Processing of the eccentring for shell elements DKT, DST, DKQ, DSQ and Q4G

Summarized:

The shell elements [R3.07.03] are intended for three-dimensional thin structure computations. The average average of these structures always does not coincide with the plane of diagram or plane of mesh. One thus introduces the notion of eccentring of the average average compared to the plane of diagram. It is usable for elements with taking into account of the transverse shears, or without this assumption.

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1 Introduction

With an aim of being able to analyze the structure behavior hurled of plate type, or curved surfaces approached by facets, whose average average is excentré compared to the plane of load application, one compared to the introduces the notion of eccentring of the average average surface of mesh. The fields of displacement varying linearly in the thickness of the plate originate in the surface of mesh, i.e. on the level of the surface of mesh, the only degrees of freedom of translation are necessary to the description of displacement.

The introduction of the kinematics into the statement of the work of strain makes it possible to obtain the stiffness of membrane, of bending and transverse shears of the excentré element from those of the element are equivalent nonexcentré and of the distance from eccentring. All computations (except specific postprocessing) are thus made in a reference of diagram attached to the plane of the mesh. By defaults the results are thus obtained in the reference of the mesh. For certain postprocessings, it is possible to have automatically these results in other references insofar as the user indicates the position of the plane of postprocessing compared to the plane of the mesh.

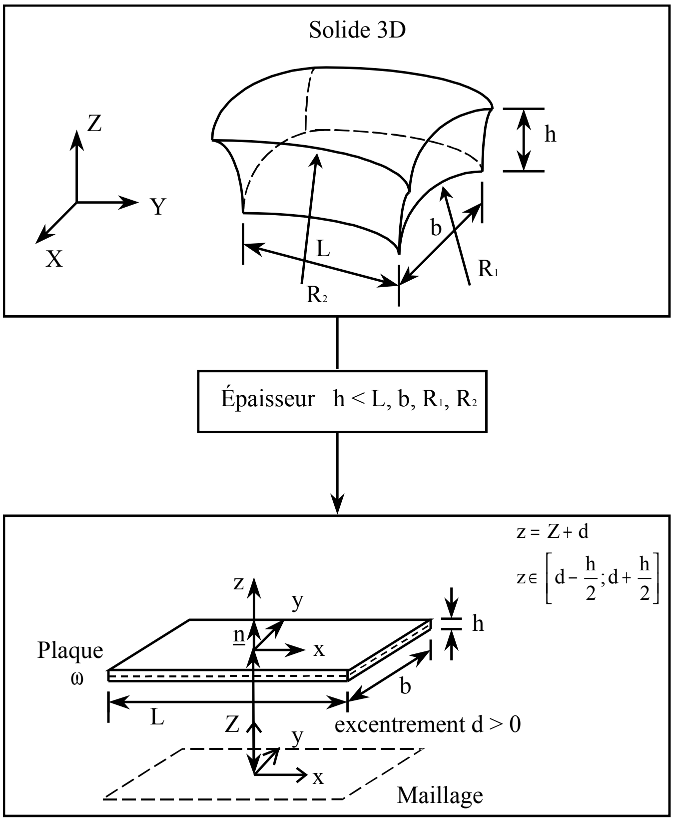
The distance from eccentring between the plane of the mesh and the average average of the plate is given in `AFFE_CARA_ELEM` to the same level as the thickness. A positive d eccentring means that the mean surface of the plate is actually at a distance $d\,\mathbf{n}$ from the shell element with a grid, the direction $\mathbf{n}$ being given by the norm to the element (see [§4.1] documentation of reference [R3.07.03] of the shell elements for the construction of this norm).

The adopted notations are those of the note [R3.07.03] on shell elements `DKT`, `DST`, `DKQ`, DSQ and `Q4G`.

2 Geometry

2.1 formulation

For the offset shell elements, the surface of reference is given by the plane of diagram or plane of the mesh (plane $x\,y$ for example). The average average of the element is positioned compared to this surface of reference. The thickness $h(x,y)$ must be small compared to other dimensions (extensions, radii of curvature) of structure to modelling. The figure [Figure 2.1-a] below illustrates our matter. Concerning the value of the eccentring d, and the conditions of linearization of bending adopted in the theory, one will take d so that an element of thickness $d+h$ remains in the theory of the plates.



Appear 2.1-a

One attaches to the plane of diagram (the plane of the mesh) a local orthonormal reference $0xyz$ associated with the plane of the mesh different from the total reference $OXYZ$. The position of the points of the plate is given by the Cartesian coordinates (x,y) in the plane of diagram (plane of the mesh) and rise z compared to this plane.

2.2 Kinematics

the cross-sections which are the sections perpendicular to the average average of the plate remain right. The material points located on a norm at not deformed mean surface remain on a line in the deformed configuration. It results from this approach that the fields of displacement vary linearly in the thickness of the plate. If one indicates by u, v, w displacements of a point of the plane of diagram $q(x, y, z)$ according to x, y and z, the kinematics of Hencky-Mindlin gives us:

$$\begin{pmatrix} u_x(x,y,z) \\ u_y(x,y,z) \\ u_z(x,y,z) \end{pmatrix} = \begin{pmatrix} u(x,y) \\ v(x,y) \\ w(x,y) \end{pmatrix} + z \begin{pmatrix} \theta_y(x,y) \\ -\theta_x(x,y) \\ 0 \end{pmatrix} = \begin{pmatrix} u(x,y) \\ v(x,y) \\ w(x,y) \end{pmatrix} + z \begin{pmatrix} \beta_x(x,y) \\ \beta_y(x,y) \\ 0 \end{pmatrix}$$

where: u, v, w are displacements of the plane of diagram;

θ_x and θ_y are respectively rotations of this plane compared to respectively the axis x and the axis y.

One prefers to introduce two rotations $\beta_x(x, y) = \theta_y(x, y), \beta_y(x, y) = -\theta_x(x, y)$. The three-dimensional strains in any point, with the kinematics introduced previously, are thus given by:

$$\left| \begin{array}{l} \varepsilon_{xx} = e_{xx} + z\kappa_{xx} \\ \varepsilon_{yy} = e_{yy} + z\kappa_{yy} \\ 2\varepsilon_{xy} = \gamma_{xy} = 2e_{xy} + 2z\kappa_{xy} \\ 2\varepsilon_{xz} = \gamma_x \\ 2\varepsilon_{yz} = \gamma_y \end{array} \right.$$

where: e_{xx}, e_{yy} and e_{xy} are the membrane strains of mean surface;

γ_x and the γ_y strains associated with the transverse shears;

κ_{xx} κ_{yy}, κ_{xy} the strains of bending of mean surface, which are written:

$$\left| \begin{array}{l} e_{xx} = \dfrac{\partial u}{\partial x} \\ e_{yy} = \dfrac{\partial v}{\partial y} \\ 2e_{xy} = \dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y} \\ \kappa_{xx} = \dfrac{\partial \beta_x}{\partial x} \\ \kappa_{yy} = \dfrac{\partial \beta_y}{\partial y} \\ 2\kappa_{xy} = \dfrac{\partial \beta_x}{\partial y} + \dfrac{\partial \beta_y}{\partial x} \\ \gamma_x = \beta_x + \dfrac{\partial w}{\partial x} \\ \gamma_y = \beta_y + \dfrac{\partial w}{\partial y} \end{array} \right.$$

Note:

- *in the theories of plate, the introduction of* β_x *and* β_y *makes it possible to symmetrize the formulations of the strains and the balance equations [R3.07.03]. In the theories of shell, one uses rather* θ_x *and* θ_y *the associated couples* M_x *and* M_y *compared to* x *and* y *,*
- *the degrees of freedom which one chose are displacements and rotations of the plane of diagram and not those of the average average. Indeed if one considers the superposition of several plates offset to produce a material sandwich it can correspond to the nodes of the mesh one field of displacement and not the various fields of displacements of the layers composing the material.*

2.3 Constitutive law

the behavior of the plates is a behavior 3D in "plane stresses". **The transverse stress** σ_{zz} **is taken null** because negligible compared to the other components of the tensor of the stresses (assumption of the plane stresses). The most general constitutive law is written then as follows:

$$\begin{pmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\\ \sigma_{xz}\\ \sigma_{yz}\end{pmatrix} = \mathbf{C}(\varepsilon,\alpha)\begin{pmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\\ \gamma_{x}\\ \gamma_{y}\end{pmatrix} = \mathbf{C}\mathbf{e} + z\mathbf{C}\kappa + \mathbf{C}\gamma \text{ with } \mathbf{e} = \begin{pmatrix}e_{xx}\\ e_{yy}\\ 2e_{xy}\\ 0\\ 0\end{pmatrix}, \quad \kappa = \begin{pmatrix}\kappa_{xx}\\ \kappa_{yy}\\ 2\kappa_{xy}\\ 0\\ 0\end{pmatrix} \quad \text{et } \gamma = \begin{pmatrix}0\\ 0\\ 0\\ \gamma_{x}\\ \gamma_{y}\end{pmatrix}$$

where: $C(\varepsilon,\alpha)$ is the local tangent stiffness matrix in plane stresses;
α represents all the local variables when the behavior is nonlinear.

For behaviors (for example of multi-layer) for which the distortions are coupled with the strains of membrane and bending, $C(\varepsilon,\alpha)$ puts itself in the form:

$$C=\begin{pmatrix}H & H_{c\gamma}\\ H^T_{c\gamma} & H_{\gamma}\end{pmatrix}$$

whe re: $H(\varepsilon,\alpha)$ is a symmetric 3×3 matrix;
$H_{\gamma}(\varepsilon,\alpha)$ a symmetric 2×2 matrix;
$H_{c\gamma}(\varepsilon,\alpha)$ a matrix 3×2 of coupling between the effects of membrane or bending and transverse shears.

If it is uncoupled, one A. $H_{c\gamma}(\varepsilon,\alpha)=0$ determination of $H_{\gamma}(\varepsilon,\alpha)$ in the frame of the theory of Reissner ([§ 2.2.3.2] of [R3.07.03]) is given in appendix. It is shown that it is equivalent to that of the not offset plates.

3 Principle of the virtual wors

3.1 Work of strain

the general statement of the work of strain 3D for the excentré shell element of the distance d compared to the datum-line is worth:

$$W_{\text{def}}=\int_S \int_{d-h/2}^{d+h/2} (\varepsilon_{xx}\sigma_{xx}+\varepsilon_{yy}\sigma_{yy}+\gamma_{xy}\sigma_{xy}+\gamma_x\sigma_{xz}+\gamma_y\sigma_{yz})\,dV$$

where S is mean surface, $dV=dxdydz$ and where the position in the thickness of the plate varies between $d \quad h/2$ and $d+h/2$.

3.1.1 Statement of the forces resulting

By adopting the kinematics from [R3.07.03], one identifies the work of the internal forces:

$$W_{\text{def}}=\int_S (e_{xx}N_{xx}+e_{yy}N_{yy}+2e_{xy}N_{xy}+\kappa_{xx}M_{xx}+\kappa_{yy}M_{yy}+2\kappa_{xy}M_{xy}+\gamma_x T_x+\gamma_y T_y)\,dS$$

where:

$$\mathbf{N}=\begin{pmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{pmatrix}=\int_{d-h/2}^{d+h/2}\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix}dz$$

$$\mathbf{M}=\begin{pmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{pmatrix}=\int_{d-h/2}^{d+h/2}\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix}z\,dz$$

$$\mathbf{T}=\begin{pmatrix} T_x \\ T_y \end{pmatrix}=\int_{d-h/2}^{d+h/2}\begin{pmatrix} \sigma_{xz} \\ \sigma_{yz} \end{pmatrix}dz$$

where: N_{xx} N_{yy}, N_{xy} are the forces resulting from membrane (in N/m);
M_{xx} M_{yy}, M_{xy} are the forces resulting from bending or moments compared to the plane from diagram (in N);
T_x, T_y are the forces resulting from shears or shears (in N/m).

3.1.2 Relation forces resulting generalized strains

the statement from the work of strain is also written:

$$W_{\text{def}}=\int_S \int_{d-h/2}^{d+h/2} [\boldsymbol{\varepsilon}\mathbf{C}(\boldsymbol{\varepsilon},\alpha)\boldsymbol{\varepsilon}]dV=\int_S \int_{d-h/2}^{d+h/2} [\mathbf{eCe}+z\mathbf{eC}\boldsymbol{\kappa}+\mathbf{eC}\boldsymbol{\gamma}+z\boldsymbol{\kappa}\mathbf{Ce}+z^2\boldsymbol{\kappa}\mathbf{C}\boldsymbol{\kappa}+z\boldsymbol{\kappa}\mathbf{C}\boldsymbol{\gamma}+\boldsymbol{\gamma}\mathbf{C}(\mathbf{e}+z\boldsymbol{\kappa}+\boldsymbol{\gamma})]dSdz$$

where: $C(\varepsilon,\alpha)$ is the local tangent stiffness matrix (symmetric matrix).

This is still written:

$$W_{def} = \int_S \int_{-h/2}^{h/2} [\mathbf{eCe} + (\zeta+d)\mathbf{eC}\kappa + \mathbf{eC}\gamma + \kappa(\zeta+d)\mathbf{Ce} + (\zeta+d)^2 \kappa\mathbf{C}\kappa + \kappa(\zeta+d)\mathbf{C}\gamma + \gamma\mathbf{C}(\mathbf{e} + (\zeta+d)\kappa + \gamma)] dS d\zeta$$

By means of the statement obtained for Wdef in the preceding paragraph, one and the finds the relation following between the resulting forces généraliées strains:

$$\mathbf{N} = \mathbf{H_m e} + (\mathbf{H_{mf}} + d\mathbf{H_m})\kappa + \mathbf{H_{m\gamma}}\gamma$$
$$\mathbf{M} = (\mathbf{H_{mf}} + d\mathbf{H_m})\mathbf{e} + (\mathbf{H_f} + 2d\mathbf{H_{mf}} + d^2\mathbf{H_m})\kappa + (\mathbf{H_{f\gamma}} + d\mathbf{H_{m\gamma}})\gamma$$
$$\mathbf{T} = \mathbf{H_{m\gamma}^T e} + (\mathbf{H_{f\gamma}^T} + d\mathbf{H_{m\gamma}^T})\kappa + \mathbf{H_{ct}}\gamma$$

with:

$$\mathbf{H_m} = \int_{-h/2}^{+h/2} \mathbf{H} d\zeta \qquad \mathbf{H_{mf}} = \int_{-h/2}^{+h/2} \mathbf{H}\zeta d\zeta \qquad \mathbf{H_f} = \int_{-h/2}^{+h/2} \mathbf{H}\zeta^2 d\zeta$$
$$\mathbf{H_{ct}} = \int_{-h/2}^{+h/2} \mathbf{H}_\gamma d\zeta \qquad \mathbf{H_{m\gamma}} = \int_{-h/2}^{+h/2} \mathbf{H}_{c\gamma} d\zeta \qquad \mathbf{H_{f\gamma}} = \int_{-h/2}^{+h/2} \mathbf{H}_{c\gamma}\zeta d\zeta$$

and:

$$\mathbf{e} = \begin{pmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{pmatrix}, \qquad \kappa = \begin{pmatrix} \kappa_{xx} \\ \kappa_{yy} \\ 2\kappa_{xy} \end{pmatrix}, \qquad \gamma = \begin{pmatrix} \gamma_x \\ \gamma_y \end{pmatrix}$$

The matrixes H_m, H_f and H_{ct} are the stiffness matrixes out of membrane, bending and transverse shears, respectively, for the nonexcentré shell element. The matrix H_{mf} is a stiffness matrix of coupling between the membrane and bending for the nonexcentré shell element. It is null if the shell element is symmetric compared to its average average. The matrix $H_m\gamma$ is a stiffness matrix of coupling between the membrane and the transverse distortion. The matrix $H_f\gamma$ is a stiffness matrix of coupling between bending and the transverse distortion. These matrixes are null for a null eccentring, except in the case as of multi-layer where they remain non-zero.

