Computation of the characteristics of an unspecified beam of cross section

Summarized:

One presents the principle of the computation of the various quantities characteristic of the sections of beams. Those are established starting from the geometrical characteristics of the cross section of the beam.

These values are with being provided to operand SECTION: "GENERALE" of operator AFFE_CARA_ELEM [U4.42.01]. To determine them, of the numerical methods are presented, and put in work in command MACR_CARA_POUTRE.

In the case of the sections "RECTANGLE" and "CERCLE", one calculates directly in AFFE_CARA_ELEM the characteristics using simplified formulas which one clarifies here.

Contents

1 geometrical characteristics

Assumption:
One treats here only cross sections of the homogeneous and isotropic beams (same characteristics of material for all the points and in all the directions). Command `MACR_CARA_POUTRE` can also calculate the geometrical characteristics of a set of disjoined sections.

1.1 Unspecified section

1.1.1 Principle

Is a section (S) of surface S in the plane $(0, y, z)$ whose origin O is the center of gravity G of the section, [Figure 1].

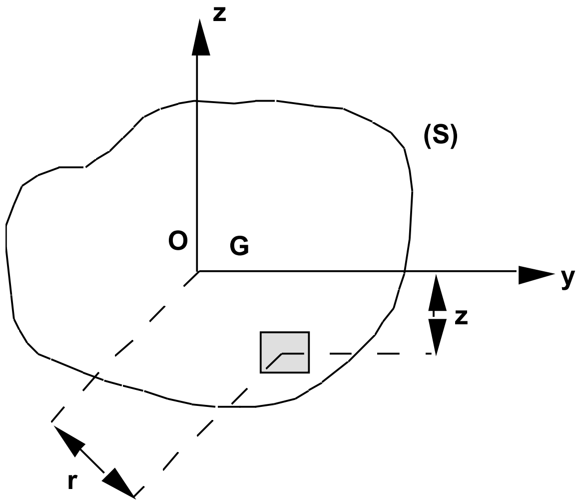


Figure 1 : section in the plane $(0, y, z)$

the geometrical main moment of inertia from (S) ratio with the axis (Oy) (which passes by the center of gravity) is expressed by:

$$I_y = \int_s z^2 \, dS \text{ with } [I] = \int_s OM \otimes OM \, dS$$

One defines in a similar way the geometrical moment compared to (Oz) by:

$$I_z = \int_s y^2 \, dS$$

When the centrifugal geometrical moment (often called produced inertia of area) defined by is null, $I_{yz} = \int_s y \, z \, dS$ the axes (Oy) and (Oz) are principal axes of the section (S). One places oneself for the continuation on this assumption; I_y and I_z are then called the principal geometrical moments.

Generally, we must place ourselves in the principal axes of a section of beam for all that relates to its characteristics since the beam elements of *Code_Aster* are formulated in this reference. On the basis of an origin located at the center of gravity, it is enough, to pass from an unspecified system of axes (G, y', z') to the system of principal axes (G, y, z), to carry out a rotation of angle θ such as [Figure2]:

$$\theta = \frac{1}{2} Arctg \left(\frac{2 \, I_{y'z'}}{I_{z'} - I_{y'}} \right)$$

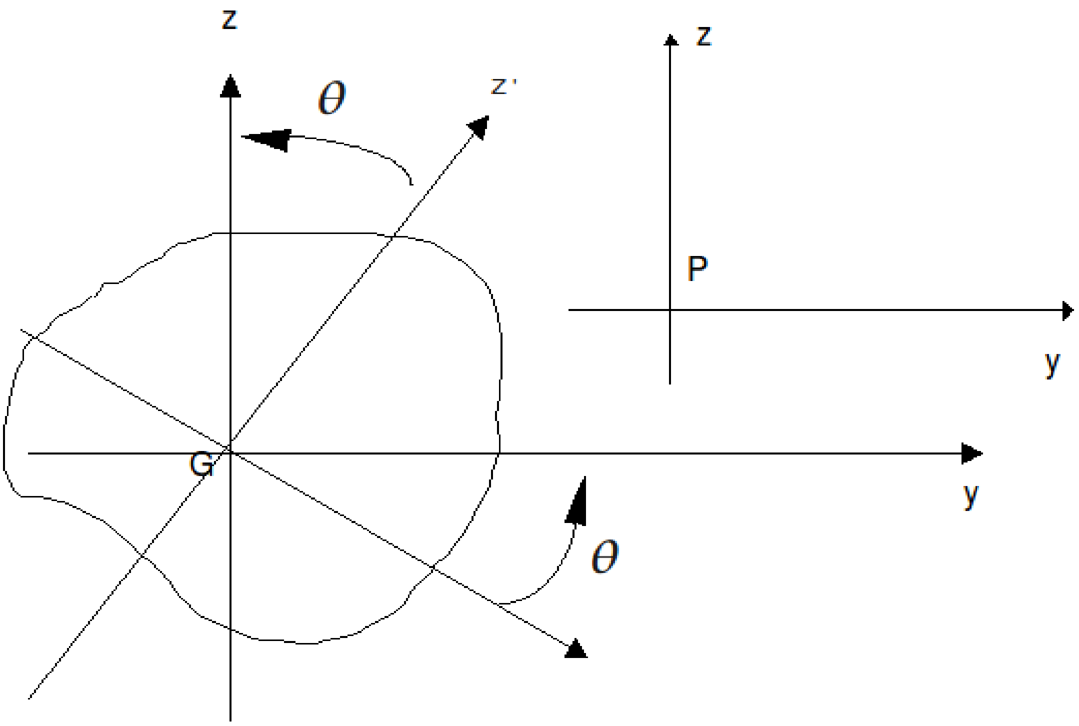


Figure 2 : Principal axes and unspecified.

The polar geometrical moment compared to the center of gravity is given by: $I_p=\int_s r^2\, dS$ where R is the distance from the element dS at the center of gravity [Figure 1]. One from of deduced naturally $I_p=I_y+I_z$.

The polar geometrical moment intervenes in the computation of the torsion stiffness of the beams of circular section (torsion of Coming Saint). For the other forms of sections, one will define a constant of the same torsion dimension.
Moreover, the geometrical moments can be calculated in another reference (P, y, z), of unspecified P origin different from the center of gravity G (formula of Huygens):

$$I_y^P=I_y^G+(\mathbf{GP}.Z)^2\ .S=\int_s z^2 dS+(\mathbf{GP}.Z)^2.S$$

$$I_z^P=I_z^G+(\mathbf{GP}.Y)^2\ .S=\int_s y^2 dS+(\mathbf{GP}.Y)^2.S$$

$$I_{yz}^P=I_{yz}^G+(\mathbf{GP}.Y)(\mathbf{GP}.Z)\ .S=\int_s yz\, dS+(\mathbf{GP}.Y)(\mathbf{GP}.Z).S$$

in a general way, the formula of Huygens gives:

$$\begin{aligned}[I]&=\int_s(\mathbf{PG}+\mathbf{GM})\otimes(\mathbf{PG}+\mathbf{GM})\\&=\int_s \mathbf{PG}\otimes\mathbf{PG}+\int_s \mathbf{GM}\otimes\mathbf{GM}\\&\quad+2\int_s \mathbf{PG}\otimes\mathbf{GM}\\&=S(\mathbf{PG}\otimes\mathbf{PG})+\int_s \mathbf{GM}\otimes\mathbf{GM}\end{aligned}$$

1.1.2 Computation of the geometrical characteristics using MACR_CARA_POUTRE

This macro-command allows the determination of the characteristics of a cross section of beam from a 2D mesh of the section [U4.42.02]. It makes it possible to build an array of values, usable in command AFFE_CARA_ELEM (SECTION : "GENERALE" [U4.42.01]).

The geometrical characteristics can be calculated on the complete mesh, half mesh with symmetry compared àformule Y or with Z, quarter of mesh with two symmetries compared to Y and with Z [Figure2].
These characteristics are calculated in the array for all the mesh and each mesh group of the list specified by the user (case of a network of beams).

The data correspond to a half or a quarter of the section if key keys `SYME_Y` or `SYME_Z` are present.

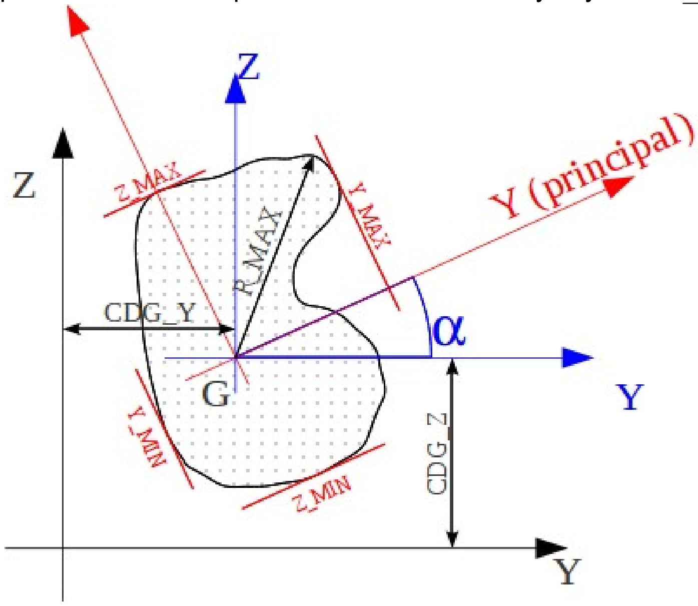


Figure 3 : Definition of the geometrical characteristics.

The results are gathered in four groups:

- In the reference OYZ of description of the mesh 2D for the mesh provided by the user
  - area: `A_M`
  - position of the center of gravity: `CDG_Y_M, CDG_Z_M`
  - moments and product of inertia of area, at the center of gravity G in reference `GYZ` : `IY_G_M`, `IZ_G_M`, `IYZ_G_M`
- In the same total reference, for the mesh obtained by symmetrization if `SYME_Y` or `SYME_Z` :
  - area: `A`
  - position of the center of gravity: `CDG_Y, CDG_Z`
  - moments and product of inertia of area, at the center of gravity G in reference `GYZ` : `IY_G`, `IZ_G`, `IYZ_G`
- In the principal reference of inertia Gyz . cross-section, whose denomination corresponds to that used with the description of the neutral fiber beam elements Gx [U4.24.01].
  - principal main moments of inertia of area in the reference Gyz , usable for the computation of the flexural rigidity of beam: `IY` and `IZ`
  - angle of flow of the reference GYZ to the principal reference of inertia Gyz : `ALPHA`
  - characteristic distances, compared to the center of gravity G of the section for computations of maximum stresses: `Y_MAX`, `Y_MIN`, `Z_MAX`, `Z_MIN` and `R_MAX`.
- In the total reference, in a point P provided by the user:
  - `Y_P`, Z_P : not computation of the main moments of inertia
  - `IY_P`, `IZ_P`, `IYZ_P` : main moments of inertia in reference `PYZ`
  - `IY_P`, `IZ_P` : main moments of inertia in the Pyz `reference`.

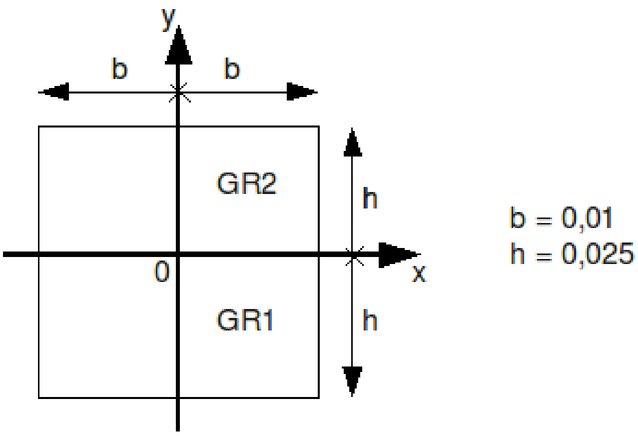
1.1.3 Computations carried out

the list of the commands called by `MACR_CARA_POUTRE` is indicated in the document [U4.42.02].
The preceding quantities are obtained by the call to `POST_ELEM`, for the option `"CARA_GEOM"`. Moreover, one can add to it key words `SYME_Y`, `SYME_Z`, and `ORIG_INER` which defines the point P .

