Shell elements: Modelization Q4GG

Summarized

This document presents the formulation of the thick shell elements, used in modelization Q4GG.

For this modelization, two standard finite elements are available, according to meshes:

- Quadrangular finite element (Q4G)
- triangular Finite element (T3G)

This thick modelization of shell element, like modelizations DKT, DKQ, DST, DSQ, DKTG and Q4G [7], is intended for computations in small strains and large displacements of curved or plane thin structures. In fact plane elements do not take into account the geometrical curvature of structures, contrary with the shell elements (COQUE_3D) which are curved: it results from it from parasitic bendings which can be reduced by means of more elements in order to be able to approach the curved geometries correctly. The formulation is thus simplified by it and the reduced number of degrees of freedom.

Note:

The formulation of quadrangular element Q4G used in modelization Q4GG is described in the document [7]. Consequently, one limits oneself here to the case of triangular element T3G.

Contents

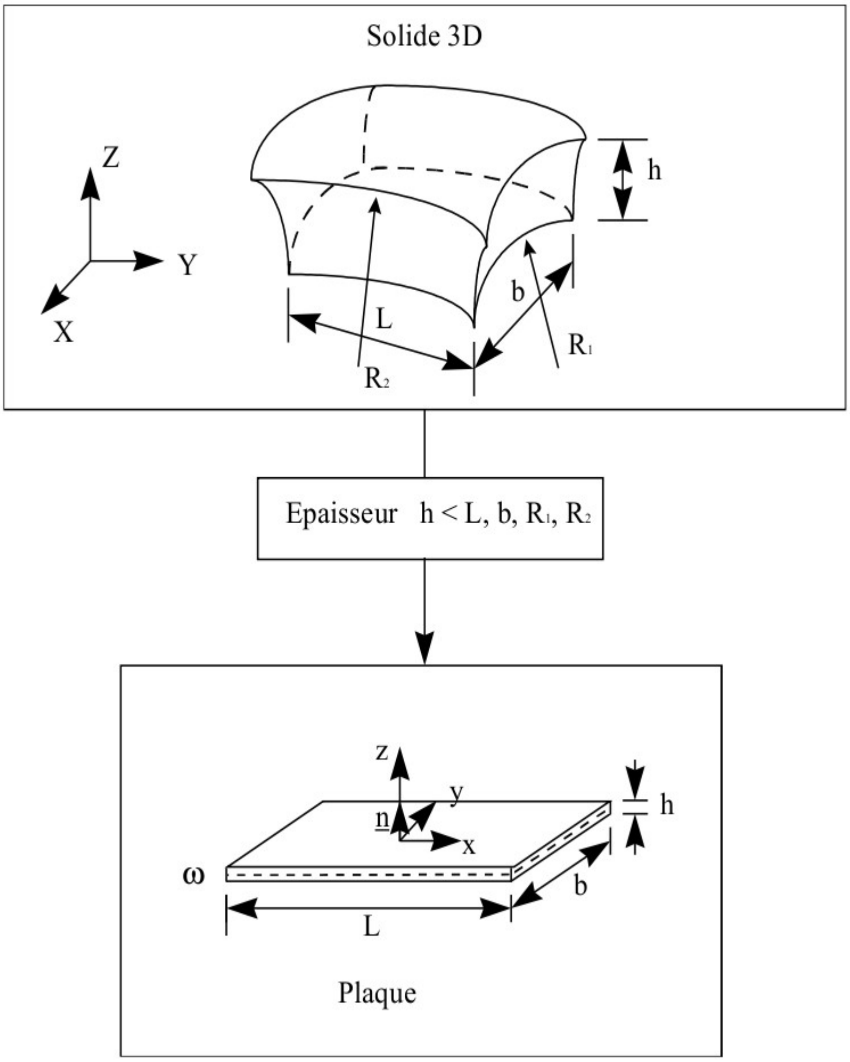
1 Introduction

One presents here the details of the thick formulation of shell element `T3G,` such as she was proposed in the publication [6]. This element is built in a way similar to element `Q4G` [7].

2 Geometry

2.1 formulation of the elements plates

For the shell elements one defines a surface of reference, or mean surface, planes (plane xy for example) and a thickness $h(x, y)$. This thickness must be small compared to other dimensions (extensions, radii of curvature) of structure to modelling. The figure below illustrates our matter.



One attaches to mean surface ω a local orthonormal reference $Oxyz$ associated with the tangent plane with structure different from the total reference $OXYZ$. The position of the points of the plate is given by the Cartesian coordinates (x, y) mean surface and rise z compared to this surface.

2.2 Kinematics

the basic kinematics of element `T3G` is that of Hencky-Mindlin (see [7] for the details), where the strain tensor 3D, ε , are defined as follows:

$$\varepsilon_{xx}=e_{xx}+z\,\kappa_{xx} \qquad \varepsilon_{yy}=e_{yy}+z\,\kappa_{yy} \qquad \varepsilon_{xy}=e_{xy}+z\,\kappa_{xy}$$

$$\varepsilon_{xz}=\gamma_x \qquad \varepsilon_{yz}=\gamma_y \qquad (1)$$

where

$$e_{xx}=\frac{\partial u}{\partial x} \qquad e_{yy}=\frac{\partial v}{\partial y} \qquad e_{xy}=\frac{1}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)$$

$$\kappa_{xx}=\frac{\partial \beta_x}{\partial x} \qquad \kappa_{yy}=\frac{\partial \beta_y}{\partial y} \qquad \kappa_{xy}=\frac{1}{2}\left(\frac{\partial \beta_x}{\partial y}+\frac{\partial \beta_y}{\partial x}\right)$$

$$\beta_x=\theta_y \qquad \beta_y=-\theta_x$$

$$\gamma_x=\beta_x+\frac{\partial w}{\partial x} \qquad \gamma_y=\beta_y+\frac{\partial w}{\partial y}$$

(formula

$e_{\alpha\beta}$ being the tensor of membrane extensions, $\kappa_{\alpha\beta}$ the tensor of variation of curvature, β_x , β_y the above definite auxiliary variables and γ_x , γ_y the transverse distortions. The basic variables are displacements u , v , w and the rotations θ_x , θ_y θ_z . In the formulations considered here normal rotation at mean surface, θ_z , is introduced exclusively for reasons of compatibility between the nodal degrees of freedom for NON-plane geometries.