For an isotropic homogeneous elastic behavior, these matrixes have as a statement:

$$\mathbf{H_m} = \frac{Eh}{1-v^2}\begin{pmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{pmatrix}, \qquad \mathbf{H_f} = \frac{Eh^3}{12(1-v^2)}\begin{pmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{pmatrix}, \qquad \mathbf{H_{ct}} = \frac{kEh}{2(1+v)}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and $H_{mf} = H_m\gamma = H_f\gamma = 0$ because there is a material symmetry compared to the plane $\zeta = 0$.

For the determination of the shear coefficient k one returns to [§2.2.3] of [R3.07.03].

The system of relation between the resulting forces and the generalized strains can be also written:

$$\mathbf{N} = \mathbf{H_m e} + \mathbf{H'_{mf}}\kappa + \mathbf{H}_{\mathbf{m}\gamma}\gamma$$
$$\mathbf{M} = \mathbf{H'_{mf} e} + \mathbf{H'_f}\kappa + \mathbf{H}'_{\mathbf{f}\gamma}\gamma$$
$$\mathbf{T} = \mathbf{H}^{\mathrm{T}}_{\mathbf{m}\gamma}\mathbf{e} + \mathbf{H}'^{\mathrm{T}}_{\mathbf{f}\gamma}\kappa + \mathbf{H_{ct}}\gamma$$

with:

$$\mathbf{H'_{mf}} = \mathbf{H_{mf}} + d\mathbf{H_m}$$
$$\mathbf{H'_f} = \mathbf{H_f} + 2d\mathbf{H_{mf}} + d^2\mathbf{H_m}$$
$$\mathbf{H}'_{\mathbf{f}\gamma} = \mathbf{H}_{\mathbf{f}\gamma} + d\mathbf{H}_{\mathbf{m}\gamma}$$

Thus, in the case of a plate having material symmetry compared to the plane $\zeta=0$, one has $H_{mf}=0$ but $H'_{\mathbf{mf}}=dH_m$. The eccentring of the plate involves a coupling between the terms of membrane and bending.

Note:

> *The relations binding H_m H_f, H_{mf} with H and H_{ct} with H_γ are valid whatever the tangent elastic constitutive law, with unelastic strains (thermoelasticity, plasticity,…).*

For a plate made up of N orthotropic layers in elasticity, the matrixes H_m H_f, H_{mf} and H_{ct} are written:

$$\mathbf{H_m}=\sum_{i=1}^{N}\mathbf{H_{mi}}, \quad \mathbf{H_{mf}}=\sum_{i=1}^{N}(\mathbf{H_{mf\,i}}+\eta_i\mathbf{H_{mi}}), \quad \mathbf{H_f}=\sum_{i=1}^{N}(\mathbf{H_{fi}}+2\eta_i\mathbf{H_{mf\,i}}+\eta_i^2\mathbf{H_{mi}}), \quad \mathbf{H_{ct}}=\sum_{i=1}^{N}\mathbf{H_{ct\,i}}$$

where: $\eta_i=\frac{1}{2}(z_{i+1}+z_i)$

H_{mi} H_{fi} H_{mfi}, $H_{\gamma i}$ represent the matrixes of membrane, bending, coupling membrane bending and transverse shears for the layer i. One notices the analogy between these statements with the form established above:

$$H'_{\mathbf{mf}}=H_{\mathbf{mf}}+dH_m$$
$$H'_f=H_f+2\mathrm{dH}_{\mathbf{mf}}+d^2H_m$$

One from of deduced whereas the eccentring for such a plate is obtained in substituent η_i+d with η_i.

3.1.3 Elastic internal energy of plate

Taking into account the preceding remarks, the elastic internal energy of the plate is more usually expressed for this kind of geometry in the following way:

$$\Phi_{\mathrm{int}}=\frac{1}{2}\int_S[e(H_m e+H'_{\mathbf{mf}}\kappa+H_{m\gamma}\gamma)+\kappa(H'_{\mathbf{mf}}e+H'_f\kappa+H'_{f\gamma}\gamma)+\gamma(H^T_{m\gamma}e+H^{T}_{f\gamma}+H_{\mathbf{ct}}\gamma)]dS$$

3.1.4 Notice

One can choose to express the forces resulting from bending or moments compared to the average average from the element and either compared to the datum-line. In this case one obtains:

$$N=\begin{pmatrix}N_{xx}\\N_{yy}\\N_{xy}\end{pmatrix}=\int_{d-h/2}^{d+h/2}\begin{pmatrix}\sigma_{xx}\\\sigma_{yy}\\\sigma_{xy}\end{pmatrix}dz \quad M'=\begin{pmatrix}M'_{xx}\\M'_{yy}\\M'_{xy}\end{pmatrix}=\int_{d-h/2}^{d+h/2}\begin{pmatrix}\sigma_{xx}\\\sigma_{yy}\\\sigma_{xy}\end{pmatrix}(z-d)\,dz\ ,\quad T=\begin{pmatrix}T_x\\T_y\end{pmatrix}=\int_{d-h/2}^{d+h/2}\begin{pmatrix}\sigma_{xz}\\\sigma_{yz}\end{pmatrix}dz$$

and the statement of the work of the internal forces becomes:

$$W_{\text{def}}=\int_S\left(e_{xx}N_{xx}+e_{yy}N_{yy}+2e_{xy}N_{xy}+\kappa_{xx}(M'_{xx}+dN_{xx})+\kappa_{yy}(M'_{yy}+dN_{yy})\right)ds+\int_S\left(2\kappa_{xy}(M'_{xy}+dN_{xy})+\gamma_x T_x+\gamma_y T_y\right)dS$$

One from of then deduced by means of the statement 3D from the work of strain that:

$$\begin{aligned}N&=H_m e+(H_{\mathbf{mf}}+dH_m)\kappa+H_{m\gamma}\gamma\\M'+dN&=(H_{\mathbf{mf}}+dH_m)e+(H_f+2\mathrm{dH}_{\mathbf{mf}}+d^2H_m)\kappa+(H_{f\gamma}+dH_{m\gamma})\gamma\\T&=H^T_{m\gamma}e+(H^T_{f\gamma}+dH^T_{m\gamma})\kappa+H_{\mathbf{ct}}\gamma\end{aligned}$$

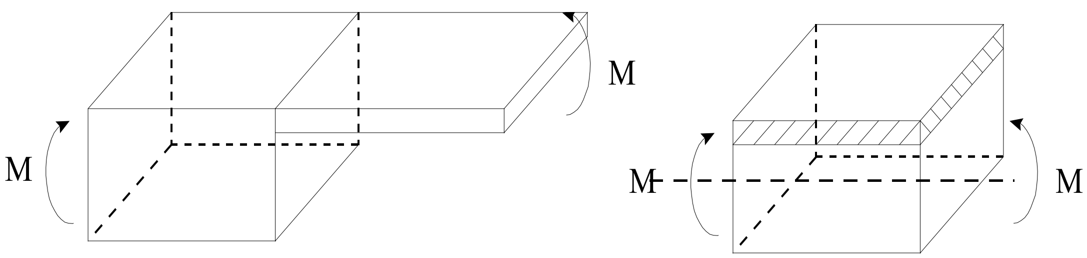
That is to say still:

$$\begin{aligned}N&=H_m e+(H_{\mathbf{mf}}+dH_m)\kappa+H_{m\gamma}\gamma\\M'&=H_{\mathbf{mf}}e+(H_f+dH_{\mathbf{mf}})\kappa+H_{f\gamma}\gamma\ .\\T&=H^T_{m\gamma}e+(H^T_{f\gamma}+dH^T_{m\gamma})\kappa+H_{\mathbf{ct}}\gamma\end{aligned}$$

The statement of the internal energy of the plate remains unchanged of course as for it. In the case of elasticity, she is always written:

$$\Phi_{\text{int}}=\frac{1}{2}\int_S\left[e(H_m e+H'_{\mathbf{mf}}\kappa+H_{m\gamma}\gamma)+\kappa(H'_{\mathbf{mf}}e+H'_f\kappa+H'_{f\gamma}\gamma)+\gamma(H^T_{m\gamma}e+H'^T_{f\gamma}\kappa+H_{\mathbf{ct}}\gamma)\right]dS$$

Question from the choice of the plane interesting to use for the statement of the moments can vary from a situation to another.



In the case of the figure of right, the approach developed above is preferable because the statement of the loadings is defined compared to the average average of each plate. In the case of the figure of left, if one wishes to replace the multi-layer shell by two offset shells, the reference axis is the average average of the multi-layer shell. One thus may find it beneficial with all to define compared to the plane of diagram. It is this approach which is adopted in the code. All the loadings applied are regarded as being defined by default in the reference of diagram or plane of the mesh. If ever certain loadings are defined compared to other planes (average average, higher average or inferior) is to the user to make the adapted changes of reference, with the hand or by the means of the command file by specifying the plane of load application when that is possible (see [§5]), to bring back itself to a loading defined in the plane of the mesh.

3.2 Work of the forces and couples external

work of the forces and couples being exerted on the plate is expressed in the following way:

$$W_{\text{ext}}=\int_S \int_{d-h/2}^{d+h/2} F_v . U\, dV+\int_S F_s . U\, dS+\int_C \int_{d-h/2}^{d+h/2} F_c . U\, dz\, ds$$

where: F_v F_s, F_c are the voluminal, surface forces and of contour being exerted on the plate, respectively.
C is the part of the contour of the plate to which the forces of contour F_c are applied.

With the kinematics of [§2.2], one determines as follows:

$$W_{\text{ext}}=\int_S (f_x u+f_y v+f_z w+c_x\theta_x+c_y\theta_y)\, dS+\int_C (\varphi_x u+\varphi_y v+\varphi_z w+\chi_x\theta_x+\chi_y\theta_y)\, ds$$

$$¿\int_S (f_x u+f_y v+f_z w+c_y\beta_x-c_x\beta_y)\, dS+\int_C (\varphi_x u+\varphi_y v+\varphi_z w+\chi_y\beta_x-\chi_x\beta_y)\, ds$$

where are present on the plate:

- f_x, f_y, f_z surface forces acting according to x, y and z ;
- $f_i=\int_{-h/2}^{+h/2} F_v . e_i dz+F_s . e_i$ where e_x and e_y are the basic vectors of the tangent plane and e_z their normal vector;
- $\mathbf{c}_x, \mathbf{c}_y$: surface couples acting around the axes x and y ;
- $c_i=\int_{-h/2}^{+h/2} [(z+d)e_z\wedge F_v].e_i dz+[(d\pm\frac{h}{2})e_z\wedge F_s].e_i$ where e_x e_y, e_z are the basic vectors previously definite.

and where are present on the contour of the plate:

- ϕ_x, ϕ_y, ϕ_z linear forces acting according to x, y and z ;
- $\phi_i=\int_{-h/2}^{+h/2} F_c . e_i dz$ where e_x e_y, e_z are the basic vectors previously definite;
- χ_x, χ_y linear couples around the axes x and y ;
- $\chi_i=\int_{-h/2}^{+h/2} [(z+d)e_z\wedge F_c].e_i dz$ where e_x e_y, e_z are the basic vectors previously definite.

Note:

The moments compared to z are null. The forces and the couples are expressed in the reference of the mesh. All the calculations are done by default in the reference of diagram. So forces or couples are expressed in another reference (that of the average average of the plate for example) the user will have to make conversions with the hand if it uses the options by default or to specify the plane of load application (see the paragraph [§ 5]).

3.3 Principle of virtual work and balance equations

This paragraph is unchanged compared to the paragraph [§3.3] of [R3.07.03].

4 Numerical discretization of the variational formulation resulting from the principle of virtual work

4.1 Introduction

the variational formulation for internal energy enables us to write:

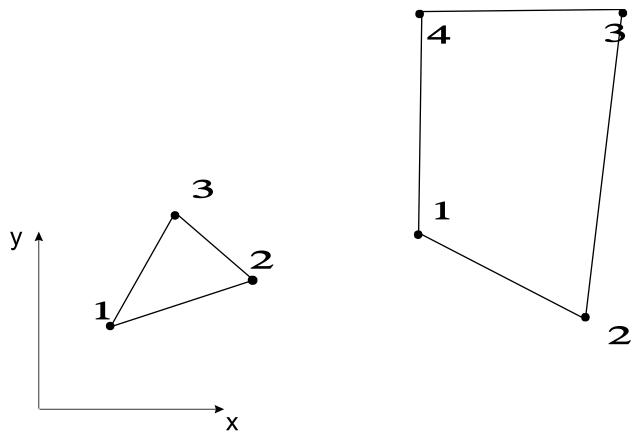
$$\delta W_{\text{int}}=\int_S [\delta e(H_m e+H'_{\mathbf{mf}}\kappa+H_{m\gamma}\gamma)+\delta\kappa(H'_{\mathbf{mf}}e+H'_f\kappa+H'_{f\gamma}\gamma)+\delta\gamma(H^T_{m\gamma}e+H'^T_{f\gamma}\kappa+H_{\mathbf{ct}}\gamma)]dS$$

with:

$$e=\begin{pmatrix}u_{,x}\\ v_{,y}\\ u_{,y}+v_{,x}\end{pmatrix},\qquad \kappa=\begin{pmatrix}\beta_{x,x}\\ \beta_{y,y}\\ \beta_{x,y}+\beta_{y,x}\end{pmatrix},\qquad \gamma=\begin{pmatrix}w_{,x}+\beta_x\\ w_{,y}+\beta_y\end{pmatrix}$$

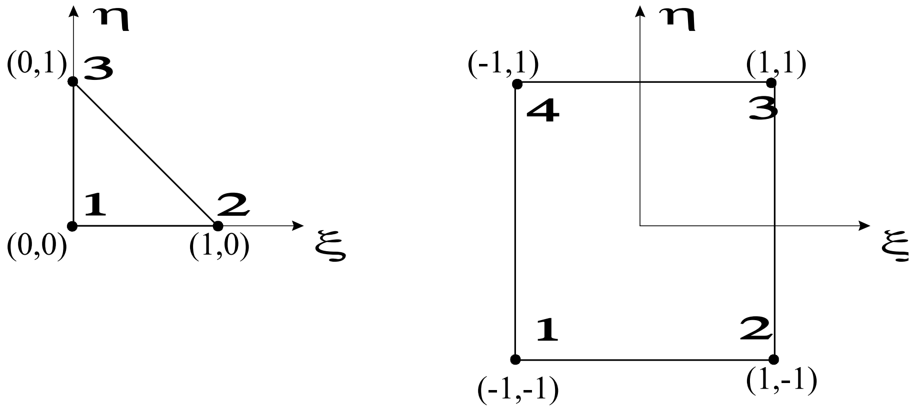
the five degrees of freedom are displacements in the plane of the mesh u and v, except plane w and the two rotations β_x and β_y.