Computations are carried out in POST_ELEM, for all the mesh, then possibly for each mesh group, in the following way:

- Buckle on the elements 2D (modelization D_PLAN), with call of elementary option "MASS_INER". One obtains a CHAM_ELEM with a value by element (1 Gauss point) containing the components:
  $$\int_{élément} dS\,, \int_{\substack{x \\ élément}} dS\,, \int_{élément} dS \quad \int_s x^2\, dS\,, \int_s y^2\, dS\,, \int_s xy\, dS\,,$$
- Summation of the preceding elementary quantities to obtain: A_M, CDG_Y_M, CDG_Z_M, IY_G_M, IZ_G_M, IYZ_G_M
- Computation of A, CDG_Y, CDG_Z, IY_G, IZ_G, IYZ_G (taken into account of SYME_Y, SYME_Z)
- Computation of IY, IZ, ALPHA
- Computation of Y_MAX, Z_MAX, Y_MIN, Z_MIN, R_MAX
- If one clarifies a particular P point (key word ORIG_INER), one calculates also the characteristics in the total reference of origin P : PYZ

1.1.4 Examples of use: Rectangle full (treaty by test SSLL107G)



Characteristic geometrical obtained

```
            LIEUA_MCDG_Y_MCDG_Z_MIY_G_MIZ_G_MIYZ_G_M
0.0000031.00E-034.24E-18-3.39E-182.08E-073.33E-082.65E-23
GR15.00E-042.20E-17-1.25E-022.60E-081.67E-083.97E-23
GR25.00E-04-8.47E-181.25E-022.60E-081.67E-085.62E-23

                                                     LIEUACDG_YCDG_ZIY_GIZ_GIYZ_GIYIZALPHA

0.0000031.00E-034.24E-18-3.39E-182.08E-073.33E-082.65E-233.33E-082.08E-079.00E+01

GR15.00E-042.20E-17-1.25E-022.60E-081.67E-083.97E-231.67E-082.60E-089.00E+01

GR25.00E-04-8.47E-181.25E-022.60E-081.67E-085.62E-231.67E-082.60E-089.00E+01



         LIEUY_PZ_PIY_PIZ_PIYZ_PIY_PRIN_PIZ_PRIN_P
0.0000030.00E+000.00E+002.08E-073.33E-082.65E-233.33E-082.08E-07

GR10.00E+000.00E+001.04E-071.67E-08-9.79E-231.67E-081.04E-07

GR20.00E+000.00E+001.04E-071.67E-08
3.31E-241.67E-081.04E-07

                                               LIEUY_MAXZ_MAXY_MINZ_MINR_MAX
0.0000032.50E-021.00E-02-2.50E-02-1.00E-022.69E-02
GR12.50E-022.25E-02-2.50E-022.50E-033.36E-02
GR22.50E-02-2.50E-03-2.50E-02-2.25E-023.36E-02

LIEUJXAYAZEYEZPCTYPCTZRT
0.000003 -------                                                  rectangular
1.93871E-2
```

```
GR13.43E-081.20E+001.20E+009.00E-17-3.97E-182.60E-17-1.25E-021.56391E-2
```

1.2 GR23.43E-081.20E+001.20E+00-4.03E-171.19E-16-1.27E-161.25E-021.56391E-2 Typical case of the sections and circular

the geometrical characteristics are directly calculated in `AFFE_CARA_ELEM` starting from the data of the user.

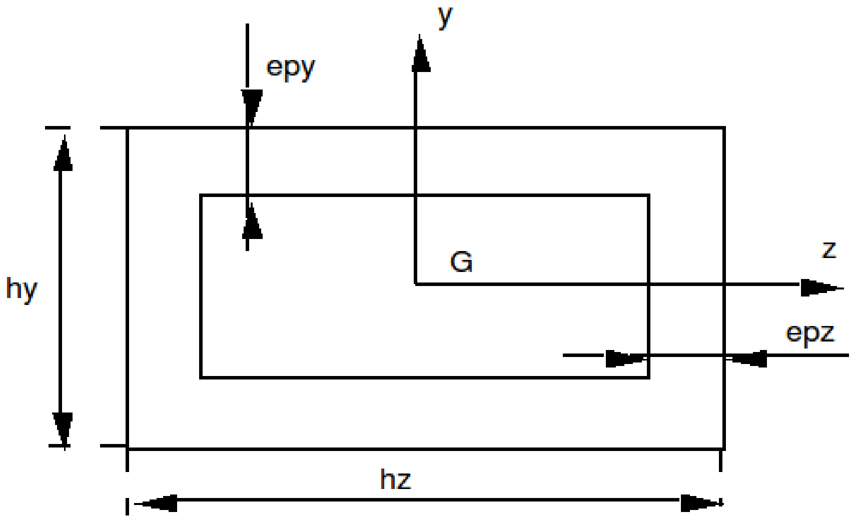


Figure 4 : rectangular section.

In the case of the rectangular beam (Operand `SECTION: "RECTANGLE"`), given computation:

$$I_y=\frac{1}{12}\left[h_y h_z^3-\left(h_y-2\,ep_y\right)\left(h_z-2\mathrm{ep}_z\right)^3\right]$$

$$I_z=\frac{1}{12}\left[h_z h_y^3-\left(h_z-2\,ep_z\right)\left(h_y-2\mathrm{ep}_y\right)^3\right]$$

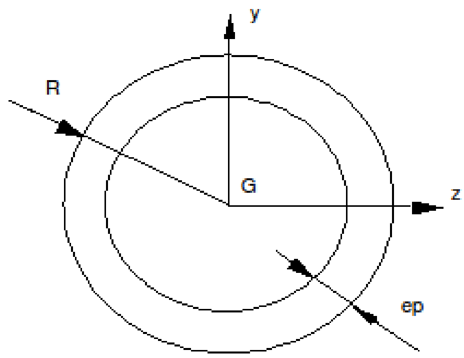


Figure 5 : circular section

For the circular section (Operand `SECTION: "CERCLE"`), one obtains:

$$I_y=I_z=\frac{p}{4}\left[R^4-(R-ep)^4\right] \quad I_p=\frac{p}{2}\left[R^4-(R-ep)^4\right]$$

2 The shear coefficients and the shear center

It is a question of evaluating the coefficients $A_y=\frac{1}{k_y}, A_z=\frac{1}{k_z}$ intervening in the models of beams of Timoshenko with taking into account of the shear strains. For the EULER beams, these coefficients do not intervene [U4.42.01 §7.4.2] and [R3.08.01 §2.3.1]. These coefficients are obtained for a linear elastic behavior.
In the case of the unspecified sections, the shear coefficients are with being provided by the user in `AFFE_CARA_ELEM`, if the selected element is a TIMOSHENKO beam (models `POU_D_T`, `POU_C_T`, `POU_D_TG` and `POU_D_TGM`).
In the case of the circular or rectangular sections, the shear coefficients are calculated by analytical methods of [§2.1].
In all the cases, they can be calculated by `MACR_CARA_POUTRE`, starting from the plane mesh of the section. The numerical method used is exposed to [§2.3]. This method applies to unspecified sections (of homogeneous and isotropic material). In appendix 2, one describes an extension of this method to the case of a network of parallel beams maintained between two rigid bottoms.
The position of the center of torsion (or shear center) is obtained only by numerical methods (cf [§2.3]). For the rectangular and circular sections, as for all the sections with 2 symmetry planes, the center of torsion is confused with the center of gravity of the section.

2.1 Analytical methods

One describes three analytical methods allowing to calculate shear coefficients, applicable to the unspecified sections.
The first two methods differ by the definition which they propose of the shear coefficient, but rest on the same assumption which consists in applying the form of the distribution of the shearing stresses in the section.

2.1.1 Assumption of distribution of the shears: formulate JOURAWSKI

Consider for example the case of a beam of cross-section S, subjected to shears $V_y=\int_s \sigma_{xy} dS$.
One writes the equilibrium of a prismatic part of the beam, ranging between the cross-sections S_x and S_{x+a} the plane of cut located at the Y-coordinate y and y_{max} (ref. [biberon88]). The forces acting on this part of beam are the vectors forced on the sides S_x and S_{x+a}, and those acting on the face located in y.

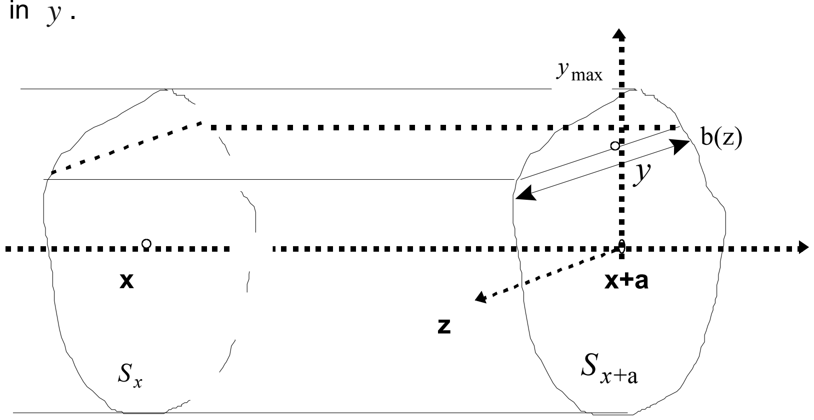


Figure 6 : Section of beam.

By applying the theorem of the resultant, one obtains:

$$\int_{S_{x+a}} \boldsymbol{\sigma}_{xx}(x,y',z)\,dy'dz-\int_{S_x} \boldsymbol{\sigma}_{xx}(x,y',z)\,dy'dz \quad = N(x+a,y)-N(x,y)$$

$$=\int_x^{x+a}\int_{\frac{-b(y)}{2}}^{\frac{b(y)}{2}} \boldsymbol{\sigma}_{xy}(\alpha,y,z)\,d\alpha\,dz$$

To evaluate the term of right, JOURAWSKI proposed to consider only the average of the shears according to Z:

$$\bar{\boldsymbol{\sigma}}_{xy}(x,y)=\frac{1}{b(y)}\int_{-\frac{b}{2}}^{\frac{b}{2}}\boldsymbol{\sigma}_{\boldsymbol{xy}}(x,y,z)\,dz$$

then

$$\int_{x}^{x+a}\int_{-\frac{b(y)}{2}}^{\frac{b(y)}{2}}\boldsymbol{\sigma}_{\boldsymbol{xy}}(\alpha,y,z)\,d\alpha\,dz=\int_{x}^{x+a}b(y)\bar{\boldsymbol{\sigma}}_{xy}(\alpha,y)\,d\alpha$$

and while making tend a towards 0 ,

$$\frac{\partial N}{\partial x}=b(y)\bar{\boldsymbol{\sigma}}_{xy}(x,y)$$

the balance equations of beam and the distribution of bending stresses (in elasticity) give:

$$N(x,y)=\int_{S_x}\boldsymbol{\sigma}_{\boldsymbol{xx}}(x,y,z)\,dydz=\int_{S_x}\frac{M_z(x).y}{I_z}dydz$$
$$=\frac{M_z(x)}{I_z}m(y)\,\text{avec}\,m(y)=\int_{y}^{y_{\max}}t\quad b(t)\quad dt$$

thus

$$\bar{\boldsymbol{\sigma}}_{xy}(x,y)=\frac{m(y)}{I_z b(y)}\frac{\partial M_z}{\partial x}=\frac{m(y)}{I_z b(y)}V_y$$

the distribution of the shears according to y is thus given by the formula of JOURAWSKI:

$$\bar{\boldsymbol{\sigma}}_{xy}(x,y)=\frac{m(y)}{I_z b(y)}V_y\,\text{avec}\,m(y)=\int_{y}^{y_{\max}}t\quad b(t)\quad dt \qquad [\ 1\ ]$$

in accordance with [U4.24.01], with the notations of [Figure6]. The quantity $m(y)$ represents the statical moment on behalf of section (hatched) understood enters y and y_{max} :

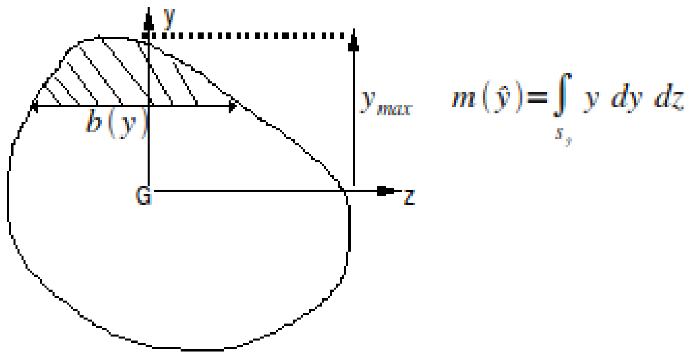


Figure 7 : section of beam

This distribution checks well the boundary conditions following y of the three-dimensional problem: the shears are quite null on bottom fibers and higher ($y=y_{\min}$, or $y=y_{\max}$). But she takes account only of the average of the shears according to z .

