Beam element with 7 degrees of freedom for the taking into account of Summarized

warping:

This document presents the element POU_D_TG which is a finite element of straight beam with taking into account of the warping of the sections. It allows the computation of the beams mean cross-sectional areas and opened profile, with constrained or free torsion.

With regard to bending, the normal force and edges, this element is based on the element POU_D_T, which is a beam element right with transverse shears (models of Timoshenko).

For the element POU_D_TG, the section is supposed to be constant (of an unspecified form) and the material is homogeneous and isotropic, of linear elastic behavior.
Element POU_D_TGM is to be used for the nonlinear behaviors.

This documentation of reference leans on the general documentation of reference of the beams, in linear elasticity [R3.08.01]. It describes specificities of the beam element right with warping POU_D_TG.

Contents

1 Field of application

the development of the beam elements of Timoshenko with warping (modelization POU_D_TG) in *Code_Aster* was carried out initially with an aim of calculating the behavior of the pylons. It was mainly a question of calculating formed structures by beams with open mean profile (corner), for which warping is important. The nonlinear behaviors are to be used with element POU_D_TGM (beam multifibers). These nonlinear behaviors relate only to the tension, bending. The shears due to the shears, as well as warping and the bi--moment (force related to warping) remain dependant by an elastic behavior, fault of being able to express a nonlinear behavior on these quantities. The description of torsion with warping is valid for the use of elements POU_D_TG and POU_D_TGM with the linear operators (MECA_STATIQUE, DYNA_LINE_TRAN,…) or not linear (STAT_NON_LINE, DYNA_NON_LINE,…).

2 Notations

the notations used here correspond to those used in [R3.08.01] and [R3.08.03]. One gives here the correspondence between this notation and that of the documentation of use. DX, DY, DZ, DRX, DRY, DRZ et GRX are the names of the degrees of freedom associated with the components with displacement u, v, w, θ_x, θ_y, θ_z, $\theta_{x,x}$. They are expressed in total reference, except the degree of freedom associated with the warping GRX, which is expressed in local coordinate system.

Notation used	**Meaning**	**Notation of the documentation of area**
S	use of the section	A
I_y, I_z	geometrical moments of bending compared to the axes x and y.	IY, IZ
C	constant of torsion	JX
I_ω	warping constant	JG
k_y, k_z	shear coefficients	$\frac{1}{AY}$ $\frac{1}{AZ}$
e_y, e_z	eccentring of the center of torsion/shears compared to the center of gravity of the cross-section	EY, EZ
N	normal force to the section	N
V_y, V_z	shears along the axes y and z	VY, VZ
M_x, M_y, M_z	moments around the axes x, y and z	MT, MFY, MFZ
M_ω	bi--moment	BX
u, v, w	translations on the axes x, y and z	DX DY DZ
θ_x, θ_y, θ_z	rotations around the axes x, y and z	DRX DRY DRZ
$\theta_{x,x}$	rotary derivative of torsion according to x	GRX
E	Young modulus	E
ν	Poisson's ratio	NU
$G=\frac{E}{2(1+\nu)}=\mu$	modulates of Coulomb (identical to the coefficient of Lamé)	G

3 Kinematical specific to torsion with warping

the kinematics used to represent the displacement of the sections of beam is identical to that of the straight beams of Timoshenko [R3.08.01] with regard to the traction and compression, and bending - shears. Only torsion here is detailed.

Two possibilities are to be considered for the modelization of the behavior in torsion of the noncircular sections [feeding-bottle1], which always produces a warping of the cross-section.

- Torsion is free (torsion of Saint-Coming) : the warping of the cross-sections is non-zero (it can even be important for an open mean section), but it is independent of the position on the axis x of the beam, (constant according to x) and there is no axial stress which had with torsion.
- Torsion is constrained (Vlassov): warping is non-zero, and moreover of nonuniform axial stresses (from which the force resulting bi--moment is called) exist in the beam.

Element POU_D_TG makes it possible to treat these two configurations: torsion can be free or constrained. The user will have access to warping in both cases, on the other hand the bi--moment will be non-zero only in the case of constrained torsion. It should be noted that at the place of the connection of the beams, the transmission of warping depends on the type of connection. In general, torsion in an assembly of beams is constrained. Warping can then be blocked at the points of connection.

Note:
With elements without modelization of warping (POU_D_T , POU_D_E), one can treat the case of free torsion (displacements other than warping will be correct), but not the case of constrained torsion.

One can uncouple the effects of torsion and bending in a local coordinate system (relocated principal reference of inertia) having for origin the center of torsion. The center of torsion is the point which remains fixed when the section is subjected to the only twisting moment. It is also called shear center because a force applied in this point does not produce rotation around x .

Displacements in the plane of the section will thus be expressed in this reference. Axial displacements remain expressed in the principal reference of inertia related to the center of gravity G , to keep a decoupling of displacements of bending and traction and compression.

The displacement of an unspecified point of the cross-section is written then in general form (free or constrained torsion):

$$\begin{Bmatrix} u(x,y,z) \\ v(x,y,z) \\ w(x,y,z) \end{Bmatrix} = \begin{Bmatrix} u_G(x) \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} z\theta_y(x) \\ 0 \\ w(x) \end{Bmatrix} + \begin{Bmatrix} -y\theta_z(x) \\ v(x) \\ 0 \end{Bmatrix} + \begin{Bmatrix} \omega(y,z)\theta_{x,x}(x) \\ -(z-z_c)\theta_x(x) \\ (y-y_c)\theta_x(x) \end{Bmatrix}$$

Displacement = membrane + inflection y + inflection z + torsion with warping

the components are expressed in the principal reference of inertia (centered in G): x is directed along the axis of the beam, y and z are the two other principal axes of inertia.

The term $\omega(y,z)\theta_{x,x}(x)$ represents axial displacement due to the warping of the cross-section. $\omega(y,z)$ is the function of warping (expressed in m^2 , but which does not have obvious physical interpretation).

The strains of an unspecified point of the section are then:

$$\begin{Bmatrix} \varepsilon_{xx}(x,y,z) \\ 2\varepsilon_{xy}(x,y,z) \\ 2\varepsilon_{xz}(x,y,z) \end{Bmatrix} = \begin{Bmatrix} u_{G_{,x}}(x) \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} z\theta_{y,x}(x) \\ 0 \\ \gamma_{xz}(x) \end{Bmatrix} + \begin{Bmatrix} -y\theta_{z,x}(x) \\ \gamma_{xy}(x) \\ 0 \end{Bmatrix} + \begin{Bmatrix} \omega(y,z)\theta_{x,xx}(x) \\ (\omega_{,y}-(z-z_c))\theta_{x,x}(x) \\ (\omega_{,z}+(y-y_c))\theta_{x,x}(x) \end{Bmatrix}$$

$$\gamma_{xy}(x) = v_{,x} - \theta_z$$

$$\gamma_{xz}(x) = w_{,x} + \theta_y$$

$$\text{Déformation} = \text{membrane} + \text{flexion}/y + \text{flexion}/z + \text{torsion avec gauchissement}$$

The term $\omega(y,z)\theta_{x,xx}(x)$ is null in the case of free torsion: one has indeed $\theta_{x,xx}(x)=0$, since warping is independent of x . It is considerable in the case of constrained torsion.

