A finite element of cable-pulley

Summarized :

The pulleys play an important role in structures of cables such as the air electric lines. It is thus useful to know to model them in one at the same time realistic and powerful way. One presents the finite element formulation of a length of cable passing by a pulley: statements of the internal forces and the stiffness matrix. The pulley can be fixed or supported by a flexible structure. Its position on the cable with the equilibrium is not known a priori: this position is that for which on both sides the tension of the cable is the same one.

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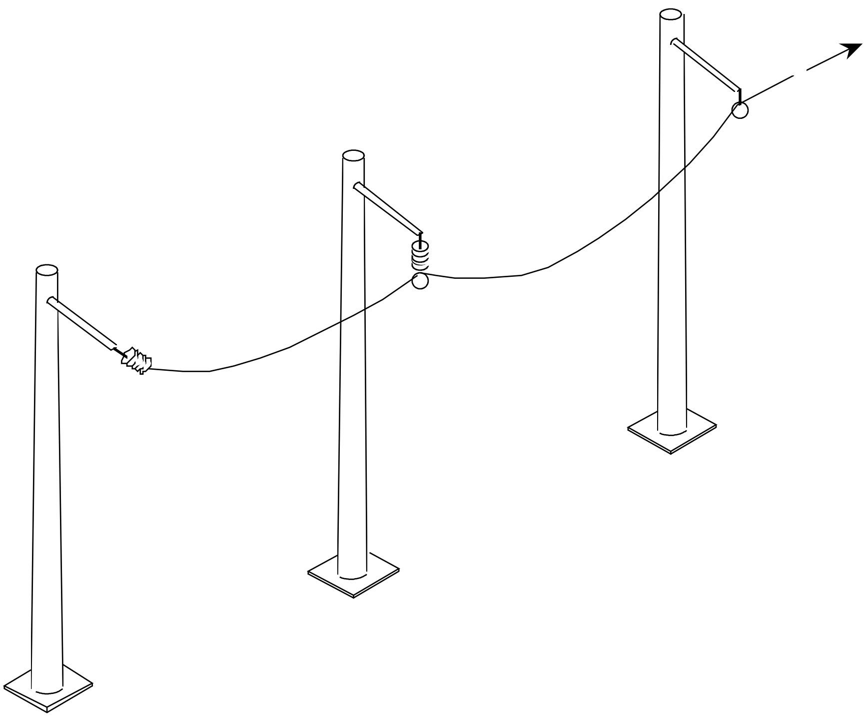
1 Notations

A	area of the cross-section of the cable.
E	Young modulus.
$\mathbf{F}$	vector of the internal forces of the element.
$\mathbf{F_n}$	internal force with the node n .
H	horizontal component of the tension [§An1].
$\mathbf{I_3}$	stamp unit of order 3.
$\mathbf{K}$	stiffness matrix of the element.
l	current length of the element.
l_0	length at rest.
l_i	initial length.
$\mathbf{l_1}$	$\overrightarrow{N_3 N_1}$
l_1	euclidian norm of $\mathbf{l_1}$
$\mathbf{l_2}$	$\overrightarrow{N_3 N_2}$
l_2	euclidian norm of $\mathbf{l_2}$.
N	current tension of the cable constituting the element.
N_i	initial tension (prevoltage).
S	marks with arrows of a range of cable [§An1].
s	length of a range [§An1].
T	current temperature.
T_i	initial temperature.
$\mathbf{u}$	vector-displacement of the nodes compared to the initial position.
$\mathbf{u}_n$	displacement of the node n compared to its initial position.
w	weight per unit of length.
$\mathbf{x}_n$	vector-position of the node n in initial configuration.
α	thermal coefficient of thermal expansion.
ε	current strain compared to the initial configuration.