By applying this formula to a full rectangular section, one finds a distribution parabolic according to y . By applying it to a beam of circular section, one finds a distribution parabolic in there and in z , which varies more slowly according to whether z according to y .

This remains valid for the other full sections. For sections comprising of holes, it is necessary to take care to consider only the matter in the computation of $b(y)$.

2.1.2 Method of TIMOSHENKO

In the beginning, TIMOSHENKO (ref. [biberon99]) proposed a simple definition of the shear coefficient, as being the relationship between the average transverse shearing stress in the noted section $\bar{\sigma}_{CT}$ and its maximum value $\left(\sigma_{CT_{Max}}\right)$. Owing to the fact that we always have the shears by:

$$V = \int_s \boldsymbol{\sigma}_{CT} dS \quad [2]$$

we deduce:

$$\bar{\sigma}_{CT} = \frac{V}{S}$$

.

Knowing that TIMOSHENKO proposes to write:

$$k = \frac{\bar{\sigma}_{CT}}{\sigma_{CT_{Max}}} \quad [3]$$

to determine k, it is enough to express the shears $V = \int_s \boldsymbol{\sigma}_{CT} dS$ [éq22] according to $\sigma_{CT_{Max}}$. In the general case of the unspecified sections, there will be naturally two coefficients k_y and k_z, for each of the two principal axes.

It remains to determine $\sigma_{CT_{Max}}$. For that, TIMOSHENKO makes an assumption on the transverse distribution of the shearing stresses: the transverse shearing stress has a parabolic distribution in the direction of the shears which produce it, with its maximum value in the center and of the zero values to edges. This is true according to the formula of JOURAWSKI for a rectangular section. By extension, the method extends this assumption of parabolic distribution to an unspecified section

This method is not applied in *Code_Aster*, except for the hollow rectangular sections. One uses the following method in the other cases.

2.1.3 "Energy" method

Actually, the definition suggested by TIMOSHENKO proves little used in practice today; one prefers to him a formulation based on internal energy due to the shears in the section. This one is written:

$$U_{CT} = \int_s \frac{1}{2} \frac{\boldsymbol{\sigma}_{CT^2}}{G} dS$$

where G is the shear modulus (equal to m).

The new definition of the shear coefficient is sometimes allotted to MINDLIN and is expressed by:

$$U_{CT} = \frac{1}{2} \frac{V^2}{k SG} \quad [4]$$

So by substitution, one thus defines for a homogeneous section of material the shear coefficient by:

$$k = \frac{\left[\int_s \boldsymbol{\sigma}_{CT} dS\right]^2}{S \int_s \boldsymbol{\sigma}_{CT^2} dS} = \frac{V^2}{S \int_s \boldsymbol{\sigma}_{CT^2} dS} \quad [5]$$

By making an assumption on the distribution of stress in the section, one can thus estimate the value of k. From the formula of JOURAWSKI [éq11], the preceding statement can be written [biberon55]:

$$k = \frac{I^2}{S\int_S \frac{m^2(y)}{b^2(y)} dS} \quad [6]$$

2.1.4 Method of COWPER

One can also take into account the three-dimensional effects to determine the coefficient k ; various formulations were proposed, by COWPER [biberon3]3 and were taken again in particular by BLEVINS [biberon22], while being melted on the resolution of the three-dimensional problem of Saint-Coming.
In this case, the coefficient k is a function of the Poisson's ratio, in general an approximation with the first order. COWPER uses the three-dimensional equations of elasticity in the dynamic case to propose a statement of k giving good performances in static and low frequency dynamics. The approximation which makes it possible to lead to the formula suggested consists in considering a distribution of stress not parabolic, but resulting from the static problem (solved analytically) of the cantilever beam transversely charged at its loose lead. It should be noted that the distribution obtained is strictly identical to the problem with a transverse loading uniformly distributed.

2.2 Typical case of the sections rectangular and circular

One distinguishes the full beams and the tubes.

For the full rectangular section, the shear coefficient is determined by the method based on internal energy of shears with parabolic distribution of the stresses:

$$k_y = \frac{I_z^2}{S\int_S \frac{m_z^2(y)}{b_z^2(y)} dy} \quad [7]$$

Applied to the rectangular section, one obtains $k_y = k_z = \frac{5}{6}$. It should be noted that this value also corresponds to the method of COWPER when the Poisson's ratio is taken equal to zero.

For the rectangular tube, *Code_Aster* uses the method of TIMOSHENKO which leads to $k_y = k_z = \frac{2}{3}$. In the case of the beams with full circular section, one uses the energy method which leads to $k_y = k_z = \frac{9}{10}$. This value is also obtained by the method of COWPER when the Poisson's ratio is equal to $\frac{1}{2}$.
For the circular tubes, one distinguishes the tubes with fine wall and those with thick wall. If one notes $m = \frac{r_i}{r_e}$ the ratio of the internal radius to the external radius, a tube is with fine wall when $m > 0.9$, if not it is with thick wall.
The shear coefficient of the circular fine wall tube is given by the method of COWPER, by considering that $m=1$ and for a null Poisson's ratio, that is to say $k_y = k_z = \frac{1}{2}$.
For the circular thick wall tubes, one uses an approximate formula of the method of COWPER which is written: $k = \frac{1}{1,093 + 0,634m + 1,156m^2 - 0,905m^3}$
Let us notice that this formula does not ensure continuity with the borderline cases of the full cylinder ($m=0$) and the cylinder infinitely thin wall ($m=1$).

If the preceding choices (carried out by `AFFE_CARA_ELEM` in the case of circular and rectangular sections) are not appropriate, it is always possible to numerically calculate the shear coefficients using `MACR_CARA_POUTRE`, of which the method is specified in the § according to.

2.3 Numerical method of computation of the shear coefficients and the shear center

2.3.1 Computation of the shear coefficients:

This method takes as a starting point [biberon11], page 62. It allows the simultaneous determination of the constants of shears and the center of torsion. It is put in work in `MACR_CARA_POUTRE`, from a plane mesh of the section. It functions currently only for homogeneous and isotropic sections (for nonhomogeneous sections, the method is similar [biberon11] but nonavailable in *Code_Aster*).

As for the energy method, one compares for shears V internal energy U_1 due to the shears in the section with energy U2 associated with the model with MINDLIN:

$$U_1 = \int_s \frac{1}{2}\frac{\sigma_{xy}^2+\sigma_{xz}^2}{G}dS=U_2=\frac{1}{2}\frac{V_z^2}{k_z SG}$$

The shear coefficient is expressed by: $k_z = \frac{1}{2}\frac{V_z^2}{SGU_1}$

It is thus necessary to calculate U_1 and thus the shearing stresses (in elasticity) in the section to estimate the value of k. One places oneself in the principal reference of inertia (G, y, z), and one supposes that the beam is subjected only to one shears V_z. It results from it that:

$$\sigma_{xx}=z\frac{M_y(x)}{I_y}$$
$$\frac{\partial\sigma_{xx}}{\partial x}=z\frac{V_z}{I_y}$$

The balance equations make it possible to write:

$$\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{xy}}{\partial y}+\frac{\partial\sigma_{xz}}{\partial z}=0=\frac{\partial\sigma_{xy}}{\partial y}+\frac{\partial\sigma_{xz}}{\partial z}+z\frac{V_z}{I_y}$$

In addition, the kinematics of the beam in bending/shears is:

$$u(x, y, z)=u(x)+z\,\theta_y(x)+\tilde{u}(y, z)$$
$$v(x, y, z)=0$$
$$w(x, y, z)=w(x)$$

$\tilde{u}(y, z)$ representing axial displacement due to the warping of the section. The strains are written:

$$\varepsilon_{xx}=\frac{\partial u(x)}{\partial x}+z\frac{\partial\theta_y(x)}{\partial x}$$
$$2\varepsilon_{xy}=\frac{\partial\tilde{u}(y,z)}{\partial y}$$
$$2\varepsilon_{xz}=\theta_y(x)+\frac{\partial\tilde{u}(y,z)}{\partial z}+\frac{\partial w(x)}{\partial x}$$

By means of the behavior model of linear elasticity, the stresses are written:

$$\sigma_{xy}=2\mu\varepsilon_{xy}=\mu\frac{\partial\tilde{u}(y,z)}{\partial y}$$

$$\sigma_{xz}=2\mu\varepsilon_{xz}=\mu\left(\theta_y(x)+\frac{\partial\tilde{u}(y,z)}{\partial z}+\frac{\partial w(x)}{\partial x}\right)$$

The components of shears thus check:

$$\frac{\partial\sigma_{xy}}{\partial z}-\frac{\partial\sigma_{xz}}{\partial y}=0$$

This relation makes it possible to introduce the stress function ψ_z such as the shearing stresses into the section are written:

$$\sigma_{xy}=\mu\frac{\partial\psi_z}{\partial y}$$

$$\sigma_{xz}=\mu\frac{\partial\psi_z}{\partial z}$$

The balance equation then makes it possible to obtain the function ψ_z by resolution of a harmonic problem quasi - which is written:

$$G\Delta\psi_z+f=0 \text{ dans } S \text{ avec } f=\frac{zV_z}{I_y}$$

$$G\frac{\partial\psi_z}{\partial n}=0 \text{ sur } \partial S$$

$$\psi_z=0 \text{ en un point}$$

This makes it possible to calculate ψ_z then the shears. In practice, in MACR_CARA_POUTRE, one uses THER_LINEAIRE to solve the problem, while comparing ψ_z to the temperature. One chooses $V=1$ and $G=1$ (G does not intervene any more in the statement of the shear coefficient). The boundary conditions of this problem of steady thermal are:

- source f being worth $\frac{z\mathbf{V}_z}{I_y}$
- null flux on ∂S
- temperature null in a point of S

One can then determine $U_1^z=\int_s\frac{1}{2}\frac{\sigma_{xy}^2+\sigma_{xz}^2}{G}dS=\frac{1}{2}\int_s G\left(\nabla\psi_z\right)^2 dS$ by an elementary computation on all the elements of the section, with option "CARA_CISA" (computation of the gradient), then summation on these elements. One calculates $k_z=\frac{1}{2}\frac{V_z^2}{\mathrm{SGU}_1^z}$

same computation then is carried out with $V_y=1$ determining $k_y=\frac{1}{2}\frac{V_y^2}{\mathrm{SGU}_1^y}$

result provided is $A_y=\frac{1}{k_y}, A_z=\frac{1}{k_z}$.

2.3.2 Computation of the coordinates of the shear center

the shear center C is the point of the section where the shearing stresses due to shears generate one null twisting moment. This point is also called center of torsion, because there remains fixed when the section is only subjected to one twisting moment.

The twisting moment compared to the point G is worth $M_{xG} = \int_s (\sigma_{xz}.y - \sigma_{xy}.z) dS$:

The twisting moment compared to the sought C point is:

$$M_{xC} = \int_s (\sigma_{xz}.(y-y_c) - \sigma_{xy}(z-z_c)) dS = M_{xG} - y_c V_z + z_c V_y$$

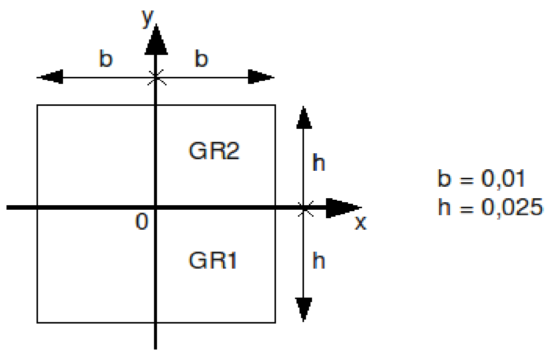
To determine the coordinates of the shear center, preceding computation is used [biberon11]:
From the shearing stresses determined for $V=1$ and $V_y=0$, one calculates: $M^z_{xG} = \int_s (\sigma_{xz}.y - \sigma_{xy}.z) dS$

One obtains: $y_c = \dfrac{M^z_{xG}}{V_z} = M^z_{xG}$

For $V=1$ and $V_y=0$, one obtains: $z_c = -\dfrac{M^y_{xG}}{V_y} = -M^y_{xG}$

2.3.3 Example

Let us take again the example of the rectangular section [§1.1.4].



