Shell elements voluminal SHB with 6,15 and 20 nodes

Summarized:

We present in this document 3 new shell elements voluminal intended to supplement the modelization SHB, which comprises already element SHB8 having for mesh support a hexahedron with 8 nodes [R3.07.07]. These 3 elements are:

- the element SHB6 which has as a mesh support a pentahedron with 6 nodes,
- the element SHB15 which has as a mesh support a pentahedron with 15 nodes,
- the element SHB20 which has as a mesh support a hexahedron with 20 nodes.

Just as the SHB8, these 3 elements have a called privileged direction thickness. Thus, they can be used to represent thin structures while correctly taking into account the phenomena through the thickness (bending, elastoplasticity), grace a numerical integration to 5 Gauss points in this privileged direction.

Like the SHB8, and in order to reduce time computation, these elements under-are integrated but, contrary to him, they do not have modes of Hourglass (modes of strain to energy null) and thus do not require a mechanism of stabilization. Nevertheless, to avoid blockings (in particular in transverse shears), the SHB6 is project following the method of the supposed strains (assumed strain).
The quadratic elements neither are stabilized, nor projected.

In addition to their cost of relatively weak computation and their good performances in elastoplasticity, these elements have another advantage. Since they are based on a three-dimensional formulation and that they have only degrees of freedom of translation, it is easy to couple them with elements 3D voluminal, which is very useful in systems where voluminal shells and elements must cohabit.

Contents

1 Introduction

element of type solid - voluminal shell of hexahedral geometry to five Gauss points was already established in ASTER. The good performances of this element, named SHB8, were put in obviousness by Abed-Meraim and Combescure [bib1], [bib2] like by Legay in [bib3]. This element represents a thick shell obtained from a purely three-dimensional formulation. It has eight nodes and five points of integration distributed according to the direction of the thickness. The three-dimensional constitutive law was also amended to approach the behavior of the shells and to avoid certain lockings (shears, membrane). To eliminate the modes with energy null due to under-integration, an effective technique of stabilization was used while following the approach of Belytschko and Bindeman [bib4]. In the same way, the operator discretized gradient was modified for the elimination of various blockings. Thus, the version obtained of this element has the following advantages:

- capacity with modelling mean three-dimensional structures with few elements of mesh thanks to tolerated important slenderness (significant time-saver of computations),
- simplified mesh of complex geometries where solid shells and elements must cohabit (reinforcements or flanges for example) without having the classical problems of connections of meshes made of different element types.

This hexahedral element was introduced into *Code_Aster* in version 7 (see [R3,07,07]). However, hexahedral element SHB8 does not make it possible to net geometries of complex forms unspecified. The development of a similar element but of prismatic geometry was thus necessary. One describes in the beginning of this document this prismatic element (element SHB6).

The research tasks of Caironi and Abed-Meraim [bib5] proved that element SHB6 did not present modes of hourglass, and after having established it, they as showed as this one presented a severe numerical blocking, in particular in the requests in transverse shears of the element. Element SHB6 established in Aster aims at eliminating these numerical blockings by means of the method “assumed strain”. The principle of this method consists in projecting the operator gradient discretized B on a suitable subspace in order to avoid the various problems involved in blocking. Several projections were tested before finding that which eliminates the maximum of lockings.

Element SHB6 is the object of the §2.

The §3 presents an extension of this family of finite elements of type solid-shell: two finite elements of prismatic and hexahedric geometry but of quadratic formulation named SHB15 and SHB20. They are respectively elements with 15 and 20 nodes. They under-are also integrated by 15 and 20 Gauss points and have a direction privileged according to the thickness of the element. These elements not having blockings are not projected.

In addition, initial element SHB8 had been coupled with the only constitutive laws elastic and elastoplastic with isotropic hardening of type Von-Put. The field of application of element SHB8 as well as the other finite elements solid-shell SHB6, SHB15 and SHB20 was extended to the other constitutive laws of Code_Aster. The §4 presents the theoretical principle of this coupling.

Finally the §5 treats establishment of these elements in *Code_Aster*.

2 Kinematical element

2.1 SHB6 of the element

element SHB6 is a pentahedron with 6 nodes. The five points of integration are selected along the direction ζ in the reference of the local coordinates of the element of reference: ξ,η,ζ (or $\hat{x}_1,\hat{x}_2,\hat{x}_3$ for certain statements). The shape of the element of reference as well as the points of integration are represented on [Figure 1].

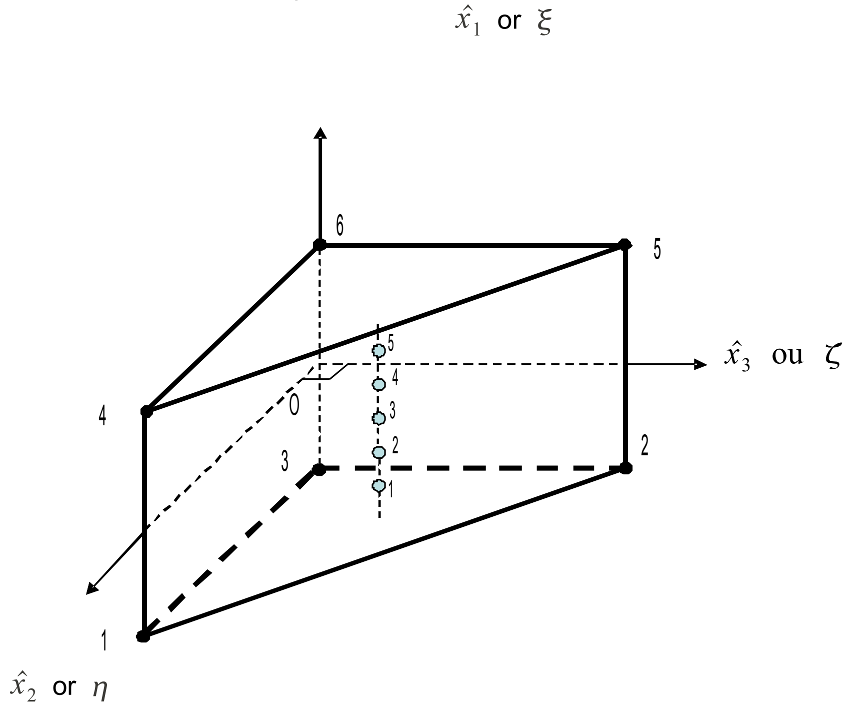


Figure 1: Geometry of the element of reference and points of integration

This element is isoparametric and has the same linear interpolation and the same kinematics as the pentahedral elements with 6 standard nodes.

2.2 Discretization

2.2.1 Discretization of the field of displacement

the spatial coordinates x_i of the element are connected to the nodal coordinates x_{iI} by means of the isoparametric shape functions N_I by the formulas:

$$\boldsymbol{x}_i=\boldsymbol{x}_{iI}\boldsymbol{N}_I(\hat{x}_1,\hat{x}_2,\hat{x}_3)=\sum_{i=1}^{6}\boldsymbol{x}_{iI}\boldsymbol{N}_I(\hat{x}_1,\hat{x}_2,\hat{x}_3)$$

In the continuation, and except contrary mention, one will adopt summation convention for the repeated indices. The indices in small letters i vary from one to three and represent the directions of the spatial coordinates. Those in capital letters I vary from one to six and correspond to the nodes of the element.

The same shape functions are used to define the field of displacement of the element u_i according to nodal displacements u_{iI} :

$$\boldsymbol{u_i}=\boldsymbol{U_{iI}}\,\boldsymbol{N_I}(\hat{x}_1,\hat{x}_2,\hat{x}_3)=\sum_{i=1}^{6}\boldsymbol{U_{iI}}\,\boldsymbol{N_I}(\hat{x}_1,\hat{x}_2,\hat{x}_3)$$

To continue computations, one gives oneself linear isoparametric shape functions $N_i(\xi,\eta,\zeta)=N_i(\hat{x}_1,\hat{x}_2,\hat{x}_3)$ associated with the prismatic element with six nodes:

$$N_1=\frac{1}{2}\hat{x}_2\left(1-\hat{x}_1\right)\qquad N_4=\frac{1}{2}\hat{x}_2\left(1+\hat{x}_1\right)$$

$$N_2=\frac{1}{2}\hat{x}_3\left(1-\hat{x}_1\right)\qquad N_5=\frac{1}{2}\hat{x}_3\left(1+\hat{x}_1\right)$$

$$N_3=\frac{1}{2}\left(1-\hat{x}_2-\hat{x}_3\right)\left(1-\hat{x}_1\right)\qquad N_6=\frac{1}{2}\left(1-\hat{x}_2-\hat{x}_3\right)\left(1+\hat{x}_1\right)$$

$$\hat{x}_1=[-1,1];\qquad \hat{x}_2=[0,1];\qquad \hat{x}_3=[0,1-\hat{x}_2]$$

The origin of the reference is confused with the right corner of the triangle of the median plane of the element.

2.2.2 Operator gradient discretized

the gradient $u_{i,j}$ of the field of displacement is a function of displacements $\mathbf{U}_{iI}$ of the nodes I in the direction i

$$u_{i,j}=\mathbf{U}_{iI}N_{I,j}$$

the linear strain tensor is given by the symmetric part of the displacement gradient:

$$\varepsilon_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i})$$

One now will build vectors allowing to express the matrix B connecting the strains to displacements in a particular form.

In a way similar to Belytschko-Bindeman [bib6], one introduces the three vectors $\underline{\mathbf{b}}_i$, derived from the shape functions at the origin of the coordinates:

$$\boldsymbol{b_i^T}=\boldsymbol{N}_{,i}(0)=\left.\frac{\partial\boldsymbol{N}}{\partial x_i}\right]_{\hat{x}_1=\hat{x}_2=\hat{x}_3=0}\qquad i=1,2,3$$

These 3 vectors are constant and are given by the statement:

$$\boldsymbol{b}_i^T = \begin{pmatrix} j_{i1} & j_{i2} & j_{i3} \end{pmatrix} \cdot \begin{bmatrix} 0 & 0 & \frac{-1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & \frac{-1}{2} \\ 0 & \frac{1}{2} & \frac{-1}{2} & 0 & \frac{1}{2} & \frac{-1}{2} \end{bmatrix}$$

where the coefficients j_{kl} are the coefficients of the jacobian matrix evaluated in the beginning.

Also let us introduce the following vectors:

$\mathbf{s}^T$ = (111111)
$\mathbf{h}_1^T$ = (- 100100)
$\mathbf{h}_2^T$ = (0-10010)
$\mathbf{X}_i^T = (x_{i1}, x_{i2}, x_{i3}, x_{i4}, x_{i5}, x_{i6})$

the three vectors $\mathbf{X}_i^T$ represent the nodal coordinates of the six nodes. The two vectors $\mathbf{h}_\alpha^T$ respectively represent the functions h_1 and h_2 for each of the six nodes, which are defined by:

$$h_1 = \hat{x}_1 \hat{x}_2 \qquad h_2 = \hat{x}_1 \hat{x}_3$$

Let us introduce finally the two following vectors:

$$\boldsymbol{\gamma}_\alpha = \frac{1}{2}\left[\mathbf{h}_\alpha - \sum_{j=1}^{3}\left(\mathbf{h}_\alpha^T . \mathbf{X}_j\right)\mathbf{b}_j\right] \qquad \alpha = 1,2$$

One can check by algebraic considerations that the following conditions of orthogonality are satisfied:

$$\begin{aligned} &\mathbf{b}_i^T \cdot \mathbf{h}_\alpha = 0 \\ &\mathbf{b}_i^T \cdot \mathbf{s} = 0 \\ &\mathbf{h}_\alpha{}^T \cdot \mathbf{s} = 0 \\ &\mathbf{b}_i^T \cdot \mathbf{X}_j = \delta_{ij} \qquad i,j = 1,2,3 \quad \alpha,\beta = 1,2 \\ &\mathbf{h}_\alpha{}^T \cdot \mathbf{h}_\beta = 2\delta_{\alpha\beta} \qquad \textbf{(1)} \\ &\gamma_\alpha^T \cdot \mathbf{X}_j = 0 \\ &\gamma_\alpha{}^T \cdot \mathbf{h}_\beta = \delta_{\alpha\beta} \end{aligned}$$

where δ_{ij} is the Kronecker symbol.

These vectors will make it possible to express the matrix B connecting the strains to displacements in a particular form used thereafter.

The gradient of the field of displacement can be now written after computations in the form (without any approximation [bib6]):

$$u_{i,j}=\left(\boldsymbol{b}_j^T+\boldsymbol{h}_{\alpha,j}\gamma_\alpha^T\right)\cdot\boldsymbol{U}_i$$

The symmetric operator gradient (noted ∇_s) discretized connecting the strain tensor to the vector of nodal displacements

$$\nabla_s\boldsymbol{u}=\boldsymbol{B}\cdot\boldsymbol{u}$$

takes the matric shape then:

$$\mathbf{B}=\begin{bmatrix}\mathbf{b}_x^T+h_{\alpha,x}\gamma_\alpha^T & 0 & 0\\ 0 & \mathbf{b}_y^T+h_{\alpha,y}\gamma_\alpha^T & 0\\ 0 & 0 & \mathbf{b}_z^T+h_{\alpha,z}\gamma_\alpha^T\\ \mathbf{b}_y^T+h_{\alpha,y}\gamma_\alpha^T & \mathbf{b}_x^T+h_{\alpha,x}\gamma_\alpha^T & 0\\ \mathbf{b}_z^T+h_{\alpha,z}\gamma_\alpha^T & 0 & \mathbf{b}_x^T+h_{\alpha,x}\gamma_\alpha^T\\ 0 & \mathbf{b}_z^T+h_{\alpha,z}\gamma_\alpha^T & \mathbf{b}_y^T+h_{\alpha,y}\gamma_\alpha^T\end{bmatrix} \quad \textbf{(2)}$$

2.3 Stiffness matrix and stabilization

2.3.1 Stiffness matrix

the stiffness matrix of the element is given by:

$$\boldsymbol{K}_e=\int_{V_e}\boldsymbol{B}^T\cdot\boldsymbol{C}\cdot\boldsymbol{B}\,dV$$

The five points of integration considered $\mathbf{P_i}$ are on the same vertical line. Their coordinates and their weights of integration are the following:

	$\hat{\mathbf{x}}_2$	$\hat{\mathbf{x}}_3$	$\hat{\mathbf{x}}_1$	ω
$\mathbf{P}_1$	1/3	1/3	-0.906179845938664	0.236926885056189
$\mathbf{P}_2$	1/3	1/3	-0.538469310105683	0.478628670499366
$\mathbf{P}_3$	1/3	1/3	0	0.568888888888889
$\mathbf{P}_4$	1/3	1/3	0.538469310105683	0.478628670499366
$\mathbf{P}_5$	1/3	1/3	0.906179845938664	0.236926885056189

Thus, the statement of the stiffness $\mathbf{K}_e$ is:

$$K_e=\sum_{j=1}^{5}\omega(P_j)J(P_j)B^T(P_j).C.B(P_j) \quad \textbf{(3)}$$

where $J(P_j)$ is the Jacobian, calculated at the Gauss point j, of the transformation between the element of reference and the current element. The matrix of elastic behavior $\mathbf{C}$ selected has the following form:

$$\mathbf{C}=\begin{bmatrix} \bar{\lambda}+2\mu & \bar{\lambda} & 0 & 0 & 0 & 0 \\ \bar{\lambda} & \bar{\lambda}+2\mu & 0 & 0 & 0 & 0 \\ 0 & 0 & E & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix}$$

where E is the Young modulus, ν the Poisson's ratio, $\mu=\frac{E}{2(1+\nu)}$ the shear modulus and $\bar{\lambda}=\frac{E\nu}{1-\nu^2}$ the coefficient of modified Lamé. This model is specific with elements SHB. It resembles that which one would have in the case of the assumption of the plane stresses, put except for the term (3,3).

Even if this choice involves an artificial anisotropic behavior, it makes it possible to satisfy all the tests without introducing blocking.

2.3.2 Analyzes modes "hourglass" for element SHB6

the modes of "hourglass" are kinematical modes which are due to under-integration and are associated with an energy null whereas they induce a non-zero strain. This anomaly is explained by the difference, that induced under-integration, between the core of the continuous operator of stiffness discretized and that. Let us start initially by noticing that the operator discretized gradient under-integrated associated with the five points of integration defined above 2) takes the shape of **the equation (** with $\alpha=1,2$.

Now let us analyze the core of the stiffness matrix obtained by under-integration. According to (3), that returns under investigation from the row of the matrix B insofar as the matrix of behavior C is not singular. In other words, it is enough to search the modes of déplacementà $\boldsymbol{d}$ strain null, i.e. checking:

$$\nabla_s(\boldsymbol{u})=\boldsymbol{B}\cdot\boldsymbol{d}=\mathbf{0} \qquad (4)$$

We will seek from now on which are the modes of strains which give a strain energy null. Strain energy is written $w(\boldsymbol{\epsilon})=\frac{1}{2}\int_V \boldsymbol{\epsilon}\cdot\boldsymbol{C}\cdot\epsilon\, dV$ and like $\boldsymbol{\epsilon}=\boldsymbol{B}\,.\,\boldsymbol{d}$. we thus have:

$$w(\boldsymbol{\epsilon})=\frac{1}{2}\int_V \boldsymbol{d}^{\boldsymbol{T}}\boldsymbol{B}^{\boldsymbol{T}}\cdot\boldsymbol{C}\cdot\boldsymbol{B}\,\boldsymbol{d}\,dV=\boldsymbol{d}^{\boldsymbol{T}}\left[\frac{1}{2}\int_V \boldsymbol{B}^{\boldsymbol{T}}\cdot\boldsymbol{C}\cdot\boldsymbol{B}\,dV\right]\boldsymbol{d}$$

and if we consider the following approximation: $\boldsymbol{B}$ is calculated at the points of Gauss quadrature, we obtain:

$$w(\boldsymbol{\epsilon})=\frac{1}{2}\boldsymbol{d}^{\boldsymbol{T}}\boldsymbol{K}_{\boldsymbol{e}}\boldsymbol{d}$$

Thus to search the modes of strains to energy null is to search the core $\boldsymbol{K}_{\boldsymbol{e}}$

$$K_e\cdot\boldsymbol{X}=\mathbf{0}\Leftrightarrow\boldsymbol{B}(\xi_{Gj})\cdot\boldsymbol{X}=\mathbf{0}$$

Thus to search the modes of hourglass is to search the vectors $\boldsymbol{X}$ such as:

$$\boldsymbol{B}(\xi_{Gj})\cdot\boldsymbol{X}=\mathbf{0}\quad\forall\,\xi_{Gj} \qquad (5)$$

It is natural to find in the core of the stiffness $\boldsymbol{K}_e$ the modes associated with motions with rigid bodies. For a three-dimensional element such as the prism with 6 nodes, these motions rigidifying are composed of three translations and three rotations. Thus the core of the continuous operator of stiffness is of dimension six and is reduced to the only following modes:

$$\begin{pmatrix} \underline{\mathbf{S}} & \underline{\mathbf{0}} & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \underline{\mathbf{S}} & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \underline{\mathbf{0}} & \underline{\mathbf{S}} \end{pmatrix} \quad et \quad \begin{pmatrix} \underline{\mathbf{y}} & \underline{\mathbf{z}} & \underline{\mathbf{0}} \\ -\underline{\mathbf{x}} & \underline{\mathbf{0}} & \underline{\mathbf{z}} \\ \underline{\mathbf{0}} & -\underline{\mathbf{x}} & -\underline{\mathbf{y}} \end{pmatrix} \qquad (6)$$

One easily checks that each of the six vectors columns above satisfy the equation **(5)** and thus belongs to the core of $\boldsymbol{K}_e$. It is enough, to see it, to use the statement **(2)** $\boldsymbol{B}$ and the conditions of orthogonality **(1)**. The first three vectors columns correspond to the translations according to the axes Ox, Oy and Oz respectively. The three other vectors are relating to rotations around the axes Oz, Oy and Ox.