2.3 Support finite element

the finite element has like support a mesh `T3`, with 6 degrees of freedom per node, which makes 18 degrees of freedom on the whole. The principal characteristics are given in the table below. One defines the same shape functions for displacements, $(u,\ v,\ w)$, as for rotations, $(\theta_x\ ,\ \theta_x)$.

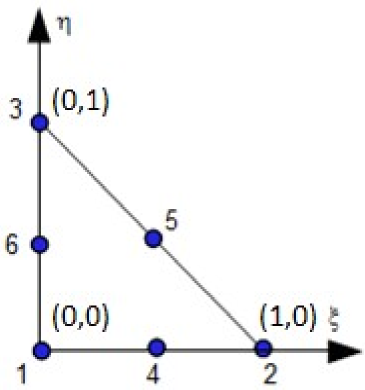
`TRIA3`	$N_i(i=1,n)$	
	$i=1\,\grave{a}\,3$ $N^1(\xi,\eta)=\lambda=1-\xi-\eta$ $N^2(\xi,\eta)=\xi$ $N^3(\xi,\eta)=\eta$	• Three collocation points (4,5,6) to determine γ_x and γ_y • a point of integration in $(\xi=\frac{1}{3};\eta=\frac{1}{3})$

Table 2.1: Finite element of reference T3G used in formulation Q4GG to triangular base

2.4 Membrane-Bending

the present formulation is based on 5 degrees of freedom by neglecting any effect of rotation perpendicular to the average average. This third rotation, represented by θ_z in the local coordinate system, is not presented in this formulation, to see [7] for more details.

The membrane extension and the variation of curvature are discretized in an equivalent way. One and the uses the same shape functions for displacements rotations, which leads to:

$$U_i = N^r u_i^r \quad , \quad \beta_i = N^r \beta_i^r \quad (r = 1\, \grave{a}\, 3)$$

u_i^r β_i^r being the nodal values of displacements and rotations (transformed), respectively. While following the equations (1) and (2) one a:

$$e_{ij} = B_{ijk}^r u_k^r , \qquad \kappa_{ij} = B_{ijk}^r \beta_k^r ,$$

$$B_{ijk}^r = \frac{1}{2} \frac{\partial N^r}{\partial \xi_l} \left(J_{lj}^{-1} \delta_{ik} + J_{li}^{-1} \delta_{jk} \right)$$

Where one introduces the jacobian of the transformation on the element of reference:

$$J_{ij} = \frac{\partial x_i}{\partial \xi_j} = \frac{\partial N^r}{\partial \xi_j} x_i^r$$

Thus, for the functions defined in Table 2.1 and by choosing the local coordinate system of an element such as $x_1 = y_1 = y_2 = 0$, one obtains:

$$\mathbf{J} = \begin{pmatrix} x_2 & x_3 \\ 0 & y_3 \end{pmatrix} \quad ; \quad \mathbf{J}^{-1} = \frac{1}{x_2 y_3} \begin{pmatrix} y_3 & -x_3 \\ 0 & x_2 \end{pmatrix}$$

$$\frac{\partial N^r}{\partial \xi_k} = \begin{pmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and with final the uniform membrane strain tensor on the element:

$$\begin{pmatrix} e_{11} \\ e_{22} \\ 2 e_{12} \end{pmatrix} = \frac{1}{x_2 y_3} \begin{pmatrix} -y_3 & 0 & y_3 & 0 & 0 & 0 \\ 0 & x_3 - x_2 & 0 & -x_3 & 0 & x_2 \\ x_3 - x_2 & -y_3 & -x_3 & y_3 & x_2 & 0 \end{pmatrix} \begin{pmatrix} u^1 \\ v^1 \\ u^2 \\ v^2 \\ u^3 \\ v^3 \end{pmatrix} \qquad (3)$$

$$\begin{pmatrix}\kappa_{11}\\ \kappa_{22}\\ 2\kappa_{12}\end{pmatrix}=\frac{1}{x_2 y_3}\begin{pmatrix}0 & -y_3 & 0 & y_3 & 0 & 0\\ -(x_3-x_2) & 0 & -x_3 & 0 & -x_2 & 0\\ y_3 & x_3-x_2 & -y_3 & -x_3 & 0 & x_2\end{pmatrix}\begin{pmatrix}\theta_x^1\\ \theta_y^1\\ \theta_x^2\\ \theta_y^2\\ \theta_x^3\\ \theta_y^3\end{pmatrix} \quad (4)$$

2.5 transverse Distortion

2.5.1 Description of a locking in transverse shears

to avoid locking in shears, one must judiciously choose the discretization of the transverse distortion. For example, while imposing,

$$\gamma_x=\beta_x+\frac{\partial w}{\partial x} \quad \text{and} \quad \gamma_y=\beta_y+\frac{\partial w}{\partial y} \quad (5)$$