The shear coefficients obtained are identical to the analytical value (6/5).

LIEU	*AY*	*AZ*	*EY*	*EZ*
all	1.20E+00	1.20E+00	– 8.72E-19	3.16E-18

the components of the vector $CG=(EY, EZ)$ expressed in the principal reference (G, y, z) are null: the shear center/torsion is actually confused with the center of gravity.

2.4 Computation of the shear coefficients of a network

the method described in appendix 2 makes it possible to calculate shear coefficients of a beam equivalent to a set of parallel beams embedded on a rigid bottom and embedded or rotulées on another.

3 The constants related to torsion

the constant of noted torsion JX must make it possible to take account of the warping of the cross-sections (not circulars) during a strain in torsion. It is used in the models of straight beams treated by *Aster* (EULER, TIMOSHENKO and warped TIMOSHENKO or `POU_D_E`, `POU_D_T`, `POU_C_T` and `POU_D_TG`). In the case of the circular sections, the sections are not warped and the constant of torsion is equal to the polar geometrical moment I_p. The constant of torsion C is defined like the moment necessary to produce a rotation of 1 radian per unit of length divided by the shear modulus, that is to say:

$$C = \frac{M_x}{\mu \frac{\partial \theta_x}{\partial x}} \quad [8]$$

JX has the same dimension which the geometrical main moments of inertia I_y and I_z is m^4.

For a circular section, the definition [éq88] is coherent since we have:

$$M_x = \mu I_p \frac{\partial \theta_x}{\partial x}$$

The determination deformule JX in the general case is made in a numerical way (`MACR_CARA_POUTRE`) and is reduced to a computation of Laplacian in 2D. The method presented here is detailed in the ref. [biberon11] [§3.6.3] for the simply related sections. An original method for computation of the constants of torsion with perforated sections is detailed in appendix. The results here are given.

3.1 Computation of C in the case of the unspecified sections

the complete resolution of the problem is in appendix. One gives here simply the results.
According to the assumptions of the theory of the pure torsion of Saint-Coming, there is no strain of line average and not of lengthening along the longitudinal axis. Torsion is free, i.e. that it does not generate axial stresses. In other words, the sections can warp freely. If one remains in small displacements, one admits that the swing angle of the cross-sections is worth:

$$q_x(x) = \frac{\partial \theta_x}{\partial x} x = \Theta . x \quad [9]$$

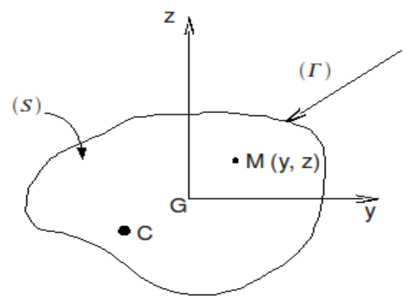


Figure 8 : general section.

If the point c is the center of torsion (which by definition remains motionless when the beam is subjected to a torsion), the field of displacement $\mathbf{u}(M)$ is given by [biberon11]:

$$\mathbf{u}(M)=\begin{bmatrix}u\\v\\w\end{bmatrix}=\begin{bmatrix}\frac{\partial\theta_x}{\partial x}x\\0\\0\end{bmatrix}\wedge\begin{bmatrix}x\\(y-y_c)\\(z-z_c)\end{bmatrix}+\begin{bmatrix}\frac{\partial\theta_x}{\partial x}\xi(y,z)\\0\\0\end{bmatrix}=\begin{pmatrix}\frac{\partial\theta_x}{\partial x}\xi(y,z)\\-\frac{\partial\theta_x}{\partial x}x(z-z_c)\\\frac{\partial\theta_x}{\partial x}x(y-y_c)\end{pmatrix}$$

where $\xi(y,z)$ is the function of warping.

The Hooke's law is written:

$$\boldsymbol{\sigma}=2\mu\boldsymbol{\varepsilon}+(E-2\mu)Trace(\boldsymbol{\varepsilon})\mathbf{I}$$

where $\mathbf{I}$ is the matrix unit and the tensor of the strains is worth $\boldsymbol{\varepsilon}=\frac{1}{2}\left(\text{grad}(\mathbf{u})+{}^T\text{grad}(\mathbf{u})\right)$

By neglecting the terms of the second order, in $\frac{\partial^2\theta_x}{\partial x^2}$, one ends in:

$$\boldsymbol{\sigma}=\mu\frac{\partial\theta_x}{\partial x}\begin{bmatrix}0&\frac{\partial\xi}{\partial y}-z&\frac{\partial\xi}{\partial z}+y\\\frac{\partial\xi}{\partial y}-z&0&0\\\frac{\partial\xi}{\partial z}+y&0&0\end{bmatrix}=\mu\frac{\partial\theta_x}{\partial x}\begin{bmatrix}0&\frac{\partial\varphi}{\partial z}&-\frac{\partial\varphi}{\partial y}\\\frac{\partial\varphi}{\partial z}&0&0\\-\frac{\partial\varphi}{\partial y}&0&0\end{bmatrix}\qquad[10]$$

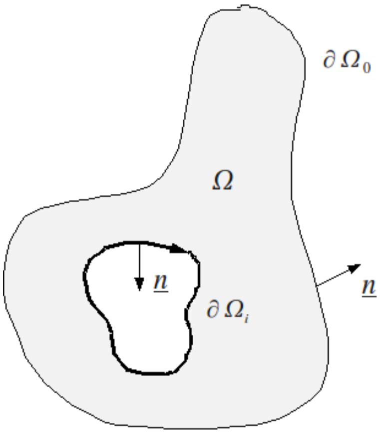
One posed: $\varphi(y,z)$ stress function. It is noted that the relation of equilibrium $\text{div}\,s=0$ is then checked. While deriving, one obtains:

$$\frac{\partial^2\varphi}{\partial y^2}=-\frac{\partial^2\xi}{\partial z\partial y}-1$$
$$\frac{\partial^2\varphi}{\partial z^2}=\frac{\partial^2\xi}{\partial y\partial z}-1.$$

By adding the two equations, one leads to:

$$\Delta\varphi=-2\qquad[11]$$

It remains to establish the boundary conditions. One notes n the norm directed towards outside at the border (Γ) which can be multiplement related:



Without external loading, one must have $\boldsymbol{\sigma}\otimes\mathbf{n}=0$, which can be written:

$\mu\frac{\partial\theta_x}{\partial x}\begin{bmatrix}\frac{\partial\varphi}{\partial z}n_y-\frac{\partial\varphi}{\partial y}n_z\\0\\0\end{bmatrix}=0$ where n_y and n_z are the two components of the norm.

This writing can be thus put in the form $\mathbf{n}\wedge\mathbf{grad}\,\varphi=0$ which implies that the vectors $\mathbf{n}$ and $\mathbf{grad}\,j$ are colinéaires. It thus follows that $\varphi(y,z)$ is constant on each related component of the border (Γ). One can impose for example that $\varphi(y,z)$ is null on external contour:

$$\varphi=0 \text{ on } \partial\Omega_0=\Gamma$$

$$\varphi=\varphi_i \text{ on } \partial\Omega_i$$

If the sections behavior of holes, the constants φ_i are undetermined. To allow the resolution of the complete problem, equations should be added. Those are obtained from the circulation of the function of warping on each closed contour. The following conditions are obtained:

$$\oint_{\partial\Omega_i}\frac{\partial\varphi}{\partial\mathbf{n}}dl=2\mathrm{A}(\partial\Omega_i)$$

where $A(\partial\Omega_i)$ is the area surrounded by the border $(\partial\Omega_i)$. These conditions are brought back to classical imposed flux conditions (where $l(\partial\Omega_i)$ the length of the border represents $(\partial\Omega_i)$):

$$\frac{\partial\varphi}{\partial\underline{n}}=\frac{2\mathrm{A}(\partial\Omega_i)}{l(\partial\Omega_i)}$$

Finally, the problem to be solved is written:

$$\Delta\varphi=-2 \text{ on } \Omega$$
$$\varphi=0 \text{ on } \partial\Omega_0=\Gamma$$
$$\varphi=\varphi_i \text{ on } (\partial\Omega_i)$$
$$\frac{\partial\varphi}{\partial\underline{n}}=\frac{2\mathrm{A}(\partial\Omega_i)}{l(\partial\Omega_i)}$$

Once solved this problem, one obtains the constant of torsion by:

$$CT=2\int_{\Omega}\varphi\, ds+2\sum_{i=1}^{n-1}\varphi_i A\left(\partial\,\Omega_i\right).$$

3.2 Computation of the constant of torsion in MACR_CARA_POUTRE

This computation is carried out in MACR_CARA_POUTRE by the resolution of a problem of thermal. It is necessary for that the user specifies with MACR_CARA_POUTRE the group of mesh which defines external edge, and if the section comprises holes, the mesh groups which define the contour of each one of them.
One then solves a linear problem of thermal (THER_LINEAIRE) on a plane mesh of the section to find the function j. One places oneself first of all in the principal reference of inertia (CREA_MAILLAGE), starting from the coordinates of the center of gravity and the directional sense of the principal reference calculated previously.

One defines then the boundary conditions in AFFE_CHAR_THER :
- The source term is worth – 2
- The temperature of external edge is imposed and is worth 0 (TEMP_IMPO)
- If the section comprises holes (presence of one or more mesh groups defining them):
  - On each group of mesh defining a hole, the temperature is constant (TEMP_UNIF)
  - it flux is worth 2 times the area of hole divided by the length of its edge. These quantities are calculated before.
- The computation deformule JX is carried out in POST_ELEM by the key word CARA_TORSION of factor key word the CARA_POUTRE. In this case, one calculates on each element the integral of j, (option CARA_TORSION on the plane thermal elements), then one carries out the sum on all the elements.

Example:
Still let us take again the example of the rectangular section [§1.1.4]. The shear coefficients obtained are:

PLACE	JX [éq1313]	JX (Aster)
all	9.9805E-08	9.9681E-08

3.3 Computation of the radius of torsion in an unspecified section

the radius of torsion is calculated thanks to the computation of the stress function on the mesh of the section. Rt is added in the array produced by MACR_CARA_POUTRE [U4.42.02].
The resolution of a steady thermal problem of unknown j makes it possible to determine the constant of torsion and the shearing stresses.
The determination of the radius of torsion Rt is the resolution of: $Rt=grad(\varphi).n$ (or n represents the normal vector external with edge considered of the section).
Rt vary along external contour; indeed, for an unspecified section, the shears due to torsion vary on edge. One chooses to take the value of Rt leading to the shears maximum on external edge, i.e. the maximum value of Rt (in absolute value) on external contour.
Moreover, if the section is alveolate, there is several "several radius torsion": $Rt=2*A(k)/L(k)$ (where $A(k)$ the area of the cell and k its $L(k)$ perimeter represents). If one is satisfied to search the maximum value of the shears, it is necessary to take the maximum of the values Rt obtained on external edge and the cells.
The radius of torsion is given in MACR_CARA_POUTRE only by commands python. During the unfolding of MACR_CARA_POUTRE command POST_ELEM is called, a new parameter Rt is thus created for this command.