The isotropic elastic constitutive law is written (by making the assumption of the plane stresses in the directions y and z):

$$\begin{Bmatrix} \sigma_{xx}(x,y,z) \\ \sigma_{xy}(x,y,z) \\ \sigma_{xz}(x,y,z) \end{Bmatrix} = \begin{Bmatrix} E.\varepsilon_{xx}(x,y,z) \\ G.2\varepsilon_{xy}(x,y,z) \\ G.2\varepsilon_{xz}(x,y,z) \end{Bmatrix}$$

The forces generalized in the section are expressed according to the stresses for a homogeneous section by [feeding-bottle1]:

$N(x)=\int_S \sigma_{xx}(x,y,z)\,ds$ Normal force

$V_y(x)=\int_S \sigma_{xy}(x,y,z)\,ds$ Shears following y

$V_z(x)=\int_S \sigma_{xz}(x,y,z)\,ds$ Shears according to z

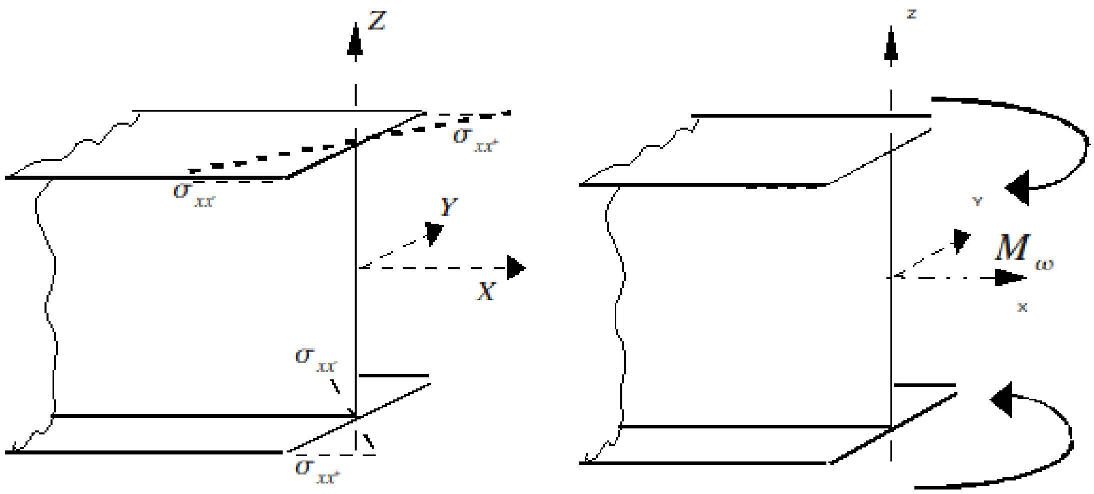
$M_y(x)=\int_S z.\sigma_{xx}(x,y,z)\,ds$ Bending moment around y

$M_z(x)=\int_S -y.\sigma_{xx}(x,y,z)\,ds$ Bending moment around z

$M_x(x)=\int_S \left((y-y_c).\sigma_{xz}(x,y,z)-(z-z_c).\sigma_{xy}(x,y,z)\right)ds$ Twisting moment

$M_\omega(x)=\int_S \omega.\sigma_{xx}(x,y,z)\,ds$ Bi--moment (associate with warping)

$M_\omega(x)$ represents the generalized force associated with warping. It is expressed in $N.m^2$. One can give of it an illustration as in [feeding-bottle1] for a beam to section in I (the bi--moment acts here according to z only):



For an isotropic and homogeneous elastic behavior in the section, the generalized forces are thus expressed directly according to displacements by the following relations:

$$
\begin{aligned}
&N(x)=E.S.u_{,x}\\
&V_y(x)=Gk_y S\left(v_{,x}-\theta_z\right)\\
&V_z(x)=Gk_z S\left(w_{,x}+\theta_y\right)\\
&M_y(x)=E.I_y\theta_{y,x}\\
&M_z(x)=E.I_z\theta_{z,x}\\
&M_x(x)=G.J.\theta_{x,x}\\
&M_\omega(x)=E.I_\omega.\theta_{x,xx}
\end{aligned}
$$

where k_y, k_z are the shear coefficients. Warping does not intervene on the level of the shears, because those are expressed in the reference related to the shear center. Indeed, the function of warping ω is such as:

$$
\begin{aligned}
&\int_S \omega(y,z)\,ds=0\\
&\int_S y.\omega(y,z)\,ds=0\\
&\int_S z.\omega(y,z)\,ds=0
\end{aligned}
$$

And the warping constant is expressed according to ω by: $\int_S \omega^2(y,z)\,ds=I_\omega$

4 Beam element right with warping: stiffness matrixes and of mass

the elementary matrixes of stiffness and mass for element POU_D_TG are identical to those of the beam element right of Timoshenko (POU_D_T) with regard to the terms of traction and compression and bending - shears [R3.08.01]. The approach is identical, one recalls simply result.

This implies that, in the case of free torsion, one preserves the properties of exactitude of the solution at the nodes for the degrees of freedom of bending and traction and compression.

On the other hand, we will see that with regard to obstructed torsion, one carries out an approximation which does not make it possible to find this property in the general case.

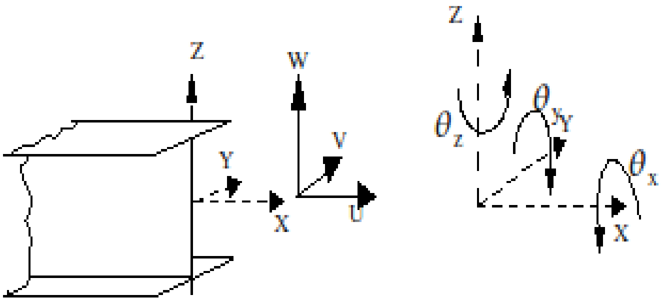
The stiffness matrixes are always calculated with option "RIGI_MECA", and the mass matrixes with option "MASS_MECA". But option "MASS_MECA_DIAG" (diagonalized mass matrix) was not carried out for this element (this option is especially useful for the fast problem of dynamics, which is not the preferential field of application of this element).

The degrees of freedom of the element are those of the beams of Timoshenko, plus a degree of freedom per node making it possible to calculate the terms relating to warping:

In each of the two nodes of the element, the degrees of freedom are:

u, v, w	translations on the axes x, y, z	$DX\ DY\ DZ$
θ_x, θ_y, θ_z	rotations around the axes x, y, z	$DRX\ \ DRY\ \ DRZ$
$\theta_{x,x}$	rotary derivative of torsion according to x	GRX

the local coordinates are expressed in the principal reference of inertia. Element POU_D_TG thus comprises 14 degrees of freedom. The element of reference is defined by: $0<x<L$



4.1 Traction and compression the degree of freedom are u or DX

the stiffness matrix of the element is: $K = \dfrac{ES}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

The mass matrix (coherent) is written: $M = \dfrac{\rho\ SL}{6}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

4.2 Bending in the Gxz plane, degrees of freedom: Ω, θψ or Dz, Dry Martini

the stiffness matrix is written for the motion of bending in the principal plane of inertia (Gxz) :

$$\mathbf{K}=\frac{12\ EI_y}{L^3(1+\varphi_y)}\begin{pmatrix} 1 & -\frac{L}{2} & -1 & -\frac{L}{2} \\ & \frac{(4+\varphi_y)\ L^2}{12} & \frac{L}{2} & \frac{(2-\varphi_y)\ L^2}{12} \\ \text{Sym} & & 1 & \frac{L}{2} \\ & & & \frac{(4+\varphi_y)\ L^2}{12} \end{pmatrix}$$

The transverse shears are taken into account by the term: $\varphi_y=\frac{12\,EI_y}{k_z SGL^2}$