Are according to the degrees of freedom:

$$\begin{pmatrix}\gamma_x\\ \gamma_y\end{pmatrix}=\begin{pmatrix}\theta_y+\frac{\partial w}{\partial x}\\ -\theta_x+\frac{\partial w}{\partial y}\end{pmatrix}=\begin{pmatrix}N^\alpha\theta_y^\alpha+\frac{\partial N^\alpha}{\partial \xi_k}J_{k1}^{-1}w^\alpha\\ -N^\alpha\theta_x^\alpha+\frac{\partial N^\alpha}{\partial \xi_k}J_{k2}^{-1}w^\alpha\end{pmatrix}$$

the limit of the thin plates, where $h\rightarrow 0$ and consequently, $\gamma_x\rightarrow 0$, $\gamma_y\rightarrow 0$ lead to the following equations in any point (ξ,η) :

$$\begin{pmatrix}\gamma_x\\ \gamma_y\end{pmatrix}=\begin{pmatrix}\theta_y^1+\frac{1}{x_2}(w_2-w_1)+(\theta_y^2-\theta_y^1)\xi+(\theta_y^3-\theta_y^1)\eta\\ -\theta_x^1+\frac{1}{x_2 y_3}((x_3-x_2)w_1-x_3 w_2+x_2 w_3)-(\theta_x^2-\theta_x^1)\xi-(\theta_x^3-\theta_x^1)\eta\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix},\quad \forall\xi,\eta \quad (6)$$

the conditions in the equation (6) lead directly to the relations $\theta_x^3=\theta_x^2=\theta_x^1$ and $\theta_y^3=\theta_y^2=\theta_y^1$. Rotations of bending are thus all equal and inevitably null if rigid body motions are correctly removed system. Locking in shears thus is observed: a thin plate ($h\approx 0$) with the kinematics of the transverse distortion described in the equation (5) cannot become deformed in bending.

2.5.2 Remedy for lockings by approach T3G

to cure lockings and in a way similar to formulation `Q4G` [7], formulation `T3G` uses an interpolation of the transverse distortion independent of displacements and rotations. The relations (5) are slackened, thanks to three collocation points. Firstly, it is supposed that the spatial variation of the transverse distortion is linear with a constant tangential component on each stops. Secondly, as in `Q4G`, compatibility is imposed on the mediums of the three stop and only in the tangential direction.

The values of the transverse distortion along stop can be calculated directly as follows:

$$\gamma_s^k = \frac{w_j - w_i}{L_k} + \frac{1}{2}(\beta_s^{ki} + \beta_s^{kj}) \qquad (7)$$

where γ_s^k is the tangential component of the transverse distortion on stops k, which is between the nodes i and j. L_k is the length of stops k ; β_s^{ki} and β_s^{kj} are the values projected of rotations (transformed) on the tangent of stops k, corresponding to the nodes i and j, respectively:

$$\beta_s^{ki} = \beta_x^i \cos\Omega_k + \beta_y^i \sin\Omega_k$$

where Ω_k represents the angle of stops k compared to the X-coordinate of the local coordinate system. The equation (7) can be rewritten as follows:

$$\begin{pmatrix}\gamma_s^1\\ \gamma_s^2\\ \gamma_s^3\end{pmatrix} = \begin{pmatrix} \frac{1}{2}C_1 & \frac{1}{2}S_1 & -\frac{1}{L_1} & \frac{1}{2}C_1 & \frac{1}{2}S_1 & \frac{1}{L_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}C_2 & \frac{1}{2}S_2 & -\frac{1}{L_2} & \frac{1}{2}C_2 & \frac{1}{2}S_2 & \frac{1}{L_2} \\ \frac{1}{2}C_3 & \frac{1}{2}S_3 & \frac{1}{L_3} & 0 & 0 & 0 & \frac{1}{2}C_3 & \frac{1}{2}S_3 & -\frac{1}{L_3} \end{pmatrix} \begin{pmatrix}\beta_x^1\\ \beta_y^1\\ w_1\\ \beta_x^2\\ \beta_y^2\\ w_2\\ \beta_x^3\\ \beta_y^3\\ w_3\end{pmatrix} \qquad (8)$$

or $C_k = \cos\Omega_k$ and $S_k = \sin\Omega_k$

In addition, one imposes the following form of the spatial variation in the reference of reference of the transverse distortion:

$$\begin{aligned}\gamma_x &= a_x + b_x\xi + c_x\eta\\ \gamma_y &= a_y + b_y\xi + c_y\eta\end{aligned} \qquad (9)$$

where a_x, a_y, b_x, b_y, c_x, c_y are the parameters to be determined starting from the assumptions according to which the tangential components of γ are constant on each stops.

One thus obtains:

$$\gamma_s^1 = \gamma_x|_{\eta=0} = a_x + b_x\xi \qquad \rightarrow \qquad a_x = \gamma_s^1 \;,\; b_x = 0$$

$$\gamma_s^2 = (\gamma_x C_2 + \gamma_y S_2)|_{\eta+\xi=1} = [a_x C_2 + a_y S_2 + c_x C_2 + (b_y S_2 - c_x C_2 - c_y S_2)\xi]$$

$$\rightarrow \qquad a_x C_2 + a_y S_2 + c_x C_2 = \gamma_s^2 \quad ; \quad b_y S_2 - c_x C_2 - c_y S_2 = 0$$

$$\gamma_s^3 = (\gamma_x C_3 + \gamma_y S_3)_{\xi=0} = a_x C_3 + a_y S_3 + (c_x C_3 + c_y S_3)\eta$$

$$\rightarrow \qquad a_x C_3 + a_y S_3 = \gamma_s^3 \quad ; \quad c_x C_3 + c_y S_3 = 0$$

What leads to the final result

$$a_x=\gamma_s^1 \quad , \quad a_y=-\frac{C_3}{S_3}\gamma_s^1+\frac{1}{S_3}\gamma_s^3$$

$$b_x=0 \quad , \quad b_y=-\frac{1}{QS_2}\gamma_s^1+\frac{1}{S_2}\gamma_s^2-\frac{1}{S_3}\gamma_s^3$$

$$c_x=-\gamma_s^1+Q\gamma_s^2-Q\frac{S_2}{S_3}\gamma_s^3 \quad , \quad c_y=\frac{C_3}{S_3}\gamma_s^1-Q\frac{C_3}{S_3}\gamma_s^2+Q\frac{C_3 S_2}{S_3^2}\gamma_s^3 \qquad (10)$$

where $Q=1/(C_2-S_2\frac{C_3}{S_3})$

From the equations (9) and (10) can write:

$$\begin{pmatrix}\gamma_x\\ \gamma_y\end{pmatrix}=\begin{pmatrix} 1-\eta & Q\eta & -\frac{S_2}{S_3}Q\eta \\ -\frac{C_3}{S_3}-\frac{1}{QS_2}\xi+\frac{C_3}{S_3}\eta & \frac{1}{S_2}\xi-\frac{C_3}{S_3}Q\eta & \frac{1}{S_3}-\frac{1}{S_3}\xi+\frac{C_3 S_2}{S_3^2}Q\eta \end{pmatrix}\begin{pmatrix}\gamma_s^1\\ \gamma_s^2\\ \gamma_s^3\end{pmatrix} \qquad (11)$$