Elements `DKT` and `DST` are triangular isoparametric elements. The elements `DKQ`, `DSQ` and `Q4gamma` are quadrilateral isoparametric elements. They are represented below:



Appear 4.1-a: Real elements

the elements of reference are presented below:



Appear 4.1-b: Elements of reference triangle and quadrangle

One defines the reduced reference of the element as the reference $(\xi . \eta)$ of the element of reference. The local coordinate system of the element, in the plane of diagram (x , y) is defined by the user, by the key word ANGLE_REP. The direction $X1$ of this local coordinate system is the projection of a direction of reference $\underline{d}$ as regards the element. This direction of reference $\underline{d}$ is chosen by the user who defines it by two nautical angles in the total reference. The norm N with the plane of the element ($12\wedge 13$ for a numbered triangle 123 and $12\wedge 14$ a numbered quadrangle 1234) fixes the second direction. The cross product of the two vectors previously definite $Y1=N\wedge X1$ makes it possible to define the local trihedron in which will be expressed the generalized forces representing the stress state. The user will have to take care that the selected reference axis is not found parallel with the norm of certain shell elements. By default, the direction of reference $\underline{d}$ is the axis X of the total reference of definition of the mesh.

Note:

For the shell elements QUAD4 , the use of a noncoplanar element can lead to irregularities ([bib1]). In this case, the user is alerted.

4.2 Discretization of the field of displacement

the jacobian matrix $J(\xi , \eta)$ is:

$$\mathbf{J} = \begin{pmatrix} x_{,\xi} & y_{,\xi} \\ x_{,\eta} & y_{,\eta} \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{N} N_{i,\xi} x_i & \sum_{i=1}^{N} N_{i,\xi} y_i \\ \sum_{i=1}^{N} N_{i,\eta} x_i & \sum_{i=1}^{N} N_{i,\eta} y_i \end{pmatrix} = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}$$

Moreover:

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} = \mathbf{j} \begin{pmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{pmatrix} \quad \text{avec} \quad \mathbf{j} = \begin{pmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{pmatrix} = \mathbf{J}^{-1} = \frac{1}{J} \begin{pmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{pmatrix} \quad \text{où } J = \det \mathbf{J} = J_{11} J_{22} - J_{12} J_{21}$$

The field of displacement is discretized by:

$$\begin{pmatrix} u \\ v \end{pmatrix} = \sum_{k=1}^{N} N_k(\xi,\eta) \begin{pmatrix} u^k \\ v^k \end{pmatrix}$$

$$\begin{pmatrix} w \\ \beta_x \\ \beta_y \end{pmatrix} = \sum_{k=1}^{N} N_k(\xi,\eta) \begin{pmatrix} w_k \\ \beta_{xk} \\ \beta_{yk} \end{pmatrix} + [\sum_{k=N+1}^{2N} \begin{pmatrix} 0 \\ P_{xk}(\xi,\eta) \\ P_{yk}(\xi,\eta) \end{pmatrix} \alpha_k]$$

In this last statement, the term between hooks is present for the elements of type DKT, DST, DKQ or DSQ, but not for the elements Q4 γ.

4.3 Taken into account of the transverse distortion

One recalls that the essential difference between elements DKT, DKQ on the one hand and DST, DSQ, Q4 γ on the other hand comes owing to the fact that for the first the transverse distortion is null is still γ = 0. The difference between Q4 γ and the elements DST and DSQ comes from a choice different of interpolation for the representation of the transverse shears. The introduction of the eccentring leads to a particular processing of the transverse shears.

One replaces in the statement of the internal energy established with [§4.1] γ by $\bar{\gamma}$ where are $\bar{\gamma}$ to them strains of substitution checking $\bar{\gamma}=\gamma$ in a weak way (integral on the sides of the element), and such as:

$$\mathbf{N} = \mathbf{H_m e} + \mathbf{H'_{mf}}\kappa + \mathbf{H_{m\gamma}}\bar{\gamma}$$
$$\mathbf{M} = \mathbf{H'_{mf} e} + \mathbf{H'_f}\kappa + \mathbf{H'_{f\gamma}}\bar{\gamma}$$
$$\mathbf{T} = \mathbf{H^T_{m\gamma} e} + \mathbf{H'^T_{f\gamma}}\kappa + \mathbf{H_{ct}}\bar{\gamma}$$

One checks thus that on the sides ij of the element, one a: $\int_i^j (\bar{\gamma}_s - \gamma_s) ds = 0$ with $\gamma_s = w_{,s} + \beta_s$.

4.3.1 For the elements Q4 γ

One linearly discretizes the constant $\bar{\gamma}$ field by side so that:

$$\bar{\gamma}^{\xi} = \begin{pmatrix} \bar{\gamma}_\xi \\ \bar{\gamma}_\eta \end{pmatrix} = \begin{pmatrix} \frac{1-\eta}{2}\gamma_\xi^{12} + \frac{1+\eta}{2}\gamma_\xi^{34} \\ \frac{1-\xi}{2}\gamma_\eta^{23} + \frac{1-\xi}{2}\gamma_\eta^{41} \end{pmatrix}$$

By means of then relations:

$$\left| \begin{array}{l} \int_{-1}^{+1} (\bar{\gamma}_\xi - (w_{,\xi} + \beta_\xi)) d\xi = 0 \ ; \\ \int_{-1}^{+1} (\bar{\gamma}_\eta - (w_{,\eta} + \beta_\eta)) d\eta = 0 \end{array} \right.$$

it is established that:

$$\left|\begin{array}{l}\gamma_\xi^{ij}=\frac{1}{2}(w_j-w_i+\beta_{\xi i}+\beta_{\xi j})\\ \gamma_\eta^{kp}=\frac{1}{2}(w_p-w_k+\beta_{\eta p}+\beta_{\eta k})\end{array}\right.$$ for $(ij)=(12,34)$ and $(kp)=(23,41)$.

By deferring the two results above in the statement of $\bar{\gamma}^\xi$, one establishes that:

$$\bar{\gamma}^\xi=\begin{pmatrix}\bar{\gamma}_\xi\\ \bar{\gamma}_\eta\end{pmatrix}=\mathbf{B}'_\xi\mathbf{u}_\xi$$

where: $\mathbf{u}_\xi=\begin{pmatrix}w_1\\ \beta_{\xi 1}\\ \beta_{\eta 1}\\ \vdots\\ w_N\\ \beta_{\xi N}\\ \beta_{\xi N}\end{pmatrix}$ and $\mathbf{B}'_\xi=(\mathbf{B}'_{\xi 1},\cdots,\mathbf{B}'_{\xi N})$ with $\mathbf{B}'_{\xi k}=\begin{pmatrix}N_{k,\xi} & \xi_k N_{k,\xi} & 0\\ N_{k,\eta} & 0 & \eta_k N_{k,\eta}\end{pmatrix}$

Moreover, like:

$$\begin{pmatrix}\beta_{\xi i}\\ \beta_{\eta i}\end{pmatrix}=\begin{pmatrix}J_{11} & J_{12}\\ J_{21} & J_{22}\end{pmatrix}\begin{pmatrix}\beta_{xi}\\ \beta_{yi}\end{pmatrix}$$

one from of deduced that

$$\bar{\gamma}^\xi=\mathbf{B}_\xi\mathbf{u}_f$$

where: $\mathbf{u}_f=\begin{pmatrix}w_1\\ \beta_{x1}\\ \beta_{y1}\\ \vdots\\ w_N\\ \beta_{xN}\\ \beta_{yN}\end{pmatrix}$ and $\mathbf{B}_\xi=(\mathbf{B}_{\xi 1},\cdots,\mathbf{B}_{\xi N})$

with: $\mathbf{B}_{\xi k}=\begin{pmatrix}N_{k,\xi} & \xi_k N_{k,\xi}J_{11} & \xi_k N_{k,\xi}J_{12}\\ N_{k,\eta} & \eta_k N_{k,\eta}J_{21} & \eta_k N_{k,\eta}J_{22}\end{pmatrix}$

Finally:

$$\bar{\gamma}=\begin{pmatrix}\bar{\gamma}_x\\ \bar{\gamma}_y\end{pmatrix}=\begin{pmatrix}j_{11} & j_{12}\\ j_{21} & j_{22}\end{pmatrix}\bar{\gamma}^\xi=\mathbf{B}_c\mathbf{u}_f$$ with $\mathbf{B}_{c[2\times 3N]}=\mathbf{j}\mathbf{B}_\xi$

Note::

This processing is equivalent to that of the elements $Q4\ \gamma$ not offset of [§ 4.3.2.1] of [R3.07.03].

4.3.2 For the elements of type DKT, DST, DKQ, DSQ

With regard to the transverse distortions, one knows that:

$$T_x = M_{xx,x} + M_{xy,y} \text{ et } T_y = M_{yy,y} + M_{xy,x} \text{ with } \mathbf{M} = \mathbf{H'_{mf}}\mathbf{e} + \mathbf{H'_f}\boldsymbol{\kappa} + \mathbf{H'_{f\gamma}}\bar{\boldsymbol{\gamma}}$$

One from of deduced that:

$$\mathbf{T} = \overline{\mathbf{H}}^c_{\mathbf{m}}\mathbf{u}_{,xx} + \overline{\mathbf{H}}^c_{\mathbf{f}}\boldsymbol{\beta}_{,xx}$$

Computation of: $\overline{\mathbf{H}}^c_{\mathbf{m}}\overline{\mathbf{H}}^c_{\mathbf{f}}$

where: $\boldsymbol{\beta}^T_{,xx} = \begin{pmatrix}\beta_{x,xx} & \beta_{x,yy} & \beta_{x,xy} & \beta_{y,xx} & \beta_{y,yy} & \beta_{y,xy}\end{pmatrix}$

$\mathbf{u}^T_{,xx} = \begin{pmatrix}u_{,xx} & u_{,yy} & u_{,xy} & v_{,xx} & v_{,yy} & v_{,xy}\end{pmatrix}$

with: $$\overline{\mathbf{H}}^c_{\mathbf{m}} = \begin{pmatrix} H^{mf}_{11} & H^{mf}_{33} & 2H^{mf}_{13} & H^{mf}_{13} & H^{mf}_{23} & H^{mf}_{12}+H^{mf}_{33} \\ H^{mf}_{13} & H^{mf}_{23} & H^{mf}_{12}+H^{mf}_{33} & H^{mf}_{33} & H^{mf}_{22} & 2H^{mf}_{23} \end{pmatrix}$$

$$\overline{\mathbf{H}}^c_{\mathbf{f}} = \begin{pmatrix} H^{f}_{11} & H^{f}_{33} & 2H^{f}_{13} & H^{f}_{13} & H^{f}_{23} & H^{f}_{12}+H^{f}_{33} \\ H^{f}_{13} & H^{f}_{23} & H^{f}_{12}+H^{f}_{33} & H^{f}_{33} & H^{f}_{22} & 2H^{f}_{23} \end{pmatrix}$$

where are H^{mf}_{ij} to them the terms (i, j) of $\mathbf{H'_{mf}}$ and where are H^{f}_{ij} to them the terms (i, j) of $\mathbf{H'_f}$.

Like:

$$\beta_{x,xx} = \sum_{k=1}^{N} N_{k,xx}(\xi,\eta)\beta_{xk} + \sum_{k=N+1}^{2N} P_{xk,xx}(\xi,\eta)\alpha_k = \sum_{k=1}^{N} N_{k,xx}(\xi,\eta)\beta_{xk} + \sum_{k=N+1}^{2N} (j^2_{11}P_{xk,\xi\xi} + 2j_{11}j_{12}P_{xk,\xi\eta} + j^2_{12}P_{xk,\eta\eta})\alpha_k$$

$$\beta_{x,yy} = \sum_{k=1}^{N} N_{k,yy}(\xi,\eta)\beta_{xk} + \sum_{k=N+1}^{2N} P_{xk,yy}(\xi,\eta)\alpha_k = \sum_{k=1}^{N} N_{k,yy}(\xi,\eta)\beta_{xk} + \sum_{k=N+1}^{2N} (j^2_{21}P_{xk,\xi\xi} + 2j_{21}j_{22}P_{xk,\xi\eta} + j^2_{22}P_{xk,\eta\eta})\alpha_k$$

$$\beta_{x,xy} = \sum_{k=1}^{N} N_{k,xy}(\xi,\eta)\beta_{xk} + \sum_{k=N+1}^{2N} P_{xk,xy}(\xi,\eta)\alpha_k = \sum_{k=1}^{N} N_{k,xy}(\xi,\eta)\beta_{xk} + \sum_{k=N+1}^{2N} (j_{11}j_{21}P_{xk,\xi\xi} + [j_{11}j_{22} + j_{12}j_{21}]P_{xk,\xi\eta} + j_{11}j_{21}P_{xk,\eta\eta})\alpha_k$$

$$\beta_{y,xx} = \sum_{k=1}^{N} N_{k,xx}(\xi,\eta)\beta_{yk} + \sum_{k=N+1}^{2N} P_{yk,xx}(\xi,\eta)\alpha_k = \sum_{k=1}^{N} N_{k,xx}(\xi,\eta)\beta_{yk} + \sum_{k=N+1}^{2N} (j^2_{11}P_{yk,\xi\xi} + 2j_{11}j_{12}P_{yk,\xi\eta} + j^2_{12}P_{yk,\eta\eta})\alpha_k$$

$$\beta_{y,yy} = \sum_{k=1}^{N} N_{k,yy}(\xi,\eta)\beta_{yk} + \sum_{k=N+1}^{2N} P_{yk,yy}(\xi,\eta)\alpha_k = \sum_{k=1}^{N} N_{k,yy}(\xi,\eta)\beta_{yk} + \sum_{k=N+1}^{2N} (j^2_{21}P_{yk,\xi\xi} + 2j_{21}j_{22}P_{yk,\xi\eta} + j^2_{22}P_{yk,\eta\eta})\alpha_k$$

$$\beta_{y,xy} = \sum_{k=1}^{N} N_{k,xy}(\xi,\eta)\beta_{yk} + \sum_{k=N+1}^{2N} P_{yk,xy}(\xi,\eta)\alpha_k = \underbrace{\sum_{k=1}^{N} N_{k,xy}(\xi,\eta)\beta_{yk}}_{\beta^1_{xx}} + \sum_{k=N+1}^{2N} (j_{11}j_{21}P_{yk,\xi\xi} + [j_{11}j_{22} + j_{12}j_{21}]P_{yk,\xi\eta} + j_{11}j_{21}P_{yk,\eta\eta})\alpha_k$$