For the mass matrix, $w\ (x,\ t)$ and $\theta_y(x,\ t)$ are discretized on the basis of function tests introduced for the computation of the stiffness matrix, that is to say:

$$w(x,\ t)=\xi_1(x)w_1(t)+\ \xi_2(x)\theta_{y_1}(t)+\xi_3(x)w_2(t)+\xi_4(x)\theta_{y_2}(t)$$
$$\theta_y(x,\ t)=\xi_5(x)w_1(t)+\ \xi_6(x)\theta_{y_1}(t)+\xi_7(x)w_2(t)+\xi_8(x)\theta_{y_2}(t)$$

The interpolation function used for the translations (ξ_1 with ξ_4) are polynomials of Hermit of degree 3, that which are used for rotations (ξ_5 with ξ_8) are of degree 2: for $0<x<L$, they are defined by [R3.08.01]:

$$\xi_1(x)=\frac{1}{1+\varphi_y}\left[2\left(\frac{x}{L}\right)^3-3\left(\frac{x}{L}\right)^2-\varphi_y\frac{x}{L}+(1+\phi_y)\right] \qquad \xi_5(x)=\frac{6}{L(1+\varphi_y)}\frac{x}{L}\left[1-\frac{x}{L}\right]$$
$$\xi_2(x)=\frac{L}{1+\varphi_y}\left[-\left(\frac{x}{L}\right)^3+\frac{4+\varphi_y}{2}\left(\frac{x}{L}\right)^2-\frac{2+\varphi_y}{2}\left(\frac{x}{L}\right)\right] \qquad \xi_6(x)=\frac{1}{1+\varphi_y}\left[3\left(\frac{x}{L}\right)^2-(4+\varphi_y)\left(\frac{x}{L}\right)+(1+\varphi_y)\right]$$
$$\xi_3(x)=\frac{1}{1+\varphi_y}\left[-2\left(\frac{x}{L}\right)^3+3\left(\frac{x}{L}\right)^2+\varphi_y\left(\frac{x}{L}\right)\right] \qquad \xi_7(x)=\frac{-6}{L(1+\varphi_y)}\frac{x}{L}\left[1-\frac{x}{L}\right] \qquad [1]$$
$$\xi_4(x)=\frac{L}{1+\varphi_y}\left[-\left(\frac{x}{L}\right)^3+\frac{2-\varphi_y}{2}\left(\frac{x}{L}\right)^2+\frac{\varphi_y}{2}\left(\frac{x}{L}\right)\right] \qquad \xi_8(x)=\frac{1}{1+\varphi_y}\left[3\left(\frac{x}{L}\right)^2+(-2+\varphi_y)\left(\frac{x}{L}\right)\right]$$

the form of the mass matrix is:

$$\mathbf{M}=\frac{\rho S}{\left(1+\phi_y\right)^2}\begin{pmatrix} \frac{13L}{35}+\frac{7L\phi_y}{10}+\frac{L\phi_y^2}{3} & -\frac{11L^2}{210}-\frac{11L^2\phi_y}{120}-\frac{L^2\phi_y^2}{24} & \frac{9L}{70}+\frac{3L\phi_y}{10}+\frac{L\phi_y^2}{6} & \frac{13L^2}{420}+\frac{3L^2\phi_y}{40}+\frac{L^2\phi_y^2}{24} \\ & \frac{L^3}{105}+\frac{L^3\phi_y}{60}+\frac{L^3\phi_y^2}{120} & -\frac{13L^2}{420}-\frac{3L^2\phi_y}{40}-\frac{L^2\phi_y^2}{24} & -\frac{L^3}{140}-\frac{L^3\phi_y}{60}-\frac{L^3\phi_y^2}{120} \\ & & \frac{13L}{35}+\frac{7L\phi_y}{10}+\frac{L\phi_y^2}{3} & -\frac{11L^2}{210}+\frac{11L^2\phi_y}{120}+\frac{L^2\phi_y^2}{24} \\ sym & & & \frac{L^3}{105}+\frac{L^3\phi_y}{60}+\frac{L^3\phi_y^2}{120} \end{pmatrix}$$

$$+\frac{\rho I_y}{\left(1+\phi_y\right)^2}\begin{pmatrix} \frac{6}{5L} & -\frac{1}{10}+\frac{\phi_y}{2} & -\frac{6}{5L} & -\frac{1}{10}+\frac{\phi_y}{2} \\ & \frac{2L}{15}+\frac{L\phi_y}{6}+\frac{L\phi_y^2}{3} & \frac{1}{10}-\frac{\phi_y}{2} & -\frac{L}{30}-\frac{L\phi_y}{6}+\frac{L\phi_y^2}{6} \\ & & \frac{6}{5L} & \frac{1}{10}-\frac{\phi_y}{2} \\ sym & & & \frac{2L}{15}+\frac{L\phi_y}{6}+\frac{L\phi_y^2}{3} \end{pmatrix}$$

4.3 Bending in the Gxy plane, degrees of freedom: Γ, θζ or Dy, Drz

Of the same, for the motion of bending around the axis (Gz), in the principal plane of inertia (Gxy), the stiffness matrix is written:

$$\mathbf{K}=\frac{12\ EI_z}{L^3\left(1+\varphi_z\right)}\begin{pmatrix} 1 & \frac{L}{2} & -1 & \frac{L}{2} \\ & \frac{\left(4+\varphi_z\right)L^2}{12} & -\frac{L}{2} & \frac{\left(2-\varphi_z\right)L^2}{12} \\ & & 1 & -\frac{L}{2} \\ \text{sym} & & & \frac{\left(4+\varphi_z\right)L^2}{12} \end{pmatrix}$$

The transverse shears are taken into account by the term:

$$\varphi_z=\frac{12EI_z}{k_y SGL^2}$$

To compute: the mass matrix, $v(x,\ t)$ and $\theta_z(x,\ t)$ are discretized by:

$$v(x,\ t)=\xi_1(x)\ v_1(t)-\xi_2(x)\theta_{z_1}(t)+\ \xi_3(x)\ v_2(t)-\xi_4(x)\theta_{z_2}(t)$$
$$\theta_z(x,\ t)=-\xi_5(x)\ v_1(t)+\xi_6(x)\theta_{z_1}(t)-\xi_7(x)\ v_2(t)+\xi_8(x)\theta_{z_2}(t)$$

We obtain the following mass matrix then:

$$\mathbf{M}=\frac{\rho S}{\left(1+\varphi_z\right)^2}\begin{pmatrix} \frac{13L}{35}+\frac{7L\varphi_z}{10}+\frac{L\varphi_z^2}{3} & \frac{11L^2}{210}+\frac{11L^2\varphi_z}{120}+\frac{L^2\varphi_z^2}{24} & \frac{9L}{70}+\frac{3L\varphi_z}{10}+\frac{L\varphi_z^2}{6} & -\frac{13L^2}{420}-\frac{3L^2\varphi_z}{40}-\frac{L^2\varphi_z^2}{24} \\ & \frac{L^3}{105}+\frac{L^3\varphi_z}{60}+\frac{L^3\varphi_z^2}{120} & \frac{13L^2}{420}+\frac{3L^2\varphi_z}{40}+\frac{L^2\varphi_z^2}{24} & -\frac{L^3}{140}-\frac{L^3\varphi_z}{60}-\frac{L^3\varphi_z^2}{120} \\ & & \frac{13L}{35}+\frac{7L\varphi_z}{10}+\frac{L\varphi_z^2}{3} & \frac{11L^2}{210}-\frac{11L^2\varphi_z}{120}-\frac{L^2\varphi_z^2}{24} \\ sym & & & \frac{L^3}{105}+\frac{L^3\varphi_z}{60}+\frac{L^3\varphi_z^2}{120} \end{pmatrix}$$

$$+\frac{\rho I_z}{\left(1+\varphi_z\right)^2}\begin{pmatrix} \frac{6}{5L} & \frac{1}{10}-\frac{\varphi_z}{2} & -\frac{6}{5L} & \frac{1}{10}-\frac{\varphi_z}{2} \\ & \frac{2L}{15}+\frac{L\varphi_z}{6}+\frac{L\varphi_z^2}{3} & -\frac{1}{10}+\frac{\varphi_z}{2} & -\frac{L}{30}-\frac{L\varphi_z}{6}+\frac{L\varphi_z^2}{6} \\ & & \frac{6}{5L} & -\frac{1}{10}+\frac{\varphi_z}{2} \\ sym & & & \frac{2L}{15}+\frac{L\varphi_z}{6}+\frac{L\varphi_z^2}{3} \end{pmatrix}$$