We search from now on, in addition to the preceding rigid modes, of the modes which also cancel the operator discretized gradient given in **(2)**. Let us take a base of eighteen following vectors:

$$\begin{bmatrix} \underline{\mathbf{S}} & \underline{\mathbf{0}} & \underline{\mathbf{0}} & \underline{\mathbf{y}} & \underline{\mathbf{z}} & \underline{\mathbf{0}} & \underline{\mathbf{x}} & \underline{\mathbf{0}} & \underline{\mathbf{0}} & \underline{\mathbf{y}} & \underline{\mathbf{z}} & \underline{\mathbf{0}} & \underline{\mathbf{h}}_1 & \underline{\mathbf{0}} & \underline{\mathbf{0}} & \underline{\mathbf{h}}_2 & \underline{\mathbf{0}} & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \underline{\mathbf{S}} & \underline{\mathbf{0}} & -\underline{\mathbf{x}} & \underline{\mathbf{0}} & \underline{\mathbf{z}} & \underline{\mathbf{0}} & \underline{\mathbf{y}} & \underline{\mathbf{0}} & \underline{\mathbf{x}} & \underline{\mathbf{0}} & \underline{\mathbf{z}} & \underline{\mathbf{0}} & \underline{\mathbf{h}}_1 & \underline{\mathbf{0}} & \underline{\mathbf{0}} & \underline{\mathbf{h}}_2 & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \underline{\mathbf{0}} & \underline{\mathbf{S}} & \underline{\mathbf{0}} & -\underline{\mathbf{x}} & -\underline{\mathbf{y}} & \underline{\mathbf{0}} & \underline{\mathbf{0}} & \underline{\mathbf{z}} & \underline{\mathbf{0}} & \underline{\mathbf{x}} & \underline{\mathbf{y}} & \underline{\mathbf{0}} & \underline{\mathbf{0}} & \underline{\mathbf{h}}_1 & \underline{\mathbf{0}} & \underline{\mathbf{0}} & \underline{\mathbf{h}}_2 \end{bmatrix}$$

One can show easily that the vectors above are linearly independent within the space of dimension eighteen. Elementary computations using the conditions of orthogonality **(1)** show that the last twelve vectors columns do not check the equation **(5)**.

That wants to say that there are not other modes only the rigid modes which cancel the operator discretized gradient given in **(2)**. In other words, element `SHB6` does not present hourglass mode.

2.3.3 Projection by "local Assumed strain method"

the first stage is to place itself in the local coordinate system of the element defined by the reference $(\hat{x}_1, \hat{x}_2, \hat{x}_3)$ describes in *Figure 1*. The strains from now on will thus be calculated in this reference. The operator discretized gradient $\boldsymbol{B}$ will be project on under suitable space in order to avoiding the various problems of blocking. This method is variationnellement coherent with the principle of Hu-Washizu if the interpolation of the stress is judiciously selected (Simo and Hughes [15]). However, it is very difficult to select in a general and systematic way the good strain field applied. The strain fields applied should present neither voluminal blocking nor blocking in shears.

We present here an easy choice and acceptable. The operator $\boldsymbol{B}$ is first of all separate in two parts $\underline{\underline{\mathbf{B}}}_1$ and $\underline{\underline{\mathbf{B}}}_2$. The matrix $\underline{\underline{\mathbf{B}}}_1$ contains the gradients in the average plane of the shell and the perpendicular strain, $\underline{\underline{\mathbf{B}}}_2$ contains the gradients associated with the shear strains transverse.

$$\underline{\underline{\mathbf{B}}} = \begin{bmatrix} \underline{\mathbf{b}}_1^T + \mathrm{h}_{\alpha,x_1}\underline{\boldsymbol{\gamma}}_\alpha^T & \underline{\mathbf{0}} & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \underline{\mathbf{b}}_2^T + \mathrm{h}_{\alpha,x_2}\underline{\boldsymbol{\gamma}}_\alpha^T & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \underline{\mathbf{0}} & \underline{\mathbf{b}}_3^T + \mathrm{h}_{\alpha,x_3}\underline{\boldsymbol{\gamma}}_\alpha^T \\ \underline{\mathbf{b}}_2^T + \mathrm{h}_{\alpha,x_2}\underline{\boldsymbol{\gamma}}_\alpha^T & \underline{\mathbf{b}}_1^T + \mathrm{h}_{\alpha,x_1}\underline{\boldsymbol{\gamma}}_\alpha^T & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \underline{\mathbf{0}} & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \underline{\mathbf{0}} & \underline{\mathbf{0}} \end{bmatrix} + \begin{bmatrix} \underline{\mathbf{0}} & \underline{\mathbf{0}} & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \underline{\mathbf{0}} & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \underline{\mathbf{0}} & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \underline{\mathbf{0}} & \underline{\mathbf{0}} \\ \underline{\mathbf{b}}_3^T + \mathrm{h}_{\alpha,x_3}\underline{\boldsymbol{\gamma}}_\alpha^T & \underline{\mathbf{0}} & \underline{\mathbf{b}}_1^T + \mathrm{h}_{\alpha,x_1}\underline{\boldsymbol{\gamma}}_\alpha^T \\ \underline{\mathbf{0}} & \underline{\mathbf{b}}_3^T + \mathrm{h}_{\alpha,x_3}\underline{\boldsymbol{\gamma}}_\alpha^T & \underline{\mathbf{b}}_2^T + \mathrm{h}_{\alpha,x_2}\underline{\boldsymbol{\gamma}}_\alpha^T \end{bmatrix}$$

$$= \underline{\underline{\mathbf{B}}}_1 + \underline{\underline{\mathbf{B}}}_2$$

The blockings noted in the element come from the transverse shears. One will seek a diagram of integration which allows of under - to integrate this part of energy. With this intention one seeks to control each component entering the energy of transverse shears. Being given the shape of the matrix $\underline{\underline{\mathbf{B}}}$ we thus have 12 non-zero terms which intervene in the strain. They will be controlled by the introduction of the parameter c into the matrixes $\underline{\underline{\mathbf{B}}}_2$. The matrix $\underline{\underline{\mathbf{B}}}_2$ becomes then $\overline{\underline{\underline{\mathbf{B}}}}_2$:

$$\overline{\underline{\underline{\mathbf{B}}}}_2 = c\begin{bmatrix} \underline{\mathbf{0}} & \underline{\mathbf{0}} & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \underline{\mathbf{0}} & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \underline{\mathbf{0}} & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \underline{\mathbf{0}} & \underline{\mathbf{0}} \\ \underline{\mathbf{b}}_3^T + \mathrm{h}_{\alpha,x_3}\,\underline{\boldsymbol{\gamma}}_\alpha^T & \underline{\mathbf{0}} & \underline{\mathbf{b}}_1^T + \mathrm{h}_{\alpha,x_1}\,\underline{\boldsymbol{\gamma}}_\alpha^T \\ \underline{\mathbf{0}} & \underline{\mathbf{b}}_3^T + \mathrm{h}_{\alpha,x_3}\,\underline{\boldsymbol{\gamma}}_\alpha^T & \underline{\mathbf{b}}_2^T + \mathrm{h}_{\alpha,x_2}\,\underline{\boldsymbol{\gamma}}_\alpha^T \end{bmatrix}$$

The stiffness matrix is written now:

$$\boldsymbol{K}_e = \int_V \overline{\boldsymbol{B}}^T \cdot \boldsymbol{C} \cdot \overline{\boldsymbol{B}}\, dV = \int_V \overline{\boldsymbol{B}}_1{}^T \cdot \boldsymbol{C} \cdot \overline{\boldsymbol{B}}_1 dV + \int_V \overline{\boldsymbol{B}}_2{}^T \cdot \boldsymbol{C} \cdot \overline{\boldsymbol{B}}_1 dV + \int_V \overline{\boldsymbol{B}}_1{}^T \cdot \boldsymbol{C} \cdot \overline{\boldsymbol{B}}_2 dV + \int_V \overline{\boldsymbol{B}}_2{}^T \cdot \boldsymbol{C} \cdot \overline{\boldsymbol{B}}_2 dV = \\ = \boldsymbol{K}_{e1} + \boldsymbol{K}_{e2} + \boldsymbol{K}_{e3} + \boldsymbol{K}_{e4}$$

The matrixes $\boldsymbol{K}_{e1,}\boldsymbol{K}_{e2,}\boldsymbol{K}_{e3,}\boldsymbol{K}_{e4}$ are integrated with five the Gauss points previously definite ones. Additive decomposition given higher $\underline{\underline{\mathbf{B}}} = \underline{\underline{\mathbf{B}}}_1 + \underline{\underline{\mathbf{B}}}_2$, for the operator discretized gradient, makes that the cross terms $\mathbf{K}_{e2}$ and $\mathbf{K}_{e3}$ are cancelled. Following many the test numerical, it was selected to characterize the matrix $\overline{\underline{\underline{\mathbf{B}}}}_2$ by the coefficient: $c = 0{,}45$, which plays here the part of a factor of reduction of the shears.

This choice gives to the element a good behavior in the cases of reference. It is clear that this strategy, as that installation for the cubic elements voluminal shells are adapted only to the quasi isotropic behavior of the selected material.

2.4 Geometrical stiffness matrix Ksigma

the matrix $\boldsymbol{K}_\sigma$ aims to solve the problems of buckling. We point out here that buckling modes are the eigenvectors of the problem to the eigenvalues generalized according to:

$$\left(\boldsymbol{K} + \mu\,\boldsymbol{K}_\sigma\right) \cdot \boldsymbol{u} = \mathbf{0} \Leftrightarrow \boldsymbol{K} \cdot \boldsymbol{u} = \lambda\,\boldsymbol{K}_\sigma \cdot \boldsymbol{u}$$

with $\lambda = -\mu$, and μ is the multiplying coefficient of the loading.

By introducing the quadratic strain $\underline{\mathbf{e}}^Q$ such as:

$$e_{ij}^Q(\delta\boldsymbol{u}, \Delta\boldsymbol{u}) = \sum_{k=1}^{3} \delta u_{k,i} . \Delta u_{k,j}$$

One can define this geometrical stiffness matrix by:

$$\delta \boldsymbol{u}^T \cdot \boldsymbol{K}_\sigma \cdot \Delta \boldsymbol{u} = \int_{\Omega_0} \boldsymbol{\sigma} : \boldsymbol{e}^Q(\delta \boldsymbol{u}, \Delta \boldsymbol{u}) \, d\Omega = \int_{\Omega_0} \boldsymbol{\sigma} : \nabla \delta \boldsymbol{u}^T \nabla \Delta \boldsymbol{u} \quad d\Omega$$

In order to express this matrix in discretized space, let us introduce the operators quadratic gradient discretized $\underline{\underline{\mathbf{B}}}^Q$ (in matric notation) such as:

$$\underline{\mathbf{e}}^Q(\delta\mathbf{u},\Delta\mathbf{u}) = \begin{bmatrix} e_{11}^Q \\ e_{22}^Q \\ e_{33}^Q \\ e_{12}^Q + e_{21}^Q \\ e_{13}^Q + e_{31}^Q \\ e_{23}^Q + e_{32}^Q \end{bmatrix} = \begin{bmatrix} \underline{\delta\mathbf{u}}^T \cdot \underline{\underline{\mathbf{B}}}_{11}^Q \cdot \underline{\Delta\mathbf{u}} \\ \underline{\delta\mathbf{u}}^T \cdot \underline{\underline{\mathbf{B}}}_{22}^Q \cdot \underline{\Delta\mathbf{u}} \\ \underline{\delta\mathbf{u}}^T \cdot \underline{\underline{\mathbf{B}}}_{33}^Q \cdot \underline{\Delta\mathbf{u}} \\ \underline{\delta\mathbf{u}}^T \cdot \underline{\underline{\mathbf{B}}}_{12}^Q \cdot \underline{\Delta\mathbf{u}} \\ \underline{\delta\mathbf{u}}^T \cdot \underline{\underline{\mathbf{B}}}_{13}^Q \cdot \underline{\Delta\mathbf{u}} \\ \underline{\delta\mathbf{u}}^T \cdot \underline{\underline{\mathbf{B}}}_{23}^Q \cdot \underline{\Delta\mathbf{u}} \end{bmatrix}$$

The various terms $\underline{\underline{\mathbf{B}}}_{ij}^Q$ are given by the following equations:

$$\underline{\underline{\mathbf{B}}}_{11}^Q = \begin{bmatrix} \underline{\mathbf{B}}_1\underline{\mathbf{B}}_1^T & \underline{\underline{\mathbf{0}}} & \underline{\underline{\mathbf{0}}} \\ \underline{\underline{\mathbf{0}}} & \underline{\mathbf{B}}_1\underline{\mathbf{B}}_1^T & \underline{\underline{\mathbf{0}}} \\ \underline{\underline{\mathbf{0}}} & \underline{\underline{\mathbf{0}}} & \underline{\mathbf{B}}_1\underline{\mathbf{B}}_1^T \end{bmatrix} ; \quad \underline{\underline{\mathbf{B}}}_{22}^Q = \begin{bmatrix} \underline{\mathbf{B}}_2\underline{\mathbf{B}}_2^T & \underline{\underline{\mathbf{0}}} & \underline{\underline{\mathbf{0}}} \\ \underline{\underline{\mathbf{0}}} & \underline{\mathbf{B}}_2\underline{\mathbf{B}}_2^T & \underline{\underline{\mathbf{0}}} \\ \underline{\underline{\mathbf{0}}} & \underline{\underline{\mathbf{0}}} & \underline{\mathbf{B}}_2\underline{\mathbf{B}}_2^T \end{bmatrix} ; \quad \underline{\underline{\mathbf{B}}}_{33}^Q = \begin{bmatrix} \underline{\mathbf{B}}_3\underline{\mathbf{B}}_3^T & \underline{\underline{\mathbf{0}}} & \underline{\underline{\mathbf{0}}} \\ \underline{\underline{\mathbf{0}}} & \underline{\mathbf{B}}_3\underline{\mathbf{B}}_3^T & \underline{\underline{\mathbf{0}}} \\ \underline{\underline{\mathbf{0}}} & \underline{\underline{\mathbf{0}}} & \underline{\mathbf{B}}_3\underline{\mathbf{B}}_3^T \end{bmatrix}$$

$$\underline{\underline{\mathbf{B}}}_{12}^Q = \begin{bmatrix} \underline{\mathbf{B}}_1\underline{\mathbf{B}}_2^T + \underline{\mathbf{B}}_2\underline{\mathbf{B}}_1^T & \underline{\underline{\mathbf{0}}} & \underline{\underline{\mathbf{0}}} \\ \underline{\underline{\mathbf{0}}} & \underline{\mathbf{B}}_1\underline{\mathbf{B}}_2^T + \underline{\mathbf{B}}_2\underline{\mathbf{B}}_1^T & \underline{\underline{\mathbf{0}}} \\ \underline{\underline{\mathbf{0}}} & \underline{\underline{\mathbf{0}}} & \underline{\mathbf{B}}_1\underline{\mathbf{B}}_2^T + \underline{\mathbf{B}}_2\underline{\mathbf{B}}_1^T \end{bmatrix}$$

$$\underline{\underline{\mathbf{B}}}_{13}^Q = c^2 \begin{bmatrix} \underline{\mathbf{B}}_1\underline{\mathbf{B}}_3^T + \underline{\mathbf{B}}_3\underline{\mathbf{B}}_1^T & \underline{\underline{\mathbf{0}}} & \underline{\underline{\mathbf{0}}} \\ \underline{\underline{\mathbf{0}}} & \underline{\mathbf{B}}_1\underline{\mathbf{B}}_3^T + \underline{\mathbf{B}}_3\underline{\mathbf{B}}_1^T & \underline{\underline{\mathbf{0}}} \\ \underline{\underline{\mathbf{0}}} & \underline{\underline{\mathbf{0}}} & \underline{\mathbf{B}}_1\underline{\mathbf{B}}_3^T + \underline{\mathbf{B}}_3\underline{\mathbf{B}}_1^T \end{bmatrix}$$

$$\underline{\underline{\mathbf{B}}}_{23}^Q = c^2 \begin{bmatrix} \underline{\mathbf{B}}_2\underline{\mathbf{B}}_3^T + \underline{\mathbf{B}}_3\underline{\mathbf{B}}_2^T & \underline{\underline{\mathbf{0}}} & \underline{\underline{\mathbf{0}}} \\ \underline{\underline{\mathbf{0}}} & \underline{\mathbf{B}}_2\underline{\mathbf{B}}_3^T + \underline{\mathbf{B}}_3\underline{\mathbf{B}}_2^T & \underline{\underline{\mathbf{0}}} \\ \underline{\underline{\mathbf{0}}} & \underline{\underline{\mathbf{0}}} & \underline{\mathbf{B}}_2\underline{\mathbf{B}}_3^T + \underline{\mathbf{B}}_3\underline{\mathbf{B}}_2^T \end{bmatrix}$$

with the vectors $\underline{\mathbf{B}}_i$ (I = 1,2,3) definite like:

$$\underline{\mathbf{B}}_i = \left(\underline{\mathbf{b}}_i + h_{\alpha,i}\underline{\boldsymbol{\gamma}}_\alpha\right)$$

Note: We must multiply the matrixes $\underline{\underline{\mathbf{B}}}_{13}^{\mathbf{Q}}$ and $\underline{\underline{\mathbf{B}}}_{23}^{\mathbf{Q}}$ by the coefficient $c^2 = 0,45^2 = 0,2025$ because element SHB6 is project by the technique " *local Assumed strain method"* to see section 2.3.3 .