with:

$$\beta^{1}{}_{,xx}=\sum_{k=1}^{N}\begin{pmatrix}0 & j_{11}^{2}N_{k,\xi\xi}+2j_{11}j_{12}N_{k,\xi\eta}+j_{12}^{2}N_{k,\eta\eta} & 0\\ 0 & j_{21}^{2}N_{k,\xi\xi}+2j_{21}j_{22}N_{k,\xi\eta}+j_{22}^{2}N_{k,\eta\eta} & 0\\ 0 & j_{11}j_{21}N_{k,\xi\xi}+[j_{11}j_{22}+j_{12}j_{21}]N_{k,\xi\eta}+j_{12}j_{22}N_{k,\eta\eta} & 0\\ 0 & 0 & j_{11}^{2}N_{k,\xi\xi}+2j_{11}j_{12}N_{k,\xi\eta}+j_{12}^{2}N_{k,\eta\eta}\\ 0 & 0 & j_{21}^{2}N_{k,\xi\xi}+2j_{21}j_{22}N_{k,\xi\eta}+j_{22}^{2}N_{k,\eta\eta}\\ 0 & 0 & j_{11}j_{21}N_{k,\xi\xi}+[j_{11}j_{22}+j_{12}j_{21}]N_{k,\xi\eta}+j_{12}j_{22}N_{k,\eta\eta}\end{pmatrix}\begin{pmatrix}w_{k}\\ \beta_{xk}\\ \beta_{yk}\end{pmatrix}$$

the first contribution to $\beta_{,xx}$ in the statement above and:

$$u_{,xx}=\sum_{k=1}^{n}\begin{pmatrix} j_{11}^2 N_{k,\xi\xi}+2j_{11}j_{12}N_{k,\xi\eta}+j_{12}^2 N_{k,\eta\eta} & 0\\ j_{21}^2 N_{k,\xi\xi}+2j_{21}j_{22}N_{k,\xi\eta}+j_{22}^2 N_{k,\eta\eta} & 0\\ j_{11}j_{21}N_{k,\xi\xi}+[j_{11}j_{22}+j_{12}j_{21}]N_{k,\xi\eta}+j_{12}j_{22}N_{k,\eta\eta} & 0\\ 0 & j_{11}^2 N_{k,\xi\xi}+2j_{11}j_{12}N_{k,\xi\eta}+j_{12}^2 N_{k,\eta\eta}\\ 0 & j_{21}^2 N_{k,\xi\xi}+2j_{21}j_{22}N_{k,\xi\eta}+j_{22}^2 N_{k,\eta\eta}\\ 0 & j_{11}j_{21}N_{k,\xi\xi}+[j_{11}j_{22}+j_{12}j_{21}]N_{k,\xi\eta}+j_{12}j_{22}N_{k,\eta\eta}\end{pmatrix}\begin{pmatrix}u_k\\ v_k\end{pmatrix}$$

that is to say still in matric form that:

$$\mathbf{T}=\bar{\mathbf{H}}_m^c\begin{pmatrix}u_{,xx}\\ u_{,yy}\\ u_{,xy}\\ v_{,xx}\\ v_{,yy}\\ v_{,xy}\end{pmatrix}+\bar{\mathbf{H}}_f^c\begin{pmatrix}\beta_{x,xx}\\ \beta_{x,yy}\\ \beta_{x,xy}\\ \beta_{y,xx}\\ \beta_{y,yy}\\ \beta_{y,xy}\end{pmatrix}$$

$$\mathbf{T}=\bar{\mathbf{H}}_m^c\sum_{k=1}^{N}\mathbf{P}_{\mathbf{cm}}^{\mathbf{k}}\begin{pmatrix}u_k\\ v_k\end{pmatrix}+\bar{\mathbf{H}}_f^c\sum_{k=1}^{N}\mathbf{P}_{\mathbf{c\beta}}^{\mathbf{k}}\begin{pmatrix}w_k\\ \beta_{xk}\\ \beta_{yk}\end{pmatrix}+\bar{\mathbf{H}}_f^c\sum_{k=N+1}^{2N}\alpha_k\begin{pmatrix}C_k(j_{11}^2 P_{k,\xi\xi}+2j_{11}j_{12}P_{k,\xi\eta}+j_{12}^2 P_{k,\eta\eta})\\ C_k(j_{21}^2 P_{k,\xi\xi}+2j_{21}j_{22}P_{k,\xi\eta}+j_{22}^2 P_{k,\eta\eta})\\ C_k(j_{11}j_{21}P_{k,\xi\xi}+[j_{11}j_{22}+j_{12}j_{21}]P_{k,\xi\eta}+j_{11}j_{21}P_{k,\eta\eta})\\ S_k(j_{11}^2 P_{k,\xi\xi}+2j_{11}j_{12}P_{k,\xi\eta}+j_{12}^2 P_{k,\eta\eta})\\ S_k(j_{21}^2 P_{k,\xi\xi}+2j_{21}j_{22}P_{k,\xi\eta}+j_{22}^2 P_{k,\eta\eta})\\ S_k(j_{11}j_{21}P_{k,\xi\xi}+[j_{11}j_{22}+j_{12}j_{21}]P_{k,\xi\eta}+j_{11}j_{21}P_{k,\eta\eta})\end{pmatrix}$$

$$\mathbf{T}=\bar{\mathbf{H}}_m^c\sum_{k=1}^{N}\mathbf{P}_{\mathbf{cm}}^{\mathbf{k}}\begin{pmatrix}u_k\\ v_k\end{pmatrix}+\bar{\mathbf{H}}_f^c\sum_{k=1}^{N}\mathbf{P}_{\mathbf{c\beta}}^{\mathbf{k}}\begin{pmatrix}w_k\\ \beta_{xk}\\ \beta_{yk}\end{pmatrix}+\sum_{k=N+1}^{2N}\bar{\mathbf{H}}_f^c T_2\begin{pmatrix}C_k P_{k,\xi\xi}\\ C_k P_{k,\eta\eta}\\ C_k P_{k,\xi\eta}\\ S_k P_{k,\xi\xi}\\ S_k P_{k,\eta\eta}\\ S_k P_{k,\xi\eta}\end{pmatrix}\alpha_k$$

$$\mathbf{T}=\bar{\mathbf{H}}_m^c\sum_{k=1}^{N}\mathbf{P}_{\mathbf{cm}}^{\mathbf{k}}\begin{pmatrix}u_k\\ v_k\end{pmatrix}+\bar{\mathbf{H}}_f^c\sum_{k=1}^{N}\mathbf{P}_{\mathbf{c\beta}}^{\mathbf{k}}\begin{pmatrix}w_k\\ \beta_{xk}\\ \beta_{yk}\end{pmatrix}+\bar{\mathbf{H}}_f^c\mathbf{T_2}\sum_{k=N+1}^{2N}\mathbf{T_{ck}}\alpha_k$$

$$\mathbf{T}=\bar{\mathbf{H}}_m^c\mathbf{P_{cm}}\mathbf{U_m}+\bar{\mathbf{H}}_f^c\mathbf{P_{c\beta}}\mathbf{U_{f\beta}}+\bar{\mathbf{H}}_f^c\mathbf{T_2}\mathbf{T_\alpha}\alpha=\mathbf{B_{cm}}\mathbf{U_m}+\mathbf{B_{c\beta}}\mathbf{U_{f\beta}}+\mathbf{B_{c\alpha}}\alpha$$

Where:

$$\mathbf{U}_m = \begin{pmatrix} u_1 \\ v_1 \\ \vdots \\ u_N \\ v_N \end{pmatrix}$$

$$\mathbf{T}_\alpha = (\mathbf{T}_{c(N+1)} \cdots \mathbf{T}_{c2N})$$

$$\mathbf{T}_2 = \begin{pmatrix} \mathbf{t}_2 & 0 \\ 0 & \mathbf{t}_2 \end{pmatrix} \quad \text{with} \quad \mathbf{t}_2 = \begin{pmatrix} j_{11}^2 & j_{12}^2 & 2 j_{11} j_{12} \\ j_{21}^2 & j_{22}^2 & 2 j_{21} j_{22} \\ j_{11} j_{21} & j_{12} j_{22} & j_{11} j_{22} + j_{12} j_{21} \end{pmatrix}$$

$$\mathbf{u}_{f\beta} = \begin{pmatrix} w_1 \\ \beta_{x1} \\ \beta_{y1} \\ \vdots \\ w_N \\ \beta_{xN} \\ \beta_{yN} \end{pmatrix}$$

One can also write:

$$\mathbf{T} = \overline{\mathbf{H}}_\mathbf{m}^c \mathbf{u}_{,xx} + \overline{\mathbf{H}}_\mathbf{f}^c \beta_{,xx} = \mathbf{B}_{\mathbf{cm}} \mathbf{U}_\mathbf{m} + \mathbf{B}_{c\beta} \mathbf{U}_{f\beta} + \mathbf{B}_{c\alpha} \alpha$$

By means of the relation $\int_i^j (\bar{\gamma}_s - \gamma_s) ds = 0$ with $\gamma_s = w_{,s} + \beta_s$ for each side ij of the element, one can obtain τ η ε α κ since this relation is still written:

$$w_j - w_i + \frac{L_k}{2}(C_k \beta_{xi} + S_k \beta_{yi} + C_k \beta_{xj} + S_k \beta_{yj}) + \frac{2}{3} L_k \alpha_k = L_k \bar{\gamma}_{sk}$$

where:

$$\begin{aligned} \bar{\gamma}_{sk} &= \left(C_k \;\; S_k\right) \bar{\gamma} = (C_k \;\; S_k) H_{ct}^{-1} \left[T - H_{m\gamma}^T e - H'^T_{f\gamma} \kappa \right] \\ &= (C_k \;\; S_k) H_{ct}^{-1} \left[\left(B_{cm} - H_{m\gamma}{}^T B_m \right) U_m + \left(B_{c\beta} - H'_{f\gamma}{}^T B_{f\beta} \right) U_{f\beta} + \left(B_{c\alpha} - H'^T_{f\gamma} B_{f\alpha} \right) \alpha \right] \end{aligned}$$

The relation above is still written in matric form:

$$\mathbf{A}_\alpha \alpha = (\mathbf{A}_w + \mathbf{A}_\beta) \mathbf{U}_{f\beta} + \mathbf{A}_\mathbf{m} \mathbf{U}_\mathbf{m}$$

with:

$$\mathbf{A}_\alpha = \frac{2}{3} \begin{pmatrix} L_{N+1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & L_{2N} \end{pmatrix} - \begin{pmatrix} L_{N+1} C_{N+1} & L_{N+1} S_{N+1} \\ \vdots & \vdots \\ L_{2N} C_{2N} & L_{2N} S_{2N} \end{pmatrix} \mathbf{H}_{ct}^{-1} (\mathbf{B}_{c\alpha} - \mathbf{H}'_{f\gamma}{}^T \mathbf{B}_{f\alpha})$$

$$A_w=-\frac{1}{2}\begin{pmatrix} -2 & L_{N+1}C_{N+1} & L_{N+1}S_{N+1} & 2 & L_{N+1}C_{N+1} & L_{N+1}S_{N+1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & L_{k+1}C_{k+1} & L_{k+1}S_{k+1} & 2 & L_{k+1}C_{k+1} & L_{k+1}S_{k+1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & L_{2N-1}C_{2N-1} & L_{2N-1}S_{2N-1} & 2 & L_{2N-1}C_{2N-1} & L_{2N-1}S_{2N-1} \\ 2 & L_{2N}C_{2N} & L_{2N}S_{2N} & 0 & 0 & \cdots & \cdots & 0 & 0 & -2 & L_{2N}C_{2N} & L_{2N}S_{2N} \end{pmatrix}$$

$$A_\beta=\begin{pmatrix} L_{N+1}C_{N+1} & L_{N+1}S_{N+1} \\ \vdots & \vdots \\ L_{2N}C_{2N} & L_{2N}S_{2N} \end{pmatrix}\mathbf{H}_{ct}^{-1}(\mathbf{B}_{c\beta}-\mathbf{H}'^{T}_{f\gamma}\mathbf{B}_{f\beta})$$

$$\mathbf{A}_m=\begin{pmatrix} L_{N+1}C_{N+1} & L_{N+1}S_{N+1} \\ \vdots & \vdots \\ L_{2N}C_{2N} & L_{2N}S_{2N} \end{pmatrix}\mathbf{H}_{ct}^{-1}(\mathbf{B}_{cm}-\mathbf{H}_{m\gamma}^{T}\mathbf{B}_{m})$$

As follows:

$$\boldsymbol{\alpha}=\mathbf{P}_\beta\mathbf{U}_{f\beta}+\mathbf{P}_m\mathbf{U}_m$$

with:

$$\mathbf{P}_\beta=\mathbf{A}_\alpha^{-1}(\mathbf{A}_w+\mathbf{A}_\beta)$$

$$\mathbf{P}_m=\mathbf{A}_\alpha^{-1}\mathbf{A}_m$$

what implies:

$$\mathbf{T}=(\mathbf{B}_{cm}+\mathbf{B}_{c\alpha}\mathbf{P}_m)\mathbf{U}_m+(\mathbf{B}_{c\beta}+\mathbf{B}_{c\alpha}\mathbf{P}_\beta)\mathbf{U}_{f\beta}$$

Note:

For the elements of type DKT and DST , one A. $\mathbf{B}_{cm}=\mathbf{B}_{c\beta}=0$ *It results from it from the simplified statements of the preceding equations.*

4.4 Elemental stiffness matrix

4.4.1 Elemental stiffness matrix for the elements Q4 γ

One takes again the forms of the stiffness matrixes given to [§4.4.1] of documentation of reference [R3.07.03] and one replaces $H_{\mathbf{mf}}$ by $H'_{\mathbf{mf}}$, H_f by H'_f and $H_{f\gamma}$ par. $H'_{f\gamma}$ One will note that in [R3.07.03] the results were presented without term of coupling transverse membrane shears or transverse bending shears. They here are added.

4.4.2 Elemental stiffness matrix for elements DKT, DKQ

One takes again the forms of the stiffness matrixes given to [§4.4.1] of documentation of reference [R3.07.03] and one replaces $H_{\mathbf{mf}}$ by $H'_{\mathbf{mf}}$, H_f par. H'_f Since the relation $\bar{\gamma}=0$ is satisfied the couplings transverse membrane shears or transverse bending shears are non-existent.