4.4 Torsion and warping, degrees of freedom: θξ, θξ, x or Drx, Grx

With regard to torsion, the formulation is obviously different from that of the beams without warping of the reference [R3.08.01]. The virtual wor of the internal forces is written for torsion [feeding-bottle1]:

$$W_{\mathrm{int}}=\int_o^L \theta^*_{x,x}.G.J.\theta_{x,x}+\theta^*_{x,xx}.E.I_\omega.\theta_{x,xx}\,dx$$

The interpolation functions of the rotation of torsion must be of class $C2$, since they must make it possible to interpolate derivative second of rotation.

By means of the balance equations, one shows in [feeding-bottle1] that the analytical solution utilizes interpolation function hyperbolic in x. This then makes it possible to get exact results with the nodes. It is not the choice made for *Code_Aster* : one chose, by preoccupation with a simplicity for numerical integration like avoiding the numerical problems of evaluating of the function hyperbolic, of the polynomials of degree 3 of type Hermit, of the same kind as those used for bending [éq1]. One writes them here on the element of reference [- 1,1] according to [feeding-bottle1] (instead of $0<x<L$ previously):

$$\xi=\frac{2x}{L}-1,\quad -1\le\xi\le1,\qquad \begin{aligned} N_1(\xi)&=\frac{1}{4}(1-\xi)^2(2+\xi)\\ N_2(\xi)&=\frac{L}{8}(1-\xi)(1-\xi^2)\\ N_3(\xi)&=\frac{1}{4}(1+\xi)^2(2-\xi)\\ N_4(\xi)&=\frac{L}{8}(1+\xi)(-1+\xi^2)\end{aligned}$$

The interpolation for the rotation of torsion and its derivative is:

$$\theta_x(\xi)=N_1(\xi)\theta^1_x+N_2(\xi)\theta^1_{x,x}+N_3(\xi)\theta^2_x+N_4(\xi)\theta^2_{x,x}$$
$$\theta_{x,x}(\xi)=N_{1,x}(\xi)\theta^1_x+N_{2,x}(\xi)\theta^1_{x,x}+N_{3,x}(\xi)\theta^2_x+N_{4,x}(\xi)\theta^2_{x,x}$$

The reference [feeding-bottle1] notes that this approximation corresponds to a borderline case of the hyperbolic interpolation, obtained for $\sqrt{\frac{GJ}{EI_\omega}} \to 0$. However, this parameter not being without dimension, it is difficult to define a priori the values for which the approximation is acceptable. The numerical tests carried out show that one converges quickly towards the solution when the size of the elements decreases.
The stiffness matrix corresponding to this approximation is written then:

$$K = K_T + K_\omega = \frac{GJ}{30\,L}\begin{pmatrix} 36 & 3\mathrm{L} & -36 & 6\mathrm{L} \\ & 4\mathrm{L}^2 & -3\mathrm{L} & -L^2 \\ & & 36 & -3\mathrm{L} \\ \mathrm{sym} & & & 4\mathrm{L}^2 \end{pmatrix} + \frac{EI_\omega}{L^3}\begin{pmatrix} 12 & 6\mathrm{L} & -12 & 6\mathrm{L} \\ & 4\mathrm{L}^2 & -6\mathrm{L} & 2\mathrm{L}^2 \\ & & 12 & -6\mathrm{L} \\ \mathrm{sym} & & & 4\mathrm{L}^2 \end{pmatrix}$$

The mass matrix can be obtained in several ways [feeding-bottle1]:

- the most complete method would consist in calculating the terms of inertia with the interpolation functions above, by taking account of the additional term:
- $W_{iner} = -\int_o^L \theta^*_{\mathrm{x,x}} . \rho . I_\omega . \ddot{\theta}_{\mathrm{x,x}}\, dx$
- in *Code_Aster*, the simplest method was selected: the mass matrix is identical to that of element POU_D_T. One preserves the already definite terms for the traction and compression and bending - shears and one uses a linear approximation for torsion. The coefficients of the mass matrix associated with warping are null with this approach.

4.5 Eccentring of the axis of torsion compared to the neutral axis

At the center of torsion C, the effects of bending and torsion are uncoupled, one can thus use the results established in the preceding chapter.

The coordinates of the point C are with being provided to AFFE_CARA_ELEM : one gives the components of the vector $\boldsymbol{GC}$ (G being the center of gravity of the cross-section) in the principal reference of inertia:

$$\mathbf{GC} = \begin{pmatrix} 0 \\ e_y \\ e_z \end{pmatrix}$$

One can numerically determine them starting from the plane mesh of the section using operator MACR_CARA_POUTRE [R3.08.03].

Once the determined C point, one finds as in [R3.08.01] the components of displacement at the center of gravity G by considering the relation of rigid body:

$$u(G) = u(C) + \mathbf{GC} \wedge \Omega \quad \text{with} \quad \Omega = \begin{pmatrix} \theta_x \\ 0 \\ 0 \end{pmatrix} \text{ vector rotation, } \begin{cases} u_G = u_C \\ v_G = v_C + e_z \theta_x \\ w_G = w_C - e_y \theta_x \end{cases}.$$

the change of variables is written in the same way that for POU_D_T, with 2 additional degrees of freedom:

$$\begin{Bmatrix} u_{x_{c_1}} \\ u_{y_{c_1}} \\ u_{z_{c_1}} \\ \theta_{x_{c_1}} \\ \theta_{y_{c_1}} \\ \theta_{z_{c_1}} \\ \theta_{x,x_{c_1}} \\ u_{x_{c_2}} \\ u_{y_{c_2}} \\ u_{z_{c_2}} \\ \theta_{x_{c_2}} \\ \theta_{y_{c_2}} \\ \theta_{z_{c_2}} \\ \theta_{x,x_{c_2}} \end{Bmatrix} = \underbrace{\begin{bmatrix} \begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -e_z & 0 & 0 & 0 \\ 0 & 0 & 1 & e_y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{matrix} & 0 \\ 0 & \begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -e_z & 0 & 0 & 0 \\ 0 & 0 & 1 & e_y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{matrix} \end{bmatrix}}_{\mathbf{P}} \begin{Bmatrix} u_{x_1} \\ u_{y_1} \\ u_{z_1} \\ \theta_{x_1} \\ \theta_{y_1} \\ \theta_{z_1} \\ \theta_{x,x_1} \\ u_{x_2} \\ u_{y_2} \\ u_{z_2} \\ \theta_{x_2} \\ \theta_{y_2} \\ \theta_{z_2} \\ \theta_{x,x_2} \end{Bmatrix}$$