3.4 Constant of torsion of the sections circular and rectangular

Of the statements simplified for these two types of sections are described here. The computation constants of torsion is then directly carried out in AFFE_CARA_ELEM.
For the circular section the preceding statements remain valid. By taking a function of torsion of the form $\varphi(y,\ z)=\frac{1}{2}(R^2-x^2-y^2)$ one finds indeed:

$$CT=I_p=\frac{\pi}{2}\left(R_0^4-R_1^4\right)$$

For the rectangular section, computation is naturally more complex but can be carried out by choosing a function which is cancelled indeed with edges, of the form:

$$\varphi(y,\ z)=\sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty}A_{ij}\cos\left[(2i+1)\frac{\pi y}{h_y}\right]\cos\left[(2j+1)\frac{\pi z}{h_z}\right]$$

The resolution involves with a constant of torsion which is written:

$$CT=\frac{h_y^3h_z^3}{h_y^2+h_z^2}c\left(\frac{h_y}{h_z}\right)=h_yh_z^3\left[\frac{c\left(\frac{h_y}{h_z}\right)}{1+\left(\frac{h_z}{h_y}\right)^2}\right] \quad [12]$$

where $c\left(\frac{h_y}{h_z}\right)$ expresses itself in the form of a series which takes the following values:

$\frac{h_y}{h_z}$	1	2	4	8	$+\infty$
$c\left(\frac{h_y}{h_z}\right)$	$0{,}281$	$0{,}286$	$0{,}299$	$0{,}312$	$1/3$

In fact, *Code_Aster* employs a formula simplified (ref. [biberon11]) for the full rectangular section which is written:

$$CT=\frac{h_yh_z^3}{16}\left(\frac{16}{3}-3.36\frac{h_z}{h_y}+0.280\left(\frac{h_z}{h_y}\right)^5\right) \quad [13]$$

It is valid if $h_y>h_z$; in the other case it is enough to exchange respective cores of h_y and h_z . The agreement between the two statements is very good as indicates it the following table:

$\frac{h_y}{h_z}$	1	2	4	8	∞
$\frac{JX}{h_yh_z^3}$ according to [éq1212]	0,1405	0,2288	0,2814	0,3072	1/3
$\frac{JX}{h_yh_z^3}$ according to *Aster* [éq1313]	0,1408	0,2289	0,2809	0,3071	1/3

For the hollow rectangular beam, there exists an approximate solution which is written (ref. [biberon22] and [biberon66]):

$$JX=\frac{2\,ep_y\,ep_z\left(h_y-ep_y\right)^2\left(h_z-ep_z\right)^2}{h_y\,ep_y-ep_y^2+h_z\,ep_z-ep_z^2}$$

with the notations of [Figure4]: section in the plane $(0, y, z)$.

3.5 The effective radius of torsion

the effective radius of torsion R_T makes it possible to calculate the transverse shearing stress of maximum torsion σ_{T_M} according to the twisting moment. One will be able to consult on this subject the drafting of MASSONET on this aspect (ref. [biberon55]). We have as follows:

$$\sigma_{T_M}=M_x\frac{R_T}{JX}$$

In the case of the circular cylinders, R_T is equal to the radius (external if it is a tube) of the section.
For the rectangular sections, the problem is definitely more complex. *Code_Aster* imposes the radius of torsion of the full section by:

$$R_T=\frac{JX\,4\left(3\,h_y+1.8\,h_z\right)}{h_y^2+h_z^2}$$

This approximate statement remains valid if the beam is not flattened too much. DHATT and BATOZ (ref. [biberon11]) give a statement having a field of validity more extended, but actually it is acted as any rigor of a series whose numerical values are given by MASSONET (ref. [biberon55]).
For the hollow rectangular beam, *Code_Aster* imposes a statement which is valid only if the wall is thin and of constant thickness ep_z, that is to say:

$$R_T=\frac{JX}{ep_z\left(h_y-2\,ep_y\right)\left(h_z-2\,ep_z\right)}$$

It is about a "adaptation" of the formula:

$$R_T=\frac{JX}{2\,e\,A}$$

where e is the thickness of the wall (constant) and A the area contained inside the line average one. This last statement is known under the name of first formula of BREDT (cf ref. [biberon11] and [biberon55]).

4 Computation of the warping constant

the warping constant is used by the model beam with warping (modelization `POU_D_TG` and `POU_D_TGM`), which it is important to take into account for the beams with open mean sections (cf [R3.08.04]).

This coefficient (noted I_w in [R3.08.04], in m^6) intervenes in the statement of the virtual wor of the internal forces on the terms of torsion:

$$W_{\text{int}}=\int_0^2\left(\theta^*_{x,x}\,\mu.C.\theta_{x,x}+\theta^*_{x,xx}.\ E\ .I_\omega.\theta_{x,xx}\right)dx$$

By taking again the approach of [§3.1], and while placing itself in a reference related to the center of torsion C, the kinematics of the torsion of an unspecified section is:

$$\boldsymbol{u}(M)=\begin{bmatrix}u\\v\\w\end{bmatrix}=\begin{bmatrix}\dfrac{\partial\theta_x}{\partial x}x\\0\\0\end{bmatrix}\wedge\begin{bmatrix}x\\y\\z\end{bmatrix}+\begin{bmatrix}\dfrac{\partial\theta_x}{\partial x}\xi(y,z)\\0\\0\end{bmatrix}=\left|\begin{matrix}\dfrac{\partial\theta_x}{\partial x}\ \xi(y,z)\\-\dfrac{\partial\theta_x}{\partial x}xz\\\dfrac{\partial\theta_x}{\partial x}xy\end{matrix}\right|$$

where $\xi(y,z)$ is the function of warping (which cancels only in the case of a circular section).
The statement of the stress field east (in elasticity):

$$\sigma_{xx}=E\,\varepsilon_{xx}=E\,\xi(y,z)\frac{\partial^2\theta_x}{\partial x^2}$$
$$\sigma_{xy}=2\mu\varepsilon_{xy}=\mu\frac{\partial\theta_x}{\partial x}\left(\frac{\partial\xi(y,z)}{\partial y}-z\right)$$
$$\sigma_{xz}=2\mu\varepsilon_{xz}=\mu\frac{\partial\theta_x}{\partial x}\left(\frac{\partial\xi(y,z)}{\partial z}-y\right)$$

Unlike [§4], the terms of the second order in $\dfrac{\partial^2\theta_x}{\partial x^2}$ are not neglected any more.

The first relation of equilibrium $(\text{div}\,\boldsymbol{\sigma})_x=\sigma_{xx,x}+\sigma_{xy,y}+\sigma_{xz,z}=0$ then implies the following condition on the function of warping: $\Delta\xi=0$

In addition, without loading external on contour of the section, one must have $\sigma\otimes n$, which can be written: $\dfrac{\partial\xi}{\partial y}n_y+\dfrac{\partial\xi}{\partial z}n_z=z.n_y-y.n_z$, where n_y and n_z are the two components of the norm, or in vectorial form: $\textbf{grad}\,\xi.\mathbf{n}=\dfrac{\partial\xi}{\partial\mathbf{n}}=(\mathbf{n}\wedge\mathbf{CM}).\,x$

This determines the function of warping except for a constant. To raise this indetermination, for example the statement of the normal force is written (for a section where torsion produces warping):

$$N=\int_S\sigma_{xx}\,ds=\int_S E\,\xi\frac{\partial^2\theta_x}{\partial x^2}ds=0$$

thus the additional condition on the function of warping is $\int_S \xi\, ds=0$. In practice, in MACR_CARA_POUTRE, one places oneself above all in a reference related to the center of torsion c. One calculates then ξ who must check:

$$\Delta x=0$$

$$\mathbf{grad}\, x.n=\frac{\partial \xi}{\partial n}=(n\wedge \mathbf{CM}).x$$

$$\int_S x\, ds=0$$

The inertia of warping I_ω is obtained then by: $I_\omega=\int_S \xi^2\, ds$

MACR_CARA_POUTRE calls on the following elementary commands:

- Translation of the coordinates of the nodes in the reference related to the center of torsion (calculated previously in array TCARS):
```
CREA_MAILLAGE (MAILLAGE = my,
    coordinate = _F (ARRAY = TCARS, NOM_ORIG = "TORSION"))
```
- Assignment of a model (thermal plane), of a material field:
```
 AFFE_MODELE (MAILLAGE = my,
    AFFE = _F (TOUT = "OUI', PHENOMENE = "THERMAL",
            MODELISATION=' PLAN'))
AFFE_MATERIAU (MAILLAGE = my,
    AFFE = _F (TOUT = "OUI' MATER: mat))
```
- Boundary conditions on external contour $G0$: $\frac{\partial \xi}{\partial y} n_y+\frac{\partial \xi}{\partial z} n_z=z.n_y-y.n_z$
```
F1=DEFI_FONCTION (NOM_PARA =, VALE = (0. , 0. , 10. , -10. ))
F2=DEFI_FONCTION (NOM_PARA =, VALE = (0. , 0. , 10. , 10. ))
CH1 = AFFE_CHAR_THER_F (MODELS = MOD,
    FLUX_REP = _F (GROUP_MA = G0, FLUX_X = F1, FLUX_Y = F2))
```
- Condition on the field solution: $\int_S \xi\, ds=0$: creation of a unit source term on all the mesh, and of the associated second member vector. LIAISON_CHAMNO then makes it possible to impose the desired condition.
```
 CHS = AFFE_CHAR_THER (= SOURCE
    MODELS = _F (TOUT = "OUI' SOUR = 1.))
VS = CALC_VECT_ELEM (OPTION = "CHAR_THER" CHARGE = CHS…)
MS = CALC_MATR_ELEM (=… OPTION MODELS = "RIGI_THER")
NUM = NUME_DDL (MATR_RIGI = MS)
GOES = ASSE_VECTEUR (VECT_ELEM = VS NUME_DDL = NUM)
CH2 = AFFE_CHAR_THER (
    LIAISON_CHAMNO = _F (CHAM_NO = GOES  COEF_IMPO = 0.))
```
- Computation of the function of warping x :
```
THER_LINEAIRE (MODELS =….
    EXCIT = (
        _F (CHARGE: CH1),
        _F (CHARGE: CH2),
    )
)
```
- Computation of the warping constant $I_\omega=\int_S \xi^2\, ds$ and enrichment of the array:
```
 TCARS = POST_ELEM (MODELS =…
    CARA_POUTRE = _F (CARA_GEOM = TCARS,
                LAPL_PHI = KSI OPTION: "CARA_GAUCHI"));
```

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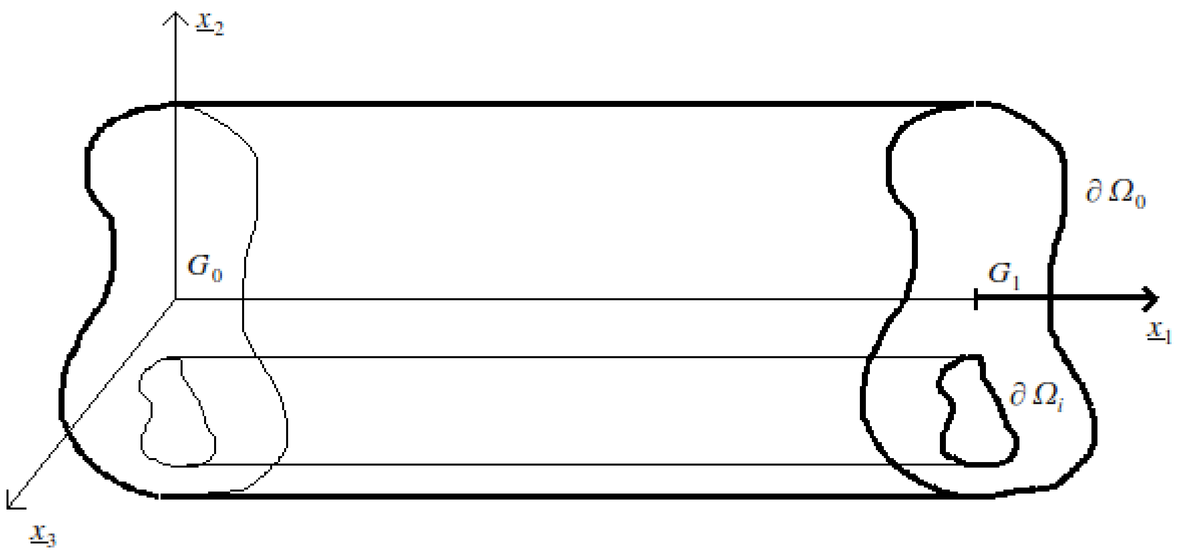
Description of the versions of the document:

VersionAster	**Author (S) Organization (S)**	**Description of the modifications**
7.4	**J.M. Proix**, N. Laurent, P.Hémon, G. Bertrand EDF-R&D/AMA, IAT St Cyr, CS-SI	
8.5	J.M.Proix, EDF-R&D/AMA Correction	of unit page 24, file REX 10783 10 Jean
10	FLÉJOU, EDF-R&D/AMA Formatted	, correction formulas, file REX 16337 Determination

Annexe 1 : of the constant of torsion for sections has borders multiplement related Is

a beam elastic, isotropic, length and of unspecified L section which can Ω be not simply related. One notes contour $\partial\Omega_0$ external of and, for Ω $\partial\Omega_1$, possible $i=1...n-1$ interior contours. His total border $\partial\Omega=\bigcup_{i=0}^{n-1}\partial\Omega_i$ is noted. One chooses

the axis according to x_1 locus of centers of gravity of the cross-sections. It is supposed to simplify the demonstration that the center of torsion is confused with the center of gravity, which makes it possible to uncouple the effects of torsion and bending. The axes and are x_2 selected x_3 following the principal directions of inertia. Figure