2 Introduction

One uses pulleys, during the construction of the air electric lines, for the operation of installation of the cables. The cable in the course of installation [Figure 2-a] is fixed at the one of the supports of stop of the canton, it rests on pulleys placed at the bottom of the insulators of the supports of alignment and it is retained by a force on the level of the second support of stop. While exploiting this force - or by moving its point of application - one adjusts the deflection of the one of the ranges, that which is subjected to stresses of environment. Then one removes the pulleys and one fixes the cable at the insulators. The length of the cable in the various ranges is then built-in and it determines the later behavior of line under statical stresses (wind, overload of white frost) and in dynamic mode (motion due to the forces of Laplace created by flows of short-circuit).

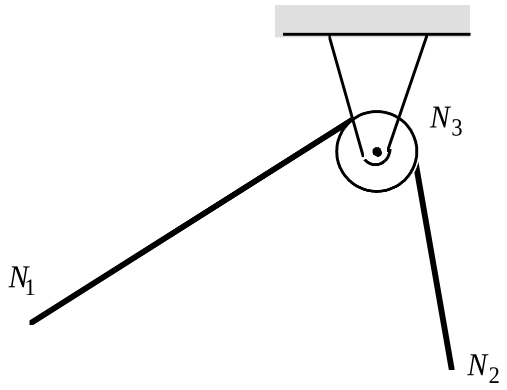


Appear 2-a: Pose of a cable in a canton with two ranges

the finite element of cable-pulley presented here allows to model the operation of installation and thus to calculate, in a natural way, the length of cable in the various ranges.

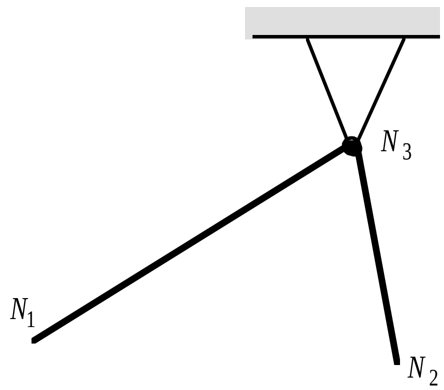
The idea of this finite element came to us some time after the conversation [bib1] and we presented his formulation in [bib2].

3 Assumptions and definition of a finite element of cable - pulley



Appears 3-a: Length of cable passing by a pulley - reality

Let us take a length of cable $N_1 N_2$ passing by the pulley N_3 [Figure 3-a]. This pulley is not inevitably fixed and can, for example and as it is the case in the example of [§8], being assembled at the end of a cable.



Appear 3-b: Length of cable passing by a pulley - modelled

the pulley is supposed to be specific [Figure 3-b]. One makes moreover following assumptions:

The equilibrium position of N_3 is not known, but it is necessarily on the section N_1 N_2 deformed from its initial position.

In modelization of the lines, horizontal motions are of low amplitude and this assumption is generally not restrictive;

The two bits N_3 N_1 and N_3 N_2 are always rectilinear, like cable elements of the 1st order.

It rises from this assumption that:

$$\mathbf{l_1}=\mathbf{x_1}+\mathbf{u_1}-\mathbf{x_3}-\mathbf{u_3}$$ éq 3-1
$$\mathbf{l_2}=\mathbf{x_2}+\mathbf{u_2}-\mathbf{x_3}-u_3$$

$$l_1=\sqrt{\mathbf{l_1^T l_1}}$$ éq the 3-2
$$l_2=\sqrt{\mathbf{l_2^T l_2}}$$

overall length of the two bits is:

- in the current location, where the temperature is T :

$$l=l_1+l_2 \quad ;$$

- in the initial position, indicated by the index i , where the tension is N_i and the temperature T_i :

$$l_i=\left(l_1\right)_i+\left(l_2\right)_i \quad ;$$

- in a not forced position, indicated by index 0, where the temperature is T_0 :

$$l_0=\left(l_1\right)_0+\left(l_2\right)_0 .$$

The pulley is without friction and thus the tension is the same one in the two bits.

It rises from this assumption that the strain ε is also the same one and one takes for value of this one the measurement of the relative lengthening of the section compared to the initial state:

$$\varepsilon=\frac{l-l_i}{l_0}$$ éq 3-3

ε must remain small, so that the section A is regarded as constant.