From the elementary matrixes of mass and stiffness calculated previously in the reference (C, x, y, z) where motions of bending and torsion are decoupled, one obtains these matrixes in the reference related to the neutral axis (G, x, y, z) by the following transformations:

$$\mathbf{K} = \mathbf{P}^T \mathbf{K_c} \mathbf{P}$$

$$\mathbf{M} = \mathbf{P}^T \mathbf{M_c} \mathbf{P}.$$

5 Geometrical stiffness - prestressed Structure

This matrix is calculated by the option RIGI_GEOM. It is used for dealing with problems of buckling or prestressed structure vibrations. In the case of a prestressed structure, therefore subjected to initial forces (known and independent of time), one cannot neglect in the balance equation the terms introduced by the change of geometry of the unconstrained state in a prestressed state. This change of geometry modifies the balance equation only by the addition of a function term of displacements and prestressing with which the matrix associated is called geometrical stiffness matrix and who expresses himself by:

$$W_G=\int_{V_o}\frac{\partial u_k^{3D}}{\partial x_i}\,\sigma_{ij}^o\,\frac{\partial v_k^{3D}}{\partial x_j}\,dV$$

where σ_{ij}^o the tensor of prestressing indicates. This term appears naturally if one introduces the tensor of the strains of GREEN-LAGRANGE into the virtual wor of the strain:

$$E_{xx}=\varepsilon_{xx}+\eta_{xx}=\frac{\partial u_x^{3D}}{\partial x}+\frac{1}{2}\left[\left(\frac{\partial u_x^{3D}}{\partial x}\right)^2+\left(\frac{\partial u_y^{3D}}{\partial x}\right)^2+\left(\frac{\partial u_z^{3D}}{\partial x}\right)^2\right]$$

$$2E_{xy}=2\varepsilon_{xy}+2\eta_{xy}=\frac{\partial u_x^{3D}}{\partial y}+\frac{\partial u_y^{3D}}{\partial x}+\left[\frac{\partial u_x^{3D}}{\partial x}\frac{\partial u_x^{3D}}{\partial y}+\frac{\partial u_y^{3D}}{\partial x}\frac{\partial u_y^{3D}}{\partial y}+\frac{\partial u_z^{3D}}{\partial x}\frac{\partial u_z^{3D}}{\partial y}\right]$$

$$2E_{xz}=2\varepsilon_{xz}+2\eta_{xz}=\frac{\partial u_x^{3D}}{\partial z}+\frac{\partial u_z^{3D}}{\partial x}+\left[\frac{\partial u_x^{3D}}{\partial x}\frac{\partial u_x^{3D}}{\partial z}+\frac{\partial u_y^{3D}}{\partial x}\frac{\partial u_y^{3D}}{\partial z}+\frac{\partial u_z^{3D}}{\partial x}\frac{\partial u_z^{3D}}{\partial z}\right]$$

In the statement of these strains, the quadratic terms $\left(\frac{\partial u_x^{3D}}{\partial x}\right)^2$, $\frac{\partial u_x^{3D}}{\partial x}\frac{\partial u_x^{3D}}{\partial y}$ et $\frac{\partial u_x^{3D}}{\partial x}\frac{\partial u_x^{3D}}{\partial z}$ are neglected here, according to the assumption usually carried out by most authors [feeding-bottle3]. For a model of beam, the stress tensor initial is reduced in the local axes of the beam to the components σ_{xx}, σ_{xy} and σ_{xz}. One uses the kinematics introduced with [§2]:

$$\begin{cases}u_x^{3D}(x,y,z) &= u_G(x)+z\,\theta_y(x)-y\,\theta_z(x)+\omega(y,z)\theta_{x,x}(x)\\ u_y^{3D}(x,y,z) &= v_C(x)-(z-z_c)\,\theta_x(x)\\ u_z^{3D}(x,y,z) &= w_C(x)+(y-y_c)\,\theta_x(x)\end{cases}$$

and the statement of the forces generalized according to the stresses:

$$N^0=\int_S\sigma_{xx}^o\,ds\qquad V_y^0=\int_S\sigma_{xy}^o\,ds\qquad V_z^0=\int_S\sigma_{xz}^o\,ds\qquad M_y^0=\int_S z\,\sigma_{xx}^0\,ds\qquad M_z^0=\int_S -y\,\sigma_{xx}^0\,ds$$

It is supposed, moreover, that N^0 V_y^0, V_z^0 are constant in the discretized element (what is inaccurate for example for a vertical beam subjected to its inertia loading). The moments are supposed to vary linearly:

$$M_y^0=\left(M_{y2}^0-M_{y1}^0\right)\frac{x}{L}+M_{y1}^0\qquad\frac{\partial M_y^0}{\partial x}-V_z^0=0$$

$$M_z^0=\left(M_{z2}^0-M_{z1}^0\right)\frac{x}{L}+M_{z1}^0\qquad\frac{\partial M_z^0}{\partial x}+V_y^0=0$$

These assumptions make it possible to express W_G for a straight beam with warping in the following way:

$$
\begin{aligned}
W_G = & \int_o^L N^0 \left(v_{,x}\delta v_{,x} + w_{,x}\delta w_{,x}\right) + N^0\left(\frac{I_y+I_z}{A} + y_c^2 + z_c^2\right)\theta_{x,x}\delta\theta_{x,x} \\
& + z_c N^0\left(v_{,x}\delta\theta_{x,x} + \theta_{x,x}\delta v_{,x}\right) - y_c N^0\left(w_{,x}\delta\theta_{x,x} + \theta_{x,x}\delta w_{,x}\right) \\
& - M_y^0\left(w_{,x}\delta\theta_{,x} + \theta_{,x}\delta w_{,x}\right) - M_z^0\left(v_{,x}\delta\theta_{,x} + \theta_{,x}\delta v_{,x}\right) \\
& - \left(y_c V_y^0 - z_c V_z^0\right)\left(\theta_{x,x}\delta\theta_x + \theta_x\delta\theta_{x,x}\right) + V_y^0\left(w_{,x}\delta\theta_x + \theta_x\delta w_{,x}\right) \\
& - V_z^0\left(v_{,x}\delta\theta_x + \theta_x\delta v_{,x}\right) + \left(\left(2\mathrm{y}_c - \frac{I_{yr2}}{I_z}\right)M_y^0 + \left(-2\mathrm{z}_c + \frac{I_{zr2}}{I_y}\right)M_z^0\right)\theta_{x,x}\delta\theta_{x,x} \\
& + \left(-\frac{I_{yr2}}{I_z}\frac{dM_z^0}{dx} + \frac{I_{zr2}}{I_y}\frac{dM_y^0}{dx}\right)\left(\theta_{x,x}\delta\theta_x + \theta_x\delta\theta_{x,x}\right)
\end{aligned}
$$

with the terms:

$$
I_{yr2} = \int_S y\left(y^2+z^2\right)ds
$$

$$
I_{zr2} = \int_S z\left(y^2+z^2\right)ds
$$

who represent it not - symmetry of the section. If the section has two axes of symmetry (thus C is confused with G), these terms are null.

Attention, these terms (which name IYR2 and IZR2 in the command AFFE_CARA_ELEM) are not currently calculated by MACR_CARA_POUTRE. The user must thus inform them from tubulate values for each type of section (corner, right-angled,…).

Moreover, to be able to deal with the problems of discharge of thin beams, requested primarily by bending moments and normal force, it is necessary to add the assumption of rotations moderated in torsion [feeding-bottle2], [feeding-bottle3].