9: Beam 9 with unspecified section. The beam

is charged on its section by one twisting moment $x_1=L$. In addition $M_{G_1}=M_t x_1$
, the side surface of the cylinder is not charged and the body forces are null. One of deduced immediately that the torsor of the internal forces at the point is G_0 the problem $M_{G_0}=M_t x_1$
of elasticity posed previously seems incompletely definite. Indeed, the boundary conditions on the cross-sections and are $x_1=L$ incomplete $x_1=0$ because there is not a condition in each point, but on average. There is thus, a priori *, an infinity of* solutions. The assumption of Saint-Coming consists in seeking a solution such as the tensor of the stresses is form: The principle

$$\boldsymbol{\sigma}=\begin{bmatrix}\sigma_{11} & \sigma_{12} & \sigma_{13}\\ \sigma_{12} & 0 & 0\\ \sigma_{13} & 0 & 0\end{bmatrix}$$

of Saint-Coming is valid far from the sections of application of the forces. Indeed, except in cases of particular loadings, the four presumedly null terms diminish exponentially with. To solve x_1
this problem of elasticity, a stress formulation is chosen. The equations to be written are thus those of equilibrium and those of compatibility. The balance equations

lead $\operatorname{div}\boldsymbol{\sigma}=\underline{0}$ to the three following scalar equations: [14] [15

$$\partial_1\sigma_{11}+\partial_2\sigma_{12}+\partial_3\sigma_{13}=0 \quad]14$$

$$\partial_1\sigma_{12}=0$$
$$\partial_1\sigma_{13}=0 \quad 15$$

the simplified notation:) the equations $\partial_i \sigma_{jk} = \dfrac{\partial \sigma_{jk}}{\partial x_i}$

of Beltrami, which take account of the compatibility equations are written: [16] [17

$$-\Delta\sigma_{11} - \partial_{11}\sigma_{11} = 0 \quad]16$$

$$-\Delta\sigma_{12} - \frac{1}{1+\nu}\partial_{12}\sigma_{11} = 0 \quad]17$$

$$-\Delta\sigma_{13} - \frac{1}{1+\nu}\partial_{13}\sigma_{11} = 0 \quad]18$$

$$-\partial_{22}\sigma_{11} + \nu\,\Delta\sigma_{11} = 0 \quad]19$$

$$-\partial_{23}\sigma_{11} = 0 \quad]20$$

$$-\partial_{33}\sigma_{11} + \nu\,\Delta\sigma_{11} = 0 \quad \text{equations}21$$

[éq16], [éq1916 and [éq2119] shows21 that, and is $\partial_{11}\sigma_{11}$ $\partial_{22}\sigma_{22}=0$ solutions $\partial_{33}\sigma_{11}$ of a homogeneous linear system and thus that. With L $\partial_{11}\sigma_{11} = \partial_{22}\sigma_{11} = \partial_{33}\sigma_{11} = 0$ "equation [éq20], one from of20 deduced that. By taking account of $\sigma_{11} = a_0 + a_1 x_1 + (b_1 x_1 + b_0) x_2 + (c_1 x_1 + c_0) x_3$ the fact that L" one deals with the problem of free torsion, one will take no one from σ_{11} now. The equations [éq15] and [éq1815] show18 that and do not depend σ_{12} σ_{13} on. The equation x_1 [éq14] is written14: where and

$$\partial_2\left[\sigma_{12} + f(x_3)\right] = \partial_3\left[-\sigma_{13} - g(x_2)\right]$$

are f two g arbitrary functions. According to the theorem of Schwartz, there exists such as $\varphi(x_{2,} x_3)$: The equations

$$\begin{cases} \partial_2\varphi = -\sigma_{13} - f(x_2) \\ \partial_3\varphi = \sigma_{12} + g(x_3) \end{cases} = \begin{cases} \sigma_{12} = \partial_3\varphi - f(x_3) \\ \sigma_{13} = -\partial_2\varphi + g(x_2) \end{cases}$$

[éq17] and [éq1817] give18: maybe where

$$\begin{cases} \partial_3 \Delta\varphi = 0 \\ \partial_2 \Delta\varphi = 0 \end{cases}$$

is $\Delta\varphi = \partial_3 f - \partial_2 f + K$

K a constant of integration. As and are f arbitrary g , one will take them identically null. The problem to be solved is thus a problem of Laplacian: on then $\Delta\varphi = K$: It Ω remains

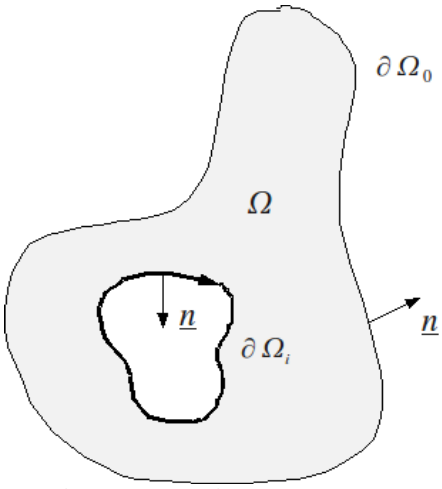
$$\boldsymbol{\sigma} = \begin{bmatrix} 0 & \partial_3\varphi & -\partial_2\varphi \\ \partial_3\varphi & 0 & 0 \\ -\partial_2\varphi & 0 & 0 \end{bmatrix}$$

to write the boundary conditions, who will allow us to write conditions on for and $\varphi\,\Omega$ on φ the boundary conditions K

are to be written on all the border. On the cross-sections (and the same $x_1=L$ in), one a: $x_1=0$ [22] Is

$$\int_{\Omega} \sigma_{12}\, ds=0$$
$$\int_{\Omega} \sigma_{13}\, ds=0$$
$$\int_{\Omega} x_2 \sigma_{13}-x_3 \sigma_{12}\, ds=M_t \quad 22$$

the norm $\underline{n}$ external with. One A. $\partial\Omega$ One poses $\boldsymbol{\sigma}\otimes\mathbf{n}=0$. One can $\boldsymbol{\tau}=\sigma_{12}\mathbf{x_2}+\sigma_{13}\mathbf{x_3}$ also write; $\boldsymbol{\tau}=\sigma_{12}\mathbf{x_2}+\sigma_{13}\mathbf{x_3}$ τ the tangential part of the stress in the cross-section is called. That is to say the point N_i running of contour for. $\varphi\,\Omega$ The condition $i=0,\dots,n-1$, on side surface, stated higher, can be written. Figure $\boldsymbol{\tau}\, \boldsymbol{dN}_\mathbf{i}=\underline{0}$



10: Definition 10 of the norms Thus, on

the side surface of a beam, the vector shear stress is tangent τ with contour. •The equation leads $\boldsymbol{\tau}\wedge \boldsymbol{dN}_\mathbf{i}=\underline{0}$ to a condition which must observe on contour φ : •The equation $d\varphi=0$ [éq22] led22 to that one $M_t=\int_{\Omega} -x_2\partial_2\varphi-x_3\partial_3\varphi\, ds$ can also write: The problem

$$M_t=2\int_{\Omega}\varphi\, ds+\int_{\partial\Omega}\varphi(x_3\, dx_2-x_2\, dx_3)$$

to solve obtain is thus φ : on on

$$\Delta\varphi=K \text{ with } \Omega$$
$$d\varphi=0 \quad \partial\Omega$$

the stress It remains $M_t=2\int_{\Omega}\varphi\, ds+\int_{\partial\Omega}\varphi(x_3\, dx_2-x_2\, dx_3)$

to identify the constant of torsion C. the constitutive law of the beams in torsion is: (cf [§ $M_t=CG\dfrac{\partial\theta_x}{\partial x}$ 3]). To solve the preceding problem more easily, one poses and, $\psi=\dfrac{\varphi}{C\dfrac{\partial\theta_x}{\partial x}}$ the problem $K=-2\mathrm{C}\dfrac{\partial\theta_x}{\partial x}$ to be solved becomes then: on on

$$\Delta\psi=-2 \text{ With } \Omega$$
$$d\psi=0 \text{ such } \partial\Omega$$
$$M_t=-2\mathrm{C}\frac{\partial\theta_x}{\partial x}\left[\int_\Omega \psi\, ds+\int_{\partial\Omega}\psi\left(x_3\,dx_2-x_2\,dx_3\right)\right]$$

a notation, one obtains contour $C=2\int_\Omega \psi\, ds+\int_{\partial\Omega}\psi\left(x_3\,dx_2-x_2\,dx_3\right)$

consists $\partial\Omega$ of several contours: an external contour and interior $\partial\Omega_0$ $n-1$ contours. The condition $\partial\Omega_i$ leads $d\psi=0$ to the following n conditions: on for $\psi=\psi_i$. $\partial\Omega_i$ Are $i=0,\ldots,n-1$ ψ_i constant unknowns. By noting that, and thus φ , is defined ψ except for a constant, one can fix Des. One will thus take ψ_i . Remain $\psi_0=0$ to determine for. ψ_i For that $i=0,\ldots,n-1$

, one will study the warping of the cross-section of X-coordinate. Let us recall x_1 that the tensor of the stresses is written (cf [§4]): One poses

$$\boldsymbol{\sigma}=G\frac{\partial\theta_x}{\partial x}\begin{bmatrix}0 & \dfrac{\partial\xi}{\partial y}-z & \dfrac{\partial\xi}{\partial z}+y\\ \dfrac{\partial\xi}{\partial y}-z & 0 & 0\\ \dfrac{\partial\xi}{\partial z}+y & 0 & 0\end{bmatrix}=G\frac{\partial\theta_x}{\partial x}\begin{bmatrix}0 & \dfrac{\partial\psi}{\partial z} & -\dfrac{\partial\psi}{\partial y}\\ \dfrac{\partial\psi}{\partial z} & 0 & 0\\ -\dfrac{\partial\psi}{\partial y} & 0 & 0\end{bmatrix}$$

; is ($\underline{\boldsymbol{\tau}}=\underline{\mathrm{grad}}\,\psi\wedge\underline{x}_1$ except for $\underline{\boldsymbol{\tau}}$ a constant) the tangential part of the vector forced in the cross-section. One thus has

: One makes $\underline{\mathbf{grad}}\,\xi=\underline{\boldsymbol{\tau}}-\underline{\mathbf{GM}}\wedge\underline{\mathbf{x}}_1$

circulate this equation along, one a: $\partial\Omega_i$ One

$$0=\oint_{\partial\Omega_i}\boldsymbol{\tau}\,dl-\oint_{\partial\Omega_i}x_3\,dx_2-x_2\,dx_3$$

helped oneself here owing to the fact that the circulation of the gradient on a closed curve is null. It is noted

that the first integral can be written in term of flux of beam ψ . In addition $\partial\Omega_i$, the second integral is equal to. Finally $2\mathrm{A}\left(\partial\Omega_i\right)$

, the problem is written: on on

$$\Delta\psi=-2 \text{ on } \Omega$$
$$\psi=0 \text{ Once } \partial\Omega_0$$
$$\psi=\psi_i \quad \partial\Omega_i$$
$$\oint_{\partial\Omega_i}\frac{\partial \mathrm{y}}{\partial\underline{n}}\,dl=2\mathrm{A}\left(\partial\Omega_i\right)$$

solved this problem, one a: the last $J=2\int_{\Omega}\psi\, ds+2\sum_{i=1}^{n-1}\psi_i A\left(\partial\Omega_i\right)$

condition: is difficult $\oint_{\partial\Omega_i}\frac{\partial y}{\partial \underline{n}}dl=2\mathrm{A}\left(\partial\Omega_i\right)$ to treat in a numerical way. Actually, the two conditions on each border bordering a hole are written: on variational

$$\psi=\psi_i \quad \partial\Omega_i$$
$$\frac{\partial\psi}{\partial\mathbf{n}}=\frac{2\mathrm{A}\left(\partial\Omega_i\right)}{l(\partial\Omega_i)}$$

