A finite element of cable-pulley

Summarized:

The pulleys play an important role in structures of cables such as the air electric lines. It is thus useful to know to model them in one at the same time realistic and powerful way. One presents the finite element formulation of a length of cable passing by a pulley: statements of the internal forces and the stiffness matrix. The pulley can be fixed or supported by a flexible structure. Its position on the cable with the equilibrium is not known a priori: this position is that for which on both sides the tension of the cable is the same one.
Contents

1 Notations ................................................................. 3
2 Introduction ............................................................... 4
3 Assumptions and definition of a finite element of cable-poulie ........................................ 5
4 Internal forces of a finite element of cable-poulie ................................................................. 7
5 Stamp rigidité ............................................................ 8
6 Matrix of masse ......................................................... 9
7 Introduction of the element of cable-pulley into Code_Aster .................................................. 9
8 Example of application ................................................ 10
9 Conclusion ................................................................. 11
10 Bibliographie ........................................................... 11
11 Description of the versions of the document .......................................................... 11
Appendix 1 Computation of the matrix of rigidité ............................................................... 12
Appendix 2 Appears of equilibrium of an inextensible cable weighing subjected to a tension given in extrémité ............................................................... 13
1 Notations

\( A \) \hspace{1cm} \text{area of the cross-section of the cable.}

\( E \) \hspace{1cm} \text{Young modulus.}

\( F \) \hspace{1cm} \text{vector of the internal forces of the element.}

\( F_n \) \hspace{1cm} \text{internal force with the node} \ n \ .

\( H \) \hspace{1cm} \text{horizontal component of the tension [§An1].}

\( I_3 \) \hspace{1cm} \text{stamp unit of order 3.}

\( K \) \hspace{1cm} \text{stiffness matrix of the element.}

\( l \) \hspace{1cm} \text{current length of the element.}

\( l_0 \) \hspace{1cm} \text{length at rest.}

\( l_i \) \hspace{1cm} \text{initial length.}

\( l_1 \) \hspace{1cm} \frac{N_3 N_1}{l_1} \text{euclidian norm of} \ l_1 \text{.}

\( l_2 \) \hspace{1cm} \frac{N_3 N_2}{l_2} \text{euclidian norm of} \ l_2 \text{.}

\( N \) \hspace{1cm} \text{current tension of the cable constituting the element.}

\( N_i \) \hspace{1cm} \text{initial tension (prevoltage).}

\( S \) \hspace{1cm} \text{marks with arrows of a range of cable [§An1].}

\( s \) \hspace{1cm} \text{length of a range [§An1].}

\( T \) \hspace{1cm} \text{current temperature.}

\( T_i \) \hspace{1cm} \text{initial temperature.}

\( u \) \hspace{1cm} \text{vector-displacement of the nodes compared to the initial position.}

\( u_n \) \hspace{1cm} \text{displacement of the node} \ n \ \text{compared to its initial position.}

\( w \) \hspace{1cm} \text{weight per unit of length.}

\( x_n \) \hspace{1cm} \text{vector-position of the node} \ n \ \text{in initial configuration.}

\( \alpha \) \hspace{1cm} \text{thermal coefficient of thermal expansion.}

\( \varepsilon \) \hspace{1cm} \text{current strain compared to the initial configuration.}
2 Introduction

One uses pulleys, during the construction of the air electric lines, for the operation of installation of the cables. The cable in the course of installation [Figure 2-a] is fixed at the one of the supports of stop of the canton, it rests on pulleys placed at the bottom of the insulators of the supports of alignment and it is retained by a force on the level of the second support of stop. While exploiting this force - or by moving its point of application - one adjusts the deflection of the one of the ranges, that which is subjected to stresses of environment. Then one removes the pulleys and one fixes the cable at the insulators. The length of the cable in the various ranges is then built-in and it determines the later behavior of line under statical stresses (wind, overload of white frost) and in dynamic mode (motion due to the forces of Laplace created by flows of short-circuit).