It will be noted that, in the frame of the finite element method, the linear loads do not prevent the tension from being constant of N_1 with N_2 . These forces are indeed concentrated with the nodes N_1 and N_2 on **the axis** of the pulley N_3 .

The behavior model is elastic:

$$N=EA\left[e-\alpha\left(T-T_i\right)\right]+N_i$$ éq 3-4

One calls **finite element of cable-pulley**, a length of satisfactory N_1 N_2 N_3 cable to the preceding assumptions.

4 Internal forces of a finite element of cable-pulley

Let us point out the following definition: one calls internal forces of a structure finite element the forces which it is necessary to exert in its nodes to maintain it in its current deformed configuration.

In the case of a finite element of cable-pulley, the internal forces result immediately from the static. One has indeed [Figure 4-a]:

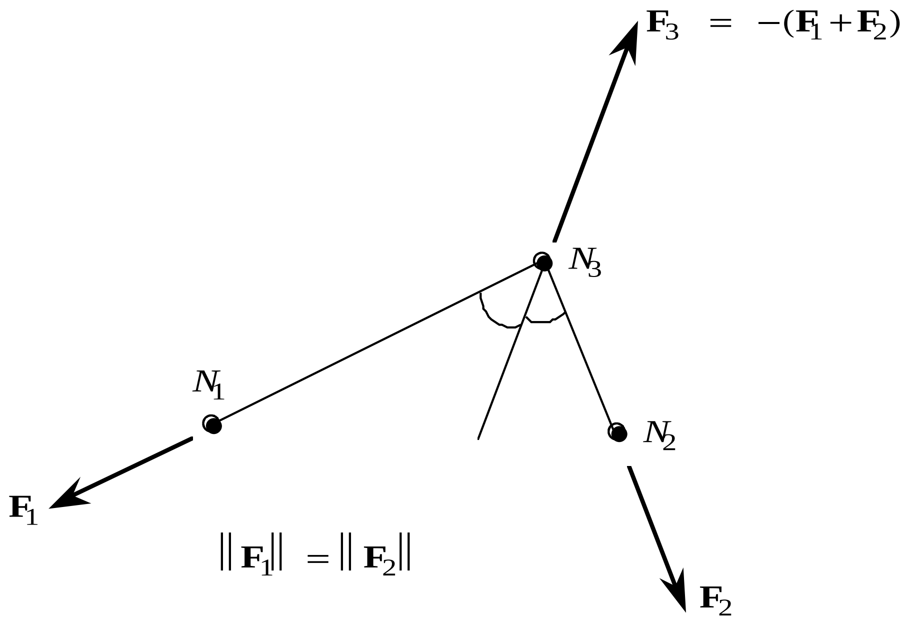
$$\mathbf{F}_1=\frac{N}{l_1}\mathbf{l}_1 \quad \text{éq 4-1}$$

$$\mathbf{F}_2=\frac{N}{l_2}\mathbf{l}_2$$

and, to ensure the equilibrium:

$$\mathbf{F}_3=-\left(\mathbf{F}_1+\mathbf{F}_2\right).$$

$\mathbf{F}_1$ and $\mathbf{F}_2$ having even modulus, $\mathbf{F}_3$ is directed according to the bisectrix of the angle $\left(N_1 N_3 N_2\right)$.



Appear 4-a: Internal forces of an element of cable-pulley

$\mathbf{F}_3$ is applied to **the axis** of the pulley.