With these notations, the contribution to the geometrical stiffness matrix $\underline{\underline{\mathbf{k}}}_\sigma$, the Gauss point ξ_j east given by:

$$\underline{\underline{\mathbf{k}}}_{\boldsymbol{\sigma}}(\xi_j)=\sigma_{11}(\xi_j)\underline{\underline{\mathbf{B}}}_{11}^{Q}(\xi_j)+\sigma_{22}(\xi_j)\underline{\underline{\mathbf{B}}}_{22}^{Q}(\xi_j)+\sigma_{33}(\xi_j)\underline{\underline{\mathbf{B}}}_{33}^{Q}(\xi_j)$$
$$+\sigma_{12}(\xi_j)\underline{\underline{\mathbf{B}}}_{12}^{Q}(\xi_j)+\sigma_{13}(\xi_j)\underline{\underline{\mathbf{B}}}_{13}^{Q}(\xi_j)+\sigma_{23}(\xi_j)\underline{\underline{\mathbf{B}}}_{23}^{Q}(\xi_j)$$

By integration on Gauss points of the element, the geometrical stiffness matrix is obtained by the formula:

$$\underline{\underline{K}}_{\sigma}=\sum_{j=1}^{5}\omega(\xi_j)J(\xi_j)\underline{\underline{k}}_{\sigma}(\xi_j)$$

2.5 Follower forces and matrix Kp

the following compressive forces are present in the tangent matrix via the matrix $\underline{\underline{\mathbf{K}}}_{P}$, because the following external forces depend on displacement [R3.03.04]. The following compressive forces are written:

$$\int_{\partial\Omega}p\,\boldsymbol{n}^{\boldsymbol{T}}\cdot\boldsymbol{u}\,dS=\int_{\partial\Omega_0}p\det[\boldsymbol{F}(\boldsymbol{u})]\boldsymbol{n}_{\boldsymbol{0}}^{\boldsymbol{T}}\boldsymbol{F}(\boldsymbol{u})^{-\mathrm{T}}dS_0=p\,\boldsymbol{F}_{\boldsymbol{0}}-p\,\boldsymbol{K}_{\boldsymbol{p}}\cdot\boldsymbol{u}$$
$$\boldsymbol{F}(\boldsymbol{u})=\mathbf{1}+\nabla\boldsymbol{u}$$

by means of notations:

- $\underline{\mathbf{n}}_0{}^T=(n_1,n_2,n_3)$, norm on the surface external of the element in the reference configuration;
- $\tilde{\underline{\mathbf{b}}}_i$, vector of size 3, derived from the shape functions to the 3 nodes of the face of the element charged in pressure;
- S_0 area of the face charged in pressure. For element SHB6, this surface S_0 is worth $\frac{1}{2}$.

The preceding formulation leads to a nonsymmetrical matrix. It is known that one can nevertheless use a symmetric formulation if the external forces due to the pressure derive from a potential. It is the case if the compressive forces do not work on the border of the modelled field. It is thus considered that the symmetric part of the matrix is enough. The symmetrized matrix takes the following shape:

$$\boldsymbol{K}_{\boldsymbol{p}}=S_0\begin{bmatrix}0 & \tilde{\boldsymbol{b}}_2^T n_1-\tilde{\boldsymbol{b}}_1^T n_2 & \tilde{\boldsymbol{b}}_3^T n_1-\tilde{\boldsymbol{b}}_1^T n_3\\ 0 & \tilde{\boldsymbol{b}}_2^T n_1-\tilde{\boldsymbol{b}}_1^T n_2 & \tilde{\boldsymbol{b}}_3^T n_1-\tilde{\boldsymbol{b}}_1^T n_3\\ 0 & \tilde{\boldsymbol{b}}_2^T n_1-\tilde{\boldsymbol{b}}_1^T n_2 & \tilde{\boldsymbol{b}}_3^T n_1-\tilde{\boldsymbol{b}}_1^T n_3\\ \tilde{\boldsymbol{b}}_1^T n_2-\tilde{\boldsymbol{b}}_2^T n_1 & 0 & \tilde{\boldsymbol{b}}_3^T n_2-\tilde{\boldsymbol{b}}_2^T n_3\\ \tilde{\boldsymbol{b}}_1^T n_2-\tilde{\boldsymbol{b}}_2^T n_1 & 0 & \tilde{\boldsymbol{b}}_3^T n_2-\tilde{\boldsymbol{b}}_2^T n_3\\ \tilde{\boldsymbol{b}}_1^T n_2-\tilde{\boldsymbol{b}}_2^T n_1 & 0 & \tilde{\boldsymbol{b}}_3^T n_2-\tilde{\boldsymbol{b}}_2^T n_3\\ \tilde{\boldsymbol{b}}_1^T n_3-\tilde{\boldsymbol{b}}_3^T n_1 & \tilde{\boldsymbol{b}}_2^T n_3-\tilde{\boldsymbol{b}}_3^T n_2 & 0\\ \tilde{\boldsymbol{b}}_1^T n_3-\tilde{\boldsymbol{b}}_3^T n_1 & \tilde{\boldsymbol{b}}_2^T n_3-\tilde{\boldsymbol{b}}_3^T n_2 & 0\\ \tilde{\boldsymbol{b}}_1^T n_3-\tilde{\boldsymbol{b}}_3^T n_1 & \tilde{\boldsymbol{b}}_2^T n_3-\tilde{\boldsymbol{b}}_3^T n_2 & 0\end{bmatrix}$$

It is a matrix (9,9), which it is necessary to multiply by displacements of the 3 nodes of the face to which one applies a pressure.

3 Elements SHB15 and SHB20

In this paragraph, one presents of the finite elements to the voluminal modelizations shells quadratic SHB15 and SHB20.

Element SHB15 is a purely three-dimensional prism with fifteen nodes with three degrees of freedom in displacement with each node, and it also has a called privileged direction “thickness” which is normal with the average plan of the prism. Reduced numerical integration is used (3 Gauss points in the plane). Integration through the thickness leans on 5 Gauss points.

Element SHB20 is a purely three-dimensional hexahedron with twenty nodes with three degrees of freedom in displacement with each node, and it has also a called privileged direction “thickness” which is normal with the average plane of the hexahedron. Reduced numerical integration is used (4 Gauss points in the plane). Integration through the thickness leans on 5 Gauss points.

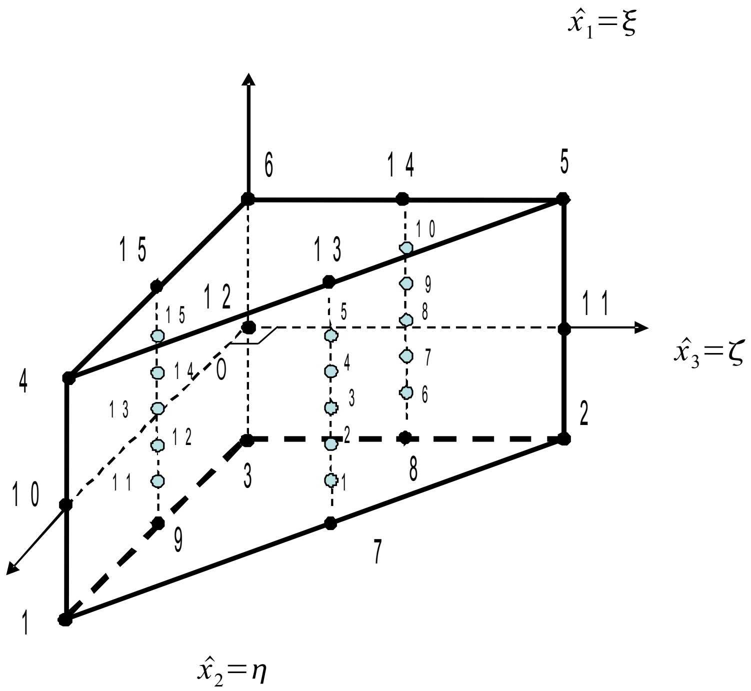
Contrary to the linear elements these finite elements have neither stabilization nor projection.

3.1 Kinematics and interpolation of elements SHB15 and SHB20

3.1.1 Element SHB15

element SHB15 is formulated in the local axes of the average plane. F igure 3.1.1-a represents the geometry of an element of reference SHB15 and its points of integration.

The reference of the local coordinates of the element of reference is defined by: ξ, η, ζ ou $\hat{x}_1, \hat{x}_2, \hat{x}_3$



F igure 3.1.1-a . Geometry of the element of reference SHB15 and its points of integration

Coordonnées des noeuds:

$1(-1,1,0)$; $2(-1,0,1)$; $3(-1,0,0)$; $4(1,1,0)$; $5(1,0,1)$; $6(1,0,0)$;

$7\left(-1,\frac{1}{2},\frac{1}{2}\right)$; $8\left(-1,0,\frac{1}{2}\right)$; $9\left(-1,\frac{1}{2},0\right)$;

$10(0,1,0)$; $11(0,0,1)$; $12(0,0,0)$; $13\left(1,\frac{1}{2},\frac{1}{2}\right)$; $14\left(1,0,\frac{1}{2}\right)$; $15\left(1,\frac{1}{2},0\right)$.

element SHB15 is an isoparametric quadratic element. The spatial coordinates $\mathbf{x}_i$ are connected to the nodal coordinates $\mathbf{x}_{iI}$ by means of the shape functions $\mathbf{N}_I$ by the formulas:

$$\boldsymbol{x}_i=\boldsymbol{x}_{iI}\,\boldsymbol{N}_I(\hat{x}_1,\hat{x}_2,\hat{x}_3)=\sum_{i=1}^{15}\boldsymbol{x}_{iI}\,\boldsymbol{N}_I(\hat{x}_1,\hat{x}_2,\hat{x}_3)$$

The same shape functions are used to define the field of displacement of the element $\mathbf{u}_i$ in terms of nodal displacements $\boldsymbol{U}_{iI}$:

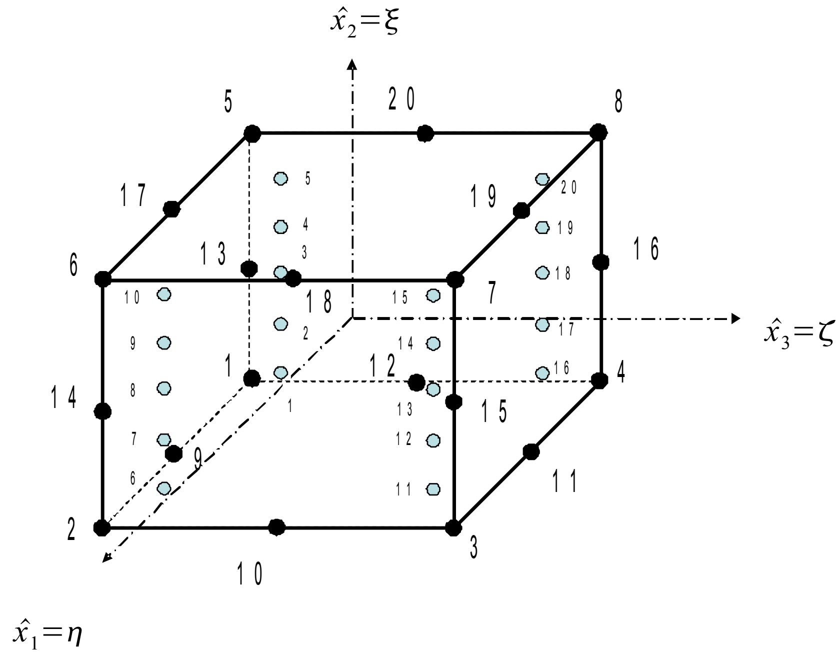
:

$$\boldsymbol{u}_i=\boldsymbol{u}_{iI}\,\boldsymbol{N}_I(\hat{x}_1,\hat{x}_2,\hat{x}_3)=\sum_{i=1}^{15}\boldsymbol{U}_{iI}\,\boldsymbol{N}_I(\hat{x}_1,\hat{x}_2,\hat{x}_3) \qquad \textbf{(6)}$$

3.1.2 Element SHB20

element SHB20 is formulated in the local axes of the average plane. Appear 3.1.2-a represents the geometry of an element of reference SHB20 and its points of integration.

The reference of the local coordinates of the element of reference is defined by: ξ,η,ζ ou $\hat{x}_1,\hat{x}_2,\hat{x}_3$



Appear 3.1.2-a. Geometry of the element of reference SHB20 and its points of integration

Coordonnées des noeuds:

$1(-1,-1,-1)$	$2(1,-1,-1)$	$3(1,1,-1)$	$4(-1,1,-1)$
$5(-1,-1,1)$	$6(1,-1,1)$	$7(1,1,1)$	$8(-1,1,1)$
$9(0,-1,-1)$	$10(1,0,-1)$	$11(0,1,-1)$	$12(-1,0,-1)$
$13(-1,-1,0)$	$14(1,-1,0)$	$15(1,1,0)$	$16(-1,1,0)$
$17(0,-1,1)$	$18(1,0,1)$	$19(0,1,1)$	$20(-1,0,1)$

element SHB20 is also an isoparametric quadratic element. The spatial coordinates $\mathbf{x}_i$ are connected to the nodal coordinates $\mathbf{x}_{iI}$ by means of the shape functions $\mathbf{N}_I$ by the formulas:

$$\boldsymbol{x_i}=\boldsymbol{x_{iI}}\,\boldsymbol{N_I}(\hat{x}_{1,}\hat{x}_{2,}\hat{x}_3)=\sum_{i=1}^{20}\boldsymbol{x_{iI}}\,\boldsymbol{N_I}(\hat{x}_{1,}\hat{x}_{2,}\hat{x}_3)$$

The same shape functions are used to define the field of displacement of the element $\mathbf{u}_i$ in terms of nodal displacements $\boldsymbol{U_{iI}}$:

$$\boldsymbol{u_i}=\boldsymbol{u_{iI}}\,\boldsymbol{N_I}(\hat{x}_{1,}\hat{x}_{2,}\hat{x}_3)=\sum_{i=1}^{20}\boldsymbol{U_{iI}}\,\boldsymbol{N_I}(\hat{x}_{1,}\hat{x}_{2,}\hat{x}_3) \quad (6)$$

3.2 Operator gradient discretized

3.2.1 Element SHB15

the interpolation of the field of displacement of the element (**6**) will allow us to define strain rate and to write the relations connecting the strains to nodal displacements. One starts initially by writing the gradient $\mathbf{u}_{i,j}$ of the field of displacement:

$$\boldsymbol{u_{i,j}}=\boldsymbol{U_{iI}}\,\boldsymbol{N_{I,j}} \quad (7)$$

the strain tensor ε_{ij} is given then by the symmetric part of the displacement gradient:

$$\varepsilon_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i}) \quad (8)$$

to continue computations, one gives oneself quadratic isoparametric shape functions $N_i(\hat{x}_{1,}\hat{x}_{2,}\hat{x}_3)$, associated with the prismatic element with fifteen nodes:

$$\begin{aligned}
&N_1=\frac{1}{2}\hat{x}_2\left(1-\hat{x}_1\right)(2\hat{x}_2-2-\hat{x}_1) \qquad N_4=\frac{1}{2}\hat{x}_2\left(1+\hat{x}_1\right)(2\hat{x}_2-2+\hat{x}_1)\\
&N_2=\frac{1}{2}\hat{x}_3\left(1-\hat{x}_1\right)(2\hat{x}_3-2-\hat{x}_1) \qquad N_5=\frac{1}{2}\hat{x}_3\left(1+\hat{x}_1\right)(2\hat{x}_3-2+\hat{x}_1) \qquad (9)\\
&N_3=-\frac{1}{2}\left(1-\hat{x}_2-\hat{x}_3\right)\left(1-\hat{x}_1\right)\hat{x}_1 \qquad N_6=\frac{1}{2}\left(1-\hat{x}_2-\hat{x}_3\right)\left(1+\hat{x}_1\right)\hat{x}_1\\
&N_7=2(1-\hat{x}_1)\hat{x}_2\hat{x}_3 \qquad N_8=2(1-\hat{x}_1)\left(1-\hat{x}_2-\hat{x}_3\right)\hat{x}_3 \qquad N_9=2(1-\hat{x}_1)\hat{x}_2\left(1-\hat{x}_2-\hat{x}_3\right)
\end{aligned}$$

$$N_{10}=(1-\hat{x}_1)(1+\hat{x}_1)\,\hat{x}_2 \qquad N_{11}=(1-\hat{x}_1)(1+\hat{x}_1)\,\hat{x}_3 \qquad N_{12}=(1-\hat{x}_1)(1+\hat{x}_1)\left(1-\hat{x}_2-\hat{x}_3\right)$$

$$N_{13}=2(1+\hat{x}_1)\,\hat{x}_2\,\hat{x}_3 \qquad N_{14}=2(1+\hat{x}_1)\left(1-\hat{x}_2-\hat{x}_3\right)\hat{x}_3 \qquad N_{15}=2(1+\hat{x}_1)\,\hat{x}_2\left(1-\hat{x}_2-\hat{x}_3\right)$$

$$\hat{x}_1=[-1,1]; \qquad \hat{x}_2=[0,1]; \qquad \hat{x}_3=[0,1-\hat{x}_2]$$

While combining the preceding equations one manages to develop the field of displacement as being the sum of a constant term, linear terms in x_i , and terms utilizing the functions h_α □
to simplify the writings, one will note $\xi=\hat{x}_1,\ \ \eta=\hat{x}_2,\ \ \zeta=\hat{x}_3$

$$\begin{cases} \mathrm{u}_i=a_{0i}+a_{1i}x_1+a_{2i}x_2+a_{3i}x_3+c_{1i}\,\mathrm{h}_1+c_{2i}\,\mathrm{h}_2+c_{3i}\,\mathrm{h}_3+ \\ \qquad c_{4i}\,\mathrm{h}_4+c_{5i}\,\mathrm{h}_5+c_{6i}\,\mathrm{h}_6+c_{7i}\,\mathrm{h}_7+c_{8i}\,\mathrm{h}_8+c_{9i}\,\mathrm{h}_9+c_{10i}\,\mathrm{h}_{10}+c_{11i}\,\mathrm{h}_{11} \\ i=1,2,3 \\ \mathrm{h}_1=\xi\zeta,\mathrm{h}_2=\eta\zeta,\mathrm{h}_3=\xi\eta,\mathrm{h}_4=\xi\eta\zeta,\mathrm{h}_5=\xi^2,\mathrm{h}_6=\eta^2, \\ \mathrm{h}_7=\zeta^2,\mathrm{h}_8=\xi^2\zeta,\mathrm{h}_9=\eta^2\zeta,\mathrm{h}_{10}=\xi\zeta^2,\mathrm{h}_{11}=\eta\zeta^2 \end{cases} \qquad (10)$$