This results in the following modification of the field of displacements (only for the computation of the geometrical stiffness):

$$
u_x^{3D}(x,y,z) = u_G(x) + z\left(\theta_y(x) + \theta_x(x)\,\theta_z(x)\right) - y\left(\theta_z(x) - \theta_x(x)\,\theta_y(x)\right) + \omega(y,z)\,\theta_{x,x}(x)
$$

The origin of this statement cannot be here detailed. It is the object of the thesis of TOWN OF GOYET [feeding-bottle2] on the buckling of the beams with open mean sections. The assumption of rotations of torsion moderate (and not infinitesimal) makes it possible to correctly model the discharge of a thin beam of section in torsion (coupling torsion - bending).
The assumption of moderate rotations results in adding in W_G^0 the term W_G^1 :

$$
W_G^1 = \frac{1}{2}\int_o^L -M_z^o\delta\left(\theta_x\,\theta_{y,x} + \theta_y\,\theta_{x,x}\right) + M_y^o\delta\left(\theta_x\,\theta_{z,x} + \theta_z\,\theta_{x,x}\right) + V_y^o\,\delta\left(\theta_x\theta_y\right) + V_z^o\delta\left(\theta_x\theta_z\right)
$$

Finally, one obtains the geometrical stiffness matrix while discretizing $W_G = W_G^0 + W_G^1$ using the same interpolation functions as the stiffness matrix of [§4.4]. For having calculated these matrixes, it is necessary to carry out a change of reference as with [§4.5]. One then obtains a geometrical stiffness matrix of the form:

$$
\mathbf{K_G} = \begin{pmatrix} \mathbf{A_1} & \mathbf{A_2} \\ \mathbf{A_2} & \mathbf{A_3} \end{pmatrix}
$$

The blocks of the matrix are clarified hereafter. One uses to simplify the statements:

$$N_{ey}^{0}=1.2\frac{N^{o}e_{y}}{L} \qquad N_{ez}^{0}=1.2\frac{N^{o}e_{z}}{L}$$

$$\bar{M}_{y}^{0}=\frac{M_{y1}^{0}+M_{y2}^{0}}{L} \qquad \bar{M}_{z}^{0}=\frac{M_{z1}^{0}+M_{z2}^{0}}{L}$$

$$\Delta M_{y}^{0}=\frac{M_{y1}^{0}-M_{y2}^{0}}{2} \qquad \Delta M_{z}^{0}=\frac{M_{z1}^{0}-M_{z2}^{0}}{2}$$

$$\tilde{k}=N^{o}\left(\frac{I_{y}+I_{z}}{S}+e_{y}^{2}+e_{z}^{2}\right)$$

$$\tilde{I}_{y}=-\frac{I_{yr2}}{I_{z}}+2e_{y}$$

$$\tilde{I}_{z}=\frac{I_{zr2}}{I_{y}}-2e_{z}$$

A1	2 v_1	3 w_1	4 θ_{x1}	5 θ_{y1}	6 q_{z1}	7 $q_{x,x1}$
2 v_1	$1.2\dfrac{N^o}{L}$		$N^0_{ez}+\dfrac{\bar{M}^0_y}{2}+1.2\dfrac{\Delta M^0_y}{L}$		$\dfrac{N^o}{10}$	$\dfrac{e_z N^0+L\bar{M}^0_y-M^0_{y2}}{10}$
3 w_1		$1.2\dfrac{N^o}{L}$	$-N^0_{ey}+\dfrac{\bar{M}^0_z}{2}+1.2\dfrac{\Delta M^0_z}{L}$	$-\dfrac{N^o}{10}$		$\dfrac{-e_y N^0+L\bar{M}^0_z-M^0_{z2}}{10}$
4 θ_{x1}			$\dfrac{1.2}{L}\left(\tilde{k}-\Delta M^0_z\tilde{I}_y-\Delta M^0_y\tilde{I}_z\right) -\left(e_y+\dfrac{I_{yr2}}{2I_z}\right)\bar{M}^0_z+\left(e_z+\dfrac{I_{zr2}}{2I_y}\right)\bar{M}^0_y$	$\dfrac{e_y N^0+L\bar{M}^0_z+M^0_{z2}}{10} -\dfrac{M^0_{z1}}{2}$	$\dfrac{e_z N^0-L\bar{M}^0_y-M^0_{y2}}{10} +\dfrac{M^0_{y1}}{2}$	$\dfrac{\tilde{k}+M^0_{z2}\tilde{I}_y+M^0_{y2}\tilde{I}_z}{10}$
5 θ_{y1}				$\frac{2LN^0}{15}$		$\dfrac{2e_y LN^0}{15}-\dfrac{L\left(3M^0_{z1}-M^0_{z2}\right)}{30}$
6 θ_{z1}			sym		$\frac{2LN^0}{15}$	$\dfrac{2e_z LN^0}{15}-\dfrac{L\left(3M^0_{y1}-M^0_{y2}\right)}{30}$
7 $\theta_{x,x1}$						$\dfrac{4\tilde{k}L-L\tilde{I}_y\left(3M^0_{z1}-M^0_{z2}\right)}{30} -\dfrac{L\tilde{I}_z\left(3M^0_{y1}-M^0_{y2}\right)}{30}$

A2	9 v_2	10: w_2	11: θ_{x2}	12: θ_{y2}	13: θ_{z2}	14: $\theta_{x,x2}$
2 v_1	$1.2\frac{N^o}{L}$		$-N^0_{ez}+\frac{\bar{M}^0_y}{2}-1.2\frac{\Delta M^0_y}{L}$		$\frac{N^o}{10}$	$\frac{e_z N^0-L\bar{M}^0_y+M^0_{y1}}{10}$
3 w_1		$-1.2\frac{N^o}{L}$	$N^0_{ey}+\frac{\bar{M}^0_z}{2}-1.2\frac{\Delta M^0_z}{L}$	$-\frac{N^o}{10}$		$\frac{-e_y N^0-L\bar{M}^0_z+M^0_{z1}}{10}$
4 θ_{x1}		$N^0_{ey}-\frac{\bar{M}^0_z}{2}-1.2\frac{\Delta M^0_z}{L}$	**- A1 (4,4)**	$\frac{e_y N^0-L\bar{M}^0_z-M^0_{z1}}{10}$	$\frac{e_z N^0+L\bar{M}^0_y+M^0_{y1}}{10}$	$\frac{\tilde{k}-M^0_{y1}\tilde{I}_z-M^0_{z1}\tilde{I}_y}{10}$
5 θ_{y1}		$\frac{N^o}{10}$	$\frac{-e_y N^0-L\bar{M}^0_z-M^0_{z2}}{10}$	$-\frac{LN^o}{30}$		$\frac{-e_y LN^0+L\Delta M^0_z}{30}+\frac{L^2\bar{M}^0_z}{60}$
6 θ_{z1}	$-\frac{N^o}{10}$		$\frac{-e_z N^0+L\bar{M}^0_y+M^0_{y2}}{10}$		$-\frac{LN^o}{30}$	$\frac{-e_z LN^0-L\Delta M^0_y}{30}-\frac{L^2\bar{M}^0_y}{60}$
7 $\theta_{x,x}$	$\frac{-e_z N^0-L\bar{M}^0_y+M^0_{y2}}{10}$	$\frac{e_y N^0-L\bar{M}^0_z+M^0_{z2}}{10}$	$\frac{-\tilde{k}-M^0_{y2}\tilde{I}_z-M^0_{z2}\tilde{I}_y}{10}$	$\frac{-e_y LN^0+L\Delta M^0_z}{30}-\frac{L^2\bar{M}^0_z}{60}$	$\frac{-e_z LN^0-L\Delta M^0_y}{30}+\frac{L^2\bar{M}^0_y}{60}$	$\frac{-\tilde{k}L+L\Delta M^0_y\tilde{I}_z}{30}+\frac{L\Delta M^0_z\tilde{I}_y}{30}$