The system of forces $\mathbf{F}$ interns of the element is thus:

$$\mathbf{F}=\begin{Bmatrix}\mathbf{F}_1\\ \mathbf{F}_2\\ \mathbf{F}_3\end{Bmatrix}.$$

5 Stiffness matrix

the stiffness matrix $\mathbf{K}$ of the element is derived from Fréchet of $\mathbf{F}$ in the direction of the displacement $\delta\mathbf{u}$ of the nodes:

$$\delta\mathbf{F}=\mathbf{K}\,\delta\mathbf{u}\ .$$

$\mathbf{K}$ is calculated by the following classical formula, used intensively in [bib3], [bib4] and [bib5]:

$$\delta\mathbf{F}=\lim_{\varepsilon\to 0}\frac{\delta}{\delta\varepsilon}\mathbf{F}\left(\mathbf{u}+\varepsilon\,\delta\mathbf{u}\right)$$.éq the 5-1

detail of computations is given in [§An1] and the final statement of K is the following one:

$$\mathbf{K}=\begin{bmatrix} \mathbf{K}_{11}+\frac{N}{l_1}\mathbf{I}_3 & \mathbf{K}_{12} & -\mathbf{K}_{11}-\mathbf{K}_{12}-\frac{N}{l_1}\mathbf{I}_3 \\ \mathbf{K}_{12}^T & \mathbf{K}_{22}+\frac{N}{l_2}\mathbf{I}_3 & -\mathbf{K}_{22}-\mathbf{K}_{12}^T-\frac{N}{l_2}\mathbf{I}_3 \\ -\mathbf{K}_{11}-\mathbf{K}_{12}^T-\frac{N}{l_1}\mathbf{I}_3 & -\mathbf{K}_{22}-\mathbf{K}_{12}-\frac{N}{l_2}\mathbf{I}_3 & \mathbf{K}_{11}+\mathbf{K}_{22}+\mathbf{K}_{12}+\mathbf{K}_{12}^T+\left(\frac{1}{l_1}+\frac{1}{l_2}\right)N\,\mathbf{I}_3 \end{bmatrix}$$ éq 5-2

N is given by [éq 3-4] and [éq 3-3];

$$\mathbf{K}_{11}=\left(\frac{EA}{l_0}-\frac{N}{l_1}\right)\frac{1}{l_1^2}\mathbf{l}_1\mathbf{l}_1^T;$$
$$\mathbf{K}_{12}=\frac{EA}{l_0 l_1 l_2}\mathbf{l}_1\mathbf{l}_2^T;$$
$$\mathbf{K}_{22}=\left(\frac{EA}{l_0}-\frac{N}{l_2}\right)\frac{1}{l_2^2}\mathbf{l}_2\mathbf{l}_2^T.$$

$\mathbf{K}$ is **symmetric**, because of the symmetry of $\mathbf{K}_{11}$ and $\mathbf{K}_{22}$ total symmetry per blocks.

But $\mathbf{K}$ depends on displacements from N_1, N_2 et N_3 the intermediary on $\mathbf{l_1}, \mathbf{l_2}$ et N : the finite element of cable-pulley is thus a nonlinear **element.**

6 Mass matrix

This matrix intervenes obviously only in the dynamic problems. The cable element - pulley is not used in *the Code_Aster* that for the quasi-static problems of installation of cables [§7].

Note:

One presents nevertheless in [bib2] the example of a dynamic problem comprising a cable-pulley.

The mass matrix of the element $N_1 \quad N_2 \quad N_3$ *is obtained by assembling the "coherent" mass matrixes of the cable elements with two nodes* $N_3 \quad N_1$ *and* $N_3 \quad N_2$ *[bib6] and by adding the point mass of the pulley.*

It should be noted that, during a dynamic analysis, this mass matrix must be updated because the lengths l_1 *and* l_2 *vary.*

7 Introduction of the element of cable-pulley into *the Code_Aster*

the element of cable-pulley is supported by a mesh SEG3.

In the command AFFE_MODELE , under factor key word `the AFFE`, one must define the arguments of the key words as follows:

key word	`GROUP_MA`	`PHENOMENE`	`MODELISATION`
argument	" MECHANICAL"	`mesh group of cable-pulley`	`"CABLE_POULIE"`

the constitutive material must be elastic.

In the command AFFE_CARA_ELEM , the cable-pulleys are treated like cables.