By evaluating the equation **(6)** with the nodes of the element, one arrives at the three systems of fifteen equations following:

$$\begin{cases} \underline{\mathbf{d}}_i=a_{0i}\underline{\mathbf{S}}+a_{1i}\underline{\mathbf{x}}_1+a_{2i}\underline{\mathbf{x}}_2+a_{3i}\underline{\mathbf{x}}_3+c_{1i}\underline{\mathbf{h}}_1+c_{2i}\underline{\mathbf{h}}_2+c_{3i}\underline{\mathbf{h}}_3+ \\ \qquad c_{4i}\underline{\mathbf{h}}_4+c_{5i}\underline{\mathbf{h}}_5+c_{6i}\underline{\mathbf{h}}_6+c_{7i}\underline{\mathbf{h}}_7+c_{8i}\underline{\mathbf{h}}_8+c_{9i}\underline{\mathbf{h}}_9+c_{10i}\underline{\mathbf{h}}_{10}+c_{11i}\underline{\mathbf{h}}_{11} \\ i=1,2,3 \end{cases} \qquad (11)$$

Thus the vectors $\underline{\mathbf{d}}_{\mathrm{i}}$ and $\underline{\mathbf{x}}_{\mathrm{i}}$ represent, respectively, displacements and the coordinated nodal and are given by:

$$\begin{cases} \underline{\mathbf{d}}_i^T=\left(u_{i1},u_{i2},u_{i3},u_{i4},u_{i5},u_{i6},u_{i>7},u_{i8},u_{i9},u_{i10},u_{i11},u_{i12},u_{i13},u_{i14},u_{i15}\right) \\ \underline{\mathbf{x}}_i^T=\left(x_{i1},x_{i2},x_{i3},x_{i4},x_{i5},x_{i6},x_{i7},x_{i8},x_{i9},x_{i10},x_{i11},x_{i12},x_{i13},x_{i14},x_{i15}\right) \end{cases} \qquad (12)$$

the vectors **S** and $\mathbf{h}_\alpha$ $\left(\alpha=1,2,3,\ldots,11\right)$ are given as for them by:

$$
\begin{cases}
\underline{\mathbf{S}}^T = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \\
\underline{\mathbf{h}}_1^T = \begin{pmatrix} 0 & -\frac{1}{2} & -1 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 \end{pmatrix} \\
\underline{\mathbf{h}}_2^T = \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{2} & -1 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 1 & \frac{1}{2} \end{pmatrix} \\
\underline{\mathbf{h}}_3^T = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 \end{pmatrix} \\
\underline{\mathbf{h}}_4^T = \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 \end{pmatrix} \\
\underline{\mathbf{h}}_5^T = \begin{pmatrix} 0 & \frac{1}{4} & 1 & \frac{1}{4} & 0 & 0 & 0 & 1 & 0 & 0 & \frac{1}{4} & 1 & \frac{1}{4} & 0 & 0 \end{pmatrix} \\
\underline{\mathbf{h}}_6^T = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{4} & 1 & \frac{1}{4} & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{4} & 1 & \frac{1}{4} \end{pmatrix} \\
\underline{\mathbf{h}}_7^T = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \\
\underline{\mathbf{h}}_8^T = \begin{pmatrix} 0 & -\frac{1}{4} & -1 & -\frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 1 & \frac{1}{4} & 0 & 0 \end{pmatrix} \\
\underline{\mathbf{h}}_9^T = \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{4} & -1 & -\frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 1 & \frac{1}{4} \end{pmatrix} \\
\underline{\mathbf{h}}_{10}^T = \begin{pmatrix} 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 \end{pmatrix} \\
\underline{\mathbf{h}}_{11}^T = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 1 & \frac{1}{2} \end{pmatrix}
\end{cases}
\tag{13}
$$

to arrive at an advantageous writing of the operator discretized gradient $\mathbf{B}$, one will introduce the three vectors $\mathbf{b_i}$ defined by:

$$\mathbf{b_i^T} = \mathbf{N^T}_{,\mathbf{i}}(0) = {}^{\mathbf{T}}\frac{\partial \mathbf{N}}{\partial \mathbf{x_i}}(0) \quad \text{I = 1,2,3} \tag{14}$$

If we place ourselves in $(\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2, \hat{\mathbf{x}}_3) = (0,0,0)$ then we obtain:

$$\mathbf{b_i^T} = \mathbf{N^T}_{,\mathbf{i}}(0) = \mathbf{cste}$$

where $\mathbf{N^T}$ represents: $\begin{pmatrix} \mathrm{N}_1 & \mathrm{N}_2 & \mathrm{N}_3 & . & . & \mathrm{N}_{15} \end{pmatrix}$.

$$\mathbf{b_i^T} = \mathbf{N^T}_{,\mathbf{i}}(0) = \left(\frac{\partial \mathrm{N}_1}{\partial \mathrm{x_i}}(0), \ldots, \frac{\partial \mathrm{N}_{15}}{\partial \mathrm{x_i}}(0)\right)$$

$$\frac{\partial N_I}{\partial x_j} = \left(\frac{\partial N_I}{\partial \xi} \cdot \frac{\partial \xi}{\partial x_j} + \frac{\partial N_I}{\partial \eta} \cdot \frac{\partial \eta}{\partial x_j} + \frac{\partial N_I}{\partial \zeta} \cdot \frac{\partial \zeta}{\partial x_j} \right)_{|\xi=\eta=\zeta=0} = \left(\frac{\partial N_I}{\partial \xi} . j_{1j} + \frac{\partial N_I}{\partial \eta} . j_{2j} + \frac{\partial N_I}{\partial \zeta} . j_{3j} \right)_{|\xi=\eta=\zeta=0}$$

$$avec \quad I = 1,2,...,15 \quad et \quad j = 1,2,3$$

$$\mathbf{F}^{-1}{}_{|\xi=\eta=\zeta=0} = \begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} & \frac{\partial \xi}{\partial z} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} & \frac{\partial \eta}{\partial z} \\ \frac{\partial \zeta}{\partial x} & \frac{\partial \zeta}{\partial y} & \frac{\partial \zeta}{\partial z} \end{pmatrix}_{|\xi=\eta=\zeta=0} = \begin{pmatrix} j_{11} & j_{12} & j_{13} \\ j_{21} & j_{22} & j_{23} \\ j_{31} & j_{32} & j_{33} \end{pmatrix}$$

We have:

$$N_{1,\xi} = \frac{(1-\zeta)}{2}(4\xi + 4\eta + \zeta - 2) \quad N_{1,\eta} = \frac{(1-\zeta)}{2}(4\xi + 4\eta + \zeta - 2) \quad N_{1,\zeta} = \frac{(1-\xi-\eta)}{2}(2\xi + 2\eta + 2\zeta - 1)$$

$$N_{2,\xi} = 2(1-\zeta)(1-2\xi-\eta) \quad N_{2,\eta} = -2(1-\zeta)\xi \quad N_{2,\zeta} = -2\xi(1-\xi-\eta)$$

$$N_{3,\xi} = \frac{(1-\zeta)}{2}(-2+4\xi-\zeta) \quad N_{3,\eta} = 0 \quad N_{3,\zeta} = \frac{\xi}{2}(1-2\xi+2\zeta)$$

$$N_{4,\xi} = 2(1-\zeta)\eta \quad N_{4,\eta} = 2(1-\zeta)\xi \quad N_{4,\zeta} = -2\xi\eta$$

$$N_{5,\xi} = 0 \quad N_{5,\eta} = \frac{(1-\zeta)}{2}(-2+4\eta-\zeta) \quad N_{5,\zeta} = \frac{\eta}{2}(1-2\eta+2\zeta)$$

$$N_{6,\xi} = -2(1-\zeta)\eta \quad N_{6,\eta} = 2(1-\zeta)(1-\xi-2\eta) \quad N_{6,\zeta} = -2\eta(1-\xi-\eta)$$

$$N_{7,\xi} = -1+\zeta^2 \quad N_{7,\eta} = -1+\zeta^2 \quad N_{7,\zeta} = -2\zeta(1-\xi-\eta)$$

$$N_{8,\xi} = 1-\zeta^2 \quad N_{8,\eta} = 0 \quad N_{8,\zeta} = -2\zeta\xi$$

$$N_{9,\xi} = 0 \quad N_{9,\eta} = 1-\zeta^2 \quad N_{9,\zeta} = -2\zeta\eta$$

$$N_{10,\xi} = \frac{(1+\zeta)}{2}(4\xi + 4\eta - \zeta - 2) \quad N_{10,\eta} = \frac{(1+\zeta)}{2}(4\xi + 4\eta - \zeta - 2) \quad N_{10,\zeta} = \frac{(1-\xi-\eta)}{2}(-2\xi - 2\eta + 2\zeta + 1)$$

$$N_{11,\xi} = 2(1+\zeta)(1-2\xi-\eta) \quad N_{11,\eta} = -2(1+\zeta)\xi \quad N_{11,\zeta} = 2\xi(1-\xi-\eta)$$

$$N_{12,\xi} = \frac{(1+\zeta)}{2}(-2+4\xi+\zeta) \quad N_{12,\eta} = 0 \quad N_{12,\zeta} = \frac{\xi}{2}(-1+2\xi+2\zeta)$$

$$N_{13,\xi} = 2(1+\zeta)\eta \quad N_{13,\eta} = 2(1+\zeta)\xi \quad N_{13,\zeta} = 2\xi\eta$$

$$N_{14,\xi} = 0 \quad N_{14,\eta} = \frac{(1+\zeta)}{2}(-2+4\eta+\zeta) \quad N_{14,\zeta} = \frac{\eta}{2}(-1+2\eta+2\zeta)$$

$$N_{15,\xi} = -2(1+\zeta)\eta \quad N_{15,\eta} = 2(1+\zeta)(1-2\eta-\xi) \quad N_{15,\zeta} = 2\eta(1-\xi-\eta)$$

Therefore, in $(\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2, \hat{\mathbf{x}}_3) = (0,0,0)$, we have:

$$\underline{\mathbf{b}}^T{}_i = \begin{pmatrix} j_{i1} & j_{i2} & j_{i3} \end{pmatrix}.\begin{bmatrix} -1 & 2 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 2 \\ -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Moreover, one can check by algebraic considerations that the following conditions of orthogonality are satisfied:

$$\begin{cases} \underline{\mathbf{b}}_i^T.\underline{\mathbf{h}}_\alpha = 0; \quad \underline{\mathbf{b}}_i^T.\underline{\mathbf{S}} = 0; \quad \underline{\mathbf{b}}_i^T.\underline{\mathbf{x}}_j = \delta_{ij} \\ \underline{\mathbf{h}}_1^T.\underline{\mathbf{S}} = 0; \quad \underline{\mathbf{h}}_2^T.\underline{\mathbf{S}} = 0; \quad \underline{\mathbf{h}}_3^T.\underline{\mathbf{S}} = \frac{1}{2}; \quad \underline{\mathbf{h}}_4^T.\underline{\mathbf{S}} = 0; \quad \underline{\mathbf{h}}_5^T.\underline{\mathbf{S}} = 4; \quad \underline{\mathbf{h}}_6^T.\underline{\mathbf{S}} = 4; \\ \underline{\mathbf{h}}_7^T.\underline{\mathbf{S}} = 12; \quad \underline{\mathbf{h}}_8^T.\underline{\mathbf{S}} = 0 \quad \underline{\mathbf{h}}_9^T.\underline{\mathbf{S}} = 0 \quad \underline{\mathbf{h}}_{10}^T.\underline{\mathbf{S}} = 4 \quad \underline{\mathbf{h}}_{11}^T.\underline{\mathbf{S}} = 4 \\ \underline{\mathbf{h}}_m^T.\underline{\mathbf{h}}_n = \begin{bmatrix} 3 & -\frac{1}{2} & 0 & \frac{1}{4} & 0 & 0 & 0 & \frac{5}{2} & \frac{1}{4} & 0 & 0 \\ -\frac{1}{2} & 3 & 0 & \frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} & \frac{5}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{8} & 0 & \frac{1}{8} & \frac{1}{8} & \frac{1}{2} & 0 & 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{8} & 0 & 0 & 0 & \frac{1}{8} & \frac{1}{8} & 0 & 0 \\ 0 & 0 & \frac{1}{8} & 0 & \frac{13}{4} & \frac{1}{8} & 3 & 0 & 0 & \frac{5}{2} & \frac{1}{4} \\ 0 & 0 & \frac{1}{8} & 0 & \frac{1}{8} & \frac{13}{4} & 3 & 0 & 0 & \frac{1}{4} & \frac{5}{2} \\ 0 & 0 & \frac{1}{2} & 0 & 3 & 3 & 12 & 0 & 0 & 4 & 4 \\ \frac{5}{2} & \frac{1}{4} & 0 & \frac{1}{8} & 0 & 0 & 0 & \frac{9}{4} & \frac{1}{8} & 0 & 0 \\ \frac{1}{4} & \frac{5}{2} & 0 & \frac{1}{8} & 0 & 0 & 0 & \frac{1}{8} & \frac{9}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 & \frac{5}{2} & \frac{1}{4} & 4 & 0 & 0 & 3 & \frac{1}{2} \\ 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & \frac{5}{2} & 4 & 0 & 0 & \frac{1}{2} & 3 \end{bmatrix} \\ \qquad \alpha = 1,2,\ldots,11 \qquad\qquad i,j = 1,2,3 \\ \delta \quad désigne \quad le \quad symbol \quad de \quad Kronec\ker; m,n = 1,2,\ldots,11 \end{cases} \quad (15)$$

A this stage, one can determine the constant unknowns who intervene in the writing (10) of the field of displacement by multiplying scalairement the equation (11) by $\underline{\mathbf{b}}_j^T$, $\underline{\mathbf{S}}^T$ and $\underline{\mathbf{h}}_\alpha^T$ respectively, and by means of the relations of orthogonality (15).

One obtains after computations: $a_{ji} = \underline{\mathbf{b}}_j^T . \underline{\mathbf{d}}_i \qquad c_{\alpha i} = \underline{\boldsymbol{\gamma}}_\alpha^T . \underline{\mathbf{d}}_i$

with

$$
\begin{aligned}
\underline{\boldsymbol{\gamma}}_\alpha^T = {} & n_{\alpha 1}\left(\underline{\mathbf{h}}_1^T - \left(\underline{\mathbf{h}}_1^T . \underline{\mathbf{x}}_j\right)\underline{\mathbf{b}}_j^T\right) + n_{\alpha 2}\left(\underline{\mathbf{h}}_2^T - \left(\underline{\mathbf{h}}_2^T . \underline{\mathbf{x}}_j\right)\underline{\mathbf{b}}_j^T\right) + \\
& + n_{\alpha 3}\left[\left(\underline{\mathbf{h}}_3^T - \frac{1}{30}\underline{\mathbf{S}}^T\right) - \left(\left(\underline{\mathbf{h}}_3^T - \frac{1}{30}\underline{\mathbf{S}}^T\right) . \underline{\mathbf{x}}_j\right)\underline{\mathbf{b}}_j^T\right] + n_{\alpha 4}\left(\underline{\mathbf{h}}_4^T - \left(\underline{\mathbf{h}}_4^T . \underline{\mathbf{x}}_j\right)\underline{\mathbf{b}}_j^T\right) \\
& + n_{\alpha 5} . \left[\left(\underline{\mathbf{h}}_5^T - \frac{4}{15}\underline{\mathbf{S}}^T\right) - \left(\left(\underline{\mathbf{h}}_5^T - \frac{4}{15}\underline{\mathbf{S}}^T\right) . \underline{\mathbf{x}}_j\right)\underline{\mathbf{b}}_j^T\right] + n_{\alpha 6}\left(\left(\underline{\mathbf{h}}_6^T - \frac{4}{15}\underline{\mathbf{S}}^T\right) - \left(\left(\underline{\mathbf{h}}_6^T - \frac{4}{15}\underline{\mathbf{S}}^T\right) . \underline{\mathbf{x}}_j\right)\underline{\mathbf{b}}_j^T\right) \\
& + n_{\alpha 7}\left(\left(\underline{\mathbf{h}}_7^T - \frac{4}{5}\underline{\mathbf{S}}^T\right) - \left(\left(\underline{\mathbf{h}}_7^T - \frac{4}{5}\underline{\mathbf{S}}^T\right) . \underline{\mathbf{x}}_j\right)\underline{\mathbf{b}}_j^T\right) + n_{\alpha 8}\left(\underline{\mathbf{h}}_8^T - \left(\underline{\mathbf{h}}_8^T . \underline{\mathbf{x}}_j\right)\underline{\mathbf{b}}_j^T\right) + n_{\alpha 9}\left(\underline{\mathbf{h}}_9^T - \left(\underline{\mathbf{h}}_9^T . \underline{\mathbf{x}}_j\right)\underline{\mathbf{b}}_j^T\right) \\
& + n_{\alpha 10}\left(\left(\underline{\mathbf{h}}_{10}^T - \frac{4}{15}\underline{\mathbf{S}}^T\right) - \left(\left(\underline{\mathbf{h}}_{10}^T - \frac{4}{15}\underline{\mathbf{S}}^T\right) . \underline{\mathbf{x}}_j\right)\underline{\mathbf{b}}_j^T\right) + n_{\alpha 11}\left(\left(\underline{\mathbf{h}}_{11}^T - \frac{4}{15}\underline{\mathbf{S}}^T\right) - \left(\left(\underline{\mathbf{h}}_{11}^T - \frac{4}{15}\underline{\mathbf{S}}^T\right) . \underline{\mathbf{x}}_j\right)\underline{\mathbf{b}}_j^T\right)
\end{aligned}
$$