By combining the equations (8) and (11) obtains:

$$\begin{pmatrix}\gamma_x\\ \gamma_y\end{pmatrix}=\begin{pmatrix} 1-\eta & Q\eta & -\frac{S_2}{S_3}Q\eta \\ -\frac{C_3}{S_3}-\frac{1}{QS_2}\xi+\frac{C_3}{S_3}\eta & \frac{1}{S_2}\xi-\frac{C_3}{S_3}Q\eta & \frac{1}{S_3}-\frac{1}{S_3}\xi+\frac{C_3 S_2}{S_3^2}Q\eta \end{pmatrix}\times$$

$$\begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{L_1} & \frac{1}{2} & 0 & \frac{1}{L_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}C_2 & \frac{1}{2}S_2 & -\frac{1}{L_2} & \frac{1}{2}C_2 & \frac{1}{2}S_2 & \frac{1}{L_2} \\ \frac{1}{2}C_3 & \frac{1}{2}S_3 & \frac{1}{L_3} & 0 & 0 & 0 & \frac{1}{2}C_3 & \frac{1}{2}S_3 & -\frac{1}{L_3} \end{pmatrix}\begin{pmatrix}\beta_x^1\\ \beta_y^1\\ w_1\\ \beta_x^2\\ \beta_y^2\\ w_2\\ \beta_x^3\\ \beta_y^3\\ w_3\end{pmatrix} \qquad (12)$$

above the statements were slightly simplified by means of the same local coordinate system as in [§2.4], i.e. that where $\Omega_1=0$ and thus $C_1=1$ and $S_1=0$. The relation between the degrees of freedom of the element, U_i^α and the transverse distortion in any point (ξ,η), of the equation (12) is written in the form:

$$\gamma_i = B^{ct}_{ik\alpha}(\xi,\eta) U^{\alpha}_k$$

Where

$$\mathbf{B}^{ct} = \begin{pmatrix} 1-\eta & Q\eta & -\frac{S_2}{S_3}Q\eta \\ -\frac{C_3}{S_3} - \frac{1}{QS_2}\xi + \frac{C_3}{S_3}\eta & \frac{1}{S_2}\xi - \frac{C_3}{S_3}Q\eta & \frac{1}{S_3} - \frac{1}{S_3}\xi + \frac{C_3 S_2}{S_3^2}Q\eta \end{pmatrix} \times$$

$$\begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{L_1} & \frac{1}{2} & 0 & \frac{1}{L_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}C_2 & \frac{1}{2}S_2 & -\frac{1}{L_2} & \frac{1}{2}C_2 & \frac{1}{2}S_2 & \frac{1}{L_2} \\ \frac{1}{2}C_3 & \frac{1}{2}S_3 & \frac{1}{L_3} & 0 & 0 & 0 & \frac{1}{2}C_3 & \frac{1}{2}S_3 & -\frac{1}{L_3} \end{pmatrix}$$

and

$$U = \begin{pmatrix} \beta_x^1 & \beta_y^1 & w_1 & \beta_x^2 & \beta_y^2 & w_2 & \beta_x^3 & \beta_y^3 & w_3 \end{pmatrix}^T$$

Contrary to the discretization exposed in [§2.5.1], and in the equation (5), the limit $h \rightarrow 0$ with the independent kinematics leads to three equations which couple in more balanced way displacements w, and the rotations θ_x θ_y, according to the equation (8).

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}C_1 & \frac{1}{2}S_1 & -\frac{1}{L_1} & \frac{1}{2}C_1 & \frac{1}{2}C_2 & \frac{1}{L_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}C_2 & \frac{1}{2}S_2 & -\frac{1}{L_2} & \frac{1}{2}C_2 & \frac{1}{2}S_2 & \frac{1}{L_2} \\ \frac{1}{2}C_3 & \frac{1}{S_3} & \frac{1}{L_3} & 0 & 0 & 0 & \frac{1}{2}C_3 & \frac{1}{2}S_3 & -\frac{1}{L_3} \end{pmatrix} \begin{pmatrix} \beta_x^1 \\ \beta_y^1 \\ w_1 \\ \beta_x^2 \\ \beta_y^2 \\ w_2 \\ \beta_x^3 \\ \beta_y^3 \\ w_3 \end{pmatrix} \quad (13)$$

the conditions in the equation (13) cause any blocking on the degrees of freedom of rotation. In addition, in the equation (13) used the equation (11) to deduce that

$$\begin{pmatrix} \gamma_x \\ \gamma_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

in any point, only if:

$$\begin{pmatrix}\gamma_s^1\\ \gamma_s^2\\ \gamma_s^3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

2.6 Computation of the stresses and internal forces

One presents here the application of the kinematics defined in [§2.4] in the computation of the internal forces. Two different cases will be considered:

- Linear, Model
- elasticity total of type `GLRC`.

For linear elasticity, only one point of integration in the thickness is necessary, because the strains and the forced vary linearly in the thickness.

The objectif principal of modelization `Q4GG` is the use of model `GLRC_DAMAGE` [R7.01.31] in the frame of element `T3G` and `Q4G`. Only one point of integration in the thickness is necessary, because non-linearity is taken into account directly in terms of forces and strains generalized.