Appear 2-a: Pose of a cable in a canton with two ranges

the finite element of cable-pulley presented here allows to model the operation of installation and thus to calculate, in a natural way, the length of cable in the various ranges.

The idea of this finite element came to us some time after the conversation [bib1] and we presented his formulation in [bib2].
Assumptions and definition of a finite element of cable - pulley

Let us take a length of cable $N_1N_2$ passing by the pulley $N_3$ [Figure 3-a]. This pulley is not inevitably fixed and can, for example and as it is the case in the example of [§8], being assembled at the end of a cable.

The pulley is supposed to be specific [Figure 3-b]. One makes moreover following assumptions:

1. The equilibrium position of $N_3$ is not known, but it is necessarily on the section $N_1N_2$ deformed from its initial position.
2. In modelization of the lines, horizontal motions are of low amplitude and this assumption is generally not restrictive;
3. The two bits $N_3N_1$ and $N_3N_2$ are always rectilinear, like cable elements of the 1st order.

Warning: The translation process used on this website is a “Machine Translation”. It may be imprecise and inaccurate in whole or in part and is provided as a convenience.

Licensed under the terms of the GNU FDL (http://www.gnu.org/copyleft/fdl.html)
It rises from this assumption that:

\[ l_1 = x_1 + u_1 - x_3 - u_3 \]  
\[ l_2 = x_2 + u_2 - x_3 - u_3 \]  
\[ l_1 = \sqrt{l_1^1 l_1} \]  
\[ l_2 = \sqrt{l_2^1 l_2} \]  

overall length of the two bits is:

- in the current location, where the temperature is \( T \):
  \[ l = l_1 + l_2 \]  
- in the initial position, indicated by the index \( i \), where the tension is \( N_i \) and the temperature \( T_i \):
  \[ l_i = (l_1)_i + (l_2)_i \]  
- in a not forced position, indicated by index 0, where the temperature is \( T_0 \):
  \[ l_0 = (l_1)_0 + (l_2)_0 \]

The pulley is without friction and thus the tension is the same one in the two bits.

It rises from this assumption that the strain \( \varepsilon \) is also the same one and one takes for value of this one the measurement of the relative lengthening of the section compared to the initial state:

\[ \varepsilon = \frac{l - l_i}{l_0} \]  

\( \varepsilon \) must remain small, so that the section \( A \) is regarded as constant.

It will be noted that, in the frame of the finite element method, the linear loads do not prevent the tension from being constant of \( N_1 \) with \( N_2 \). These forces are indeed concentrated with the nodes \( N_1 \) and \( N_2 \) on the axis of the pulley \( N_3 \).

The behavior model is elastic:

\[ N = E A \left[ e - \alpha \left| T - T_i \right| \right] + N_i \]  

One calls finite element of cable-pulley, a length of satisfactory \( N_1 \) \( N_2 \) \( N_3 \) cable to the preceding assumptions.
4 Internal forces of a finite element of cable-pulley

Let us point out the following definition: one calls internal forces of a structure finite element the forces which it is necessary to exert in its nodes to maintain it in its current deformed configuration.

In the case of a finite element of cable-pulley, the internal forces result immediately from the static. One has indeed [Figure 4-a]:

\[ F_1 = \frac{N}{l_1} l_1 \]

\[ F_2 = \frac{N}{l_2} l_2 \]

and, to ensure the equilibrium:

\[ F_3 = -|F_1 + F_2| . \]

\( F_1 \) and \( F_2 \) having even modulus, \( F_3 \) is directed according to the bisectrix of the angle \( \angle N_1 N_3 N_2 \).

\[ F_3 = -(F_1 + F_2) \]

\[ \|F_1\| = \|F_2\| \]

\( F_3 \) is applied to the axis of the pulley.