$$
\left[\mathbf{n}_{\alpha\beta}\right] = \begin{bmatrix}
\frac{17}{2} & 0 & 0 & -8 & 0 & 0 & 0 & -9 & 0 & 0 & 0 \\
0 & \frac{17}{2} & 0 & -8 & 0 & 0 & 0 & 0 & -9 & 0 & 0 \\
0 & 0 & \frac{256}{17} & 0 & \frac{36}{17} & \frac{36}{17} & 2 & 0 & 0 & -\frac{58}{17} & -\frac{58}{17} \\
-8 & -8 & 0 & 24 & 0 & 0 & 0 & 8 & 8 & 0 & 0 \\
0 & 0 & \frac{36}{17} & 0 & \frac{316}{187} & \frac{146}{187} & 1 & 0 & 0 & -\frac{324}{187} & -\frac{171}{187} \\
0 & 0 & \frac{36}{17} & 0 & \frac{146}{187} & \frac{316}{187} & 1 & 0 & 0 & -\frac{171}{187} & -\frac{324}{187} \\
0 & 0 & 2 & 0 & 1 & 1 & \frac{3}{2} & 0 & 0 & -\frac{3}{2} & -\frac{3}{2} \\
-9 & 0 & 0 & 8 & 0 & 0 & 0 & 10 & 0 & 0 & 0 \\
0 & -9 & 0 & 8 & 0 & 0 & 0 & 0 & 10 & 0 & 0 \\
0 & 0 & -\frac{58}{17} & 0 & -\frac{324}{187} & -\frac{171}{187} & -\frac{3}{2} & 0 & 0 & \frac{505}{187} & \frac{585}{374} \\
0 & 0 & -\frac{58}{17} & 0 & -\frac{171}{187} & -\frac{324}{187} & -\frac{3}{2} & 0 & 0 & \frac{585}{374} & \frac{505}{187}
\end{bmatrix} \qquad \alpha, \beta = 1, 2, ..., 11
$$

the field of displacement puts itself finally in the following form:

$$
\begin{aligned}
\mathrm{u}_i &= a_{0i} + a_{1i}x_1 + a_{2i}x_2 + a_{3i}x_3 + c_{1i}\,\mathrm{h}_1 + c_{2i}\,\mathrm{h}_2 + c_{3i}\,\mathrm{h}_3 + c_{4i}\,\mathrm{h}_4 + c_{5i}\,\mathrm{h}_5 + c_{6i}\,\mathrm{h}_6 \\
&\quad + c_{7i}\,\mathrm{h}_7 + c_{8i}\,\mathrm{h}_8 + c_{9i}\,\mathrm{h}_9 + c_{10i}\,\mathrm{h}_{10} + c_{11i}\,\mathrm{h}_{11} \\
\mathrm{u}_i &= a_{0i} + \underline{\mathbf{b}}_1^T.\underline{\mathbf{d}}_i x_1 + \underline{\mathbf{b}}_2^T.\underline{\mathbf{d}}_i x_2 + \underline{\mathbf{b}}_3^T.\underline{\mathbf{d}}_i x_3 + \underline{\gamma}_1^T.\underline{\mathbf{d}}_i\,\mathrm{h}_1 + \underline{\gamma}_2^T.\underline{\mathbf{d}}_i\,\mathrm{h}_2 + \\
&\quad + \underline{\gamma}_3^T.\underline{\mathbf{d}}_i\,\mathrm{h}_3 + \underline{\gamma}_4^T.\underline{\mathbf{d}}_i\,\mathrm{h}_4 + \underline{\gamma}_5^T.\underline{\mathbf{d}}_i\,\mathrm{h}_5 + \underline{\gamma}_6^T.\underline{\mathbf{d}}_i\,\mathrm{h}_6 + \underline{\gamma}_7^T.\underline{\mathbf{d}}_i\,\mathrm{h}_7 \\
&\quad + \underline{\gamma}_8^T.\underline{\mathbf{d}}_i\,\mathrm{h}_8 + \underline{\gamma}_9^T.\underline{\mathbf{d}}_i\,\mathrm{h}_9 + \underline{\gamma}_{10}^T.\underline{\mathbf{d}}_i\,\mathrm{h}_{10} + \underline{\gamma}_{11}^T.\underline{\mathbf{d}}_i\,\mathrm{h}_{11} \\
\mathrm{u}_i &= a_{0i} + (\underline{\mathbf{b}}_1^T x_1 + \underline{\mathbf{b}}_2^T x_2 + \underline{\mathbf{b}}_3^T x_3 + \underline{\gamma}_1^T\,\mathrm{h}_1 + \underline{\gamma}_2^T\,\mathrm{h}_2 + \underline{\gamma}_3^T\,\mathrm{h}_3 + \underline{\gamma}_4^T\,\mathrm{h}_4 + \underline{\gamma}_5^T\,\mathrm{h}_5 \\
&\quad + \underline{\gamma}_6^T\,\mathrm{h}_6 + \underline{\gamma}_7^T\,\mathrm{h}_7 + \underline{\gamma}_8^T\,\mathrm{h}_8 + \underline{\gamma}_9^T\,\mathrm{h}_9 + \underline{\gamma}_{10}^T\,\mathrm{h}_{10} + \underline{\gamma}_{11}^T\,\mathrm{h}_{11}).\underline{\mathbf{d}}_i
\end{aligned}
$$

(16)

By deriving the formula above compared to x_j, one obtains the displacement gradient:

$$u_{i,j} = \left(\boldsymbol{b}_j^t + \sum_{\alpha=1}^{11} h_{\alpha,j}\gamma_\alpha^T \right) \cdot \boldsymbol{d}_i \quad (17)$$

3.2.2 Element SHB20

the interpolation of the field of displacement of the element will enable us to define strain rate and to write the relations connecting the strains to nodal displacements. One starts initially by writing the gradient $\mathbf{u}_{i,j}$ of the field of displacement:

$$\mathbf{u}_{i,j} = \mathbf{u}_{iI}\mathbf{N}_{I,j} \quad (18)$$

the strain tensor $\boldsymbol{\varepsilon}_{\mathbf{ij}}$ is given then by the symmetric part of the displacement gradient:

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad (19)$$

to continue computations, one gives oneself quadratic isoparametric shape functions $\boldsymbol{N}_i(\hat{x}_1, \hat{x}_2, \hat{x}_3)$ associated with the hexahedral element with twenty nodes:

$$
\begin{aligned}
\mathbf{N}_1 &= \frac{1}{8}\left(1-\hat{\mathbf{x}}_1\right)\left(1-\hat{\mathbf{x}}_2\right)\left(1-\hat{\mathbf{x}}_3\right)\left(-2-\hat{\mathbf{x}}_1-\hat{\mathbf{x}}_2-\hat{\mathbf{x}}_3\right)\\
\mathbf{N}_2 &= \frac{1}{8}\left(1+\hat{\mathbf{x}}_1\right)\left(1-\hat{\mathbf{x}}_2\right)\left(1-\hat{\mathbf{x}}_3\right)\left(-2+\hat{\mathbf{x}}_1-\hat{\mathbf{x}}_2-\hat{\mathbf{x}}_3\right)\\
\mathbf{N}_3 &= \frac{1}{8}\left(1+\hat{\mathbf{x}}_1\right)\left(1+\hat{\mathbf{x}}_2\right)\left(1-\hat{\mathbf{x}}_3\right)\left(-2+\hat{\mathbf{x}}_1+\hat{\mathbf{x}}_2-\hat{\mathbf{x}}_3\right)\\
\mathbf{N}_4 &= \frac{1}{8}\left(1-\hat{\mathbf{x}}_1\right)\left(1+\hat{\mathbf{x}}_2\right)\left(1-\hat{\mathbf{x}}_3\right)\left(-2-\hat{\mathbf{x}}_1+\hat{\mathbf{x}}_2-\hat{\mathbf{x}}_3\right)\\
\mathbf{N}_5 &= \frac{1}{8}\left(1-\hat{\mathbf{x}}_1\right)\left(1-\hat{\mathbf{x}}_2\right)\left(1+\hat{\mathbf{x}}_3\right)\left(-2-\hat{\mathbf{x}}_1-\hat{\mathbf{x}}_2+\hat{\mathbf{x}}_3\right)\\
\mathbf{N}_6 &= \frac{1}{8}\left(1+\hat{\mathbf{x}}_1\right)\left(1-\hat{\mathbf{x}}_2\right)\left(1+\hat{\mathbf{x}}_3\right)\left(-2+\hat{\mathbf{x}}_1-\hat{\mathbf{x}}_2+\hat{\mathbf{x}}_3\right)\\
\mathbf{N}_7 &= \frac{1}{8}\left(1+\hat{\mathbf{x}}_1\right)\left(1+\hat{\mathbf{x}}_2\right)\left(1+\hat{\mathbf{x}}_3\right)\left(-2+\hat{\mathbf{x}}_1+\hat{\mathbf{x}}_2+\hat{\mathbf{x}}_3\right)\\
\mathbf{N}_8 &= \frac{1}{8}\left(1-\hat{\mathbf{x}}_1\right)\left(1+\hat{\mathbf{x}}_2\right)\left(1+\hat{\mathbf{x}}_3\right)\left(-2-\hat{\mathbf{x}}_1+\hat{\mathbf{x}}_2+\hat{\mathbf{x}}_3\right)\\
\mathbf{N}_9 &= \frac{1}{4}\left(1-\hat{\mathbf{x}}_1^{\,2}\right)\left(1-\hat{\mathbf{x}}_2\right)\left(1-\hat{\mathbf{x}}_3\right)\\
\mathbf{N}_{10} &= \frac{1}{4}\left(1-\hat{\mathbf{x}}_2^{\,2}\right)\left(1+\hat{\mathbf{x}}_1\right)\left(1-\hat{\mathbf{x}}_3\right)
\end{aligned}
\qquad (20)
$$

$$
\begin{aligned}
\mathbf{N}_{11} &= \frac{1}{4}\left(1-\hat{\mathbf{x}}_1^{\,2}\right)\left(1+\hat{\mathbf{x}}_2\right)\left(1-\hat{\mathbf{x}}_3\right) & \mathbf{N}_{12} &= \frac{1}{4}\left(1-\hat{\mathbf{x}}_2^{\,2}\right)\left(1-\hat{\mathbf{x}}_1\right)\left(1-\hat{\mathbf{x}}_3\right)\\
\mathbf{N}_{13} &= \frac{1}{4}\left(1-\hat{\mathbf{x}}_3^{\,2}\right)\left(1-\hat{\mathbf{x}}_1\right)\left(1-\hat{\mathbf{x}}_2\right) & \mathbf{N}_{14} &= \frac{1}{4}\left(1-\hat{\mathbf{x}}_3^{\,2}\right)\left(1+\hat{\mathbf{x}}_1\right)\left(1-\hat{\mathbf{x}}_2\right)\\
\mathbf{N}_{15} &= \frac{1}{4}\left(1-\hat{\mathbf{x}}_3^{\,2}\right)\left(1+\hat{\mathbf{x}}_1\right)\left(1+\hat{\mathbf{x}}_2\right) & \mathbf{N}_{16} &= \frac{1}{4}\left(1-\hat{\mathbf{x}}_3^{\,2}\right)\left(1-\hat{\mathbf{x}}_1\right)\left(1+\hat{\mathbf{x}}_2\right)\\
\mathbf{N}_{17} &= \frac{1}{4}\left(1-\hat{\mathbf{x}}_1^{\,2}\right)\left(1-\hat{\mathbf{x}}_2\right)\left(1+\hat{\mathbf{x}}_3\right) & \mathbf{N}_{18} &= \frac{1}{4}\left(1-\hat{\mathbf{x}}_2^{\,2}\right)\left(1+\hat{\mathbf{x}}_1\right)\left(1+\hat{\mathbf{x}}_3\right)\\
\mathbf{N}_{19} &= \frac{1}{4}\left(1-\hat{\mathbf{x}}_1^{\,2}\right)\left(1+\hat{\mathbf{x}}_2\right)\left(1+\hat{\mathbf{x}}_3\right) & \mathbf{N}_{20} &= \frac{1}{4}\left(1-\hat{\mathbf{x}}_2^{\,2}\right)\left(1-\hat{\mathbf{x}}_1\right)\left(1+\hat{\mathbf{x}}_3\right)
\end{aligned}
$$

$$
\hat{\mathbf{x}}_1=[-1,1];\qquad \hat{\mathbf{x}}_2=[-1,1];\qquad \hat{\mathbf{x}}_3=[-1,1]
$$

While combining the preceding equations one manages to develop the field of displacement as being the sum of a constant term, linear terms in x_i , and terms utilizing the functions h_α □

$$\begin{cases} u_i = a_{0i} + a_{1i} x_1 + a_{2i} x_2 + a_{3i} x_3 + c_{1i}\,\mathrm{h}_1 + c_{2i}\,\mathrm{h}_2 + c_{3i}\,\mathrm{h}_3 + c_{4i}\,\mathrm{h}_4 + c_{5i}\,\mathrm{h}_5 + c_{6i}\,\mathrm{h}_6 + c_{7i}\,\mathrm{h}_7 \\ \qquad + c_{8i}\,\mathrm{h}_8 + c_{9i}\,\mathrm{h}_9 + c_{10i}\,\mathrm{h}_{10} + c_{11i}\,\mathrm{h}_{11} + c_{12i}\,\mathrm{h}_{12} + c_{13i}\,\mathrm{h}_{13} + c_{14i}\,\mathrm{h}_{14} + c_{15i}\,\mathrm{h}_{15} + c_{16i}\,\mathrm{h}_{16} \\ i = 1,2,3 \\ \mathrm{h}_1 = \xi\zeta, \mathrm{h}_2 = \eta\zeta, \mathrm{h}_3 = \xi\eta, \mathrm{h}_4 = \xi^2, \mathrm{h}_5 = \eta^2, \mathrm{h}_6 = \zeta^2, \mathrm{h}_7 = \xi\eta\zeta, \mathrm{h}_8 = \xi^2\eta, \mathrm{h}_9 = \xi^2\zeta, \\ \mathrm{h}_{10} = \eta^2\xi, \mathrm{h}_{11} = \eta^2\zeta, \mathrm{h}_{12} = \zeta^2\xi, \mathrm{h}_{13} = \zeta^2\eta, \mathrm{h}_{14} = \xi^2\eta\zeta, \mathrm{h}_{15} = \xi\eta^2\zeta, \mathrm{h}_{16} = \xi\eta\zeta^2 \end{cases} \quad (21)$$

to simplify the writings, one will note $\xi = \hat{x}_1$, $\eta = \hat{x}_2$, $\zeta = \hat{x}_3$

By evaluating equation **16** with the nodes of the element, one arrives at the three systems of twenty equations following:

$$\begin{cases} \underline{\mathbf{d}}_i = a_{0i}\underline{\mathbf{S}} + a_{1i}\underline{\mathbf{x}}_1 + a_{2i}\underline{\mathbf{x}}_2 + a_{3i}\underline{\mathbf{x}}_3 + c_{1i}\underline{\mathbf{h}}_1 + c_{2i}\underline{\mathbf{h}}_2 + c_{3i}\underline{\mathbf{h}}_3 + \ldots + c_{16i}\underline{\mathbf{h}}_{16} \\ i = 1,2,3 \end{cases} \quad (22)$$

Thus the vectors $\underline{\mathbf{d}}_{\mathrm{i}}$ and $\underline{\mathbf{x}}_{\mathrm{i}}$ represent, respectively, displacements and the coordinated nodal and are given by:

$$\begin{cases} \underline{\mathbf{d}}_i^T = (u_{i1}, u_{i2}, u_{i3}, \ldots, u_{i20}) \\ \underline{\mathbf{x}}_i^T = (x_{i1}, x_{i2}, x_{i3}, \ldots, x_{i20}) \end{cases} \quad (23)$$

the vectors $\underline{\mathbf{S}}$ and $\underline{\mathbf{h}}_\alpha$ $(\alpha = 1,2,3,\ldots,16)$ are given as for them by:

$$\begin{cases} \underline{\mathbf{S}}^T = (1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1) \\ \underline{\mathbf{h}}_1^T = (1\ -1\ -1\ 1\ -1\ 1\ 1\ -1\ 0\ -1\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ -1) \\ \underline{\mathbf{h}}_2^T = (1\ 1\ -1\ -1\ -1\ -1\ 1\ 1\ 1\ 0\ -1\ 0\ 0\ 0\ 0\ 0\ -1\ 0\ 1\ 0) \\ \underline{\mathbf{h}}_3^T = (1\ -1\ 1\ -1\ 1\ -1\ 1\ -1\ 0\ 0\ 0\ 0\ 1\ -1\ 1\ -1\ 0\ 0\ 0\ 0) \\ \underline{\mathbf{h}}_4^T = (1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 0\ 1\ 0\ 1\ 1\ 1\ 1\ 1\ 0\ 1\ 0\ 1) \\ \underline{\mathbf{h}}_5^T = (1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 0\ 1\ 0\ 1\ 1\ 1\ 1\ 1\ 0\ 1\ 0) \\ \underline{\mathbf{h}}_6^T = (1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 1) \\ \underline{\mathbf{h}}_7^T = (-1\ 1\ -1\ 1\ 1\ -1\ 1\ -1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) \\ \underline{\mathbf{h}}_8^T = (-1\ -1\ 1\ 1\ -1\ -1\ 1\ 1\ 0\ 0\ 0\ 0\ -1\ -1\ 1\ 1\ 0\ 0\ 0\ 0) \\ \underline{\mathbf{h}}_9^T = (-1\ -1\ -1\ -1\ 1\ 1\ 1\ 1\ 0\ -1\ 0\ -1\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 1) \\ \underline{\mathbf{h}}_{10}^T = (-1\ 1\ 1\ -1\ -1\ 1\ 1\ -1\ 0\ 0\ 0\ 0\ -1\ 1\ 1\ -1\ 0\ 0\ 0\ 0) \\ \underline{\mathbf{h}}_{11}^T = (-1\ -1\ -1\ -1\ 1\ 1\ 1\ 1\ -1\ 0\ -1\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 1\ 0) \\ \underline{\mathbf{h}}_{12}^T = (-1\ 1\ 1\ -1\ -1\ 1\ 1\ -1\ 0\ 1\ 0\ -1\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ -1) \\ \underline{\mathbf{h}}_{13}^T = (-1\ -1\ 1\ 1\ -1\ -1\ 1\ 1\ -1\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ -1\ 0\ 1\ 0) \\ \underline{\mathbf{h}}_{14}^T = (1\ 1\ -1\ -1\ -1\ -1\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) \\ \underline{\mathbf{h}}_{15}^T = (1\ -1\ -1\ 1\ -1\ 1\ 1\ -1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) \\ \underline{\mathbf{h}}_{16}^T = (1\ -1\ 1\ -1\ 1\ -1\ 1\ -1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) \end{cases} \quad (24)$$

to arrive at an advantageous writing of the operator discretized gradient $\mathbf{B}$, one will introduce the three vectors $\mathbf{b_i}$ defined by:

$$\mathbf{b_i^T} = \mathbf{N^T}_{,\mathbf{i}}(0) = {}^{\mathbf{T}}\frac{\partial \mathbf{N}}{\partial \mathbf{x_i}}(0) \quad \mathrm{I} = 1,2,3 \qquad (25)$$