As the element of cable-pulley is nonlinear [§5] and that, for time, it is used only in static [§6], it is accessible only by the operator `STAT_NON_LINE`. Under factor key word `the COMP_ELAS`, the arguments of the key words are the following:

key word	`GROUP_MA`	`RELATION`	`DEFORMATION`
argument	mesh group of cable-pulley	`"ELAS"`	`"GROT_GDEP"`

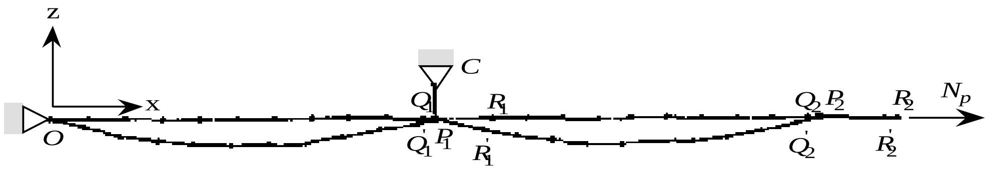
Lastly, the force of gravity acting on the nodes N_1 et N_2 of an element of cable-pulley [fig 3 - B] is following because it depends on the length of the bits $N_3 N_1$ et $N_3 N_2$. For a structure comprising at least an element of cable-pulley, one must specify this load (following) in `STAT_NON_LINE` under the key word factor `EXCIT`.

One finds an example of application in test SSNL100A [V6.02.100].

8 Example of application

This example is that of the installation of a cable with two ranges and imposed tension of adjustment.

On [Figure 7-a], O is the anchorage of the cable on the support of stop of left. P_1 is a first pulley placed at the bottom of the lifting chain P_1C, fixed in C at the support of alignment. P_2 is one second pulley placed on the support of stop of right. O, P_1 and P_2 are, to simplify, located on horizontal.



Appear 7-a: Balance of a cable with two ranges, subjected to a tension of adjustment given.

$OP_1 = P_1P_2 = 100\,m$; $w = 30\,N/m$; $E'A = 5\,x\,10^7\,N$; $N_p = 5\,000\,N$

In initial position, the cable at rest, supposed in weightlessness, is right: line in feature of axis of [Figure 7-a]. For the modelization in finite elements, this line is cut out in:

ten cable elements with two nodes enters O and Q_1 ;
an element of cable-pulley Q_1 R_1 P_1 ;
nine cable elements enters R_1 and Q_2 ;
an element of cable-pulley Q_2 R_2 P_2 .

One simultaneously subjects the cable to gravity and the tension of adjustment N_p exerted in R_2. The equilibrium position (line in feature full with [Figure 7-a]) is reached in 11 iterations by the operator `STAT_NON_LINE` of *Code_Aster*. The deflections are of $7.955\,m$ and $7.867\,m$, respectively in the range of left and that of right.

For an inextensible **cable** of a range of $100\,m$, weight linear $30\,N/m$ and subjected to a tension in the end of $5\,000\,N$, the theoretical deflection is of $7.941\,m$ [§An2].

While exploiting the tension of adjustment, one can adjust one of the two deflections to a value built-in advance.

9 Conclusion

the finite element of cable-pulley presented in this note, of a very simple mechanical formulation, has performances comparable to those of an ordinary cable element. It is very convenient and even essential for a realistic modelization of the air electric lines. It should find other applications, in particular in Robotics.

10 Bibliography

1) J.L. LILIEN : Private communication.

2) Mr. AUFAURE : A finite element of cable passing through has pulley. Computers & Structures 46,807-812 (1993).

3) J.C. SIMO, L. CONSIDERING-QUOC : A three-dimensional finite-strain rod model. Leaves II: computational aspects. *Comput. Meth. appl. Mech. Engng* 58,79-116 (1986).

4) A. Cardona, Mr. GERADIN : A beam finite element nonlinear theory with finite rotations. *Int. J. Numer. Meth. Engng.* 26,2403-2438 (1988).

5) A. CARDONA : Year integrated approach to mechanism analysis. Thesis, University of Liege, Belgium (1989).

