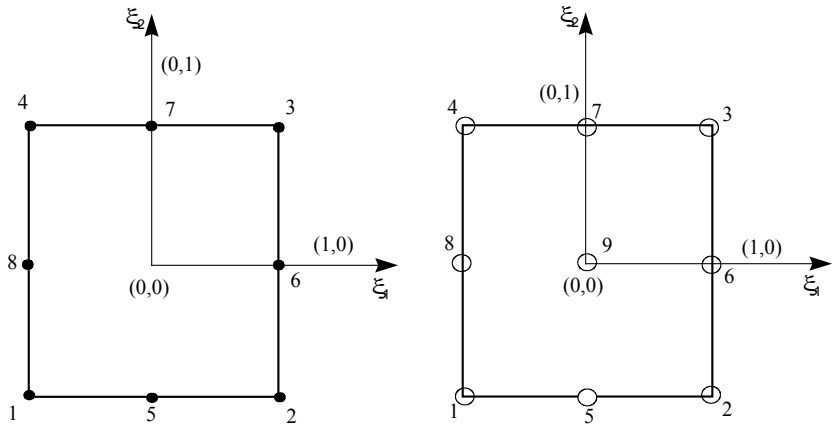
$\xi_{1i}=-1\,i=1,8,4$ ; $\xi_{2i}=-1\,i=1,5,2$ ;
: and $\xi_{1i}=0\,i=5,7$ ; . A2.2 $\xi_{2i}=0\,i=6,8$ ;
$\xi_{1i}=+1\,i=2,6,3$ . $\xi_{2i}=+1\,i=3,7,4$ .

## Shape functions for rotations

the 9 shape functions of Lagrange of the element quadrangle Q9H [A2.2-a Figure] for the interpolation of rotations are $\tilde{\theta}_\alpha$ : where

$N_i^{(2)}(x_1,\xi_2)=N_i(\xi_1)N_i(\xi_2)$ for $N_i(\xi_P)=P_{r'i}\frac{\xi_{\mathrm{Pr}}-\xi_P}{\xi_{\mathrm{Pr}}-\xi_{\mathrm{Pi}}}$ and where $p=1,2$ described $r$ all two node aligned with the node in $i$ the direction. One $\xi_P$ a:

$x_{1i}=-1\,i=1,8,4$ ; $x_{2i}=-1\,i=1,5,2$ ;
and $x_{1i}=0\,i=5,7$ ; . Appear $x_{2i}=0\,i=6,8$ ;
$x_{1i}=+1\,i=2,6,3$ . $x_{2i}=+1\,i=3,7,4$ .




**A2.2-a: Degrees of freedom for the translations and the rotations of the element quadrangle Q9H Shape functions**

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# Annexe 3 for element T7H A3.1

## Shape functions for the translations

the 6 shape functions of triangular element T7H [A3.2-a Figure] for the interpolation of displacements $u_k$ are given on page 175 of [bib8]: where

- $N_1^{(1)}(x_1,x_2)=x_2(2x_2-1)$
- $N_2^{(1)}(x_1,x_2)=\lambda(2\lambda-1)$
- $N_3^{(1)}(x_1,x_2)=x_1(2x_1-1)$
- $N_4^{(1)}(x_1,x_2)=4\lambda x_2$
- $N_5^{(1)}(x_1,x_2)=4x_1\lambda$
- $N_6^{(1)}(x_1,x_2)=4x_1x_2$

: A3.2 $\lambda=1-x_1-x_2$

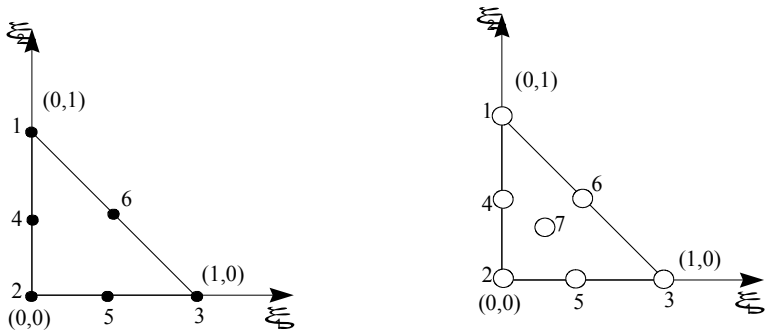
## Shape functions for rotations

the 7 shape functions of triangular element T7H [A3.2-a Figure] for the interpolation of rotations are $\tilde{q}_\alpha$ : with

- $N_1^{(2)}(x_1,x_2)=x_2(2x_2-1)+\frac{1}{9}N_7^{(2)}$
- $N_2^{(2)}(x_1,x_2)=(1-x_1-x_2)[2(1-x_1-x_2)-1]+\frac{1}{9}N_7^{(2)}$
- $N_3^{(2)}(x_1,x_2)=x_1(2x_1-1)+\frac{1}{9}N_7^{(2)}$
- $N_4^{(2)}(x_1,x_2)=4x_2(1-x_1-x_2)-\frac{4}{9}N_7^{(2)}$
- $N_5^{(2)}(x_1,x_2)=4x_1(1-x_1-x_2)-\frac{4}{9}N_7^{(2)}$
- $N_6^{(2)}(x_1,x_2)=4x_1x_2-\frac{4}{9}N_7^{(2)}$

: Appear

- $N_7^{(2)}(x_1,x_2)=27x_1x_2(1-x_1-x_2)$




**A3.2-a: Degrees of freedom for the translations and the rotations of the element triangle T7H**

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