If we place ourselves in $(\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2, \hat{\mathbf{x}}_3) = (0,0,0)$ then we obtain:

$$\mathbf{b_i^T} = \mathbf{N^T}_{,\mathbf{i}}(0) = \mathbf{cste}$$

where $\mathbf{N^T}$ represents: $\left(\mathrm{N}_1 \;\; \mathrm{N}_2 \;\; \mathrm{N}_3 \ldots \mathrm{N}_{20}\right)$

$$\mathbf{b_i^T} = \mathbf{N^T}_{,\mathbf{i}}(0) = \left(\frac{\partial \mathrm{N}_I}{\partial \mathrm{x_i}}(0), \ldots\ldots, \frac{\partial \mathrm{N}_{20}}{\partial \mathrm{x_i}}(0)\right)$$

$$\frac{\partial \mathrm{N}_I}{\partial x_{\mathrm{j}}} = \left(\frac{\partial \mathrm{N}_I}{\partial \xi} \cdot \frac{\partial \xi}{\partial x_{\mathrm{j}}} + \frac{\partial \mathrm{N}_I}{\partial \eta} \cdot \frac{\partial \eta}{\partial x_{\mathrm{j}}} + \frac{\partial \mathrm{N}_I}{\partial \zeta} \cdot \frac{\partial \zeta}{\partial x_{\mathrm{j}}} \right)_{\xi=\eta=\zeta=0} = \left(\frac{\partial \mathrm{N}_I}{\partial \xi} \cdot j_{1j} + \frac{\partial \mathrm{N}_I}{\partial \eta} \cdot j_{2j} + \frac{\partial \mathrm{N}_I}{\partial \zeta} \cdot j_{3j} \right)_{\xi=\eta=\zeta=0}$$

$$avec \quad I = 1,2,\ldots,20 \quad et \quad j = 1,2,3$$

$$\mathbf{F}^{-1}{}_{|\xi=\eta=\zeta=0} = \begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} & \frac{\partial \xi}{\partial z} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} & \frac{\partial \eta}{\partial z} \\ \frac{\partial \zeta}{\partial x} & \frac{\partial \zeta}{\partial y} & \frac{\partial \zeta}{\partial z} \end{pmatrix}_{\xi=\eta=\zeta=0} = \begin{pmatrix} j_{11} & j_{12} & j_{13} \\ j_{21} & j_{22} & j_{23} \\ j_{31} & j_{32} & j_{33} \end{pmatrix}$$

After computations one finds:

$$\underline{\mathbf{b}}_i^T = \left(j_{i1} \;\; j_{i2} \;\; j_{i3} \right) \cdot \begin{bmatrix} \frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & \frac{1}{8} & 0 & \frac{1}{4} & 0 & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & 0 & -\frac{1}{4} \\ \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{4} & 0 & \frac{1}{4} & 0 & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & 0 \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

Moreover, one can check by algebraic considerations that the following conditions of orthogonality are satisfied:

$$\underline{\mathbf{h}}_m^T.\underline{\mathbf{h}}_n=\begin{bmatrix}
\mathbf{12}&0&0&0&0&0&0&0&0&0&0&0&0&0&\mathbf{8}&0\\
0&\mathbf{12}&0&0&0&0&0&0&0&0&0&0&0&\mathbf{8}&0&0\\
0&0&\mathbf{12}&0&0&0&0&0&0&0&0&0&0&0&0&\mathbf{8}\\
0&0&0&\mathbf{16}&\mathbf{12}&\mathbf{12}&0&0&0&0&0&0&0&0&0&0\\
0&0&0&\mathbf{12}&\mathbf{16}&\mathbf{12}&0&0&0&0&0&0&0&0&0&0\\
0&0&0&\mathbf{12}&\mathbf{12}&\mathbf{16}&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&\mathbf{8}&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&\mathbf{12}&0&0&0&0&\mathbf{8}&0&0&0\\
0&0&0&0&0&0&0&0&\mathbf{12}&0&\mathbf{8}&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&\mathbf{12}&0&\mathbf{8}&0&0&0&0\\
0&0&0&0&0&0&0&0&\mathbf{8}&0&\mathbf{12}&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&\mathbf{8}&0&\mathbf{12}&0&0&0&0\\
0&0&0&0&0&0&0&\mathbf{8}&0&0&0&0&\mathbf{12}&0&0&0\\
0&\mathbf{8}&0&0&0&0&0&0&0&0&0&0&0&\mathbf{8}&0&0\\
\mathbf{8}&0&0&0&0&0&0&0&0&0&0&0&0&0&\mathbf{8}&0\\
0&0&\mathbf{8}&0&0&0&0&0&0&0&0&0&0&0&0&\mathbf{8}
\end{bmatrix}$$

$$m,n=1,2,...,16$$

$$\begin{aligned}
&\underline{\mathbf{b}}_i^T.\underline{\mathbf{h}}_\alpha=0;\quad \underline{\mathbf{b}}_i^T.\underline{\mathbf{S}}=0;\ \underline{\mathbf{b}}_i^T.\underline{\mathbf{x}}_j=\delta_{ij}\\
&\underline{\mathbf{h}}_1^T.\underline{\mathbf{S}}=0;\ \underline{\mathbf{h}}_2^T.\underline{\mathbf{S}}=0;\ \underline{\mathbf{h}}_3^T.\underline{\mathbf{S}}=0;\ \underline{\mathbf{h}}_4^T.\underline{\mathbf{S}}=16;\ \underline{\mathbf{h}}_5^T.\underline{\mathbf{S}}=16;\ \underline{\mathbf{h}}_6^T.\underline{\mathbf{S}}=16;\ \underline{\mathbf{h}}_7^T.\underline{\mathbf{S}}=0;\ \underline{\mathbf{h}}_8^T.\underline{\mathbf{S}}=0\\
&\underline{\mathbf{h}}_9^T.\underline{\mathbf{S}}=0\ \ \underline{\mathbf{h}}_{10}^T.\underline{\mathbf{S}}=0;\ \underline{\mathbf{h}}_{11}^T.\underline{\mathbf{S}}=0\ \ \underline{\mathbf{h}}_{12}^T.\underline{\mathbf{S}}=0\ \ \underline{\mathbf{h}}_{13}^T.\underline{\mathbf{S}}=0\ \ \underline{\mathbf{h}}_{14}^T.\underline{\mathbf{S}}=0\ \ \underline{\mathbf{h}}_{15}^T.\underline{\mathbf{S}}=0\ \ \underline{\mathbf{h}}_{16}^T.\underline{\mathbf{S}}=0
\end{aligned}\qquad \textbf{(26)}$$

A this stage, one can determine the constant unknowns who intervene in the writing **(21)** of the field of displacement by multiplying scalairement the equation (22) by $\underline{\mathbf{b}}_j^T$, $\underline{\mathbf{S}}^T$ and $\underline{\mathbf{h}}_\alpha^T$ respectively, and by means of the relations of orthogonality **(26**). One obtains:

$$a_{ji}=\underline{\mathbf{b}}_j^T.\underline{\mathbf{d}}_i \qquad\qquad c_{\alpha i}=\underline{\boldsymbol{\gamma}}_\alpha^T.\underline{\mathbf{d}}_i$$

with:

$$\begin{aligned}
\underline{\boldsymbol{\gamma}}_\alpha^T=\ &n_{\alpha 1}\left(\underline{\mathbf{h}}_1^T-\left(\underline{\mathbf{h}}_1^T.\underline{\mathbf{x}}_j\right)\underline{\mathbf{b}}_j^T\right)+n_{\alpha 2}\left(\underline{\mathbf{h}}_2^T-\left(\underline{\mathbf{h}}_2^T.\underline{\mathbf{x}}_j\right)\underline{\mathbf{b}}_j^T\right)+n_{\alpha 3}\left(\underline{\mathbf{h}}_3^T-\left(\underline{\mathbf{h}}_3^T.\underline{\mathbf{x}}_j\right)\underline{\mathbf{b}}_j^T\right)\\
&+n_{\alpha 4}\left[\left(\underline{\mathbf{h}}_4^T-\frac{4}{5}\underline{\mathbf{S}}^T\right)-\left(\left(\underline{\mathbf{h}}_4^T-\frac{4}{5}\underline{\mathbf{S}}^T\right).\underline{\mathbf{x}}_j\right)\underline{\mathbf{b}}_j^T\right]+n_{\alpha 5}\left[\left(\underline{\mathbf{h}}_5^T-\frac{4}{5}\underline{\mathbf{S}}^T\right)-\left(\left(\underline{\mathbf{h}}_5^T-\frac{4}{5}\underline{\mathbf{S}}^T\right).\underline{\mathbf{x}}_j\right)\underline{\mathbf{b}}_j^T\right]+\\
&+n_{\alpha 6}\left[\left(\underline{\mathbf{h}}_6^T-\frac{4}{5}\underline{\mathbf{S}}^T\right)-\left(\left(\underline{\mathbf{h}}_6^T-\frac{4}{5}\underline{\mathbf{S}}^T\right).\underline{\mathbf{x}}_j\right)\underline{\mathbf{b}}_j^T\right]+n_{\alpha 7}\left(\underline{\mathbf{h}}_7^T-\left(\underline{\mathbf{h}}_7^T.\underline{\mathbf{x}}_j\right)\underline{\mathbf{b}}_j^T\right)\\
&+n_{\alpha 8}\left(\underline{\mathbf{h}}_8^T-\left(\underline{\mathbf{h}}_8^T.\underline{\mathbf{x}}_j\right)\underline{\mathbf{b}}_j^T\right)+n_{\alpha 9}\left(\underline{\mathbf{h}}_9^T-\left(\underline{\mathbf{h}}_9^T.\underline{\mathbf{x}}_j\right)\underline{\mathbf{b}}_j^T\right)+n_{\alpha 10}\left(\underline{\mathbf{h}}_{10}^T-\left(\underline{\mathbf{h}}_{10}^T.\underline{\mathbf{x}}_j\right)\underline{\mathbf{b}}_j^T\right)\\
&+n_{\alpha 11}\left(\underline{\mathbf{h}}_{11}^T-\left(\underline{\mathbf{h}}_{11}^T.\underline{\mathbf{x}}_j\right)\underline{\mathbf{b}}_j^T\right)+n_{\alpha 12}\left(\underline{\mathbf{h}}_{12}^T-\left(\underline{\mathbf{h}}_{12}^T.\underline{\mathbf{x}}_j\right)\underline{\mathbf{b}}_j^T\right)+n_{\alpha 13}\left(\underline{\mathbf{h}}_{13}^T-\left(\underline{\mathbf{h}}_{13}^T.\underline{\mathbf{x}}_j\right)\underline{\mathbf{b}}_j^T\right)\\
&+n_{\alpha 14}\left(\underline{\mathbf{h}}_{14}^T-\left(\underline{\mathbf{h}}_{14}^T.\underline{\mathbf{x}}_j\right)\underline{\mathbf{b}}_j^T\right)+n_{\alpha 15}\left(\underline{\mathbf{h}}_{15}^T-\left(\underline{\mathbf{h}}_{15}^T.\underline{\mathbf{x}}_j\right)\underline{\mathbf{b}}_j^T\right)+n_{\alpha 16}\left(\underline{\mathbf{h}}_{16}^T-\left(\underline{\mathbf{h}}_{16}^T.\underline{\mathbf{x}}_j\right)\underline{\mathbf{b}}_j^T\right)
\end{aligned}$$

$$\left[\mathbf{n}_{\alpha\beta}\right]=\begin{bmatrix}
\frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{4} & 0\\
0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{4} & 0 & 0\\
0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{4}\\
0 & 0 & 0 & \frac{3}{8} & \frac{1}{8} & \frac{1}{8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & \frac{1}{8} & \frac{3}{8} & \frac{1}{8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & \frac{1}{8} & \frac{1}{8} & \frac{3}{8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{20} & 0 & 0 & 0 & 0 & -\frac{1}{10} & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{20} & 0 & -\frac{1}{10} & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{20} & 0 & -\frac{1}{10} & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{10} & 0 & \frac{3}{20} & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{10} & 0 & \frac{3}{20} & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{10} & 0 & 0 & 0 & 0 & \frac{3}{20} & 0 & 0 & 0\\
0 & -\frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{8} & 0 & 0\\
-\frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{8} & 0\\
0 & 0 & -\frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{8}
\end{bmatrix} \qquad \alpha,\beta=1,2,\ldots,16$$

For 2 elements SHB15 and SHB20, the operator discretized gradient connecting the strain tensor to the vector of nodal displacements is given by:

$$\underline{\underline{\nabla}}_S\left(\underline{\mathbf{u}}\right)=\underline{\underline{\mathbf{B}}}.\underline{\mathbf{d}} \tag{27}$$

where:

$$\underline{\underline{\nabla}}_S\left(\underline{\mathbf{u}}\right)=\begin{bmatrix} u_{x,x}\\ u_{y,y}\\ u_{z,z}\\ u_{x,y}+u_{y,x}\\ u_{x,z}+u_{z,x}\\ u_{y,z}+u_{z,y}\end{bmatrix} \quad , \quad \underline{\mathbf{d}}=\begin{bmatrix}\underline{\mathbf{d}}_1\\ \underline{\mathbf{d}}_2\\ \underline{\mathbf{d}}_3\end{bmatrix} \tag{28}$$

and takes the practical matric shape then:

$$\underline{\underline{\mathbf{B}}} = \begin{bmatrix} \underline{\mathbf{b}}_1^T + \mathrm{h}_{\alpha,x_1} \underline{\gamma}_\alpha^T & \underline{0} & \underline{0} \\ \underline{0} & \underline{\mathbf{b}}_2^T + \mathrm{h}_{\alpha,x_2} \underline{\gamma}_\alpha^T & \underline{0} \\ \underline{0} & \underline{0} & \underline{\mathbf{b}}_3^T + \mathrm{h}_{\alpha,x_3} \underline{\gamma}_\alpha^T \\ \underline{\mathbf{b}}_2^T + \mathrm{h}_{\alpha,x_2} \underline{\gamma}_\alpha^T & \underline{\mathbf{b}}_1^T + \mathrm{h}_{\alpha,x_1} \underline{\gamma}_\alpha^T & \underline{0} \\ \underline{\mathbf{b}}_3^T + \mathrm{h}_{\alpha,x_3} \underline{\gamma}_\alpha^T & \underline{0} & \underline{\mathbf{b}}_1^T + \mathrm{h}_{\alpha,x_1} \underline{\gamma}_\alpha^T \\ \underline{0} & \underline{\mathbf{b}}_3^T + \mathrm{h}_{\alpha,x_3} \underline{\gamma}_\alpha^T & \underline{\mathbf{b}}_2^T + \mathrm{h}_{\alpha,x_2} \underline{\gamma}_\alpha^T \end{bmatrix} \quad \textbf{(29)}$$

This writing of the operator discretized gradient using the formulas of Hallquist [4] is very convenient because the vectors $\underline{\gamma}_\alpha$, which intervene in the statement of $\underline{\underline{\mathbf{B}}}$, check the following conditions of orthogonality:

$$\underline{\gamma}_\alpha^T . \underline{\mathbf{x}}_j = 0 \quad , \quad \underline{\gamma}_\alpha^T . \underline{\mathbf{h}}_\beta = \delta_{\alpha\beta} \quad (30)$$

This makes it possible to separately handle each mode of the strain to obtain the shape of the strain field simply applied. Let us note that an element based on the formulation (29) is convergent when it is evaluated exactly. However, the evaluating of this operator $\underline{\underline{\mathbf{B}}}$, given in (29), of each point of integration makes this element expensive in computing times for the practical applications, and the simplified shape of this element is essential.

3.3 Variational formulation used for elements SHB15 and SHB20

the extension of the weak form of the variational principle of Hu-Washizu to the case of the mechanics of nonlinear solids is due to Fish and Belytschko [6]. For a simple element, one a:

$$\delta \pi (\boldsymbol{u}, \dot{\boldsymbol{\epsilon}}, \bar{\boldsymbol{\sigma}}) = \int_{V_e} \delta \dot{\boldsymbol{\epsilon}} \cdot \bar{\boldsymbol{\sigma}} \, dV + \delta \int_{V_e} \bar{\boldsymbol{\sigma}} \cdot (\nabla_s \boldsymbol{u} - \dot{\boldsymbol{\epsilon}}) dV - \delta \boldsymbol{d}^T \cdot \boldsymbol{f}^{ext} = 0 \quad \textbf{(31)}$$

where δ a variation represents, $\underline{\mathbf{u}}$ the field of displacement, $\dot{\boldsymbol{\epsilon}}$ is strain rate applied, $\bar{\underline{\underline{\boldsymbol{\sigma}}}}$ the applied stress, $\underline{\underline{\boldsymbol{\sigma}}}$ the stress evaluated by the constitutive law, $\underline{\mathbf{d}}$ external nodal displacements $\underline{\mathbf{f}}^{ext}$, nodal forces, and $\nabla_s \boldsymbol{u}$ the symmetric part of the gradient of the field of displacement.