Formulation: One considers

$$\forall v \ \in H^1\left(\Omega\right)\text{tel que}\, v|_{\partial\Omega_0}=0$$
$$\forall \mu \ \in L^2\left(\partial\Omega_i\right)$$
$$\begin{cases}\int_{\Omega}\nabla\psi\nabla v\,\Omega+\sum_i\int_{\partial\Omega_i}\lambda^i v\, dl=2\int_{\Omega}v\, d\,\Omega+2\mathrm{A}\left(\partial\Omega_i\right)\int_{\partial\Omega_i}v\, dl\\ \int_{\partial\Omega_i}\mu\psi\, dl=\psi_i\int_{\partial\Omega_i}m\, dl\\ \int_{\partial\Omega_i}\lambda^i dl=0\end{cases}$$

the function such as θ on. Matriciellement $\theta\equiv 1 \quad \partial\Omega_i$, one a: Where is

$$\psi={}^t\left[\boldsymbol{\psi}\right]\left[\boldsymbol{\Phi}\right]$$
$$v={}^{\mathbf{t}}\left[\mathbf{v}\right]\left[\boldsymbol{\Phi}\right]$$
$$\theta={}^{\mathbf{t}}\left[\boldsymbol{\theta}\right]\left[\mathbf{F}\right]$$

the vector $\left[\boldsymbol{\Phi}\right]$ whose components are the shape functions. Under these conditions, the variational approximation of the weak formulation gives: Where one

$$\begin{cases}\left[\mathbf{K}\right]\left[\boldsymbol{\psi}\right]+\sum_i {}^{\mathbf{t}}\left[\mathbf{B^i}\right]\left[\boldsymbol{\lambda}^{\mathbf{i}}\right] & = & \left[\mathbf{f}\right]+\left[\mathbf{T}\right]\\ \left[\mathbf{B}\right]\left[\boldsymbol{\psi}\right] & = & \left[\mathbf{0}\right]\end{cases}$$

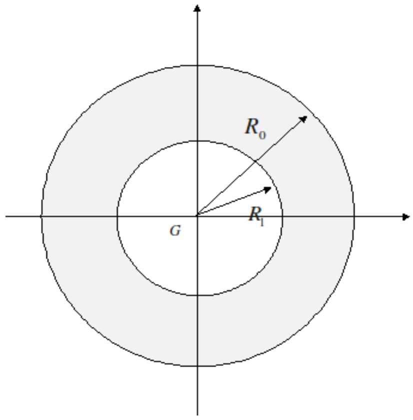
posed: One can

$$^{\mathbf{t}}\left[\mathbf{f}\right]\left[\mathbf{v}\right]=2\int_{\Omega}v\, d\,\Omega$$
$$^{t}\left[\mathbf{T}\right]\left[\mathbf{v}\right]=\frac{2\mathrm{A}\left(\partial\Omega_i\right)}{\|\partial\Omega_i\|}\int_{\partial\Omega_i}v\, dl$$
$$\left[\mathbf{B}\right]=\sum_i\left[\mathbf{B^i}\right]$$

check that the condition on flux is checked: It is seen,

$$\begin{aligned}\int_{\partial\Omega_i}\frac{\partial\psi}{\partial n}dl=\int_{\partial\Omega_i}\Delta\psi\,\theta\,dl &= \int_\Omega\Delta\psi\,\theta\,d\Omega+\int_\Omega\nabla\psi\,\nabla\theta\,d\Omega\\ &= \underbrace{-\int_\Omega 2\,\theta\,d\Omega^t+[\mathbf{f}][\boldsymbol{\theta}]^t}_{=0}+[\mathbf{T}][\boldsymbol{\theta}]-\underbrace{{}^{\mathbf{t}}[\boldsymbol{\theta}][\mathbf{B^i}]}_{=0}[\boldsymbol{\lambda}]\\ &= \frac{2\mathrm{A}(\partial\Omega_i)}{\|\partial\Omega_i\|}\int_{\partial\Omega_i}\theta\,dl\\ &= 2\mathrm{A}(\partial\Omega_i)\end{aligned}$$

with this new formulation, that the posed mathematical problem returns to a problem of linear thermal with a particular loading. This is easily programmable in Code_Aster *. By applying* the preceding method to a beam whose cross-section is the contour ranging between radius and with R_1 . R_0 There is $R_1<R_0$



the following problem to solve: The general

$$\left|\begin{array}{l}\Delta\psi=-2\\ \psi(r=R_0)=\psi_0=0\\ \psi(r=R_1)=\psi_1\\ \int_{\partial\Omega_1}\frac{\partial\psi}{\partial\mathbf{n}}dl=2\pi R_0^2\end{array}\right.$$

solution of the problem is written Here, one $\psi=\psi(r)=-\frac{r^2}{2}+A\ln(r)+B$

A. From where $\frac{\partial\psi}{\partial\mathbf{n}}\Big|_{\partial\Omega_1}=-\frac{\partial\psi}{\partial r}\mathbf{e}_r$. In addition $2\pi R_1^2-2\pi A\ln(R_1)=2\pi R_1^2\Rightarrow A=0$, one A.

Finalement $\psi(R_0)=0\Rightarrow B=\frac{R_0^2}{2}$: . One can

$$\psi(r)=-\frac{r^2}{2}+\frac{R_0^2}{2}$$

now calculate. One has $J=2\int_\Omega\psi\,ds+2\,\psi_1 A(\partial\Omega_i)$ and. All $\psi_1=-\frac{R_1^2}{2}+\frac{R_0^2}{2}$ done $A(\partial\Omega_1)=\psi R_1^2$ calculations, there is the result classical one: Determination $J=\frac{\pi}{2}(R_0^4-R_1^4)$

Annexe 2 : of the constant of shears of a beam equivalent to a set of parallel beams Position

A 2.1 : of problem: One clarifies

here a method developed in command MACR_CARA `_POUTRE to obtain` the shear coefficients and of AY AZ a beam equivalent to a set of disjoined beams (e.g. columns embedded between two bottoms). This makes it possible for example to produce model "skewers" of buildings, i.e. condensed in only one beam. For only one

beam, the definition of the shear coefficients rests on the energy method [§2.1.3]: the formulation is based on complementary energy due to the shears in the section. The shear coefficient

is: Note: $k = \dfrac{\left[\int_s \sigma_{CT} dS\right]^2}{SG\int_s \dfrac{1}{G}\sigma_{CT^2} dS}$

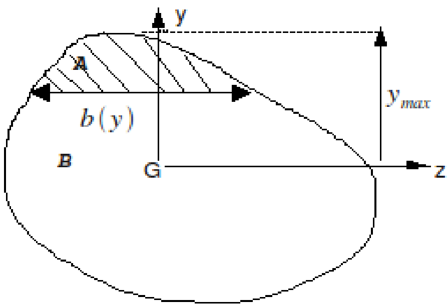
This statement

is still valid in the case of a heterogeneous beam (variable G). The distribution

of shearing stress in the section, for only one beam, is based on the formula of Jourawski, [§2.1.1] which provides the distribution of the shearing stresses due to shears in a direction and only the average of the shears in the other direction. The formula
of Jourawski is written: with

$$\sigma_{CT} = \frac{m(y)}{Ib(y)} V \textbf{ the quantity } m(y) = \int_y^{y_{max}} t \quad b(t) \quad dt$$

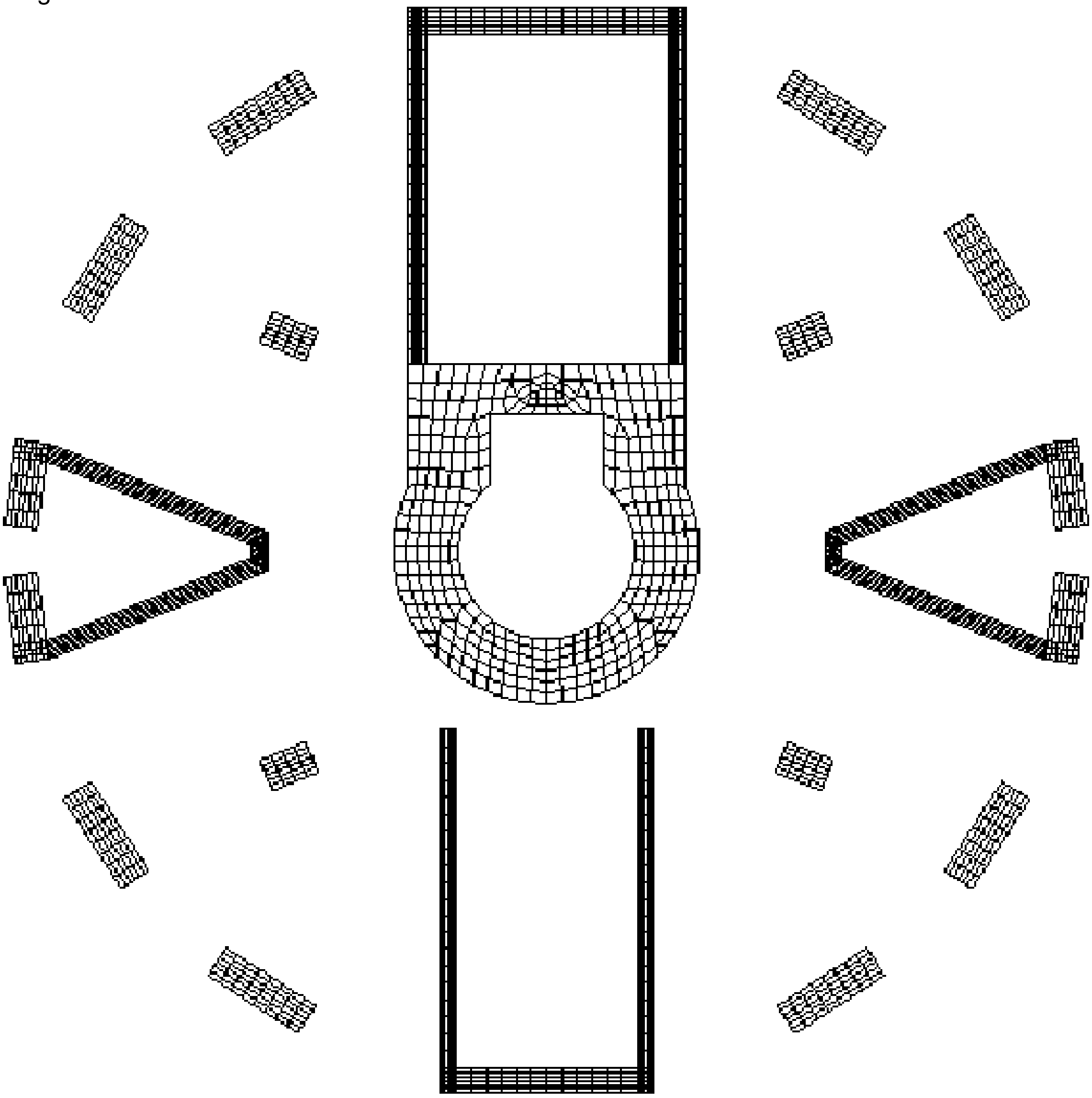
$m(y)$ the statical moment on behalf of section understood represents A enters and Then y can y_{max}



be written k in the form: The subjacent $k = \dfrac{I^2}{S\ G\int_y^{y_{max}} \dfrac{m^2(y)}{Gb^2(y)} dy}$

idea is that the section supports normal stresses resulting from the theory of the beams of Eulerian, and that one evaluates the force of sliding of on. For A B

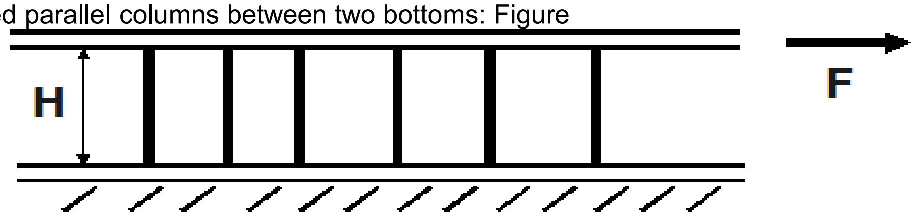
a nonrelated section, like the section out of cut of a building, the assumption of Jourawski cannot be made (except considering that all the section becomes deformed axially like the same beam with each X-coordinate). One x cannot know a priori the distribution of the shears nor of the normal stresses in each column. The following figure gives an idea of the cross-section of a building engine: Figure



11: Section 11 of a building engine Statement

A 2.2 : simplified of the shear coefficients Assumption

: the columns are embedded in bottoms: the building seen on side can be represented like a set of clamped parallel columns between two bottoms: Figure