The system of forces \( F \) interns of the element is thus:

\[ F = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} . \]
5 Stiffness matrix

the stiffness matrix $K$ of the element is derived from Fréchet of $F$ in the direction of the displacement $\delta u$ of the nodes:

$$\delta F = K \delta u.$$ 

$K$ is calculated by the following classical formula, used intensively in [bib3], [bib4] and [bib5]:

$$\delta F = \lim_{\varepsilon \to 0} \frac{\delta}{\delta \varepsilon} F(u + \varepsilon \delta u) \quad \text{eq 5-1}$$

detail of computations is given in $§\text{An1}$ and the final statement of $K$ is the following one:

$$K = \begin{bmatrix}
K_{11} + \frac{N}{l_1} I_3 & K_{12} & -K_{11} - K_{12} - \frac{N}{l_1} I_3 \\
K_{12}^T & K_{22} + \frac{N}{l_2} I_3 & -K_{22} - K_{12} - \frac{N}{l_2} I_3 \\
-K_{11} - K_{12} - \frac{N}{l_1} I_3 & -K_{22} - K_{12} - \frac{N}{l_2} I_3 & K_{11} + K_{22} + K_{12} + K_{12}^T + \left(\frac{1}{l_1} + \frac{1}{l_2}\right) N I_3
\end{bmatrix} \quad \text{eq 5-2}$$

$N$ is given by [eq 3-4] and [eq 3-3]:

$$K_{11} = \left[\frac{EA}{l_0} \frac{-N}{l_1} \frac{1}{l_1^2} l_1 \right] l_1^T;$$

$$K_{12} = \left[\frac{EA}{l_0 l_1 l_2} l_1 \right] l_2^T;$$

$$K_{22} = \left[\frac{EA}{l_0} \frac{-N}{l_2} \frac{1}{l_2^2} l_2 \right] l_2^T.$$ 

$K$ is symmetric, because of the symmetry of $K_{11}$ and $K_{22}$ total symmetry per blocks.

But $K$ depends on displacements from $N_1$, $N_2$ et $N_3$ the intermediary on $l_1$, $l_2$ et $N$: the finite element of cable-pulley is thus a nonlinear element.
6 Mass matrix

This matrix intervenes obviously only in the dynamic problems. The cable element - pulley is not used in the Code_Aster that for the quasi-static problems of installation of cables [§7].

Note:

One presents nevertheless in [bib2] the example of a dynamic problem comprising a cable-pulley. The mass matrix of the element \( N_1 \ N_2 \ N_3 \) is obtained by assembling the “coherent” mass matrices of the cable elements with two nodes \( N_3 \ N_1 \) and \( N_3 \ N_2 \) [bib6] and by adding the point mass of the pulley.

It should be noted that, during a dynamic analysis, this mass matrix must be updated because the lengths \( l_1 \) and \( l_2 \) vary.

7 Introduction of the element of cable-pulley into the Code_Aster

the element of cable-pulley is supported by a mesh SEG3.

In the command AFFE_MODELE, under factor key word the AFFE, one must define the arguments of the key words as follows:

<table>
<thead>
<tr>
<th>key word</th>
<th>GROUP_MA</th>
<th>PHENOMENE</th>
<th>MODELISATION</th>
</tr>
</thead>
<tbody>
<tr>
<td>argument</td>
<td>&quot;MECHANICAL&quot;</td>
<td>mesh group of cable-pulley</td>
<td>&quot;CABLE_POULIE&quot;</td>
</tr>
</tbody>
</table>

the constitutive material must be elastic.

In the command AFFE_CARA_ELEM, the cable-pulleys are treated like cables.