The formulation "*Assumed strain* " (projection of the operator gradient discretized $\underline{\underline{\mathbf{B}}}$ on under suitable space in order to avoiding the various problems of blocking) is based on a simplified form of the variational principle of Hu-Washizu as it was described by Simo and Hughes [7]. In this simplified form, the applied stress is selected orthogonal with the difference between the applied symmetric part of the displacement gradient and strain rate. Thus, the second term in the equation **(31**) is eliminated and one obtains:

$$\delta \pi (\boldsymbol{u}, \dot{\boldsymbol{\epsilon}}, \bar{\boldsymbol{\sigma}}) = \int_{V_e} \delta \dot{\boldsymbol{\epsilon}} \cdot \boldsymbol{\sigma} \, dV - \delta \boldsymbol{d}^T \cdot \boldsymbol{f}^{ext} = 0 \quad (32)$$

In this form, the variational principle is independent of the interpolation of the stress, since the applied stress does not intervene any more and thus need does not have to be defined. The discretized equations thus require the only interpolation of displacement $\underline{\mathbf{u}}$ and strain rate applied $\dot{\boldsymbol{\epsilon}}$ in the element. With the preceding vectorial notations one a:

$$\boldsymbol{u}(x,t) = \sum_{i=1}^{15} \boldsymbol{d}_{\boldsymbol{I}}(t) N_I(x) \text{ (element SHB15)}$$

Or $$\boldsymbol{u}(x,t)=\sum_{i=1}^{20}\boldsymbol{d}_{\boldsymbol{I}}(t)N_I(x)$$ (element SHB20) (33)

This led to:

$$\nabla_s\boldsymbol{u}(x,t)=\boldsymbol{B}(x)\boldsymbol{d}(t) \quad (34)$$

the applied strain is defined as for it by:

$$\dot{\boldsymbol{\epsilon}}(x,t)=\bar{\boldsymbol{B}}(x)\boldsymbol{d}(t) \quad (35)$$

Replacing the statement (35) in the variational principle (32), one obtains:

$$\delta\boldsymbol{d}^{\boldsymbol{T}}\left(\int_{V_e}\bar{\boldsymbol{B}}\cdot\boldsymbol{\sigma}\,dV-\boldsymbol{F}^{\boldsymbol{ext}}\right)=0 \quad (36)$$

As $\delta\underline{d}$ can be arbitrarily selected, the preceding equation leads to:

$$\underline{\mathbf{f}}^{\text{int}}=\underline{\mathbf{f}}^{ext} \quad (37)$$

with:

$$\boldsymbol{f}^{\mathbf{int}}=\int_{V_e}\bar{\boldsymbol{B}}(x)\cdot\boldsymbol{\sigma}(\dot{\boldsymbol{\epsilon}})\,dV \quad (38)$$

In the equation above, it is well specified that the stress $\underline{\underline{\boldsymbol{\sigma}}}$ is calculated by the applied constitutive law starting from strain rate. $\dot{\boldsymbol{\epsilon}}$ For the nonlinear problems, $\underline{\underline{\boldsymbol{\sigma}}}$ can also be an integral function of strain rate applied and other local variables:

$$\boldsymbol{\sigma}=F(\dot{\boldsymbol{\epsilon}},\boldsymbol{\alpha},...) \quad (39)$$

where $\underline{\underline{\boldsymbol{\alpha}}}$ represents the local variables. The formulation thus obtained is valid for problems including the two types of nonlinearities: geometrical and material. In the case of linear problems, one a:

$$\underline{\underline{\boldsymbol{\sigma}}}=\underline{\underline{\mathbf{C}}}.\underline{\bar{\boldsymbol{\varepsilon}}}=\underline{\underline{\mathbf{C}}}.\underline{\underline{\bar{\mathbf{B}}}}.\underline{\mathbf{d}} \quad (40)$$

the elastic matrix of behavior $\underline{\underline{\mathbf{C}}}$, in the case of an isotropic material, is selected like following:

$$\underline{\underline{\mathbf{C}}}=\begin{bmatrix}\frac{E}{1-\nu^2} & \frac{E\nu}{1-\nu^2} & 0 & 0 & 0 & 0\\ \frac{E\nu}{1-\nu^2} & \frac{E}{1-\nu^2} & 0 & 0 & 0 & 0\\ 0 & 0 & E & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{E}{2(1+\nu)} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{E}{2(1+\nu)} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{E}{2(1+\nu)}\end{bmatrix}$$

In this matrix, E is the modulus Young and v is the Poisson's ratio. This model is specific with elements SHB. It resembles that which one would have in the case of the assumption of the plane stresses, put except for the term (3,3). We can note that this choice involves an artificial anisotropic behavior. This choice makes it possible to satisfy all the tests without introducing blocking.

The internal forces of the element are written then simply in terms of the elemental stiffness matrix:

$$\boldsymbol{f}^{\mathrm{int}}=\boldsymbol{K}_e\cdot\boldsymbol{d} \qquad (41)$$

where:

$$\boldsymbol{K}_e=\int_{V_e}\overline{\boldsymbol{B}}^T\cdot\boldsymbol{C}\cdot\overline{\boldsymbol{B}}\,dV \qquad (42)$$

In a standard approach in displacement, strain rate applied is identified with the symmetric part of the gradient velocity, which returns to remplacerpardans $\overline{\boldsymbol{B}}$ $\boldsymbol{B}$ the preceding statements. One thus obtains simply:

$$\boldsymbol{K}_e=\int_{V_e}\boldsymbol{B}^T\cdot\boldsymbol{C}\cdot\boldsymbol{B}\,dV \qquad (43)$$

the prismatic element with 15 nodes, named “SHB15”, has 15 points of integration. Their coordinates (ξ,η,ζ) and their weights of integration are the roots of the polynomial of Gauss-Legendre given in the following table:

Not deGauss formulef ormulefo rmule	ξ	η	ζ	$\mathrm{w}(\xi,\eta,\zeta)$
P (1)	1/21/2		$\xi_{G1}=-0.906179845938664$	0.236926885056189/6
P (2)	1/21/2		$\xi_{G2}=-0.538469310105683$	0.478628670499366/6
P (3)	1/21/2		$\xi_{G3}=0$	0.568888888888889/6
P (4)	1/21/2		$\xi_{G4}=0.538469310105683$	0.478628670499366/6
P (5)	1/21/2		$\xi_{G5}=0.906179845938664$	0.236926885056189/6
P (6)	01/2		$\xi_{G6}=-0.906179845938664$	0.236926885056189/6
P (7)	01/2		$\xi_{G7}=-0.538469310105683$	0.478628670499366/6
P (8)	01/2		$\xi_{G8}=0$	0.568888888888889/6
P (9)	01/2		$\xi_{G9}=0.538469310105683$	0.478628670499366/6
P (10)	01/2		$\xi_{G10}=0.906179845938664$	0.236926885056189/6
P (11)	1/20		$\xi_{G11}=-0.906179845938664$	0.236926885056189/6
P (12)	1/20		$\xi_{G12}=-0.538469310105683$	0.478628670499366/6
P (13)	1/20		$\xi_{G13}=0$	0.568888888888889/6
P (14)	1/20		$\xi_{G14}=0.538469310105683$	0.478628670499366/6
P (15)	1/20		$\xi_{G15}=0.906179845938664$	0.236926885056189/6

Thus, the statement of the stiffness is: $\boldsymbol{K}_e=\sum_{j=1}^{15} w(P(j))\,J(P(j))\,\overline{\boldsymbol{B}}^T(P(j))\cdot\boldsymbol{C}\cdot\overline{\boldsymbol{B}}(P(j))$

The coordinates of Gauss points and their weights for element SHB20 are given in the table below:

Not deGauss	ξ	η	ζ	$w(\xi,\eta,\zeta)$
P (1)	$-1/\sqrt{3}$	$-1/\sqrt{3}$	$\zeta_{G1}=-0.906179845938664$	0.236926885056189
P (2)	$-1/\sqrt{3}$	$-1/\sqrt{3}$	$\zeta_{G2}=-0.538469310105683$	0.478628670499366
P (3)	$-1/\sqrt{3}$	$-1/\sqrt{3}$	$\zeta_{G3}=0$	0.568888888888889
P (4)	$-1/\sqrt{3}$	$-1/\sqrt{3}$	$\zeta_{G4}=0.538469310105683$	0.478628670499366
P (5)	$-1/\sqrt{3}$	$-1/\sqrt{3}$	$\zeta_{G5}=0.906179845938664$	0.236926885056189
P (6)	$1/\sqrt{3}$	$-1/\sqrt{3}$	$\zeta_{G6}=-0.906179845938664$	0.236926885056189
P (7)	$1/\sqrt{3}$	$-1/\sqrt{3}$	$\zeta_{G7}=-0.538469310105683$	0.478628670499366
P (8)	$1/\sqrt{3}$	$-1/\sqrt{3}$	$\zeta_{G8}=0$	0.568888888888889
P (9)	$1/\sqrt{3}$	$-1/\sqrt{3}$	$\zeta_{G9}=0.538469310105683$	0.478628670499366
P (10)	$1/\sqrt{3}$	$-1/\sqrt{3}$	$\zeta_{G10}=0.906179845938664$	0.236926885056189
P (11)	$1/\sqrt{3}$	$1/\sqrt{3}$	$\zeta_{G11}=-0.906179845938664$	0.236926885056189
P (12)	$1/\sqrt{3}$	$1/\sqrt{3}$	$\zeta_{G12}=-0.538469310105683$	0.478628670499366
P (13)	$1/\sqrt{3}$	$1/\sqrt{3}$	$\zeta_{G13}=0$	0.568888888888889
P (14)	$1/\sqrt{3}$	$1/\sqrt{3}$	$\zeta_{G14}=0.538469310105683$	0.478628670499366
P (15)	$1/\sqrt{3}$	$1/\sqrt{3}$	$\zeta_{G15}=0.906179845938664$	0.236926885056189
P (16)	$-1/\sqrt{3}$	$1/\sqrt{3}$	$\zeta_{G16}=-0.906179845938664$	0.236926885056189
P (17)	$-1/\sqrt{3}$	$1/\sqrt{3}$	$\zeta_{G17}=-0.538469310105683$	0.478628670499366
P (18)	$-1/\sqrt{3}$	$1/\sqrt{3}$	$\zeta_{G18}=0$	0.568888888888889
P (19)	$-1/\sqrt{3}$	$1/\sqrt{3}$	$\zeta_{G19}=0.538469310105683$	0.478628670499366
P (20)	$-1/\sqrt{3}$	$1/\sqrt{3}$	$\zeta_{G20}=0.906179845938664$	0.236926885056189

Thus, the statement of the stiffness is: $\boldsymbol{K}_e=\sum_{j=1}^{20} w(P(j))J(P(j))\bar{\boldsymbol{B}}^T(P(j))\cdot\boldsymbol{C}\cdot\bar{\boldsymbol{B}}(P(j))$

3.4 Geometrical stiffness matrix Ksigma

the matrix $\boldsymbol{K}_\sigma$ aims to solve the problems of buckling. We point out here that buckling modes are the eigenvectors of the problem to the eigenvalues generalized according to:

$$\left(\boldsymbol{K}+\mu\,\boldsymbol{K}_\sigma\right)\cdot\boldsymbol{u}=\boldsymbol{0}\Leftrightarrow\boldsymbol{K}\cdot\boldsymbol{u}=\lambda\,\boldsymbol{K}_\sigma\cdot\boldsymbol{u}$$

with $\lambda=-\mu$, and μ is the multiplying coefficient of the loading.

By introducing the quadratic strain $\underline{\mathbf{e}}^Q$ such as:

$$e_{ij}^Q(\delta\boldsymbol{u},\Delta\boldsymbol{u})=\sum_{k=1}^{3}\delta u_{k,i}.\Delta u_{k,j}$$

One can define this geometrical stiffness matrix by:

$$\delta \boldsymbol{u}^T \cdot \boldsymbol{K}_\sigma \cdot \Delta \boldsymbol{u} = \int_{\Omega_0} \boldsymbol{\sigma} : \boldsymbol{e}^Q(\delta \boldsymbol{u}, \Delta \boldsymbol{u}) \, d\Omega = \int_{\Omega_0} \boldsymbol{\sigma} : \nabla \delta \boldsymbol{u}^T \nabla \Delta \boldsymbol{u} \quad d\Omega$$

In order to express this matrix in discretized space, let us introduce the operators quadratic gradient discretized $\underline{\underline{\mathbf{B}}}^Q$ (in matric notation) such as:

$$\underline{\mathbf{e}}^Q(\delta\mathbf{u}, \Delta\mathbf{u}) = \begin{bmatrix} e^Q_{11} \\ e^Q_{22} \\ e^Q_{33} \\ e^Q_{12} + e^Q_{21} \\ e^Q_{13} + e^Q_{31} \\ e^Q_{23} + e^Q_{32} \end{bmatrix} = \begin{bmatrix} \underline{\delta\mathbf{u}}^T \cdot \underline{\underline{\mathbf{B}}}^Q_{11} \cdot \underline{\Delta\mathbf{u}} \\ \underline{\delta\mathbf{u}}^T \cdot \underline{\underline{\mathbf{B}}}^Q_{22} \cdot \underline{\Delta\mathbf{u}} \\ \underline{\delta\mathbf{u}}^T \cdot \underline{\underline{\mathbf{B}}}^Q_{33} \cdot \underline{\Delta\mathbf{u}} \\ \underline{\delta\mathbf{u}}^T \cdot \underline{\underline{\mathbf{B}}}^Q_{12} \cdot \underline{\Delta\mathbf{u}} \\ \underline{\delta\mathbf{u}}^T \cdot \underline{\underline{\mathbf{B}}}^Q_{13} \cdot \underline{\Delta\mathbf{u}} \\ \underline{\delta\mathbf{u}}^T \cdot \underline{\underline{\mathbf{B}}}^Q_{23} \cdot \underline{\Delta\mathbf{u}} \end{bmatrix}$$

The various terms $\underline{\underline{\mathbf{B}}}^Q_{ij}$ are given by the following equations:

$$\underline{\underline{\mathbf{B}}}^Q_{11} = \begin{bmatrix} \underline{\mathbf{B}}_1 \underline{\mathbf{B}}_1^{\mathbf{T}} & \underline{\underline{\mathbf{0}}} & \underline{\underline{\mathbf{0}}} \\ \underline{\underline{\mathbf{0}}} & \underline{\mathbf{B}}_1 \underline{\mathbf{B}}_1^{\mathbf{T}} & \underline{\underline{\mathbf{0}}} \\ \underline{\underline{\mathbf{0}}} & \underline{\underline{\mathbf{0}}} & \underline{\mathbf{B}}_1 \underline{\mathbf{B}}_1^{\mathbf{T}} \end{bmatrix} \; ; \; \underline{\underline{\mathbf{B}}}^Q_{22} = \begin{bmatrix} \underline{\mathbf{B}}_2 \underline{\mathbf{B}}_2^{\mathbf{T}} & \underline{\underline{\mathbf{0}}} & \underline{\underline{\mathbf{0}}} \\ \underline{\underline{\mathbf{0}}} & \underline{\mathbf{B}}_2 \underline{\mathbf{B}}_2^{\mathbf{T}} & \underline{\underline{\mathbf{0}}} \\ \underline{\underline{\mathbf{0}}} & \underline{\underline{\mathbf{0}}} & \underline{\mathbf{B}}_2 \underline{\mathbf{B}}_2^{\mathbf{T}} \end{bmatrix} \; ; \; \underline{\underline{\mathbf{B}}}^Q_{33} = \begin{bmatrix} \underline{\mathbf{B}}_3 \underline{\mathbf{B}}_3^{\mathbf{T}} & \underline{\underline{\mathbf{0}}} & \underline{\underline{\mathbf{0}}} \\ \underline{\underline{\mathbf{0}}} & \underline{\mathbf{B}}_3 \underline{\mathbf{B}}_3^{\mathbf{T}} & \underline{\underline{\mathbf{0}}} \\ \underline{\underline{\mathbf{0}}} & \underline{\underline{\mathbf{0}}} & \underline{\mathbf{B}}_3 \underline{\mathbf{B}}_3^{\mathbf{T}} \end{bmatrix}$$

$$\underline{\underline{\mathbf{B}}}^Q_{12} = \begin{bmatrix} \underline{\mathbf{B}}_1 \underline{\mathbf{B}}_2^{\mathbf{T}} + \underline{\mathbf{B}}_2 \underline{\mathbf{B}}_1^{\mathbf{T}} & \underline{\underline{\mathbf{0}}} & \underline{\underline{\mathbf{0}}} \\ \underline{\underline{\mathbf{0}}} & \underline{\mathbf{B}}_1 \underline{\mathbf{B}}_2^{\mathbf{T}} + \underline{\mathbf{B}}_2 \underline{\mathbf{B}}_1^{\mathbf{T}} & \underline{\underline{\mathbf{0}}} \\ \underline{\underline{\mathbf{0}}} & \underline{\underline{\mathbf{0}}} & \underline{\mathbf{B}}_1 \underline{\mathbf{B}}_2^{\mathbf{T}} + \underline{\mathbf{B}}_2 \underline{\mathbf{B}}_1^{\mathbf{T}} \end{bmatrix}$$

$$\underline{\underline{\mathbf{B}}}^Q_{13} = c^2 \begin{bmatrix} \underline{\mathbf{B}}_1 \underline{\mathbf{B}}_3^{\mathbf{T}} + \underline{\mathbf{B}}_3 \underline{\mathbf{B}}_1^{\mathbf{T}} & \underline{\underline{\mathbf{0}}} & \underline{\underline{\mathbf{0}}} \\ \underline{\underline{\mathbf{0}}} & \underline{\mathbf{B}}_1 \underline{\mathbf{B}}_3^{\mathbf{T}} + \underline{\mathbf{B}}_3 \underline{\mathbf{B}}_1^{\mathbf{T}} & \underline{\underline{\mathbf{0}}} \\ \underline{\underline{\mathbf{0}}} & \underline{\underline{\mathbf{0}}} & \underline{\mathbf{B}}_1 \underline{\mathbf{B}}_3^{\mathbf{T}} + \underline{\mathbf{B}}_3 \underline{\mathbf{B}}_1^{\mathbf{T}} \end{bmatrix}$$

$$\underline{\underline{\mathbf{B}}}^Q_{23} = c^2 \begin{bmatrix} \underline{\mathbf{B}}_2 \underline{\mathbf{B}}_3^{\mathbf{T}} + \underline{\mathbf{B}}_3 \underline{\mathbf{B}}_2^{\mathbf{T}} & \underline{\underline{\mathbf{0}}} & \underline{\underline{\mathbf{0}}} \\ \underline{\underline{\mathbf{0}}} & \underline{\mathbf{B}}_2 \underline{\mathbf{B}}_3^{\mathbf{T}} + \underline{\mathbf{B}}_3 \underline{\mathbf{B}}_2^{\mathbf{T}} & \underline{\underline{\mathbf{0}}} \\ \underline{\underline{\mathbf{0}}} & \underline{\underline{\mathbf{0}}} & \underline{\mathbf{B}}_2 \underline{\mathbf{B}}_3^{\mathbf{T}} + \underline{\mathbf{B}}_3 \underline{\mathbf{B}}_2^{\mathbf{T}} \end{bmatrix}$$

with the vectors $\underline{\mathbf{B}}_i$ (I = 1,2,3) definite like:

$$\underline{\mathbf{B}}_i = \left(\underline{\mathbf{b}}_i \boldsymbol{\gamma} + h_{\alpha,i\_\alpha} \right)$$