4.4.3 Elemental stiffness matrix for elements DST, DSQ

One a:

$$\delta W^e_{\text{int}} = \int_e \delta\mathbf{e}(\mathbf{H_m e} + \mathbf{H}'_{mf}\boldsymbol{\kappa} + \mathbf{H}_{m\gamma}\boldsymbol{\gamma} - \mathbf{H}_{m\gamma}\mathbf{H}^{-1}_{ct}\mathbf{T}) + \delta\boldsymbol{\kappa}(\mathbf{H}'_{mf}\mathbf{e} + \mathbf{H}'_{f}\boldsymbol{\kappa} + \mathbf{H}'_{f\gamma}\boldsymbol{\gamma} - \mathbf{H}'_{f\gamma}\mathbf{H}^{-1}_{ct}\mathbf{T}) + \delta\mathbf{T}\mathbf{H}^{-1}_{ct}\mathbf{T}\,dS =$$
$$\int_e \delta\mathbf{e}([\mathbf{H_m} - \mathbf{H}_{m\gamma}\mathbf{H}^{-1}_{ct}\mathbf{H}^{T}_{m\gamma}]\mathbf{e} + [\mathbf{H}'_{mf} - \mathbf{H}_{m\gamma}\mathbf{H}^{-1}_{ct}\mathbf{H'}^{T}_{f\gamma}]\boldsymbol{\kappa}) + \delta\boldsymbol{\kappa}([\mathbf{H'}_{mf} - \mathbf{H}'_{f\gamma}\mathbf{H}^{-1}_{ct}\mathbf{H}^{T}_{m\gamma}]\mathbf{e}\,dS$$
$$+ \int_e [\mathbf{H}'_{f} - \mathbf{H}'_{f\gamma}\mathbf{H}_{ct^{-1}}\mathbf{H'}^{T}_{f\gamma}]\boldsymbol{\kappa}) + \delta\mathbf{T}\mathbf{H}_{ct^{-1}}\mathbf{T}\,dS$$

Is still:

$$\delta W^e_{\text{int}} = \int_e \delta\mathbf{e}(\mathbf{H}''_m\mathbf{e} + \mathbf{H}''_{mf}\kappa) + \delta\kappa(\mathbf{H}''^{T}_{mf}\mathbf{e} + \mathbf{H}''_f\kappa) + \delta\mathbf{T}\mathbf{H}_{ct}{}^{-1}\mathbf{T}dS$$

where:

$$\mathbf{H}''_{\mathbf{m}} = \mathbf{H_m} - \mathbf{H}_{m\gamma}\mathbf{H}^{-1}_{ct}\mathbf{H}^{T}_{m\gamma}$$
$$\mathbf{H}''_{\mathbf{mf}} = \mathbf{H}'_{\mathbf{mf}} - \mathbf{H}_{m\gamma}\mathbf{H}^{-1}_{ct}\mathbf{H}'^{T}_{f\gamma}$$
$$\mathbf{H}''_{\mathbf{f}} = \mathbf{H}'_{\mathbf{f}} - \mathbf{H}'_{f\gamma}\mathbf{H}^{-1}_{ct}\mathbf{H}'^{T}_{f\gamma}$$

From where:

$$\delta W^e_{\text{int}} = \int_e (\delta\mathbf{U}^T_m\mathbf{B}^T_m\mathbf{H}''_m\mathbf{B}_m\mathbf{U}_m + \delta\mathbf{U}^T_m\mathbf{B}^T_m\mathbf{H}''_{mf}\mathbf{B}_f\mathbf{U}_f + \delta\mathbf{U}^T_f\mathbf{B}^T_f\mathbf{H}''^{T}_{mf}\mathbf{B}_m\mathbf{U}_m + \delta\mathbf{U}^T_f\mathbf{B}^T_f\mathbf{H}''_f\mathbf{B}_f\mathbf{U}_f$$
$$+ \delta\alpha^T\mathbf{B}^T_{c\alpha}\mathbf{H}^{-1}_{ct}\mathbf{B}_{c\alpha}\alpha + \delta\alpha^T\mathbf{B}^T_{c\alpha}\mathbf{H}^{-1}_{ct}\mathbf{B}_{cm}\mathbf{U}_m + \delta\mathbf{U}^T_m\mathbf{B}^T_{cm}\mathbf{H}^{-1}_{ct}\mathbf{B}_{c\alpha}\alpha + \delta\mathbf{U}^T_m\mathbf{B}^T_{cm}\mathbf{H}^{-1}_{ct}\mathbf{B}_{cm}\mathbf{U}_m$$
$$+ \delta\alpha^T\mathbf{B}^T_{c\alpha}\mathbf{H}^{-1}_{ct}\mathbf{B}_{c\beta}\mathbf{U}_{f\beta} + \delta\mathbf{U}^T_{f\beta}\mathbf{B}^T_{c\beta}\mathbf{H}^{-1}_{ct}\mathbf{B}_{c\alpha}\alpha + \delta\mathbf{U}^T_{f\beta}\mathbf{B}^T_{c\beta}\mathbf{H}^{-1}_{ct}\mathbf{B}_{c\beta}\mathbf{U}_{f\beta}$$
$$+ \delta\mathbf{U}^T_m\mathbf{B}^T_{cm}\mathbf{H}^{-1}_{ct}\mathbf{B}_{c\beta}\mathbf{U}_{f\beta} + \delta\mathbf{U}^T_{f\beta}\mathbf{B}^T_{c\beta}\mathbf{H}^{-1}_{ct}\mathbf{B}_{cm}\mathbf{U}_m)\,dS$$

$$= \delta\mathbf{U}^T_m\left(\int_e \mathbf{B}^T_m\mathbf{H}''_m\mathbf{B}_m\,\mathrm{dS}\right)\mathbf{U}_m + \delta\mathbf{U}^T_f\left(\int_e \mathbf{B}^T_f\mathbf{H}''_f\mathbf{B}_f\,\mathrm{dS}\right)\mathbf{U}_f$$
$$+ \delta\mathbf{U}^T_m\left(\int_e \mathbf{B}^T_m\mathbf{H}''_{\mathrm{mf}}\mathbf{B}_f\,\mathrm{dS}\right)\mathbf{U}_f + \delta\mathbf{U}^T_f\left(\int_e \mathbf{B}^T_f\mathbf{H}''^{T}_{\mathrm{mf}}\mathbf{B}_m\,\mathrm{dS}\right)\mathbf{U}_m$$
$$+ \delta\alpha^T\left(\int_e \mathbf{B}^T_{c\alpha}\mathbf{H}^{-1}_{\mathrm{ct}}\mathbf{B}_{c\alpha}\,\mathrm{dS}\right)\alpha + \delta\alpha^T\left(\int_e \mathbf{B}^T_{c\alpha}\mathbf{H}^{-1}_{\mathrm{ct}}\mathbf{B}_{cm}\,\mathrm{dS}\right)\mathbf{U_m}$$
$$+ \delta\mathbf{U}_{¿}{}_m{}^T\left(\int_e \mathbf{B}^T_{cm}\mathbf{H}^{-1}_{ct}\mathbf{B}_{c\alpha}\,dS\right)\alpha + \delta\mathbf{U}^T_m\left(\int_e \mathbf{B}^T_{\mathrm{cm}}\mathbf{H}^{-1}_{\mathrm{ct}}\mathbf{B}_{\mathrm{cm}}\,\mathrm{dS}\right)\mathbf{U}_m$$
$$+ \delta\alpha^T\left(\int_e \mathbf{B}^T_{c\alpha}\mathbf{H}^{-1}_{\mathrm{ct}}\mathbf{B}_{c\beta}\,\mathrm{dS}\right)\mathbf{U}_{f\beta} + \delta\mathbf{U}^T_{f\beta}\left(\int_e \mathbf{B}^T_{c\beta}\mathbf{H}^{-1}_{\mathrm{ct}}\mathbf{B}_{c\alpha}\,\mathrm{dS}\right)\alpha$$
$$+ \delta\mathbf{U}^T_{f\beta}\left(\int_e \mathbf{B}^T_{c\beta}\mathbf{H}^{-1}_{\mathrm{ct}}\mathbf{B}_{c\beta}\,\mathrm{dS}\right)\mathbf{U}_{f\beta}$$
$$+ \delta\mathbf{U}^T_m\left(\int_e \mathbf{B}^T_{cm}\mathbf{H}^{-1}_{ct}\mathbf{B}_{c\beta}\,dS\right)\mathbf{U}_{f\beta} + \delta\mathbf{U}^T_{f\beta}\left(\int_e \mathbf{B}^T_{c\beta}\mathbf{H}^{-1}_{ct}\mathbf{B}_{cm}\,dS\right)\mathbf{U}_m$$
$$= \delta\mathbf{U}^T_m\mathbf{K}_m\mathbf{U}_m + \delta\mathbf{U}^T_f\mathbf{K}_f\mathbf{U}_f + \delta\mathbf{U}^T_m\mathbf{K}_{mf}\mathbf{U}_f + \delta\mathbf{U}^T_f\mathbf{K}_{fm}\mathbf{U}_m + \delta\alpha^T\mathbf{K}_{\alpha\alpha}\alpha + \delta\mathbf{U}^T_m\mathbf{K}_{m\alpha}\alpha + \delta\alpha^T\mathbf{K}^T_{m\alpha}\mathbf{U}_m$$
$$+ \delta\mathbf{U}^T_{f\beta}\mathbf{K}_{\beta\alpha}\alpha + \delta\alpha^T\mathbf{K}^T_{\beta\alpha}\mathbf{U}_{f\beta} + \delta\mathbf{U}^T_m\mathbf{K}_{m\beta}\mathbf{U}_{f\beta} + \delta\mathbf{U}^T_{f\beta}\mathbf{K}_{\beta m}\mathbf{U}_m + \delta\mathbf{U}^T_{f\beta}\mathbf{K}_{\beta\beta}\mathbf{U}_{f\beta}$$

with:

$$\mathbf{K}_m = \int_s [\mathbf{B}^T_m\mathbf{H}''_m\mathbf{B}_m + \mathbf{B}^T_{cm}\mathbf{H}^{-1}_{ct}\mathbf{B}_{cm}]\mathrm{dS}$$

It is also known that $\mathbf{U}_f = (\mathbf{U}_{f\beta}, \alpha)$ from where it results that:

$$\mathbf{K}_f = \begin{pmatrix} \mathbf{K}_{f11} & \mathbf{K}_{f12} \\ \mathbf{K}_{f12}^T & \mathbf{K}_{22} \end{pmatrix} \quad \text{with:} \quad \begin{vmatrix} \mathbf{K}_{f11} = \int_s \mathbf{B}_{f\beta}^T \mathbf{H}_f'' \mathbf{B}_{f\beta} dS \\ \mathbf{K}_{f12} = \int_s \mathbf{B}_{f\beta}^T \mathbf{H}_f'' \mathbf{B}_{f\alpha} dS \\ \mathbf{K}_{f22} = \int_s \mathbf{B}_{f\alpha}^T \mathbf{H}_f'' \mathbf{B}_{f\alpha} dS \end{vmatrix}$$

$$K_{mf} = \left(K_{mf11} \quad K_{mf12} \right) \quad \text{with:} \quad \begin{vmatrix} K_{mf11} = \int_s B_m^T H_{mf}'' B_{f\beta} dS \\ K_{mf12} = \int_s B_m^T H_{mf}'' B_{f\alpha} dS \end{vmatrix}$$

$$K_{fm} = K_{mf}^T$$

Using the fact that $\alpha = P_\beta U_{f\beta} + P_m U_m$ one from of deduced that:

$$\delta W_{\text{int}} = \delta U_m^T K_m' U_m + \delta U_{f\beta}^T K_f' U_{f\beta} + \delta U_m^T K_{mf}' U_{f\beta} + \delta U_{f\beta}^T K_{fm}' U_m$$

where:

$$K_m' = K_m + P_m^T (K_{f22} + K_{\alpha\alpha}) P_m + (K_{mf12} + K_{m\alpha}) P_m + P_m^T (K_{mf12}^T + K_{m\alpha}^T)$$
$$K_f' = K_{f11} + K_{\beta\beta} + P_\beta^T (K_{f22} + K_{\alpha\alpha}) P_\beta + (K_{f12} + K_{\beta\alpha}) P_\beta + P_\beta^T (K_{f12}^T + K_{\beta\alpha}^T)$$
$$K_{mf}' = K_{mf11} + K_{m\beta} + (K_{mf12} + K_{m\alpha}) P_\beta + P_m^T (K_{f12}^T + K_{\beta\alpha}^T) + P_m^T (K_{f22} + K_{\alpha\alpha}) P_\beta ¿$$
$$T$$
$$K_{fm}' = K_{mf}^{¿}$$

This is still written:

$$\delta W_{\text{int}}^e = \left(\delta U_m , \delta U_{f\beta} \right) K \begin{pmatrix} U_m \\ U_{f\beta} \end{pmatrix}$$

where: $\mathbf{K}_{[5N \times 5N]} = \begin{pmatrix} \mathbf{K}'_{m[2N \times 2N]} & \mathbf{K}'_{mf[2N \times 3N]} \\ \mathbf{K}'^T_{mf[3N \times 2N]} & \mathbf{K}'_{f[3N \times 3N]} \end{pmatrix}$ is the elemental stiffness matrix for a excentré shell element DST.

4.5 Elementary mass matrix

the terms of the mass matrix are obtained after discretization of the following variational formulation:

$$\delta W_{mass}^{ac} = \int_{d-h/2}^{d+h/2} \int_S \rho \ddot{u} \, \delta u \, dz dS$$
$$= \int_S \rho_m (\ddot{u} \, \delta u + \ddot{v} \, \delta v + \ddot{w} \, \delta w) + (\rho_{mf} + d \rho_m)(\ddot{u} \, \delta \beta_x + \ddot{v} \, \delta \beta_y + \ddot{\beta}_x \, \delta u + \ddot{\beta}_y \, \delta v) dS +$$
$$\int_S (\rho_f + 2 d \rho_{mf} + d^2 \rho_m)(\ddot{\beta}_x \, \delta \beta_x + \ddot{\beta}_y \, \delta \beta_y) dS$$

¿

with $\rho_m = \int_{-h/2}^{+h/2} \rho \, dz$, $\rho_{mf} = \int_{-h/2}^{+h/2} \rho z dz$, and $\rho_f = \int_{-h/2}^{+h/2} \rho z^2 dz$.

Note:

If the plate is homogeneous or symmetric compared to its average average then $\rho_{mf}=0$ *.*

4.5.1 Elementary mass matrix classical

4.5.1.1 Q4gamma Element

the discretization of displacement for this isoparametric element is:

$$\mathbf{u}=\sum_{k=1}^{N}\mathbf{N_k}\left(\begin{pmatrix} u_k \\ v_k \\ w_k \\ \beta_{xk} \\ \beta_{yk}\end{pmatrix}\right) k=1,\dots,N$$

The mass matrix, in the base where the degrees of freedom are gathered according to the directions of translation and rotation, has then as a statement:

$$M=\begin{pmatrix} M_m & 0 & 0 & M_{mf} & 0 \\ 0 & M_m & 0 & 0 & M_{mf} \\ 0 & 0 & M_m & 0 & 0 \\ M_{mf}^T & 0 & 0 & M_f & 0 \\ 0 & M_{mf}^T & 0 & 0 & M_f \end{pmatrix}$$

with: $M_m=\int_S \rho_m N^T N\, dS$

$$M_{mf}=\int_S (\rho_{mf}+d\rho_m) N^T N\, dS$$

$$M_f=\int_S (\rho_f+2\mathrm{d}\rho_{mf}+d^2\rho_m) N^T N\, dS$$

where: $N=(N_1 \ln_k)$.