2.6.1 Linear elasticity

For a linear elastic behavior one defines potential energy in the following way:

$$\Phi^L=\frac{1}{2}\int_S\int_{\frac{-h}{2}}^{\frac{h}{2}}\varepsilon_{ij}C^{el}_{ijkl}\varepsilon_{kl}\,dz\,dS+\Phi^{ext}$$

$$=\frac{1}{2}\int_S[e_{ij}H^M_{ijkl}e_{kl}+\kappa_{ij}H^M_{ijkl}\kappa_{kl}+\gamma_i H^{CT}_{ij}\gamma_j]dS+\Phi^{ext} \quad (14)$$

where $\boldsymbol{\varepsilon}$ $\mathbf{C}^{el}$, respectively, the tensor $3D$ strain and the elastic tensor corresponding For a linear material, the integral according to z is carried out analytically. Φ^{ext} the contribution due to the limiting conditions. The tensors e, κ, represent the membrane extension and the variation of curvature and the vector γ the transverse distortion, as introduced into [§2.4]. In the equation (14), one makes as the assumption as the shell is symmetric compared to the average average, if not an additional term of membrane-flexure coupling is obtained. The details of computation in the équationformule (14) are provided in [7], where one finds the statements following for the matrixes $\mathbf{H}$:

$$\mathbf{H}^M=\frac{Eh}{1-\nu^2}\begin{pmatrix}1&\nu&0\\ \nu&1&0\\ 0&0&\frac{1-\nu}{2}\end{pmatrix}\quad \mathbf{H}^F=\frac{Eh^3}{12(1-\nu^2)}\begin{pmatrix}1&\nu&0\\ \nu&1&0\\ 0&0&\frac{1-\nu}{2}\end{pmatrix}\quad \mathbf{H}^{CT}=\frac{kEh}{2(1+\nu)}\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

where h is the thickness, E the Young's modulus formulates, ν the Poisson's ratio and k a parameter which can vary according to the theory applied. Generally formula $k=\frac{5}{6}$ (Reissner models).

Then, one applies the introduced kinematics to [§2.3] and [§2.4]:

$$\begin{pmatrix} e_{xx} \\ e_{yy} \\ e_{xy} \end{pmatrix} = \mathbf{B}^M \begin{pmatrix} u^1 \\ v^1 \\ u^2 \\ v^2 \\ u^3 \\ v^3 \end{pmatrix} \qquad \begin{pmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{pmatrix} = \mathbf{B}^F \begin{pmatrix} \theta_x^1 \\ \theta_y^1 \\ \theta_x^2 \\ \theta_y^2 \\ \theta_x^3 \\ \theta_y^3 \end{pmatrix} \qquad \begin{pmatrix} \gamma_x \\ \gamma_y \end{pmatrix} = \mathbf{B}^{CT} \begin{pmatrix} \theta_x^1 \\ \theta_y^1 \\ w^1 \\ \theta_x^2 \\ \theta_y^2 \\ w^2 \\ \theta_x^3 \\ \theta_y^3 \\ w^3 \end{pmatrix}$$

to obtain from the equation (14) :

$$\Phi^L = \frac{1}{2} U_\alpha^T \underbrace{\left[\int_S B_{ij\alpha}^M H_{ijkl}^M B_{kl\beta}^M dS \right]}_{K_{\alpha\beta}^M} U_\beta + \frac{1}{2} U_\alpha^T \underbrace{\left[\int_S B_{ij\alpha}^F H_{ijkl}^F B_{kl\beta}^F dS \right]}_{K_{\alpha\beta}^F} U_\beta + \frac{1}{2} U_\alpha^T \underbrace{\left[\int_S B_{ij\alpha}^{CT} H_{ijkl}^{CT} B_{kl\beta}^{CT} dS \right]}_{K_{\alpha\beta}^{CT}} U_\beta + \Phi^{ext}$$

By resorting to the principle of minimal free energy, $\partial \Phi^L = 0$, $\forall \delta U_\alpha$, one obtains:

$$\underbrace{(K_{\alpha\beta}^M + K_{\alpha\beta}^F + K_{\alpha\beta}^{CT}) U_\beta}_{\mathbf{F}^{\text{int}}} = F_\alpha^{ext}$$

where $\mathbf{F}^{\text{int}}$ and $\mathbf{F}^{\text{ext}}$ are, respectively, the internal forces and external.

The integration of all the contributions to the stiffness matrix $\mathbf{K}^M$, $\mathbf{K}^F$ and $\mathbf{K}^{CT}$ is carried out with only one point of integration in the plane of the triangle (center of gravity). This gives an exact integration for the terms $\mathbf{K}^M$ $\mathbf{K}^F$ because $\mathbf{H}^M$, $\mathbf{H}^F$, $\mathbf{B}^M$, $\mathbf{B}^F$ are constant by element (see [§4.5]). On the other hand, $\mathbf{B}^{CT}$ is a linear function (see [§2.4]) and the exact integration of $\mathbf{K}^{CT}$ would require three Gauss points.

Like element `T3G`, as proposed in [3] and discussed in [12], under-is integrated on the part of the transverse shears, it has a parasitic mode, i.e. a mode with energy zero, which is not a mode of rigid body. From two elements, the model `T3G` finds a correct “row however”.