12: together 12 of columns between two boards One calculates

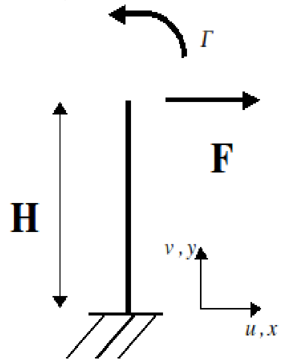
this typical case: each column is a rectangular beam of section (the principal axes of inertia of the various columns are not colinéaires). with. $H=3$ The beams 4m
are embedded at the two ends. It is then necessary to seek a relation between a force imposed F on higher bottom, and the displacement of this bottom in the same direction, i.e. to calculate the stiffness of this system in this direction. •For

one beam: The method
used is exposed for example in [bib11]. •Beam11
fixed at an end and free with the other: The system
is statically determinate and elastic. One wants to express displacement according to $u(H)$ and. F The Principle Γ of the Virtual wors is written: for any

$$f(H).v(H)=\int_0^H M^f.\kappa(v)+V^f.\gamma(v)\,dl$$

$$\kappa(v)=\frac{M(v)}{E.I}$$

$$\gamma(v)=\frac{V(v)}{G.S_r}$$



virtual displacement, and v a specific force in, (here fL $y=H$ "force normal is null). One chooses, and one calculates $f=1$ successively displacements due to a force and F a couple in. By Γ integrating $y=H$ L" preceding statement, one finds that, under the effect of, real F displacement and u rotation are worth: with and

$$\mathbf{u}(H)=\frac{\mathbf{F}.H^3}{3E.I}+\frac{\mathbf{F}.H}{G.S_r}=\frac{\mathbf{F}.H^3}{12E.I}\left(4+\frac{12EI}{G.H^2S_r}\right)$$ Under $S_r=k\cdot S$ $\boldsymbol{\theta}(H)=-\frac{\mathbf{F}.H^2}{2E.I}$

the effect of the moment, one obtains Γ : If the beam

$$\mathbf{u}_\Gamma(H)=-\frac{\boldsymbol{\Gamma}.H^2}{2E.I}\quad \boldsymbol{\theta}_\Gamma(H)=\frac{\boldsymbol{\Gamma}.H}{E.I}$$

has a rectangular section of width and thickness b , one obtains h : for F imposed

$$S_r=bhk=bh\frac{5}{6}$$

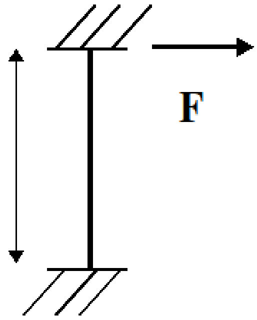
: for imposed $\mathbf{u}(H)=\frac{\mathbf{F}.H^3}{E.bh^3}\left(4+\frac{12}{5}\frac{h^2}{H^2}(1+\nu)\right)$ $\boldsymbol{\theta}(H)=-\frac{6\mathbf{F}.H^2}{E.bh^3}$ Γ : •Beam

$$\mathbf{u}_\Gamma(H)=-\frac{6\boldsymbol{\Gamma}.H^2}{E.bh^3}\quad \boldsymbol{\theta}_\Gamma(H)=\frac{12\boldsymbol{\Gamma}.H}{E.bh^3}$$

fixed at the two ends: (fixed support slipping in) the system $y=H$ is hyperstatic of degree 1. One expresses displacement according to $u(H)$ the redundant unknowns and using F Γ the preceding results. Under the effect of and, F displacement Γ U real and rotation (null because of the fixed support) are worth: The resolution

$$u(H)=\frac{F.H^3}{3E.I}+\frac{F.H}{GS_r}-\frac{\Gamma.H^2}{2E.I}=\frac{F.H^3}{E.bh^3}\left(4+\frac{12}{5}\frac{h^2}{H^2}(1+\nu)\right)$$

$$0=-\frac{F.H^2}{2EI}+\frac{\Gamma.H}{E.I}=-\frac{6F.H^2}{E.bh}+\frac{12\Gamma.H}{E.bh^2}$$



of this system makes it possible to obtain U (H) according to F: One also

$$\mathbf{u}(H)=\frac{F.H^3}{12EI}\left(1+\frac{12EI}{GH^2S_r}\right)=\frac{F.H^3}{E.bh^3}\left(1+\frac{12}{5}\frac{h^2}{H^2}(1+\nu)\right)$$

$$\mathbf{F}=\left(\frac{12EI}{H^3}\frac{1}{\left(1+\frac{12EI}{GH^2S_r}\right)}\right)\mathbf{u}(H)=\left(\frac{E.bh^3}{H^3}\frac{1}{\left(1+\frac{12}{5}\frac{h^2}{H^2}(1+\nu)\right)}\right)\mathbf{u}(H)=K.\mathbf{u}(H)$$

finds this result by considering the stiffness matrix of an "exact" beam element with 2 nodes ([bib1] or [R3.08.011]). The term above corresponds exactly at the end of stiffness of shears alone following direction X: Note:

$$K_{xx}=\left(\frac{12EI}{H^3}\frac{1}{(1+\Phi)}\right)\qquad \Phi=\frac{12EI}{GH^2S_r}=\frac{12EI}{GH^2kS}$$

The embed-free

situation differs only from one coefficient (4 instead of 1): The two

$$\mathbf{u}(H)=\frac{\mathbf{F}.H^3}{E.bh^3}\left(4+\frac{12}{5}\frac{h^2}{H^2}(1+\nu)\right)$$

opportunities are given in MACR_CARA `_POUTRE`: • the two
ends are clamped (actually one is embedded, the other is embedded in a mobile bottom: slipping fixed support) •the higher
end is free (makes some in connection rotulée with higher bottom). One can

thus propose to express the stiffness with the shears of each column in the form: with For

$$\mathbf{F}=\left(\frac{12EI}{H^3}\frac{1}{\left(\xi+\frac{12EI}{GH^2S_r}\right)}\right)\mathbf{u}(H)=\mathbf{K}.\mathbf{u}(H)$$ a set of $\xi=\begin{cases}1\,encastré-encastré\\4\,encastré-libre\end{cases}$

A 2.3 : beams the method

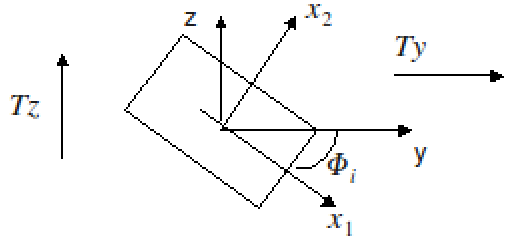
consists in calculating the stiffness of each column in the preceding way, and to compare the stiffness of the group with that of an equivalent beam embedded between two bottoms. For that one expresses the total shears applied to all the columns (for example in the direction): Each y column

$$T_y=\sum T_i=\tilde{K}_y.u_y$$

having an unspecified directional sense compared to the total axes, it is necessary for all to express the forces in T_i the total reference: and indicating

x_1 x_2 the principal axes of inertia of column I, the shears in this T_i reference are: Moreover,

$$T_y^i = T_1^i \cos(\varphi_i) + T_2^i \sin(\varphi_i)$$
$$T_z^i = -T_1^i \sin(\varphi_i) + T_2^i \cos(\varphi_i)$$



it is supposed that the total displacement of all u the columns is uniform (components and) and u_x must u_y be colinéaire with the shears. (what T is not certain: couplings are possible if there are no particular symmetries). This involves for the direction: and one y obtains

$$u_1^i = u_y \cos(\varphi_i) \quad T_1^i = K_1^i . u_1^i$$
$$u_2^i = u_y \sin(\varphi_i) \quad T_2^i = K_2^i . u_2^i$$

: with in the same way

$$T_y^i = \left(K_1^i \cos^2(\varphi_i) + K_2^i \sin^2(\varphi_i) \right) u_y^i$$
$$T_y = \sum T_y^i = \sum \left(K_1^i \cos^2(\varphi_i) + K_2^i \sin^2(\varphi_i) \right) u_y = \tilde{K}_y u_y$$

$$\tilde{K}_y = \frac{12\,EI}{H^3} \frac{1}{\left(\xi + \dfrac{12\,EI}{GH^2 S_r} \right)}$$

in the direction: thus z In addition

$$u_1^i = -u_z \sin(\varphi_i) \qquad T_z = \sum T_z^i = \sum \left(K_1^i \sin^2(\varphi_i) + K_2^i \cos^2(\varphi_i) \right) u_z = \tilde{K}_z u_z$$
$$u_2^i = u_z \cos(\varphi_i)$$

, for the equivalent beam, one makes the assumption that the stiffness with the shears S "expresses in the same way that of each beam: In fact,

$$T_y = K_y^{eq} u_y = \frac{12\,EI_z^{eq}}{H^3 \left(1 + \Phi_y \right)} u_y \text{avec } \Phi_y = \frac{12\,EI_z^{eq}}{S^{eq} H^2 G k_y^{eq}}$$

it would have to be checked that energies due to bending and L" force normal are quite negligible. The two statements of the shears lead to the statement of the equivalent shear coefficient: and in

$$k_y^{eq} = \frac{12\,EI_z^{eq}}{GS^{eq} H^2 \left(\dfrac{12\,EI_z^{eq}}{H^3 \tilde{K}_y} - 1 \right)}$$ the direction: Method z $$k_z^{eq} = \frac{12\,EI_y^{eq}}{GS^{eq} H^2 \left(\dfrac{12\,EI_y^{eq}}{H^3 \tilde{K}_z} - 1 \right)}$$

A 2.4 : used in MACR_CARA `_POUTRE By means of`

assumptions described previously , namely: •only
the stiffness due to the shears is taken into account in the computation of the shear coefficients •the equivalent
beam is embedded on two bottoms •two
design assumptions are to be envisaged concerning each column (embed-kneecap and embed-embedded). One can
propose a method of calculating in MACR_CARA `_POUTRE to obtain` shear coefficients equivalent to a set of beams of axes parallel, embedded in a bottom with one their ends, and free with the other, or embedded at the other end. Restrictions
of use: •it is
reasonable to place the equivalent beam on the center of gravity of all the columns, and in the principal reference of inertia of the group, to avoid the parasitic couplings •it is necessary

to ensure the continuity of all the degrees of freedom of the equivalent beam (translation and rotation) with the degrees of freedom of bottoms (what models the fixed support of the beam in bottom), which forces to model bottom in shell elements or, if it is with a grid in 3D, to connect it using beams or of plates. The method of calculating

is the following one: •For
each column, to do the usual calculation by geometrical characteristics and shear coefficients of the section, in the principal reference of inertia of each section (already available). •Always for each section, computation of the stiffness to the shears (the user must provide H, distance between bottoms. •Computation

$$K_1^i=\frac{12\,EI_2^i}{H^3\left(1+\Phi_1\right)}\ \text{avec}\ \Phi_1=\frac{12\,EI_2^i}{S^i H^2 G k_1^i}$$

$$K_2^i=\frac{12\,EI_1^i}{H^3\left(1+\Phi_2\right)}\ \text{avec}\ \Phi_2=\frac{12\,EI_1^i}{S^i H^2 G k_2^i}$$

of the stiffness equivalent to all the beams: •Computation

$$\tilde{K}_y=\sum\left(K_1^i\cos^2(\varphi_i)+K_2^i\sin^2(\varphi_i)\right)$$

$$\tilde{K}_z=\sum\left(K_1^i\sin^2(\varphi_i)+K_2^i\cos^2(\varphi_i)\right)$$

of the shear coefficients are equivalent: knowing

$$k_y^{eq}=\frac{12\,EI_z^{eq}}{GS^{eq}H^2\left(\dfrac{12\,EI_z^{eq}}{H^3\tilde{K}_y}-1\right)}$$

that, and S^{eq} I_y^{eq} I_z^{eq} are already calculated by MACR_CARA _POUTRE. For
key keys of the command MACR_CARA _POUTRE, one by the key word needs that the user provides, H LONGUEUR , the characteristics (constant) of the material (key word MATERIAU) and chooses boundary conditions LIAISON by the key word : LIAISON
: HINGE or LIAISON : ENCASTREMENT This computation

of course is activated only if several GROUP_MA are defined by the user (indicating that the section is made up of under disjoined parts).