For the continuation, one poses $\rho'_{mf}=\rho_{mf}+d\rho_m$ and $\rho'_f=\rho_f+2\mathrm{d}\rho_{mf}+d^2\rho_m$.

4.5.1.2 Elements of type DKT, DST

Like:

$$\begin{pmatrix} w \\ \beta_x \\ \beta_y \end{pmatrix}=\sum_{k=1}^{N} N_k(\xi,\eta)\begin{pmatrix} w_k \\ \beta_{xk} \\ \beta_{yk}\end{pmatrix}+\sum_{k=N+1}^{2N}\begin{pmatrix} 0 \\ P_{xk}(\xi,\eta) \\ P_{yk}(\xi,\eta)\end{pmatrix}\alpha_k$$

where: $\alpha=P_m U_m+P_\beta U_{f\beta}$

one from of deduced that:

$$\begin{pmatrix} w \\ \beta_x \\ \beta_y \end{pmatrix} = \sum_{k=1}^{N} \begin{pmatrix} 0 & 0 & N_k(\xi,\eta) & 0 & 0 \\ N_{kxu}(\xi,\eta) & N_{kxv}(\xi,\eta) & N_{kxw}(\xi,\eta) & N_{kxx}(\xi,\eta) & N_{kxy}(\xi,\eta) \\ N_{kyu}(\xi,\eta) & N_{kyv}(\xi,\eta) & N_{kyw}(\xi,\eta) & N_{kyx}(\xi,\eta) & N_{kyy}(\xi,\eta) \end{pmatrix} \begin{pmatrix} u_k \\ v_k \\ w_k \\ \beta_{xk} \\ \beta_{yk} \end{pmatrix}.$$

The mass matrix has then as a statement:

$$M = \begin{pmatrix} M'_m & M'_{mf} \\ M'_{fm} & M'_f \end{pmatrix}$$

The membrane part M'_m of the elementary matrix of mass is composed of the blocks kp (k ième line and p ième column) following:

$$\rho_m \begin{pmatrix} N_k N_p & 0 \\ 0 & N_k N_p \end{pmatrix} + \rho'_{mf} \begin{pmatrix} N_k N_{pxu} + N_{kxu} N_p & N_k N_{pxv} + N_{kyu} N_p \\ N_k N_{pyu} + N_{kxv} N_p & N_k N_{pyv} + N_{kyv} N_p \end{pmatrix}$$
$$+ \rho'_f \begin{pmatrix} N_{kxu} N_{pxu} + N_{kyu} N_{pyu} & N_{kxu} N_{pxv} + N_{kyu} N_{pyv} \\ N_{pxu} N_{kxv} + N_{pyu} N_{kyv} & N_{kxv} N_{pxv} + N_{kyv} N_{pyv} \end{pmatrix}$$

The bending part M'_f is composed of the blocks kp (k ième line and p ième column) following:

$$\rho_m \begin{pmatrix} N_k N_p & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \rho'_f \begin{pmatrix} N_{kxw} N_{pxw} + N_{kyw} N_{pyw} & N_{kxw} N_{pxx} + N_{kyw} N_{pyx} & N_{kxw} N_{pxy} + N_{kyw} N_{pyy} \\ N_{kxx} N_{pxw} + N_{kyx} N_{pyw} & N_{kxx} N_{pxx} + N_{kyx} N_{pyx} & N_{kxx} N_{pxy} + N_{kyx} N_{pyy} \\ N_{kxy} N_{pxw} + N_{kyy} N_{pyw} & N_{kxy} N_{pxx} + N_{kyy} N_{pyx} & N_{kxy} N_{pxy} + N_{kyy} N_{pyy} \end{pmatrix}$$

The coupling part between the membrane and bending M'_{mf} is composed of the blocks kp (k ième line and p ième column) following:

$$\rho'_{mf} \begin{pmatrix} N_k N_{pxw} & N_k N_{pxx} & N_k N_{pxy} \\ N_k N_{pyw} & N_k N_{pyx} & N_k N_{pyy} \end{pmatrix} +$$
$$\rho'_f \begin{pmatrix} N_{kxu} N_{pxw} + N_{kyu} N_{pyw} & N_{kxu} N_{pxx} + N_{kyu} N_{pyx} & N_{kxu} N_{pxy} + N_{kyu} N_{pyy} \\ N_{kxv} N_{pxw} + N_{kyv} N_{pyw} & N_{kxv} N_{pxx} + N_{kyv} N_{pyx} & N_{kxv} N_{pxy} + N_{kyv} N_{pyy} \end{pmatrix}$$

The coupling part between bending and the membrane M'_{fm} is composed of the blocks kp (k ième line and p ième column) following:

$$\rho'_{mf} \begin{pmatrix} N_{kxw} N_p & N_{kyw} N_p \\ N_{kxx} N_p & N_{kyx} N_p \\ N_{kxy} N_p & N_{kyy} N_p \end{pmatrix} + \rho'_f \begin{pmatrix} N_{kxw} N_{pxu} + N_{kyw} N_{pyu} & N_{kxw} N_{pxv} + N_{kyw} N_{pyv} \\ N_{kxx} N_{pxu} + N_{kyx} N_{pyu} & N_{kxx} N_{pxv} + N_{kyx} N_{pyv} \\ N_{kxy} N_{pxu} + N_{kyy} N_{pyu} & N_{kxy} N_{pxv} + N_{kyy} N_{pyv} \end{pmatrix}$$

4.5.2 Improved elementary mass matrix

As the deflection of a flexbeam only can be represented by a linear approximation with difficulty, one can enrich the shape functions for the terms by bending. This approach is used in *Code_Aster* for the elements of type DKT, DST and Q4G where the shape functions used in the computation of the mass matrix of bending are of order 3. The interpolation for w is written as follows:

$$w=\sum_{k=1}^{N} N_{3(k-1)+1}(\xi,\eta)w_k + N_{3(k-1)+2}(\xi,\eta)w_{,\xi k} + N_{3(k-1)+3}(\xi,\eta)w_{,\eta k}$$

4.5.2.1 Elements of type DKT

It is known that in the approximation of one Coils-Kirchhoff has $\beta_x=-w_{,x}$ and $\beta_y=-w_{,y}$ in any point of the element.
Because of discretization stated above one a:

$$w=\sum_{k=1}^{N} N_{3(k-1)+1}(\xi,\eta)w_k+(J_{11}N_{3(k-1)+2}(\xi,\eta)+J_{21}N_{3(k-1)+3}(\xi,\eta))w_{,xk}+ (J_{12}N_{3(k-1)+2}(\xi,\eta)+J_{22}N_{3(k-1)+3}(\xi,\eta))w_{,yk}$$

since:

$$\begin{pmatrix} w_{,\xi k} \\ w_{,\eta k} \end{pmatrix}=\begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}\begin{pmatrix} w_{,xk} \\ w_{,yk} \end{pmatrix}$$

This is still written:

$$\begin{aligned} w &= \sum_{k=1}^{N} N'_{3(k-1)+1}(\xi,\eta)w_k + N'_{3(k-1)+2}(\xi,\eta)\beta_{xk} + N'_{3(k-1)+3}(\xi,\eta)\beta_{yk} \\ &= \sum_{k=1}^{N} N_{kww}(\xi,\eta)w_k + N_{kwx}(\xi,\eta)\beta_{xk} + N_{kwy}(\xi,\eta)\beta_{yk} \end{aligned}$$

where:

$$\begin{aligned} N'_{3(k-1)+1}(\xi,\eta) &= N_{3(k-1)+1}(\xi,\eta) \\ N'_{3(k-1)+2}(\xi,\eta) &= -J_{11}N_{3(k-1)+2}(\xi,\eta) - J_{21}N_{3(k-1)+3}(\xi,\eta)\ . \\ N'_{3(k-1)+3}(\xi,\eta) &= -J_{12}N_{3(k-1)+2}(\xi,\eta) - J_{22}N_{3(k-1)+3}(\xi,\eta) \end{aligned}$$

As follows:

$$\begin{pmatrix} w \\ \beta_x \\ \beta_y \end{pmatrix}=\sum_{k=1}^{N}\begin{pmatrix} 0 & 0 & N_{kww}(\xi,\eta) & N_{kwx}(\xi,\eta) & N_{kwy}(\xi,\eta) \\ N_{kxu}(\xi,\eta) & N_{kxv}(\xi,\eta) & N_{kxw}(\xi,\eta) & N_{kxx}(\xi,\eta) & N_{kxy}(\xi,\eta) \\ N_{kyu}(\xi,\eta) & N_{kyv}(\xi,\eta) & N_{kyw}(\xi,\eta) & N_{kyx}(\xi,\eta) & N_{kyy}(\xi,\eta) \end{pmatrix}\begin{pmatrix} u_k \\ v_k \\ w_k \\ \beta_{xk} \\ \beta_{yk} \end{pmatrix}$$

By not taking account of the effects of inertia, the mass matrix has the following form thus:

$$\mathbf{M}=\begin{pmatrix} \mathbf{M}'_m & \mathbf{M}'_{mf} \\ \mathbf{M}'_{fm} & \mathbf{M}'_f \end{pmatrix}$$

The membrane part M'_m of the elementary matrix of mass is composed of the blocks kp (k ième line and p ième column) following:

$$\rho_m \begin{pmatrix} N_k N_p & 0 \\ 0 & N_k N_p \end{pmatrix} + \rho'_{mf} \begin{pmatrix} N_k N_{pxu} + N_{kxu} N_p & N_k N_{pxv} + N_{kyu} N_p \\ N_k N_{pyu} + N_{kxv} N_p & N_k N_{pyv} + N_{kyv} N_p \end{pmatrix}$$
$$+ \rho'_f \begin{pmatrix} N_{kxu} N_{pxu} + N_{kyu} N_{pyu} & N_{kxu} N_{pxv} + N_{kyu} N_{pyv} \\ N_{pxu} N_{kxv} + N_{pyu} N_{kyv} & N_{kxv} N_{pxv} + N_{kyv} N_{pyv} \end{pmatrix}$$

The membrane-bending part M'_{mf} is composed of the blocks kp (k ième line and p ième column) following:

$$\rho'_{mf} \begin{pmatrix} N_k N_{pxw} & N_k N_{pxx} & N_k N_{pxy} \\ N_k N_{pyw} & N_k N_{pyx} & N_k N_{pyy} \end{pmatrix}$$
$$+ \rho'_f \begin{pmatrix} N_{kxu} N_{pxw} + N_{kyu} N_{pyw} & N_{kxu} N_{pxx} + N_{kyu} N_{pyx} & N_{kxu} N_{pxy} + N_{kyu} N_{pyy} \\ N_{kxv} N_{pxw} + N_{kyv} N_{pyw} & N_{kxv} N_{pxx} + N_{kyv} N_{pyx} & N_{kxv} N_{pxy} + N_{kyv} N_{pyy} \end{pmatrix}$$

The bending-membrane part M'_{fm} is composed of the blocks kp (k ième line and p ième column) following:

$$\rho'_{mf} \begin{pmatrix} N_{kxw} N_p & N_{kyw} N_p \\ N_{kxx} N_p & N_{kyx} N_p \\ N_{kxy} N_p & N_{kyy} N_p \end{pmatrix} + \rho'_f \begin{pmatrix} N_{kxw} N_{pxu} + N_{kyw} N_{pyu} & N_{kxw} N_{pxv} + N_{kyw} N_{pyv} \\ N_{kxx} N_{pxu} + N_{kyx} N_{pyu} & N_{kxx} N_{pxv} + N_{kyx} N_{pyv} \\ N_{kxy} N_{pxu} + N_{kyy} N_{pyu} & N_{kxy} N_{pxv} + N_{kyy} N_{pyv} \end{pmatrix}$$

The term M'_f of bending is composed of the blocks kp (k ième line and p ième column) following:

$$\rho_m \begin{pmatrix} N_{kww} N_{pww} & N_{kww} N_{pwx} & N_{kww} N_{pwy} \\ N_{kwx} N_{pww} & N_{kwx} N_{pwx} & N_{kwx} N_{pwy} \\ N_{kwy} N_{pww} & N_{kwy} N_{pwx} & N_{kwy} N_{pwy} \end{pmatrix} +$$
$$\rho'_f \begin{pmatrix} N_{kxw} N_{pxw} + N_{kyw} N_{pyw} & N_{kxw} N_{pxx} + N_{kyw} N_{pyx} & N_{kxw} N_{pxy} + N_{kyw} N_{pyy} \\ N_{kxx} N_{pxw} + N_{kyx} N_{pyw} & N_{kxx} N_{pxx} + N_{kyx} N_{pyx} & N_{kxx} N_{pxy} + N_{kyx} N_{pyy} \\ N_{kxy} N_{pxw} + N_{kyy} N_{pyw} & N_{kxy} N_{pxx} + N_{kyy} N_{pyx} & N_{kxy} N_{pxy} + N_{kyy} N_{pyy} \end{pmatrix}$$

4.5.2.2 Elements of type `DST`

It is known that for these elements one has $\beta_x = \gamma_x - w_{,x}$ and $\beta_y = \gamma_y - w_{,y}$ where the distortion γ is constant on the element.