6) Mr. AUFAURE : Modelization of the cables in *the Code_Aster*, in the course of drafting.

7) H. MAX IRVINE : Cable Structures. MIT Close (1981).

11 Description of the versions of the document

Version Aster	Author (S) Organization (S)	Description of modifications
3	*Mr. AUFAURE (EDF/IMA/MMN)*	initial Text
10.1	J.M.PROIX (EDF-R&D/AMA) Modification	of GREEN in GROT_GDEP Computation of

Annexe 1the stiffness matrix One shows

here how applies the formula [éq 5-1] to the computation of the first three lines of the matrix [éq 5-2 K], those which are relating to the force. The other $\mathbf{F}_1$ lines are obtained either by permutation of the indices, or by summation of two preceding lines. The first

lines are thus obtained by calculating derivative: éq An1-

$$\lim_{e\to 0}\frac{d}{de}F_1(u+ed\,u)$$

1 and by putting

in factor the vector. According to $\delta\boldsymbol{u}$

the relations [éq 4-1], [éq 3-4] and [éq 3-3], one a: with, according to

$$\mathbf{F}_1(\mathbf{u}+\varepsilon\,\delta\,\mathbf{u})=\left\{EA\left[\frac{l_1(\mathbf{u}+\varepsilon\,d\,\mathbf{u})+l_2(\mathbf{u}+\varepsilon\,\delta\,\mathbf{u})-l_i}{l_0}-\alpha(T-T_i)\right]+N_i\right\}\frac{l_1(\mathbf{u}+\varepsilon\,\delta\,\mathbf{u})}{l_1(\mathbf{u}+\varepsilon\,\delta\,\mathbf{u})}$$

[éq 3-1]: and, according to

$$\mathbf{l}_1(\mathbf{u}+\varepsilon\,\delta\,\mathbf{u})=\mathbf{x}_1+\mathbf{u}_1+\varepsilon\,\delta\,\mathbf{u}_1-\mathbf{x}_3-\mathbf{u}_3-\varepsilon\,\delta\,\mathbf{u}_3$$

[éq 3-2]: Consequently

$$l_1(\mathbf{u}+\varepsilon\,\delta\,\mathbf{u})=\sqrt{\mathbf{l}_1^T(\mathbf{u}+\varepsilon\,\delta\,\mathbf{u})\mathbf{l}_1(\mathbf{u}+\varepsilon\,\delta\,\mathbf{u})}.$$

: and by permutation

$$\lim_{\varepsilon\to 0}\frac{d}{d\,\varepsilon}\mathbf{l}_1(\mathbf{u}+\varepsilon\,\delta\,\mathbf{u})=\delta\,\mathbf{u}_1-\delta\,\mathbf{u}_3$$

$$\lim_{\varepsilon\to 0}\frac{d}{d\,\varepsilon}\mathbf{l}_1(\mathbf{u}+\varepsilon\,\delta\,\mathbf{u})=\frac{1}{l_1(\mathbf{u})}\mathbf{l}_1^T(\mathbf{u})(\delta\,\mathbf{u}_1-\delta\,\mathbf{u}_3)$$

of indices 1 and 2: Finally

$$\lim_{\varepsilon\to 0}\frac{d}{d\,\varepsilon}\mathbf{l}_2(\mathbf{u}+\varepsilon\,\delta\,\mathbf{u})=\frac{1}{l_2(\mathbf{u})}\mathbf{l}_2^T(\mathbf{u})(\delta\,\mathbf{u}_2-\delta\,\mathbf{u}_3)$$