A3	9: v_2	10: w_2	11: θ_{x2}	12: θ_{y2}	13: θ_{z2}	14: $\theta_{x,x2}$
2 v_2	$1.2\frac{N^o}{L}$		$N_{ez}^0-\frac{\bar{M}_y^0}{2}+1.2\frac{\Delta M_y^0}{L}$		$-\frac{N^o}{10}$	$\frac{-e_z N^0+L\bar{M}_y^0-M_{y1}^0}{10}$
3 w_2		$1.2\frac{N^o}{L}$	$-N_{ey}^0-\frac{\bar{M}_z^0}{2}+1.2\frac{\Delta M_z^0}{L}$	$\frac{N^o}{10}$		$\frac{e_y N^0+L\bar{M}_z^0-M_{z1}^0}{10}$
4 θ_{x2}			$1.2\frac{\tilde{k}}{L}-1.2\left(\frac{\Delta M_z^0}{L}\tilde{I}_y+\frac{\Delta M_y^0}{L}\tilde{I}_z\right)+\left(e_y+\frac{I_{yr2}}{2I_z}\right)\bar{M}_z^0-\left(e_z+\frac{I_{zr2}}{2I_y}\right)\bar{M}_y^0$	$\frac{-e_y N^0+L\bar{M}_z^0+M_{z1}^0}{10}-\frac{M_{z2}^0}{2}$	$\frac{-e_z N^0-L\bar{M}_y^0-M_{y1}^0}{10}+\frac{M_{y2}^0}{2}$	$\frac{-\tilde{k}+M_{y1}^0\tilde{I}_z+M_{z1}^0\tilde{I}_y}{10}$
5 θ_{y2}				$\frac{2LN^\circ}{15}$		$\frac{2e_y LN^0}{15}-\frac{L\left(M_{z1}^0-3M_{z2}^0\right)}{30}$
6 θ_{z2}					$\frac{2LN^\circ}{15}$	$\frac{2e_z LN^0}{15}+\frac{L\left(M_{y1}^0-3M_{y2}^0\right)}{30}$
7 $\theta_{x,x2}$			Sym			$\frac{4\tilde{k}L-L\tilde{I}_y\left(M_{z1}^0-3M_{z2}^0\right)}{30}-\frac{L\tilde{I}_z\left(M_{y1}^0-3M_{y2}^0\right)}{30}$

6 Loadings

the various types of loading available for element POU_D_TG are:

Types or options	
CHAR_MECA_FR1D1D	distributed loading by actual values
CHAR_MECA_FF1D1D	distributed loading by function
CHAR_MECA_PESA_R	loading due to "
thermal	" gravity CHAR_MECA_TEMP_R loading
CHAR_MECA_EPSI_R	loading by imposition of a strain (of standard thermal stratification)

the loadings are in the same way calculated that for the elements without warping [R3.08.01]. There is thus nothing in particular to element POU_D_TG. The other types of loading described in [R3.08.01] are not available for this element.

With regard to warping, it is possible to give boundary conditions utilizing the degree of freedom GRX (what makes it possible to model constrained torsion: $GRX=0$), but on the other hand, nothing is designed to affect a loading of type bi--moment, whose physical interpretation is difficult to establish.

In connection with connection between elements, the transmission of warping is an open-ended question as the reference [feeding-bottle 1]1 it: the continuity of the variable GRX from one element to another (on which warping depends directly) depends by way of technology on connection between the various beams (weld in the axis, in which case warping can be transmitted completely, connection by gusset,…).

For an assembled structure such as a truss, it seems more reasonable to suppose than torsion is obstructed, therefore that warping is null at the ends. To determine the influence of this assumption, one will be able to refer to the test SSLL102 (beam of corner section) whose modelizations C and D use element POU_D_TG, with free torsion for the modelization C, and torsion obstructed for the modelization D [V3.01.102B].
It is noted that for the loading of bending, the variation on displacement is weak (2.5%), but for a loading in torsion, one obtains for this section a non-zero side displacement (discharge) from which the value differs notably according to the assumption taken:
$u_z=2.2\ 10^{-5}$ for free torsion and $u_z=2.62\ 10^{-5}$ constrained torsion.
In the same way, rotation strongly varies:
$\theta_x=3,79\ 10^{-4}$ for free torsion and $\theta_x=6,39\ 10^{-4}$ constrained torsion (GRX is null at the ends).

6.1 Distributed loadings, options: CHAR_MECA_FR1D1D and CHAR_MECA_FF1D1D

the loadings are given under key word FORCE_POUTRE, either by actual values in AFFE_CHAR_MECA (option CHAR_MECA_FR1D1D), or by functions in AFFE_CHAR_MECA_F (option CHAR_MECA_FF1D1D).
The loading is given only by distributed forces, not by moments distributed.
The second associated member with the distributed loading of traction and compression is:

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \text{ with } f_1=\int_0^1 f_{ext}(x)\left(1-\frac{x}{L}\right)dx$$

$$f_2=\int_0^1 f_{ext}(x)\,\frac{x}{L}\,dx$$

For a loading constant or varying linearly, one obtains:

$$F_{x_1} = L\left(\frac{n_1}{3}+\frac{n_2}{6}\right),$$

$$F_{x_2} = L\left(\frac{n_1}{6}+\frac{n_2}{3}\right).$$

n_1 and n_2 are the components of the axial loading as in points 1 and 2 coming from the data of the user replaced in the local coordinate system.

If t_{y_1}, t_{y2}, t_{z1} and t_{z2} are those of the shears, one a:

$$F_{y_1}=L\left(\frac{7t_{y_1}}{20}+\frac{3t_{y_2}}{20}\right)\qquad M_{z_1}=L^2\left(\frac{t_{y_1}}{20}+\frac{t_{y_2}}{30}\right)$$
$$F_{y_2}=L\left(\frac{3t_{y_1}}{20}+\frac{7t_{y_2}}{20}\right)\qquad M_{z_2}=-L^2\left(\frac{t_{y_1}}{30}+\frac{t_{y_2}}{20}\right),$$
$$F_{z_1}=L\left(\frac{7t_{z_1}}{20}+\frac{3t_{z_2}}{20}\right),\qquad M_{y_1}=-L^2\left(\frac{t_{z_1}}{20}+\frac{t_{z_2}}{30}\right),$$
$$F_{z_2}=L\left(\frac{3t_{z_1}}{20}+\frac{7t_{z_2}}{20}\right),\qquad M_{y_2}=L^2\left(\frac{t_{z_1}}{30}+\frac{t_{z_2}}{20}\right).$$

6.2 Loading of gravity, option: `CHAR_MECA_PESA_R`

the force of gravity is given by the modulus of acceleration $\mathbf{g}$ and a normalized vector $\mathbf{n}$ indicating the direction of the loading.

Remarks *(simplifying assumption)* :
The shape functions used for this computation are those of the model Eulerian-Bernoulli.