2.6.2 Behavior total nonlinear of type GLRC

During the use of a total model, the relation between the stresses and the strains is defined in term of the generalized variables, ε^G , σ^G :

$$\varepsilon^G=\begin{pmatrix}e\\ \kappa\\ \gamma\end{pmatrix}\qquad \sigma^G=\begin{pmatrix}N\\ M\\ T\end{pmatrix}$$

where the generalized strains are the membrane extension e , the variation of curvature κ , and the transverse distortion γ (see [§2.4] for more precise definitions). The generalized stresses are the membrane force $\mathbf{N}$, the bending moment $\mathbf{M}$, and the transverse shears $\mathbf{T}$:

$$\mathbf{N}=\begin{pmatrix}N_{xx}\\ N_{yy}\\ N_{xy}\end{pmatrix}=\frac{1}{2}\int_{-\frac{h}{2}}^{\frac{h}{2}}\begin{pmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{pmatrix}dz\qquad \mathbf{M}=\begin{pmatrix}M_{xx}\\ M_{yy}\\ M_{xy}\end{pmatrix}=\frac{1}{2}\int_{-\frac{h}{2}}^{\frac{h}{2}}z\begin{pmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{pmatrix}dz$$

$$\mathbf{T}=\begin{pmatrix}T_x\\ T_y\end{pmatrix}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\begin{pmatrix}\sigma_{xz}\\ \sigma_{yz}\end{pmatrix}dz$$

GLRC_DAMAGE `The model` takes into account elastoplasticity in membrane-bending, mainly due to steels, and the damage in bending, mainly due to the concrete. The free energy corresponding to the model is defined like:

$$\Phi^G=\frac{1}{2}\int_S(e-e^P)_{ij}H^M_{ijkl}(e-e^p)_{kl}\,dS+\frac{1}{2}\int_S(\kappa-\kappa^P)_{ij}H^{Fd}_{ijkl}(d_{1,}d_2)(\kappa-\kappa^p)_{kl}\,dS \qquad (15)$$

$$+\frac{1}{2}\int_S\gamma_{ij}H^{CT}_{ijkl}\gamma_{kl}\,dS$$

where e^p and κ^p are, respectively, plastic membrane extension and plastic variation of curvature; $\mathbf{H}^M$ and $\mathbf{H}^{CT}$ the elastic matrixes for the membrane and the transverse shears, the same ones as those defined in [§2.6.1]. For the part of bending one defines the matrix $\mathbf{H}^{Fd}$, which is a function of the variables of damage, d_1 and d_2 [8]. One points out that no non-linearity is considered for the part of the transverse shears. This assumption is not really justified, because the damage of the concrete inevitably induces a reduction in the stiffness in transverse shears. Since the model was conceived in the frame of the assumption of the thin plates, it is disadvised using GLRC_DAMAGE `in` situations the model where the transverse energy of shears becomes comparable to the contributions of the membrane or bending.

By imposing $\partial\Phi^G=0$, one obtains:

$$\delta\Phi^G=\delta U^\alpha\underbrace{\int_S(\hat{B}^M_\alpha)^T\begin{pmatrix}N_{xx}\\ N_{yy}\\ N_{xy}\end{pmatrix}dS}_{\mathbf{F}^M_{\text{int}}}+\delta\theta^\alpha\underbrace{\int_S(\hat{B}^F_\alpha)^T\begin{pmatrix}M_{xx}\\ M_{yy}\\ M_{xy}\end{pmatrix}+(\hat{B}^{CT}_{\theta\alpha})^T\mathbf{H}^{CT}\begin{pmatrix}\gamma_x\\ \gamma_y\end{pmatrix}dS}_{\mathbf{F}^F_{\text{int}}} \qquad (16)$$

$$+\delta w^\alpha\underbrace{\int_s(\hat{B}^{CT}_{w\alpha})^T\mathbf{H}^{CT}\begin{pmatrix}\gamma_x\\ \gamma_y\end{pmatrix}dS}_{\mathbf{F}^{CT}_{\text{int}}}$$

where the following relations are used:

$$\hat{\mathbf{B}}^M=\int_{-\frac{h}{2}}^{\frac{h}{2}}\mathbf{B}^M\,dz \quad \hat{\mathbf{B}}^F=\int_{-\frac{h}{2}}^{\frac{h}{2}}z\,\mathbf{B}^F\,dz \quad \hat{\mathbf{B}}^{CT}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\mathbf{B}^{CT}\,dz$$

the constitutive law, whose practically all characteristics rise from the definition of the free energy [éq .4.7.3-1], can be summarized, according to the equation (16) , in term of relations:

$$\mathbf{N}=\mathbf{N}(e,\kappa) \qquad \mathbf{M}=\mathbf{M}(e,\kappa) \qquad \mathbf{T}=\mathbf{H}^{CT}\gamma$$

the details of the application of model `GLRC_DAMAGE` in a frame "finite elements" are given in [4] and [8].

2.7 "Lumped" mass matrix

In this paragraph one presents the approach used to build the "lumped" mass matrix. One focuses oneself on the terms due to bending, the terms corresponding to the membrane being obtained in a classical way also applied to the elements $2\mathrm{D}$. The mass matrix $\mathbf{M}$ is defined from kinetic energy E^{cin} and of the kinematics suggested with [§2.2]:

$$E^{cin}=\frac{1}{2}\int_S\int_{-\frac{h}{2}}^{\frac{h}{2}}\rho\,\dot{u}^2\,dz\,dS=(\dot{\theta}_x^{\alpha}\ \ \dot{\theta}_y^{\alpha}\ \ \dot{w}^{\alpha})\,M^{\alpha\beta}\begin{pmatrix}\dot{\theta}_x^{\beta}\\ \dot{\theta}_y^{\beta}\\ \dot{w}^{\beta}\end{pmatrix}+E_{memb}^{cin}$$

where E_{memb}^{cin} represents the contribution of the membrane, ρ being the density and the mass matrix $\mathbf{M}$ being equal to:

$$M^{\alpha\beta}=m^{\alpha\beta}\begin{pmatrix}\frac{h^3}{12} & 0 & 0\\ 0 & \frac{h^3}{12} & 0\\ 0 & 0 & h\end{pmatrix} \tag{17}$$

$$m^{\alpha\beta}=\int_s\rho\,N^{\alpha}\,N^{\beta}\,dS \tag{18}$$

a way calculate the integral (18) master key by the squaring of Gauss, which leads to the coherent matrix coupling the contributions of the various degrees of freedom. It is generally recognized that the computation of a coherent mass matrix is not justified for computations in fast dynamics, for which the "lumped" matrix always offers a better accuracy-cost ratio.