As the element of cable-pulley is nonlinear [§5] and that, for time, it is used only in static [§6], it is accessible only by the operator STAT_NON_LINE. Under factor key word the COMP_ELAS, the arguments of the key words are the following:

<table>
<thead>
<tr>
<th>key word</th>
<th>GROUP_MA</th>
<th>RELATION</th>
<th>DEFORMATION</th>
</tr>
</thead>
<tbody>
<tr>
<td>argument</td>
<td>mesh group of cable-pulley</td>
<td>&quot;ELAS&quot;</td>
<td>&quot;GROT_GDEP&quot;</td>
</tr>
</tbody>
</table>

Lastly, the force of gravity acting on the nodes \( N_1 \) et \( N_2 \) of an element of cable-pulley [fig 3 - B] is following because it depends on the length of the bits \( N_3 N_1 \) et \( N_3 N_2 \). For a structure comprising at least an element of cable-pulley, one must specify this load (following) in STAT_NON_LINE under the key word factor EXCIT.

One finds an example of application in test SSNL100A [V6.02.100].
Example of application

This example is that of the installation of a cable with two ranges and imposed tension of adjustment.

On [Figure 7-a], $O$ is the anchorage of the cable on the support of stop of left. $P_1$ is a first pulley placed at the bottom of the lifting chain $P_1C$, fixed in $C$ at the support of alignment. $P_2$ is one second pulley placed on the support of stop of right. $O$, $P_1$ and $P_2$ are, to simplify, located on horizontal.

![Diagram of cable installation](image)

Appear 7-a: Balance of a cable with two ranges, subjected to a tension of adjustment given.

\[
OP_1 = P_1P_2 = 100 \text{ m} \quad ; \quad w = 30 \text{ N/m} \quad ; \quad E' = 5 \times 10^7 \text{ N} \quad ; \quad N_p = 5000 \text{ N}
\]

In initial position, the cable at rest, supposed in weightlessness, is right: line in feature of axis of [Figure 7-a]. For the modelization in finite elements, this line is cut out in:

- ten cable elements with two nodes enters $O$ and $Q_1$;
- an element of cable-pulley $Q_1R_1P_1$;
- nine cable elements enters $R_1$ and $Q_2$;
- an element of cable-pulley $Q_2R_2P_2$.

One simultaneously subjects the cable to gravity and the tension of adjustment $N_p$ exerted in $R_2$. The equilibrium position (line in feature full with [Figure 7-a]) is reached in 11 iterations by the operator STAT_NON_LINE of Code_Aster. The deflections are of $7.955 \text{ m}$ and $7.867 \text{ m}$, respectively in the range of left and that of right.

For an inextensible cable of a range of $100 \text{ m}$, weight linear $30 \text{ N/m}$ and subjected to a tension in the end of $5000 \text{ N}$, the theoretical deflection is of $7.941 \text{ m}$ [§An2].

While exploiting the tension of adjustment, one can adjust one of the two deflections to a value built-in advance.
9 Conclusion

the finite element of cable-pulley presented in this note, of a very simple mechanical formulation, has performances comparable to those of an ordinary cable element. It is very convenient and even essential for a realistic modelization of the air electric lines. It should find other applications, in particular in Robotics.

10 Bibliography

1) J.L. LILIEN: Private communication.
6) Mr. AUFAURE: Modelization of the cables in the Code_Aster, in the course of drafting.

11 Description of the versions of the document

<table>
<thead>
<tr>
<th>Version Aster</th>
<th>Author (S) Organization (S)</th>
<th>Description of modifications</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>Mr. AUFAURE (EDF/IMA/MMN)</td>
<td>Initial Text</td>
</tr>
<tr>
<td>10.1</td>
<td>J.M.PROIX (EDF-R&amp;D/AMA)</td>
<td>Modification</td>
</tr>
</tbody>
</table>

Warning: The translation process used on this website is a "Machine Translation". It may be imprecise and inaccurate in whole or in part and is provided as a convenience.