With these notations, the contribution to the geometrical stiffness matrix $\underline{\underline{\mathbf{k}}}_\sigma$, the Gauss point ξ_j east given by:

$$\underline{\underline{\mathbf{k}}}_{\boldsymbol{\sigma}}(\xi_j)=\sigma_{11}(\xi_j)\,\underline{\underline{\mathbf{B}}}_{11}^{Q}(\xi_j)+\sigma_{22}(\xi_j)\,\underline{\underline{\mathbf{B}}}_{22}^{Q}(\xi_j)+\sigma_{33}(\xi_j)\,\underline{\underline{\mathbf{B}}}_{33}^{Q}(\xi_j)$$
$$+\sigma_{12}(\xi_j)\,\underline{\underline{\mathbf{B}}}_{12}^{Q}(\xi_j)+\sigma_{13}(\xi_j)\,\underline{\underline{\mathbf{B}}}_{13}^{Q}(\xi_j)+\sigma_{23}(\xi_j)\,\underline{\underline{\mathbf{B}}}_{23}^{Q}(\xi_j)$$

By integration on Gauss points of the element, the geometrical stiffness matrix is obtained by the formula:

$$\boldsymbol{K}_{\boldsymbol{\sigma}}=\sum_{j=1}^{5} w(\xi_j)J(\xi_j)\boldsymbol{k}_{\boldsymbol{\sigma}}(\xi_j)$$ for element SHB15 and the element SHB20

3.5 Follower forces and matrix of Kp pressure

the following compressive forces are present in the tangent matrix via the matrix K_p, because the following external forces depend on displacement. The following compressive forces are written:

$$\int_{\partial\Omega} p\,\boldsymbol{n}^{\boldsymbol{T}}\cdot\boldsymbol{u}\,dS=\int_{\partial\Omega_0} p\det[\boldsymbol{F}(\boldsymbol{u})]\,\boldsymbol{n}_{\boldsymbol{0}}^{\boldsymbol{T}}\boldsymbol{F}(\boldsymbol{u})^{-\mathrm{T}}dS_0=p\,\boldsymbol{F}_{\boldsymbol{0}}-p\,\boldsymbol{K}_{\boldsymbol{p}}\cdot\boldsymbol{u}$$
$$\boldsymbol{F}(\boldsymbol{u})=\mathbf{1}+\nabla\boldsymbol{u}$$

by means of notations:

- $\underline{\mathbf{n}}_0{}^T=(n_1,n_2,n_3)$, norm on the surface external of the element in the reference configuration;
- $\tilde{\boldsymbol{b}}_{\boldsymbol{i}}$, vector of size 6 (for SHB15) or 8 (for SHB20), derived from the shape functions to the 6 (for SHB15) or 8 (for SHB20) nodes of the face of the element charged in pressure;
- S_0 area of the face charged in pressure.

The preceding formulation leads to a nonsymmetrical matrix. It is known that one can nevertheless use a symmetric formulation if the external forces due to the pressure derive from a potential. It is the case if the compressive forces do not work on the border of the modelled field. It is thus considered that the symmetric part of the matrix is enough. The symmetrized matrix takes the following shape:

$$\underline{\underline{\mathbf{K}}}_{\mathbf{P}}=S_0\begin{bmatrix} \mathbf{0} & \tilde{\mathbf{b}}_2^T n_1-\tilde{\mathbf{b}}_1^T n_2 & \tilde{\mathbf{b}}_3^T n_1-\tilde{\mathbf{b}}_1^T n_3 \\ \mathbf{0} & \tilde{\mathbf{b}}_2^T n_1-\tilde{\mathbf{b}}_1^T n_2 & \tilde{\mathbf{b}}_3^T n_1-\tilde{\mathbf{b}}_1^T n_3 \\ \mathbf{0} & \tilde{\mathbf{b}}_2^T n_1-\tilde{\mathbf{b}}_1^T n_2 & \tilde{\mathbf{b}}_3^T n_1-\tilde{\mathbf{b}}_1^T n_3 \\ \tilde{\mathbf{b}}_1^T n_2-\tilde{\mathbf{b}}_2^T n_1 & \mathbf{0} & \tilde{\mathbf{b}}_3^T n_2-\tilde{\mathbf{b}}_2^T n_3 \\ \tilde{\mathbf{b}}_1^T n_2-\tilde{\mathbf{b}}_2^T n_1 & \mathbf{0} & \tilde{\mathbf{b}}_3^T n_2-\tilde{\mathbf{b}}_2^T n_3 \\ \tilde{\mathbf{b}}_1^T n_2-\tilde{\mathbf{b}}_2^T n_1 & \mathbf{0} & \tilde{\mathbf{b}}_3^T n_2-\tilde{\mathbf{b}}_2^T n_3 \\ \tilde{\mathbf{b}}_3^T n_1-\tilde{\mathbf{b}}_1^T n_3 & \tilde{\mathbf{b}}_2^T n_3-\tilde{\mathbf{b}}_3^T n_2 & \mathbf{0} \\ \tilde{\mathbf{b}}_3^T n_1-\tilde{\mathbf{b}}_1^T n_3 & \tilde{\mathbf{b}}_2^T n_3-\tilde{\mathbf{b}}_3^T n_2 & \mathbf{0} \\ \tilde{\mathbf{b}}_3^T n_1-\tilde{\mathbf{b}}_1^T n_3 & \tilde{\mathbf{b}}_2^T n_3-\tilde{\mathbf{b}}_3^T n_2 & \mathbf{0} \end{bmatrix}$$

It is a matrix 18×18 or 24×24, which it is necessary to multiply by displacements of the 6 (for SHB15) or 8 (for SHB20) nodes of the face to which one applies a pressure.

4 Strategy for geometrical

4.1 nonlinear computations Non-linearities

One treats here the case of large displacements, but weak rotations and small strains. One adopts for that an up to date put Lagrangian formulation.

In nonlinear, we seek to write the equilibrium between external internal forces and force at the end of the increment of load (located by the index 2):

$$\mathbf{F}_2^{\text{int}} = \mathbf{F}_2^{\mathbf{ext}}$$

The statement of the internal forces is written:

$$\underline{\mathbf{F}}_2^{\text{int}} = \int_{\Omega_2} \underline{\underline{\mathbf{B}}}_2^T \underline{\boldsymbol{\sigma}}_2 \, dV$$

In the preceding equation the operator $\underline{\underline{\mathbf{B}}}_2$ is the operator allowing to pass from the displacement to the linear strain calculated on the geometry at the end of the step, the stress $\underline{\boldsymbol{\sigma}}_2$ is the stress of Cauchy at the end of the step and integration is made on the volume Ω_2 deformed at the end of the step.

Important remarks:

- For the element SHB6, the matrix $\underline{\underline{\mathbf{B}}}_2$ is also modified by "*local Assumed strain method*". It takes the following shape:

$$\underline{\underline{\mathbf{B}}}_2 = \begin{bmatrix} \underline{\mathbf{b}}_1^T + \mathrm{h}_{\alpha,x_1}\,\underline{\boldsymbol{\gamma}}_\alpha^T & \underline{\mathbf{0}} & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \underline{\mathbf{b}}_2^T + \mathrm{h}_{\alpha,x_2}\,\underline{\boldsymbol{\gamma}}_\alpha^T & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \underline{\mathbf{0}} & \underline{\mathbf{b}}_3^T + \mathrm{h}_{\alpha,x_3}\,\underline{\boldsymbol{\gamma}}_\alpha^T \\ \underline{\mathbf{b}}_2^T + \mathrm{h}_{\alpha,x_2}\,\underline{\boldsymbol{\gamma}}_\alpha^T & \underline{\mathbf{b}}_1^T + \mathrm{h}_{\alpha,x_1}\,\underline{\boldsymbol{\gamma}}_\alpha^T & \underline{\mathbf{0}} \\ c*\left(\underline{\mathbf{b}}_3^T + \mathrm{h}_{\alpha,x_3}\,\underline{\boldsymbol{\gamma}}_\alpha^T\right) & \underline{\mathbf{0}} & c*\left(\underline{\mathbf{b}}_1^T + \mathrm{h}_{\alpha,x_1}\,\underline{\boldsymbol{\gamma}}_\alpha^T\right) \\ \underline{\mathbf{0}} & c*\left(\underline{\mathbf{b}}_3^T + \mathrm{h}_{\alpha,x_3}\,\underline{\boldsymbol{\gamma}}_\alpha^T\right) & c*\left(\underline{\mathbf{b}}_2^T + \mathrm{h}_{\alpha,x_2}\,\underline{\boldsymbol{\gamma}}_\alpha^T\right) \end{bmatrix} \quad \text{où} \;\; \mathrm{c} = 0{,}45$$

- For elements SHB15 or SHB20, the matrix $\underline{\underline{\mathbf{B}}}_2$ does not need modification. It thus takes, the following form:

$$\underline{\underline{\mathbf{B}}}_2 = \begin{bmatrix} \underline{\mathbf{b}}_1^T + \mathrm{h}_{\alpha,x_1}\,\underline{\boldsymbol{\gamma}}_\alpha^T & \underline{\mathbf{0}} & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \underline{\mathbf{b}}_2^T + \mathrm{h}_{\alpha,x_2}\,\underline{\boldsymbol{\gamma}}_\alpha^T & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \underline{\mathbf{0}} & \underline{\mathbf{b}}_3^T + \mathrm{h}_{\alpha,x_3}\,\underline{\boldsymbol{\gamma}}_\alpha^T \\ \underline{\mathbf{b}}_2^T + \mathrm{h}_{\alpha,x_2}\,\underline{\boldsymbol{\gamma}}_\alpha^T & \underline{\mathbf{b}}_1^T + \mathrm{h}_{\alpha,x_1}\,\underline{\boldsymbol{\gamma}}_\alpha^T & \underline{\mathbf{0}} \\ \left(\underline{\mathbf{b}}_3^T + \mathrm{h}_{\alpha,x_3}\,\underline{\boldsymbol{\gamma}}_\alpha^T\right) & \underline{\mathbf{0}} & \left(\underline{\mathbf{b}}_1^T + \mathrm{h}_{\alpha,x_1}\,\underline{\boldsymbol{\gamma}}_\alpha^T\right) \\ \underline{\mathbf{0}} & \left(\underline{\mathbf{b}}_3^T + \mathrm{h}_{\alpha,x_3}\,\underline{\boldsymbol{\gamma}}_\alpha^T\right) & \left(\underline{\mathbf{b}}_2^T + \mathrm{h}_{\alpha,x_2}\,\underline{\boldsymbol{\gamma}}_\alpha^T\right) \end{bmatrix}$$

The element available to date in *Aster* is programmed in small rotations. Indeed, the increment of strain is calculated by means of only the linear strain:

$$\Delta \boldsymbol{E}=\frac{1}{2}\left(\nabla_1(\Delta \boldsymbol{u})+\nabla_1^T(\Delta \boldsymbol{u})\right)$$

The operator gradient is calculated on the geometry of beginning of step. This writing of the strain is restricted with small rotations (lower than 5 degrees).

One could without difficulty of extending the formulation to large rotations by including in the strain the terms of second order (tensor of Green-Lagrange):

$$\Delta \boldsymbol{E}=\frac{1}{2}\left(\nabla_1(\Delta \boldsymbol{u})+\nabla_1^T(\Delta \boldsymbol{u})+\nabla_1^T(\Delta \boldsymbol{u})\cdot\nabla_1(\Delta \boldsymbol{u})\right)$$

The associated stress tensor is the second tensor of Piola Kirchoff II [R5.03.22]. But this is not available in version 9 of Code_Aster.

In elasticity, the constitutive law is written:

$$\Delta \underline{\underline{C}}=\underline{\underline{C'}}\,\Delta \underline{\underline{E}}$$

where C' is the matrix of Hooke. Let us notice that for the elements SHB, this matrix is a transverse orthotropic matrix which is written in the axes of the lamina:

$$\underline{\underline{\mathbf{C}}}'=\begin{bmatrix} \lambda+2\mu & \mu & 0 & 0 & 0 & 0 \\ \mu & \lambda+2\mu & 0 & 0 & 0 & 0 \\ 0 & 0 & E & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix}$$

4.2 Non-linearities materials

Into nonlinear materials, we propose a method of particular construction of the tangent matrix $\underline{\underline{C}}^T$. It consists in supposing initially that the element is in plane stress state in the local coordinate system of each point of Gauss quadrature and the strains except plane are elastic. That involves then immediately that the total deflections except plane are equal to the elastic strain. Let us call $\underline{\underline{C}}^{CPT}$ the tangent matrix in plane stresses. The tangent matrix of behavior for the selected behavior and is written:

$$\underline{\underline{C}}^T=\begin{bmatrix} C_{xxxx}^{CPT} & C_{xxyy}^{CPT} & 0 & C_{xxxy}^{CPT} & 0 & 0 \\ C_{xyyx}^{CPT} & C_{yyyy}^{CPT} & 0 & C_{yyxy}^{CPT} & 0 & 0 \\ 0 & 0 & E & 0 & 0 & 0 \\ C_{xyxx}^{CPT} & C_{xyyy}^{CPT} & 0 & C_{xyxy}^{CPT} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix}$$

Then the stresses except plane are calculated in an elastic way. This method thus makes it possible to connect elements SHB to all the constitutive laws available in *Code Aster*.

5 Establishment of elements SHB in *Code_Aster*

5.1 Description

These elements lean on meshes 3D the voluminal PENTA6, HEXA8, PENTA15 and HEXA20.

5.2 Use

These elements are used in the following way:

5.2.1 Mesh

by means of To check the good directional sense of the sides of indicated elements (compatibility with the privileged direction) ORIE_SHB of operator MODI_MAILLAGE.

5.2.2 Modelization

the name of modelization SHB8 was preserved. It is of course an abuse language, this modelization gathering now 4 finite elements SHB6, SHB8, SHB15, SHB20.

5.2.3 Material

For a homogeneous isotropic elastic behavior in the thickness one uses key word ELAS in DEFI_MATERIAU where one defines the coefficients E, Young modulus and NU, Poisson's ratio.

To define a plastic behavior one uses the key word TENSION in DEFI_MATERIAU where the name of a curve of tension is defined. Only this kind of definition is available for time.

5.2.4 Boundary conditions and loading

One 3D imposes the boundary conditions on the degrees of freedom of volume (AFFE_CHAR_MECA / DDL_IMPO), and the forces in the total reference (FORCE_NODALE).

One defines the forces of pressure distributed on the sides of the element (under key word PRES_REP). One will have taken care to define as a preliminary meshes of skin QUAD4 and to suitably direct the outgoing norms with these meshes of skin using command
MODI_MAILLAGE key word ORIE_PEAU_3D

5.2.5 Computation in linear elasticity

MECA_STATIQUE Orders
the options of postprocessing available are SIEF_ELNO and SIEQ_ELNO.

5.2.6 Computation in linear buckling

option RIGI_MECA_GE being activated in the catalog of the element, it is possible to carry out a classical computation of buckling after assembly of the stiffness matrixes elastic and geometrical.

5.2.7 Computation in geometrical nonlinear "elasticity"

One chooses behavior ELAS under key word COMP_INCR of STAT_NON_LINE, in small strains ("PETIT") or in large displacements "GREEN" under key word DEFORMATION. In this last case, only the geometry is brought up to date at the beginning of time step, the behavior remains calculated in small strains.

The strategy used being based on the use of a tangent stiffness matrix during iterations (reactualization at the beginning of step only), one will take care to use another option only that which is activated by default, namely REAC_ITER = 0 pennies NEWTON.

Numerical integration in the thickness is carried out with 5 Gauss points, just like in nonlinear material.

5.2.8 Plastic nonlinear computation

Only the criterion of Von Mises is available to date (RELATION = "VMIS_ISOT_TRAC" under COMP_INCR). One defines the mode of computation of the strains as in the case of nonlinear elasticity (DEFORMATION = "GREEN" or "PETIT").

The strategy used being based on the use of a tangent stiffness matrix during iterations (reactualization at the beginning of step only), one will take care to use another option only that which is activated by default, namely REAC_ITER = 0 pennies NEWTON.

5.3 Establishment

options RIGI_MECA, RIGI_MECA_GE, FORC_NODA, FULL_MECA, RIGI_MECA_TANG, RAPH_MECA, SIEF_ELGA, SIEF_ELNO were activated in the catalog gener_shb3d_3.catastrophes.

No development was necessary for the compressive forces distributed and the following compressive forces. Indeed, these loadings lean on meshes of skin identical to those of the elements 3D voluminal.

5.4 Validation

the tests validating these elements are:

Tests into linear:

•SSLS101 C, D, K, L: C plate circular simply posed subjected to a uniform pressure [V3.03.101] Modelization: SHB8, Modelization D: SHB20, Modelization K: SHB6, Modelization L: SHB15.

•SSLS105 C: hemisphere doubly pinch [V3.03.105] classical test to check the convergence of element (SHB8)

•SSLS108 C with H: beam bored in bending, test allowing to check the absence of blocking [V3.03.108] Modelizations C, D: SHB8, Modelization G: SHB20, Modelizations E, F: SHB6, Modelization H: SHB15.

•SSLS123 a: sphere under external pressure [V3.03.123] to validate the loadings of pressure and the orthotropic behavior particular to this elementModelisation
A: SHB8, Modelizations C, D: SHB6.

•SSLS124 A with G: thin plate in bending with various slenderness, to delimit the field of use of the element [V3.03.124].
Modelizations A, B: SHB8, Modelization C: SHB6, Modelizations D, E: SHB20, Modelizations F, G: SHB15.

•SSLS125 A, b: buckling (modes of Eulerian) of a free cylinder under external pressure [V3.03.125] this test makes it possible to validate the nature of stiffness geometriqueModelisation
A: SHB8, Modelization b: SHB20.Tests

in nonlinear:

•SSNS101 C with G: breakdown of a cylindrical roof [V6.03.101]. This test makes it possible to validate geometrical nonlinear computation and the élastoplasticitéModélisations
C, D: SHB8, Modelization E: SHB20, Modelization F: SHB6, Modelization G: SHB15.

•SSNS102 A, b: buckling of a shell with stiffeners in large displacements and following pressure [V6.03.102].

Modelization A: SHB8, Modelization b: SHB20.

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Description of the versions of the document:

Version Aster	**Author (S), Organization (S)**	**Description of the modifications**
9.5	Trinh Vuong God (thesis) X Desroches EDF R & D AMA	initial Version