One uses the same mass matrix for `the T3G` as for `the DKT`. Besides the approximation of the decoupling of the various degrees of freedom, leading to the use of the lumped matrix, one makes approximations on the terms of inertia in order to increase time step stability.

In [5] one proposes to build the lumped matrix from the equation (18) formulates the squaring of Lobatto, whose alternatives are the trapezoidal diagram and the diagram of Simpson, where the points of integration coincide with the nodes. The construction of the mass matrix is done through a finite element of beam, linear and with two nodes, by means of the trapezoidal diagram, leading to:

$$M_0^{pout}=\frac{1}{2}\rho LA\begin{pmatrix}\frac{I}{A} & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & \frac{I}{A} & 0\\ 0 & 0 & 0 & 1\end{pmatrix} \quad (19)$$

for which the vector of degrees of freedom is written like $(\theta_1 w_1 \theta_2 w_2)^T$. A, L and I are the area of the section, the length and the main moment of inertia of the element beam, respectively. The use of the matrix $(eq.\ 19)$ formulates seeming too restrictive stability condition of the explicit diagram of integration in time, one proposes in [5] rather:

$$M_0^{pout}=\frac{1}{2}\rho LA\begin{pmatrix}\alpha & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & \alpha & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

where the parameter α is introduced so that its adjustment can maximize time step stability. According to [5] its optimal value would be $\alpha=\frac{1}{8}L^2$. By directly applying these results by analogy to the plates one replaces the matrix of the equation (19) by:

$$M^{\alpha\beta}=m^{\alpha\beta}\begin{pmatrix}\frac{hA^e}{8} & 0 & 0\\ 0 & \frac{hA^e}{8} & 0\\ 0 & 0 & h\end{pmatrix} \quad (20)$$

where A^e is the area of the element considered. In the transition of the beam to the plate one supposed a certain equivalence between the length of the element beam and the area of the element plates, so that $A^e\approx L^2$. It is pointed out that the approach suggested in [5] is not rigorous from the geometrical point of view and that it is focused on the maximization of the step of stability. In the version established in *Code_Aster* one makes sure of the desired effect of the modification of (18) in (19) by means of (as that was made for the elements of family DKT [7]):

$$M^{\alpha\beta}=m^{\alpha\beta}\begin{pmatrix}max\left(\frac{h^3}{12}\ ,\ \frac{hA^e}{8}\right) & 0 & 0\\ 0 & max\left(\frac{h^3}{12}\ ,\ \frac{hA^e}{8}\right) & 0\\ 0 & 0 & h\end{pmatrix} \quad (21)$$

because the equation (20) is not interesting that for the coarse meshes, a priori more current, while the equation (17) favorable for very fine meshes. Moreover, in *Code_Aster* one decides to integrate (18) by the squaring of Gauss into three points by keeping only the diagonal terms:

$$m^{\alpha\beta}=\delta_{\alpha\beta}\int_s \rho\, N^{\alpha} N^{\beta} dS \quad (22)$$

Note:: the classical coherent mass matrix was not developed for this finite element in *Code_Aster*.

2.8 Second member corresponding to work external

the variational formulation of work external and the discretization of the potential of the external forces are presented in §4.8 [R3.07.03].
Note: The thermal loading is not treated.

3 Establishment of the shell elements in *Code_Aster*

3.1 Description

These elements (of names MEQ4GG4, MET3GG3) lean on meshes QUAD4 and plane TRIA3. These elements are not exact with the nodes and it is necessary to net with several elements to get correct results.

3.2 Introduced use and developments

These elements are used in the following way:

•AFFE_MODELE (MODELISATION=' Q4GG',…) for the triangle and the quadrangle.

```
•AFFE_CARA_ELEM (COQUE=_F (EPAISSEUR=' EP'
                ANGL_REP = (" α " " β ")
                COEF_RIGI_DRZ = "CTOR")
```

to make postprocessings (generalized forces,…) in a reference chosen by the user who is not the local coordinate system of the element, one gives a direction of reference **D** defined by two nautical angles in the total reference. The projection of this direction of reference as regards the plate fixes a direction $X1$ reference. The norm with the plane into fixed one second and the cross product of the two vectors previously definite make it possible to define the local trihedron in which will be expressed the generalized forces and the forced. The user will have to take care that the selected reference axis is not found parallel with the norm of certain shell elements of the model. By default this direction of reference is the axis X total reference of definition of the mesh.

Value CTOR corresponds to the coefficient which the user can introduce for the processing of the terms of stiffness and mass according to normal rotation with the plan of the plate. This coefficient must be sufficiently small not to disturb the energy assessment of the element and not too small so that the stiffness matrixes and of mass are invertible. A value of 10^{-5} is put by default.

For a behavior:

•Homogeneous isotropic elastic in the thickness one uses key word ELAS in DEFI_MATERIAU where one defines the coefficients E Young modulus, ν Poisson's ratio, ρ the density:

```
ELAS =_F (E = Young NU = nu , RHO = rho …)
```

•Elastoplastic endommageable of the type GLRC_DAMAGE, this total model of reinforced concrete plate is able to represent its behavior until failure. The characteristics of the concrete and reinforcements are given in DEFI_GLRC.

```
DEFI_GLRC (  RELATION = "GLRC_DAMAGE ",
              BETON    = _F (MATER =  …, EPAIS=…) ,
             THREE-DIMENSIONS FUNCTION    = _F (MATER =  …, OMX
=…))
```

AFFE_CHAR_MECA (

DDL_IMPO =_F (DX =. DY =. DZ =. DRX =. DRY =. DRZ =. degree of freedom of plate in the total reference.

FORCE_COQUE =_F (FX =. FY =. FZ =. MX =. MY =. MZ =.) They is the surface forces (membrane and bending) on shell elements. These forces can be given in the total reference or the reference user defined by ANGL_REP.

FORCE_NODALE =_F (FX =. FY =. FZ =. MX =. MY =. MZ =.) They is the forces of shell in the total reference.