Licensed under the terms of the GNU FDL (http://www.gnu.org/copyleft/fdl.html)
Annexe 1

The stiffness matrix one shows here how applies the formula [éq 5-1] to the computation of the first three lines of the matrix [éq 5-2 \( K \)], those which are relating to the force. The other \( F_i \) lines are obtained either by permutation of the indices, or by summation of two preceding lines. The first lines are thus obtained by calculating derivative: [éq An1-1]

\[
\lim_{\varepsilon \to 0} \frac{d}{de} F_1(u + \varepsilon \delta u)
\]

1 and by putting in factor the vector. According to \( \delta u \) the relations [éq 4-1], [éq 3-4] and [éq 3-3], one a: with, according to

\[
F_1(u + \varepsilon \delta u) = E A \left[ \frac{l_1(u + \varepsilon d u) + l_2(u + \varepsilon \delta u) - l_i}{l_0} - \alpha(T - T_i) \right] + N_i \frac{l_i(u + \varepsilon \delta u)}{l_1(u + \varepsilon \delta u)}
\]

[éq 3-1]: and, according to

\[
l_1(u + \varepsilon \delta u) = x_1 + u_1 + \varepsilon \delta u_1 - x_3 - u_3 - \varepsilon \delta u_3
\]

[éq 3-2]: Consequently

\[
l_1(u + \varepsilon \delta u) = \frac{1}{l_1(u)} \left[ l_1(u + \varepsilon \delta u) \right] \delta u_1 - \delta u_3
\]

: and by permutation

\[
\lim_{\varepsilon \to 0} \frac{d}{d \varepsilon} l_1(u + \varepsilon \delta u) = \delta u_1 - \delta u_3
\]

\[
\lim_{\varepsilon \to 0} \frac{d}{d \varepsilon} l_1(u + \varepsilon \delta u) = \frac{1}{l_1(u)} l_1^{T}(u) (\delta u_1 - \delta u_3)
\]

of indices 1 and 2: Finally

\[
\lim_{\varepsilon \to 0} \frac{d}{d \varepsilon} l_2(u + \varepsilon \delta u) = \frac{1}{l_2(u)} l_2^{T}(u) (\delta u_2 - \delta u_3)
\]

: While carrying

\[
\lim_{\varepsilon \to 0} \frac{d}{d \varepsilon} l_1(u + \varepsilon \delta u) = \frac{1}{l_1(u)} l_1^{T}(u) (\delta u_1 - \delta u_3)
\]

the preceding statements in [éq An1-1] and by putting in factor the vector, one easily \( \delta u \) obtains the first three lines of. Appear \( K \) of
Annexe 2

**equilibrium of an inextensible cable weighing subjected to a tension given in the end**

Take a cable [An2-a Figure] whose end is fixed at the point and whose other end is subjected to the tension. and are $N_P$, $O$ on $P$ same horizontal and distant ones of. The linear weight is. $w$ The deflection is sought. Appear $S$ Year

\[ z(x) = \frac{H}{w} \left[ \cosh \left( \frac{w}{H} \left( \frac{S}{2} - x \right) \right) \right]; \]

1 tension

\[ N(x) = H \cosh \left( \frac{w}{H} \left( \frac{S}{2} - x \right) \right). \]

2 is the horizontal $H$, constant tension along the cable since the external force distributed - the weight - is vertical. is calculated

$H$ by [éq An2-2] written in: is thus $P$

\[ \cosh \left( \frac{wS}{2H} \right) - \frac{2N_P}{wS} \frac{wS}{2H} = 0. \]

$\frac{wS}{2H}$ root of the transcendent equation: This equation

\[ \cosh X = \frac{2N_p}{wS} X. \]
has two roots [An2-b Figure] if: with: and

\[
\frac{2N_p}{ws} > p_0,
\]

: Appear

\[
p_0 = \sinh x_0
\]

Year

\[
x_0 = \coth x_0 > 0.
\]

2-b: Computation of the \( \frac{ws}{2H} \)

smallest root, which corresponds to the greatest tension of the cable, is only useful. The other root corresponds to a sag of the considerable cable, of about size of its range, being calculated

\[
\frac{ws}{2H}, \text{ the deflection results from [éq An2-1]: .}
\]

\[
S = \frac{s}{2 \frac{2H}{ws}} \left( \cosh \frac{ws}{2H} - 1 \right)
\]