The approach is similar to that used for the distributed forces, on condition that transforming initially the vector loading due to gravity in the local coordinate system with the element. One obtains in the local coordinate system of beam:

$$F_{x_i} = \int_o^L \xi_i \rho S \;\; \mathbf{g}\cdot\mathbf{x}\, dx \qquad \left(\xi_1 = 1-\frac{x}{L}\ ,\ \ \xi_2=\frac{x}{L}\right)$$

from where:

$$F_{x_1} = \rho\ \mathbf{g}\cdot\mathbf{x}\, L\left(\frac{S}{3}+\frac{S}{6}\right) \quad \text{au point 1,}$$
$$F_{x_2} = \rho\ \mathbf{g}\cdot\mathbf{x}\, L\left(\frac{S}{6}+\frac{S}{3}\right) \quad \text{au point 2}$$

Bending in the plane (Gxz) **:**

$$F_{z_1} = \rho\mathbf{g}\cdot\mathbf{z}\, L\left(\frac{7S}{20}+\frac{3S}{20}\right)$$
$$M_{y_1} = -\rho\mathbf{g}\cdot\mathbf{z}\;\; L^2\left(\frac{S}{20}+\frac{S}{30}\right)$$
$$F_{z_2} = \rho\mathbf{g}\cdot\mathbf{z}\, L\left(\frac{3S}{20}+\frac{7S}{20}\right)$$
$$M_{y_2} = \rho\mathbf{g}\cdot\mathbf{z}\, L^2\left(\frac{S}{30}+\frac{S}{20}\right)$$

Bending in the plane (Gxy) **:**

$$F_{y_1}=\rho\,\mathbf{g}\cdot\mathbf{y}\,L\left(\frac{7S}{20}+\frac{3S}{20}\right)$$
$$M_{z_1}=\rho\,\mathbf{g}\cdot\mathbf{y}\quad L^2\left(\frac{S}{20}+\frac{S}{30}\right)$$
$$F_{y_2}=\rho\,\mathbf{g}\cdot\mathbf{y}\,L\left(\frac{3S}{20}+\frac{7S}{20}\right)$$
$$M_{z_2}=-\rho\,\mathbf{g}\cdot\mathbf{y}\,L^2\left(\frac{S}{30}+\frac{S}{20}\right)$$

6.3 Thermal loading, option: CHAR_MECA_TEMP_R

to obtain this loading, it is necessary to calculate axial displacements induced by the difference in temperature $T-T_{\text{référence}}$:

$$u_1 = -L\alpha\left(T-T_{\text{référence}}\right)$$
$$u_2 = L\alpha\left(T-T_{\text{référence}}\right)$$

(α : thermal coefficient of thermal expansion)

Then, one calculates simply the induced forces par. $\mathbf{F} = \mathbf{K}\,\mathbf{u}$

As K is the local stiffness matrix with the element, one must then carry out a change of reference to obtain the values of the components loading in the total reference.

6.4 Loading by imposed strain, option: CHAR_MECA_EPSI_R

One calculates as for elements POU_D_T the loading from a strain state (this option was developed to take into account the thermal stratification in the pipework). The model takes into account only one work in traction and compression and pure bending (not of shears, not of twisting moment).
The strain is given by the user using key word PRE_EPSI in AFFE_CHAR_MECA. While being given $\frac{\partial u}{\partial x}$, $\frac{\partial\theta_y}{\partial x}$ and $\frac{\partial\theta_z}{\partial x}$ on the beam, one obtains the second elementary member associated with this loading:

with node 1:

$$F_{x_1} = E\,S_1\frac{\partial u}{\partial x},$$
$$M_{y_1} = E\,I_{y_1}\frac{\partial\theta_y}{\partial x},$$
$$M_{z_1} = E\,I_{z_1}\frac{\partial\theta_z}{\partial x},$$

with node 2:

$$F_{x_2} = E\,S_2\frac{\partial u}{\partial x},$$
$$M_{y_2} = E\,I_{y_2}\frac{\partial\theta_y}{\partial x},$$
$$M_{z_2} = E\,I_{z_2}\frac{\partial\theta_z}{\partial x}$$

7 Torsor of the forces - nodal Forces and reactions

7.1 Options available

the various options of postprocessing available for element POU_D_TG are:

Types or options	
EFGE_ELNO	torsor of the forces to the 2 nodes of each element
SIEF_ELGA	field of forces necessary to the computation of the nodal forces (option "FORC_NODA") and of the reactions (option "REAC_NODA").
FORC_NODA	nodal forces expressed in total reference
REAC_NODA	nodal reactions

7.2 the torsor of the forces

7.2.1 generalized Forces, option: EFGE_ELNO

One seeks to calculate with the two nodes of each element "beam" constituting the mesh of studied structure, the forces exerted on the element "beam" by the rest of structure. The values are given in the local base of each element. By integrating the balance equations, one obtains the forces in the local coordinate system of the element:

$$\mathbf{R}_{LOC} = K^e_{LOC} u_{LOC} + M^e_{LOC} \ddot{u}_{LOC} - f^e_{LOC}$$

where:

$$\mathbf{R}_{LOC} = \left(-N^1, -V^1_Y, -V^1_Z, -M^1_T, -M^1_Y, -M^1_Z, -M^1_\omega, N^2, V^2_Y, V^2_Z, M^2_T, M^2_Y, M^2_Z, M^2_\omega\right)$$

$\mathbf{K}^{\mathbf{e}}_{LOC}$	elementary matrix of stiffness of the element beam,
$\mathbf{M}^{\mathbf{e}}_{LOC}$	elementary matrix of mass of the element beam,
$\mathbf{f}^{\mathbf{e}}_{LOC}$	vector of the forces "distributed" on the element beam,
$\mathbf{u}_{LOC}$	vector "degree of freedom" limited to the element beam,
$\ddot{\mathbf{u}}_{LOC}$	vector "acceleration" limited to the element beam.

One changes then the signs of nodal efforts 1.
Indeed, by taking for example the case of the traction and compression, one shows [R3.08.01] that the forces in the element (option EFGE_ELNO) are obtained by:

$$\begin{bmatrix} -N(o) \\ N(L) \end{bmatrix} = [K] \begin{bmatrix} u(o) \\ u(L) \end{bmatrix} - \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

7.2.2 Generalized forces, option: SIEF_ELGA

option "SIEF_ELGA" is established for reasons of compatibility with other options. It is used only for computation of the nodal forces. It produces fields of forces by elements. It is calculated by:

$$\mathbf{R}_{LOC} = \mathbf{K}^{\mathbf{e}}_{LOC} \mathbf{u}_{LOC}$$

7.3 Computation of the nodal forces and the reactions

7.3.1 nodal Forces, option: FORC_NODA

This option calculates a force vector nodal on all structure, expressed in total reference. It produces a field at nodes in command CALC_CHAMP by assembly of the elementary terms.

For this computation, one uses the principle of the virtual wors and one writes [R5.03.01]:

$$\mathbf{F} = \mathbf{Q}^{\mathbf{T}} \sigma$$

where $\mathbf{Q}^{\mathbf{T}}$ the matrix associated with the operator divergence represents symbolically. For an element, one writes the work of the strain field virtual:

$$\left(\mathbf{Q}^{\mathbf{T}}\sigma\right)u^{*}=\int_{\Omega}\sigma\left(\mathbf{u}\right)\,\varepsilon\,\left(u^{*}\right)\quad\forall u^{*}\ \textbf{kinematically admissible}$$

For the beam elements, one calculates simply the nodal forces by assembly of the elementary nodal forces calculated by option SIEF_ELGA, which are expressed by:

$$\left[\mathbf{F}_{LOC}\right]=\left[\mathbf{K}_{LOC}\right]\left[\mathbf{U}_{LOC}\right]$$

7.3.2 Nodal reactions, option: REAC_NODA

This option, called by CALC_CHAMP, makes it possible to obtain the reactions R to the bearings, expressed in the total reference, starting from the nodal forces $\mathbf{F}$ by:

$$\mathbf{R}=\mathbf{F}-\mathbf{F}^{char}+\mathbf{F}^{iner}$$

$\mathbf{F}^{char}$ et $\mathbf{F}^{iner}$ being nodal forces respectively associated with the loadings given (specific and distributed) and with the forces with inertia.

8 Bibliography

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9 Description of the versions of the document

Version Aster	Author (S) Organization (S)	Description of the modifications
04/07/09	***J.L. FLEJOU*** *,* *J.M. PROIX (EDF-R&D/AMA)*	
2/22/2013	***J.L. FLEJOU***	Addition remarks on `POU_D_TGM`.