Like:

$$w = \sum_{k=1}^{N} N_{3(k-1)+1}(\xi,\eta) w_k + (J_{11} N_{3(k-1)+2}(\xi,\eta) + J_{21} N_{3(k-1)+3}(\xi,\eta)) w_{,xk} +$$
$$(J_{12} N_{3(k-1)+2}(\xi,\eta) + J_{22} N_{3(k-1)+3}(\xi,\eta)) w_{,yk}$$

one can also write:

$$w = \sum_{k=1}^{N} N'_{3(k-1)+1}(\xi,\eta) w_k + N'_{3(k-1)+2}(\xi,\eta)\beta_{xk} + N'_{3(k-1)+3}(\xi,\eta)\beta_{yk}$$
$$+(J_{11}\bar{\gamma}_x + J_{12}\bar{\gamma}_y)\Sigma N_{3(k-1)+2}(\xi,\eta) + (J_{21}\bar{\gamma}_x + J_{22}\bar{\gamma}_y)\Sigma N_{3(k-1)+3}(\xi,\eta)$$

where:

$$\left| \begin{array}{l} N'_{3(k-1)+1}(\xi,\eta) = N_{3(k-1)+1}(\xi,\eta) \\ N'_{3(k-1)+2}(\xi,\eta) = -J_{11} N_{3(k-1)+2}(\xi,\eta) - J_{21} N_{3(k-1)+3}(\xi,\eta) \\ N'_{3(k-1)+3}(\xi,\eta) = -J_{12} N_{3(k-1)+2}(\xi,\eta) - J_{22} N_{3(k-1)+3}(\xi,\eta) \end{array} \right.$$

$$\left| \begin{array}{l} \Sigma N_{3(k-1)+1}(\xi,\eta) = \sum_{k=1}^{N} N_{3(k-1)+1}(\xi,\eta) \\ \Sigma N_{3(k-1)+2}(\xi,\eta) = \sum_{k=1}^{N} N_{3(k-1)+2}(\xi,\eta) \\ \Sigma N_{3(k-1)+3}(\xi,\eta) = \sum_{k=1}^{N} N_{3(k-1)+3}(\xi,\eta) \end{array} \right.$$

$$\begin{pmatrix} \bar{\gamma}_x \\ \bar{\gamma}_y \end{pmatrix} = \mathbf{H}_{ct}^{-1} [(\mathbf{B}_{cm} + \mathbf{B}_{c\alpha}\mathbf{P}_m) \begin{pmatrix} u_1 \\ v_1 \\ \vdots \\ u_N \\ v_N \end{pmatrix} + (\mathbf{B}_{c\beta} + \mathbf{B}_{c\alpha}\mathbf{P}_\beta) \begin{pmatrix} w_1 \\ \beta_{x1} \\ \beta_{y1} \\ \vdots \\ w_N \\ \beta_{xN} \\ \beta_{yN} \end{pmatrix}] = \mathbf{T}_{\gamma u} \begin{pmatrix} u_1 \\ v_1 \\ \vdots \\ u_N \\ v_N \end{pmatrix} + \mathbf{T}_{\gamma w} \begin{pmatrix} w_1 \\ \beta_{x1} \\ \beta_{y1} \\ \vdots \\ w_N \\ \beta_{xN} \\ \beta_{yN} \end{pmatrix}$$

One obtains the interpolation then for w :

$$w = \sum_{k=1}^{N} N''_{5(k-1)+1}(\xi,\eta) u_k + N''_{5(k-1)+2}(\xi,\eta) v_k +$$
$$\sum_{k=1}^{N} N''_{5(k-1)+3}(\xi,\eta) w_k + N''_{5(k-1)+4}(\xi,\eta)\beta_{xk} + N''_{5(k-1)+5}(\xi,\eta)\beta_{yk}$$

where:

$$
\begin{aligned}
N''_{5(k-1)+1}(\xi,\eta) &= (J_{11}T_{\gamma u}(1,2(k-1)+1)+J_{12}T_{\gamma u}(2,2(k-1)+1))\Sigma N_{3(j-1)+2}(\xi,\eta) \\
&+ (J_{21}T_{\gamma u}(1,2(k-1)+1)+J_{22}T_{\gamma u}(2,2(k-1)+1))\Sigma N_{3(j-1)+3}(\xi,\eta) \\
N''_{5(k-1)+2}(\xi,\eta) &= (J_{11}T_{\gamma u}(1,2(k-1)+2)+J_{12}T_{\gamma u}(2,2(k-1)+2))\Sigma N_{3(j-1)+2}(\xi,\eta) \\
&+ (J_{21}T_{\gamma u}(1,2(k-1)+2)+J_{22}T_{\gamma u}(2,2(k-1)+2))\Sigma N_{3(j-1)+3}(\xi,\eta) \\
N''_{5(k-1)+3}(\xi,\eta) &= N'_{3(k-1)+1}(\xi,\eta) \\
&+ (J_{11}T_{\gamma w}(1,3(k-1)+1)+J_{12}T_{\gamma w}(2,3(k-1)+1))\Sigma N_{3(j-1)+2}(\xi,\eta) \\
&+ (J_{21}T_{\gamma w}(1,3(k-1)+1)+J_{22}T_{\gamma w}(2,3(k-1)+1))\Sigma N_{3(j-1)+3}(\xi,\eta) \\
N''_{5(k-1)+4}(\xi,\eta) &= N'_{3(k-1)+2}(\xi,\eta) \\
&+ (J_{11}T_{\gamma w}(1,3(k-1)+2)+J_{12}T_{\gamma w}(2,3(k-1)+2))\Sigma N_{3(j-1)+2}(\xi,\eta) \\
&+ (J_{21}T_{\gamma w}(1,3(k-1)+2)+J_{22}T_{\gamma w}(2,3(k-1)+2))\Sigma N_{3(j-1)+3}(\xi,\eta) \\
N''_{5(k-1)+5}(\xi,\eta) &= N'_{3(k-1)+3}(\xi,\eta) \\
&+ (J_{11}T_{\gamma w}(1,3(k-1)+3)+J_{12}T_{\gamma w}(2,3(k-1)+3))\Sigma N_{3(j-1)+2}(\xi,\eta) \\
&+ (J_{21}T_{\gamma w}(1,3(k-1)+3)+J_{22}T_{\gamma w}(2,3(k-1)+3))\Sigma N_{3(j-1)+3}(\xi,\eta)
\end{aligned}
$$

This can be still written in the following way:

$$
\begin{pmatrix} w \\ \beta_x \\ \beta_y \end{pmatrix} = \sum_{k=1}^{N} \begin{pmatrix} N_{kwu}(\xi,\eta) & N_{kwv}(\xi,\eta) & N_{kww}(\xi,\eta) & N_{kwx}(\xi,\eta) & N_{kwy}(\xi,\eta) \\ N_{kxu}(\xi,\eta) & N_{kxv}(\xi,\eta) & N_{kxw}(\xi,\eta) & N_{kxx}(\xi,\eta) & N_{kxy}(\xi,\eta) \\ N_{kyu}(\xi,\eta) & N_{kyv}(\xi,\eta) & N_{kyw}(\xi,\eta) & N_{kyx}(\xi,\eta) & N_{kyy}(\xi,\eta) \end{pmatrix} \begin{pmatrix} u_k \\ v_k \\ w_k \\ \beta_{xk} \\ \beta_{yk} \end{pmatrix}
$$

The mass matrix has the following form thus:

$$
M = \begin{pmatrix} M'_m & M'_{mf} \\ M'_{fm} & M'_f \end{pmatrix}
$$

The membrane part M'_m of the elementary matrix of mass is composed of the blocks kp (k ième line and p ième column) following:

$$
\rho_m \begin{pmatrix} N_k N_p + N_{kwu} N_{pwu} & N_{kwu} N_{pwv} \\ N_{kwv} N_{pwu} & N_k N_p + N_{kwv} N_{pwv} \end{pmatrix} + \rho'_{mf} \begin{pmatrix} N_k N_{pxu} + N_{kxu} N_p & N_k N_{pxv} + N_{kyu} N_p \\ N_k N_{pyu} + N_{kxv} N_p & N_k N_{pyv} + N_{kyv} N_p \end{pmatrix}
$$
$$
+ \rho'_f \begin{pmatrix} N_{kxu} N_{pxu} + N_{kyu} N_{pyu} & N_{kxu} N_{pxv} + N_{kyu} N_{pyv} \\ N_{pxu} N_{kxv} + N_{pyu} N_{kyv} & N_{kxv} N_{pxv} + N_{kyv} N_{pyv} \end{pmatrix}
$$

The membrane-bending part M'_{mf} is composed of the blocks kp (k ième line and p ième column) following:

$$\rho_m \begin{pmatrix} N_{kwu} N_{pww} & N_{kwu} N_{pwx} & N_{kwu} N_{pwy} \\ N_{kwv} N_{pww} & N_{kwv} N_{pwx} & N_{kwv} N_{pwy} \end{pmatrix} + \rho'_{mf} \begin{pmatrix} N_k N_{pxw} & N_k N_{pxx} & N_k N_{pxy} \\ N_k N_{pyw} & N_k N_{pyx} & N_k N_{pyy} \end{pmatrix}$$
$$+\rho'_f \begin{pmatrix} N_{kxu} N_{pxw} + N_{kyu} N_{pyw} & N_{kxu} N_{pxx} + N_{kyu} N_{pyx} & N_{kxu} N_{pxy} + N_{kyu} N_{pyy} \\ N_{kxv} N_{pxw} + N_{kyv} N_{pyw} & N_{kxv} N_{pxx} + N_{kyv} N_{pyx} & N_{kxv} N_{pxy} + N_{kyv} N_{pyy} \end{pmatrix}$$

The bending-membrane part M'_{fm} is composed of the blocks kp (k ième line and p ième column) following:

$$\rho_m \begin{pmatrix} N_{kww} N_{pwu} & N_{kww} N_{pwv} \\ N_{kwx} N_{pwu} & N_{kwx} N_{pwv} \\ N_{kwy} N_{pwu} & N_{kwy} N_{pwv} \end{pmatrix} + \rho'_{mf} \begin{pmatrix} N_{kxw} N_p & N_{kyw} N_p \\ N_{kxx} N_p & N_{kyx} N_p \\ N_{kxy} N_p & N_{kyy} N_p \end{pmatrix}$$
$$+\rho'_f \begin{pmatrix} N_{kxw} N_{pxu} + N_{kyw} N_{pyu} & N_{kxw} N_{pxv} + N_{kyw} N_{pyv} \\ N_{kxx} N_{pxu} + N_{kyx} N_{pyu} & N_{kxx} N_{pxv} + N_{kyx} N_{pyv} \\ N_{kxy} N_{pxu} + N_{kyy} N_{pyu} & N_{kxy} N_{pxv} + N_{kyy} N_{pyv} \end{pmatrix}$$

The term M'_f of bending is composed of the blocks kp (k ième line and p ième column) following:

$$\rho_m \begin{pmatrix} N_{kww} N_{pww} & N_{kww} N_{pwx} & N_{kww} N_{pwy} \\ N_{kwx} N_{pww} & N_{kwx} N_{pwx} & N_{kwx} N_{pwy} \\ N_{kwy} N_{pww} & N_{kwy} N_{pwx} & N_{kwy} N_{pwy} \end{pmatrix} +$$
$$\rho'_f \begin{pmatrix} N_{kxw} N_{pxw} + N_{kyw} N_{pyw} & N_{kxw} N_{pxx} + N_{kyw} N_{pyx} & N_{kxw} N_{pxy} + N_{kyw} N_{pyy} \\ N_{kxx} N_{pxw} + N_{kyx} N_{pyw} & N_{kxx} N_{pxx} + N_{kyx} N_{pyx} & N_{kxx} N_{pxy} + N_{kyx} N_{pyy} \\ N_{kxy} N_{pxw} + N_{kyy} N_{pyw} & N_{kxy} N_{pxx} + N_{kyy} N_{pyx} & N_{kxy} N_{pxy} + N_{kyy} N_{pyy} \end{pmatrix}$$

4.5.2.3 Elements of the type Q4 Γ

One proceeds in the same way that for the elements of type DST but with:

$$\begin{pmatrix} \bar{\gamma}_x \\ \bar{\gamma}_y \end{pmatrix} = B_c \begin{pmatrix} w_1 \\ \beta_{x1} \\ \beta_{y1} \\ \vdots \\ w_N \\ \beta_{xN} \\ \beta_{yN} \end{pmatrix}$$

where: B_c is the matrix established with [§4.3.1].

One from of deduced that:

$$\begin{pmatrix} w \\ \beta_x \\ \beta_y \end{pmatrix} = \sum_{k=1}^{N} \begin{pmatrix} 0 & 0 & N_{kww}(\xi,\eta) & N_{kwx}(\xi,\eta) & N_{kwy}(\xi,\eta) \\ 0 & 0 & 0 & N_k(\xi,\eta) & 0 \\ 0 & 0 & 0 & 0 & N_k(\xi,\eta) \end{pmatrix} \begin{pmatrix} u_k \\ v_k \\ w_k \\ \beta_{xk} \\ \beta_{yk} \end{pmatrix}$$

the mass matrix has the following form thus:

$$M = \begin{pmatrix} M'_m & 0 \\ 0 & M'_f \end{pmatrix}$$

The membrane part M'_m of the elementary matrix of mass is composed of the blocks kp (k ième line and p ième column) following:

$$\rho_m \begin{pmatrix} N_k N_p & 0 \\ 0 & N_k N_p \end{pmatrix}$$

The term M'_f of bending is composed of the blocks kp (k ième line and p ième column) following:

$$\rho_m \begin{pmatrix} N_{kww} N_{pww} & N_{kww} N_{pwx} & N_{kww} N_{pwy} \\ N_{kwx} N_{pww} & N_{kwx} N_{pwx} & N_{kwx} N_{pwy} \\ N_{kwy} N_{pww} & N_{kwy} N_{pwx} & N_{kwy} N_{pwy} \end{pmatrix} + \rho'_f \begin{pmatrix} 0 & 0 & 0 \\ 0 & N_k N_p & 0 \\ 0 & 0 & N_k N_p \end{pmatrix}$$

4.5.2.4 Notice

One neglects in the form of the elementary mass matrix without eccentring the terms of inertia of rotationformule $\int_S \rho_f (\ddot{\beta}_x \delta \beta_x + \ddot{\beta}_y \delta \beta_y) dS$ because the latter are negligible compared to the different one. Indeed a multiplicative factor of $h^2/12$ the dregs to the other terms and they become negligible for a thickness ratio over characteristic length lower than $1/10$. When the eccentring is introduced, these terms of the form $\int_S (\rho_f + 2\mathrm{d}\, \rho_{mf} + d^2 \rho_m)(\ddot{\beta}_x \delta \beta_x + \ddot{\beta}_y \delta \beta_y) dS$ are not negligible any more and are introduced into the form of the mass matrix.

5 Implemented and postprocessings

the eccentring is introduced by the key word optional EXCENTREMENT on the level of AFFE_CARA_ELEM in the same way as the thickness according to the methods defined in introduction. When this key word is not present the eccentring is worth zero by default.

5.1 Couple and load application

All the calculations are done in the reference of diagram (plane of the mesh). If one defines forces or couples compared to another reference, the user will have to make for FORCE_ARETE and FORCE_NODALE the transformations necessary to be reduced to the reference mesh. For FORCE_COQUE the user will be able to specify the plane of load application and conversion towards the reference of computation will be automatic.

One thus introduces into AFFE_CHAR_MECA the notion of plane of load application by the key word PLANE under FORCE_COQUE. This plane of application is different from the datum-line or plane from diagram on which the mesh is pressed. For this key word one will define the four following possibilities of application of the forces: "INF" "MOY" "SUP" "MAIL". "INF" "MOY" and "SUP" mean that one applies the forces in lower skin, average and higher of plate respectively. "MAIL" means that one applies the forces to the level of the datum-line or plane of the mesh. By defaults the forces will be applied as regards mesh of the plate. The forces of the type FORCE_COQUE of the TE0032 are concerned.

In local coordinate system with the element, when the forces and the couples are brackets on "MOY" one uses the simple relation of transition:

$$c_x' = c_x - d f_y$$
$$c_y' = c_y + d f_x$$

to bring back the forces and the couples in the reference of the mesh where the calculations are done.