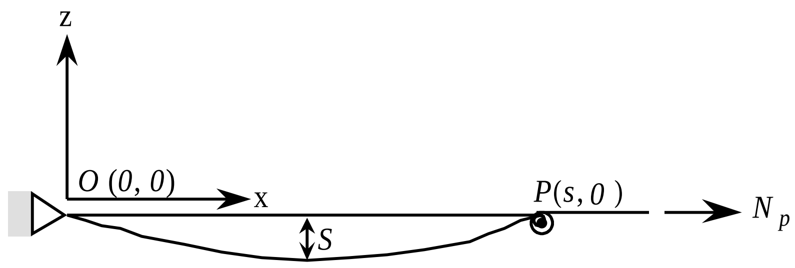
: While carrying

$$\lim_{\varepsilon\to 0}\frac{d}{d\,\varepsilon}\mathbf{l}_1(\mathbf{u}+\varepsilon\,\delta\,\mathbf{u})=\frac{1}{l_1^3(\mathbf{u})}\mathbf{l}_1^T(\mathbf{u})(\delta\,\mathbf{u}_1-\delta\,\mathbf{u}_3)$$

the preceding statements in [éq An1-1] and by putting in factor the vector, one easily $\delta\,\mathbf{u}$ obtains the first three lines of. Appear $\mathbf{K}$ of

Annexe 2equilibrium of an inextensible cable weighing subjected to a tension given in the end Take

a cable [An2-a Figure] whose end is fixed at the point and whose O other end is subjected P to the tension. and are N_p O on P same horizontal and distant ones of. The linear s weight is. w The deflection is sought. Appear S Year



2-a: Heavy cable balances some One finds

in [bib7], p 6, the following well-known formulas: appear of

equilibrium of the cable: éq An2-

$$z(x)=\frac{H}{w}\left[\cosh\frac{w}{H}\left(\frac{s}{2}-x\right)-\cosh\frac{ws}{2\mathrm{H}}\right];$$ 1 tension

: éq An2-

$$N(x)=H\cosh\frac{w}{H}\left(\frac{s}{2}-x\right).$$ 2 is the horizontal

H , constant tension along the cable since the external force distributed - the weight - is vertical. is calculated

H by [éq An2-2] written in: is thus P

$$\cosh\frac{ws}{2\mathrm{H}}-\frac{2\mathrm{N}_p}{ws}\frac{ws}{2\mathrm{H}}=0.$$

$\frac{ws}{2\mathrm{H}}$ root of the transcendent equation: This equation

$$\cosh X=\frac{2\mathrm{N}_p}{ws}X.$$

has two roots [An2-b Figure] if: with: and

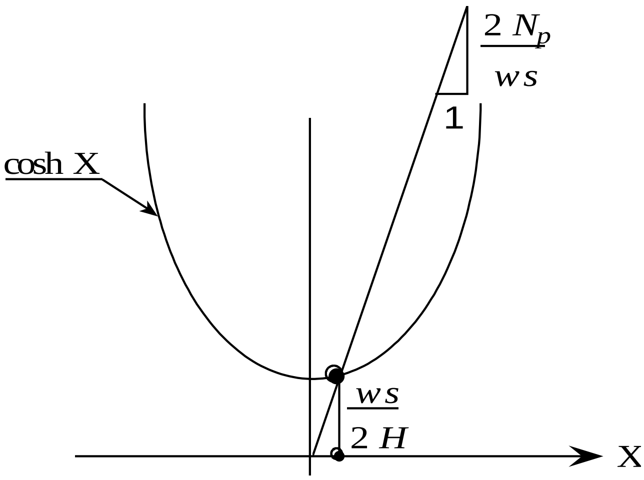
$$\frac{2\mathrm{N}_p}{ws} > p_0,$$

: Appear

$$p_0 = \sinh x_0$$

Year

$$x_0 = \text{cotanh}\, x_0\, x_0 > 0\,.$$



2-b: Computation of the $\frac{ws}{2\mathrm{H}}$

smallest root, which corresponds to the greatest tension of the cable, is only useful. The other root corresponds to a sag of the considerable cable, of about size of its range. being calculated

$\frac{ws}{2\mathrm{H}}$, the deflection results from [éq An2-1]: .

$$S = \frac{s}{2}\frac{2\mathrm{H}}{ws}\left(\cosh\frac{ws}{2\mathrm{H}} - 1\right)$$