3.3 Computation in linear elasticity

the stiffness matrix and the mass matrix (respectively options RIGI_MECA and MASS_MECA) are integrated numerically. It is not checked if the mesh is plane or not. The computation account holds owing to the fact that the terms corresponding to the degrees of freedom of plate are expressed in the local coordinate system of the element. A transition matrix makes it possible to pass from the local degrees of freedom to the total degrees of freedom.

Currently available elementary computations correspond to the options:

DEGE_ELNO : who gives the strains generalized by element to the nodes starting from displacements in the reference user: EXX, EYY, EXY, KXX, KYY, KXY, GAX, GAY.
EFGE_ELNO : who gives the forces (generalized stresses) by element to the nodes starting from displacements: NXX, NYY, NXY, MXX, MYY, MXY, QX, QY.
SIEF_ELGA : who gives the forces (generalized stresses) by element to Gauss points starting from displacements: NXX, NYY, NXY, MXX, MYY, MXY, QX, QY.
EPOT_ELEM : who gives the elastic strain energy of strain per element starting from displacements.
ECIN_ELEM : who gives kinetic energy by element.

Finally one calculates also option FORC_NODA of computation of the nodal forces for operator CALC_CHAMP.

3.4 Nonlinear computation

the stiffness matrix is there too integrated numerically. One calls on the computation option STAT_NON_LINE in which one defines in the level of the nonlinear behavior the number of layers to be used for numerical integration.

```
STAT_NON_LINE (….
 COMP_INCR =_F (RELATION = ' GLRC_DAMAGE'
….)
```

For the modelization Q4GG, the only constitutive laws used are total models (since there is only one point of integration in the thickness), connecting the strains generalized to the generalized stresses.

Currently available elementary computations correspond to the options:

- DEGE_ELNO which provides the strains generalized by element to the nodes in the reference user starting from displacements.
- SIGM_ELNO which makes it possible to obtain the stress field in the thickness by element with the nodes for all the subpoints (all the layers and all the positions: in lower skin, in the medium and in higher skin of layer).
- EFGE_ELNO which makes it possible to obtain the forces (generalized stresses) by element with the nodes in the reference user.
- VARI_ELNO which calculates the field of local variables and the forced by element with the nodes for all the layers, in the local coordinate system of the element.

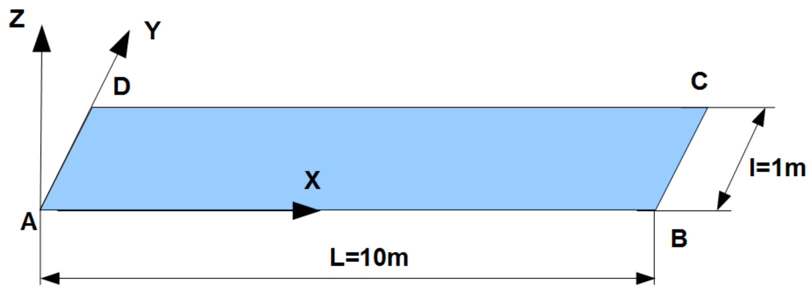
4 Validation of modelization Q4GG

4.1 SDLS123 – Straight beam with damping of Rayleigh (V2.03.123)

This test represents dynamic computation elastic with damping of Rayleigh of a straight beam, embedded at the two ends and subjected to a constant pressure. The got results are compared with the results got with Europlexus.

Two modelizations are carried out:

- Modelization A makes it possible to test the model Q4GG with QUAD4,
- Modelization B makes it possible to test Q4GG with TRIA3 the model.



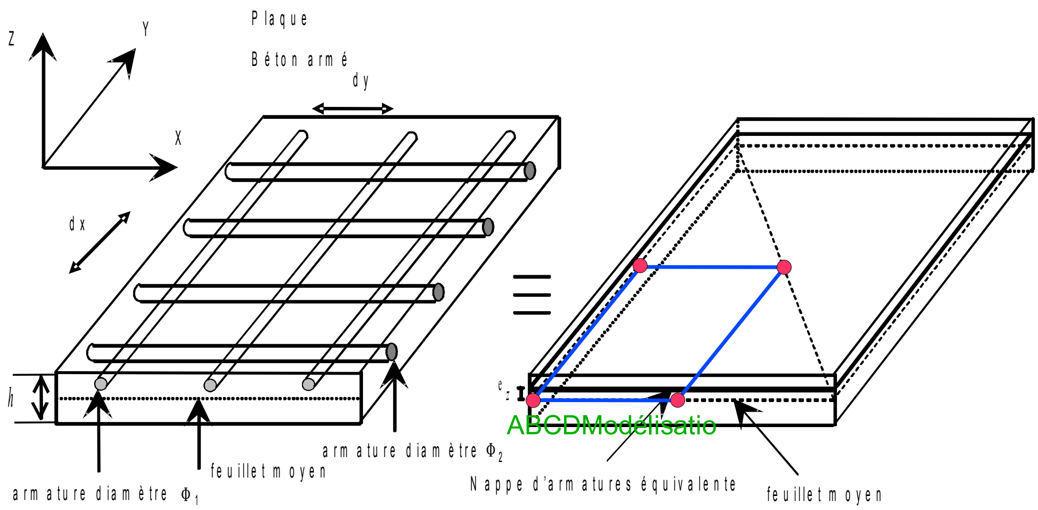
Epaisseur = 1m

4.2 SDNS108 – Dynamic response of a reinforced concrete slab (GLRC_DAMAGE models) leaned on 4 sides subjected to a concentrated loading: mode of 4.2 elastoplastic plate (V5.06.108)

This test represents the computation of a reinforced concrete slab, in bending, subjected to a concentrated loading. It makes it possible the model to validate modelization Q4GG using total material GLRC_DAMAGE by comparing it with a similar computation with Europlexus. The slab is simply leaned on the four sides.

Two modelizations are carried out:

- Modelization A makes it possible to test the model Q4GG with QUAD4,
- B makes it possible to test Q4GG with TRIA3 the model.



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