In local coordinate system with the element, when the forces and the couples are applied to "SUP" one uses the simple relation of transition:

$$c_x' = c_x - (d + h/2) f_y$$
$$c_y' = c_y + (d + h/2) f_x$$

In local coordinate system with the element, when the forces and the couples are applied to "INF" one uses the simple relation of transition:

$$c_x' = c_x - (d - h/2) f_y$$
$$c_y' = c_y + (d - h/2) f_x$$

If the forces are given in the total reference of the element, one uses relations of transition of the type: $c' = c + (d + \varepsilon h/2) n \wedge f$ where c is defined compared to reference "INF" "MOY" "SUP" with ε equal to -1,0 and 1, respectively. When there is no eccentring, the preceding formula is reduced to $c' = c + \varepsilon h/2 \mathrm{n} \wedge f$.

Note:

For the loadings of the type FORCE_ARETE or FORCE_NODALE the forces and couples can be expressed only compared to the reference of the mesh. If the user knows them only compared to the average average of the plate, it will have to carry out the change of reference to the hand to have the statement of the forces and the couples compared to the surface of mesh. The relation to be used is $c' = c + dn \wedge f$ where d is the distance between the plane of computation and the loading plan directed by the norm with the shell. It is obvious that the user has interest so that the loading plan is the plane of the mesh, but it is not always possible to make coincide these two planes as one can see it on the left part of the figure of page 6.

5.2 Application of the boundary conditions in displacement

For the boundary conditions of type displacement the user will have to pay attention to the fact that they can apply only to the reference of mesh. The relations of transition compared to conditions given on the average average are the following ones:

$$\theta_{ref}=\theta_{moy}$$
$$u_{ref}=u_{moy}-\theta_{moy}\wedge dn$$

5.3 Postprocessings

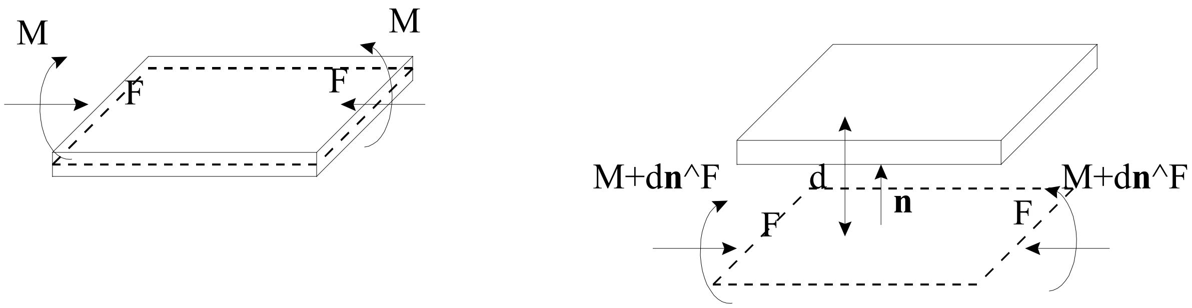
For postprocessings, the by default results of generalized the forces type are given in the reference corresponding to the plane of diagram. To have them in the other references, it will be necessary that the user indicates the plane of postprocessing and the changes of references will be automatic.

For the postprocessing of the generalized forces, one will be able to calculate them in the average average of the shell via command `POST_CHAMP/COQUE_EXCENT`.

6 Static validation and modal

6.1 initial Validation

the first part of the validation consists in testing a simple plate subjected to forces and couples and whose plane of mesh does not coincide with the plane of the average average on which the forces are applied. For the plate subjected to forces and couples, the results with and without eccentring must take account of the change of reference for the couples as indicated below.



Displacements are in the following way dependant for a point located at a height Z compared to the average average:

$$u = u_{moy} + \theta_{moy} \wedge zn = u_{ref} + \theta_{ref} \wedge (z + d) n$$

what is still written:

$$\theta_{moy} = \theta_{ref}$$
$$u_{moy} = u_{ref} + \theta_{ref} \wedge dn$$

what enables us to establish the relations of transition between displacements compared to the average average and those compared to the datum-line.

For the generalized forces, in the two preceding cases, there are the same results on the averages layer, inferior and superior of plate.

6.2 Benchmark SSLS111: eccentring for simple plates

It acts of a computation in bending of double-layered made up by two different isotropic materials. Membrane-flexure coupling is studied. The computation of reference is that of double-layered defined by `DEFI_COMPOSITE` made up of the two different isotropic materials (not symmetry according to z). The other modelization is made up of two offset plates compared to the average fiber of the plate used with `DEFI_COMPOSITE` . The results, identical of one modelization to the other, are given in term of displacements and generalized forces. Moreover one carries out on the geometry of this test a modal analysis for the two modelizations: the found eigenfrequencies are identical.

6.3 Benchmark SSLS112: eccentring for composite plates

It acts of a computation in bending of a quadricouche having a material NON-symmetry compared to its average plane. The computation of reference uses a definite quadricouches by `DEFI_COMPOSITE.` The other modelization uses two double-layered definite by `DEFI_COMPOSITE` but offset compared to the average fiber of the quadricouche. The results, identical of one modelization to the other, are given in term of displacements.

7 Conclusion

the finite elements of plate which we describe here are used in slender thin structure computations whose thickness ratio over characteristic length is lower than $1/10$. The average average of these structures does not coincide with the plane of the mesh (plane of diagram). The eccentring thus corresponds to the distance from the average average compared to the average of diagram. A positive d eccentring means that the mean surface of the plate is at a distance $d\,\boldsymbol{n}$ from the shell element with a grid, the direction $\boldsymbol{n}$ being given by the norm to the element.
The values of displacements and generalized forces obtained are given by default in the reference of the mesh. For the generalized forces, one can however define a reference of postprocessing - reference associated with the average average - different from the reference of diagram. Same way, the forces applied are regarded as being given by default in the reference of diagram. In the case of FORCE_COQUE, one can however specify a reference of load application and couples - reference associated with the average average - different from the reference of diagram.
Equivalent elements are not available in thermal; the thermomechanical sequences are thus not available for the offset shell elements.

8 Bibliographical references

[1] J.L. BATOZ, G.DHATT: “Modelization of structures by finite elements: beams and plates”, Hermes, Paris, 1992.

[2] D. BUI: “Shears in the plates and the shells: modelization and computation”, HI - 71/7784, 1992 Notes.

[3] J.G. REN: “A new theory of laminated punt”, Composite Science and Technology, Vol.26, p.225-239,1986.

[4] T.A. ROCK'N'ROLL, E. HINTON: “A finite transverse element method for the free vibration of punts allowing for shear strain”, Computers and Structures, Vol.6, p.37-44,1976.

[5] T.J.R. HUGHES: “The finite element method”, Prentice Hall, 1987. E. HINTON

[6] , T. ROCK'N'ROLL and O.C. ZIENKIEWICZ: “A one notes Farmhouse Lumping and Related Processes in the Finite Element Method”, Earthquake Engineering and Structural Dynamics, Vol4, p. 245 - 249, 1976. F. VOLDOIRE

[7] : “Modelization by thermal and thermoelastic homogenization of thin mechanical components”, CR MMN/97/091. P. MASSIN

[8] , F. VOLDOIRE, S. ANDRIEUX: “Models of thermal for the thin shells”, Handbook of Reference of the Code_Aster *[R3.11.01*]. F. VOLDOIRE

[9] : “Thermo-elastic Hollow roll”, Handbook of Validation of the Code_Aster *[V7.01* .100]. A.K.

[10] NOOR, W.S. BURTON: “Composite Assessment of shear strain theories for multilayered punts”, ASME, Applied Mechanics Review, Vol.42, N°1, p.1-13,1989. A.K.

[11] NOOR, W.S. BURTON, J.M. PETERS: “Composite Assessment of computational models for multilayered cylinders” in Analytical and Computational Models of Shells, Noor and al. Eds, ASME, CED - Vol.3, p.419-442,1989. Description

9 of the versions of the document Version

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P.MASSI N	EDF R & D /AMA initial	Text Factors

Annexe 1 of transverse correction of shears for orthotropic or stratified plates offset the matrix

is $H_{\mathbf{ct}}$ defined so that the surface density of transverse energy of shears obtained in the case of the three-dimensional distribution of the stresses resulting from the resolution of the equilibrium is equal to that of the model of plate based on the assumptions of Reissner, for a behavior in pure bending. One must thus find such as $H_{\mathbf{ct}}$: with

$$\frac{1}{2}\int_{-h/2}^{+h/2}\tau\, H_g^{-1}\tau=\frac{1}{2}\mathbf{T}\mathbf{H}_{\mathbf{ct}}^{-1}T=\frac{1}{2}\gamma\, H_{\mathbf{ct}}\gamma \text{ and } \tau=\begin{pmatrix}\sigma_{xz}\\ \sigma_{yz}\end{pmatrix} \text{ formulates } T=\int_{-h/2}^{+h/2}\tau\, dz=H_{\mathbf{ct}}\gamma$$

To obtain one uses $H_{\mathbf{ct}}$ the distribution of following τ 3D z obtained from the resolution of the balance equations without external couples: formuleavec

$$\sigma_{xz}=-\int_{-h/2}^{z}(\sigma_{xx,x}+\sigma_{xy,y})d\zeta\,;s_{yz}=-\int_{-h/2}^{z}(\sigma_{xy,x}+\sigma_{yy,y})d\zeta \text{ for } \sigma_{xz}=\sigma_{yz}=0 \text{ . If } z=\pm h/2$$

there is no coupling membrane bending (symmetry compared to), $z=0$ the stresses in the plane of the element in the case of $\sigma_{xx},\sigma_{yy},\sigma_{xy}$ have as a statement a behavior of pure bending: with

$$\sigma=zA(z)M \text{ . If } A(z)=H(z)H_f^{-1}$$

and $H(z)$ do not depend H_f on and x one can y determine. Indeed $H_{\mathbf{ct}}$: where

$$\tau(z)=D_1(z)T+D_2(z)\lambda \text{ and } T=\begin{pmatrix}T_x\\ T_y\end{pmatrix}=\begin{pmatrix}M_{xx,x}+M_{xy,y}\\ M_{xy,x}+M_{yy,y}\end{pmatrix} \text{ like } \lambda=\begin{pmatrix}M_{xx,x}-M_{xy,y}\\ M_{xy,x}-M_{yy,y}\\ M_{yy,x}\\ M_{xx,y}\end{pmatrix}$$

: .

$$\mathbf{D}_1=-\int_{-h/2}^{z}\frac{z}{2}\begin{pmatrix}A_{11}+A_{33} & A_{13}+A_{32}\\ A_{31}+A_{23} & A_{22}+A_{33}\end{pmatrix}dz \text{ It}$$

$$\mathbf{D}_2=-\int_{-h/2}^{z}\frac{z}{2}\begin{pmatrix}A_{11}-A_{33} & A_{13}-A_{32} & 2\mathrm{A}_{12} & 2\mathrm{A}_{31}\\ A_{31}-A_{23} & A_{33}-A_{22} & 2\mathrm{A}_{32} & 2\mathrm{A}_{21}\end{pmatrix}dz \text{ results}$$

from it that with $\frac{1}{2}\int_{-h/2}^{+h/2}t\,\mathbf{H}_g^{-1}t=\frac{1}{2}\begin{pmatrix}\mathbf{T}\\ \lambda\end{pmatrix}\begin{pmatrix}\mathbf{C_{11}} & \mathbf{C_{12}}\\ \mathbf{C_{12}^{\mathit{T}}} & \mathbf{C_{22}}\end{pmatrix}\begin{pmatrix}\mathbf{T}\\ \lambda\end{pmatrix}$: As

$$\mathbf{C_{11}}=\int_{-h/2}^{+h/2}\mathbf{D}_1^T\mathbf{H}_g^{-1}\mathbf{D}_1 dz\,;$$

$$\mathbf{C_{12}}=\int_{-h/2}^{+h/2}\mathbf{D}_1^T\mathbf{H}_g^{-1}\mathbf{D}_2 dz\,;$$

$$\mathbf{C_{22}}=\int_{-h/2}^{+h/2}\mathbf{D}_2^T\mathbf{H}_g^{-1}\mathbf{D}_2 dz$$

in addition one proposes $\frac{1}{2}\int_{-h/2}^{+h/2} t\,H_g^{-1}\,t=\frac{1}{2}\mathbf{T}\mathbf{H}_{\mathbf{ct}}^{\mathbf{-1}}T$ to take $H_{\mathbf{ct}}=C_{\mathbf{11}}^{-1}$ to satisfy as well as possible the two equations whatever and T . While λ comparing

thus $H_{\mathbf{ct}}$ calculated with one reveals $\bar{H}_{\mathbf{ct}}=\int_{-h/2}^{+h/2} H_g\,dz$ the coefficients of correction of following transverse shears: . For $k_1=H_{ct}^{11}/\bar{H}_{ct}^{11};k_{12}=H_{ct}^{12}/\bar{H}_{ct}^{12};k_2=H_{ct}^{22}/\bar{H}_{ct}^{22}$

a homogeneous, isotropic or anisotropic plate, one finds as follows: with $\mathbf{H}_{\mathbf{ct}}=kh\,\mathbf{H}_g$. Note: $k=5/6$

This

method is valid only when the composite plate is symmetric compared to z=0. For

- a multi-layer material, one establishes that: where

$$\mathbf{C_{11}}=\sum_{i=1}^{N}\frac{h_i}{4}(\sum_{p=1}^{i-1}h_p h_p\mathbf{A}_p^T-\frac{1}{2}z_i^2\mathbf{A}_i^T)\mathbf{H}_g^{-1}(\sum_{p=1}^{i-1}h_p h_p\mathbf{A}_p-\frac{1}{2}z_i^2\mathbf{A}_i)+$$
$$\frac{1}{24}(z_{i+1}^3-z_i^3)[\mathbf{A}_i^T\mathbf{H}_g^{-1}(\sum_{p=1}^{i-1}h_p h_p\mathbf{A}_p-\frac{1}{2}z_i^2\mathbf{A}_i)+(\sum_{p=1}^{i-1}h_p h_p\mathbf{A}_p^T-\frac{1}{2}z_i^2\mathbf{A}_i^T)\mathbf{H}_g^{-1}\mathbf{A}_i]$$
$$+\frac{1}{80}(z_{i+1}^5-z_i^5)\mathbf{A}_i^T\mathbf{H}_g^{-1}\mathbf{A}_i$$

: and $h_i=z_{i+1}-z_i, h_i=\frac{1}{2}(z_{i+1}+z_i)$ $\mathbf{A}_i$ the matrix for $\begin{pmatrix}A_{11}+A_{33} & A_{13}+A_{32}\\ A_{31}+A_{23} & A_{22}+A_{33}\end{pmatrix}$ layer I represents. The validity

- of the choice can $\mathbf{H}_{\mathbf{ct}}=\mathbf{C}_{\mathbf{11}}^{-1}$ be examined a posteriori when one has an estimate of the solution (fields of displacements and plane stresses, in particular). One can then estimate the difference between the two estimates on energy. A approach of computation in two stages for the multi-layer plates and shells (with diagonal $\mathbf{H}_{\mathbf{ct}}$ and two coefficients and k_1) k_2 was developed besides by Noor and Burton [bib10] [bib11]. In the case of
- an isotropic or anisotropic homogeneous plate the equality between two energies is satisfied in a strict sense since. $D_2=0$ The choice makes above is then valid and no examination a posteriori is necessary.