Formulation of a model of thermal for the thin shells

Abstract:

The model presented here is resulting from the asymptotic analysis of the equations of the thermal when the thickness of structure tends towards zero.

The temperature is described by 3 fields defined on the mean surface of the shell.

One shows on some examples, the capacities of the model by reference to solutions 3D.

The applications concerned are thermomechanical computations of shells, the thermal restitution of wall for the thermohydraulics of the pipework, the problems of identification.

Contents

1 Introduction

the mechanical models of thin structures (shells and plates) arrived at an extremely at least advanced stage of development for homogeneous elastic structures in the thickness. The problem is known since strong a long time and various theories came out, generally dedicated to specific problems (thick shells, buckling etc…). However a basic model, that of LOVE-KIRCHHOFF, win unanimous support in the most current applications. The difficulties lie rather in the numerical computation of this one of the fact, on the one hand, of the need for approaching the surface of the shell correctly (in particular its curvature), and on the other hand of the high order of the partial derivative equations which should be solved (4th order).

In thermal on the other hand, the situation is much less clear and a large number of approaches coexist. It is indeed only recently that the problem arose with the possibilities (and the need) for thermomechanical computations. The first models parallel to neglect conduction mean surface to retain only the heat transfers in the thickness of the shell, this approach is completely paradoxical of that of thin structures where, on the contrary, the low thickness of structure leads to simplifying assumptions on the variation in the thickness of the physical fields of variables.

The most recent modelizations take as a starting point the mechanical ideas of thin shells being attached to the second approach, one can classify them according to a completely similar order.

1) Models utilizing a more or less thorough polynomial development of the temperature in the thickness [bib2], [bib9], [bib10]. It is primarily about formulation of finite elements.
2) Models associated with the theories with surfaces to directors (Surfaces of COSSERAT) [bib5], [bib8]. The director is here the gradient of the temperature in the thickness. The problem of these approaches lies in the constitutive law to introduce. Coherence with the three-dimensional model leads to choices which are interpreted like an assumption of linear distribution of the temperature in the thickness. This formalism thus joined practically the preceding models (the introduction of several directors being identified to various orders of development of the polynomials).
3) Models of degenerated finite elements [bib11]: on the basis of a three-dimensional finite element, the introduction of stresses between the degrees of freedom located on the same norm at mean surface makes it possible by condensation to deduce an element from “thermal shell”. Practically, there still, the basic element using a parabolic interpolation according to the thickness, the shell element corresponds to a linear distribution in the thickness.

Parallel to these approaches numerical (1) and (3) or based on assumptions a priori (2), results on the shape of the field of temperature of a thin plate and problem of which it is solution were obtained by asymptotic methods [bib3], [bib1].

As for the model mechanical, those make it possible to justify the assumptions made a priori in the mean theories of shells, to even obtain the equations of the problem of shell. The results of [feeding-bottle 1] are pointed out low and will be used as a basis for the model suggested. Let us note simply here that the idea subjacent with any approach of the asymptotic type is to introduce a small parameter ε (here the thickness ratio of the plate on a dimension characteristic of this one), then having obtained the limiting problem when ε tends towards zero starting from the three-dimensional problem, to approach in the applications (where ε takes obviously a non-zero value) the solution by its limit.

From a practical point of view, the limit obtained for the equations of the steady thermal seems to be too much “poor” to be of a real interest, (one will give of it an illustration in [§2.2.2]). More precisely the values ε to reach to identify the solution with its limit are very small in the real situations met.

This is why one proposes in this note to keep **the form of the limiting solution** (parabolic distribution in the thickness) **but to lay out about it like assumption a priori on the three-dimensional solution** allowing to bring back itself to a problem posed on mean surface.

One thus has **an approximate thin structure model converging towards the model limits three-dimensional equations**. In this meaning, it is “optimal” since a linear assumption of distribution in the thickness leads to a model **not converging towards the limiting solution** and that a model based on a richer development in the thickness sees its terms of order higher than two **converging towards zero** when the shell is thin.

The plane of the note is the following:

- one starts by pointing out the equations of the steady thermal problem for three-dimensional solid and their statements in a coordinate system adapted to the cases where the solid is a “thin shell”,
- then, having pointed out to the results of an asymptotic study of these equations carried out in the case of a plate, one gives the complete description of the model suggested,
- one applies then the model to a certain number of geometries and of thermal loadings and a comparison is made compared to analytical solutions or three-dimensional numerical computations,
- finally, one gives some indications on the numerical aspects of the use of the model in a computation by surface and linear finite elements.

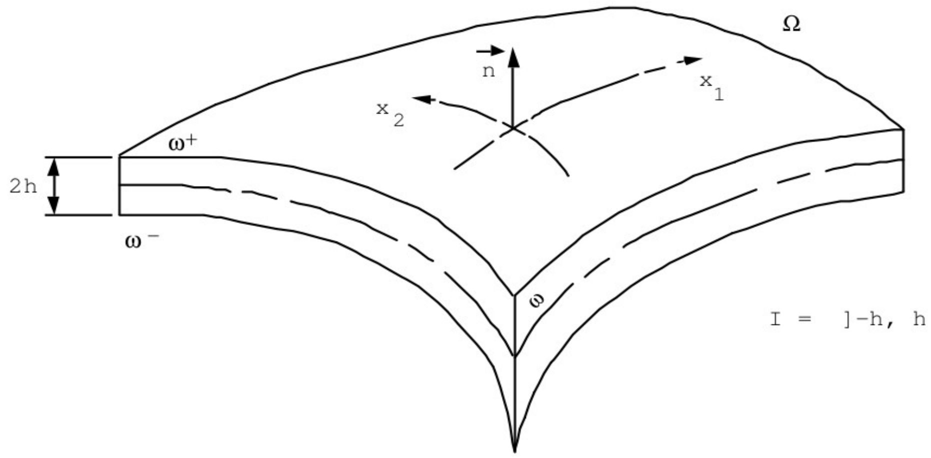
2 Presentation of the model

2.1 Position of the thermal problem in the shells

In this paragraph, we first of all will point out the description of the geometry of the shells, seen like thin three-dimensional solids. One will pose then the problem of thermal conduction.

2.1.1 Description of the geometry

a shell is defined as being a solid Ω , thin perpendicular to a mean surface ω . $2h$ The thickness of the shell is noted; one chooses a coordinate system $(x_{1,}x_2)$ on surface ω . One notes g the associated metric tensor, $\vec{n}$ the normal vector, c the tensor curvature of ω .



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shell Ω is described by the coordinate system (x_γ, x_3) , x_3 according to $\vec{n}$: $\Omega = \omega \times]-h, h[$

(The Greek indices α, β, γ are dedicated to the surface coordinates on ω).

This description is appropriate of course for a shell of thickness $2h$ lower than the smallest radius of curvature of ω .

In an unspecified point (x_γ, x_3) of the shell Ω , the metric tensor G is expressed according to the fundamental tensors g and c of mean surface ω by:

$$\begin{cases} G_{\alpha\beta}(x_\gamma, x_3) = g_{\alpha,\beta}(x_\gamma) - 2x_3 c_{\alpha\beta}(X_\gamma) \\ G_{\alpha 3} = 0 \quad , \quad G_{33} = 1 \end{cases}$$ **éq 2.1.1-1**

and

$$\begin{aligned} \sqrt{\det G} &= \sqrt{\det G}(1 - x_3 \operatorname{tr} \ c) \\ &= \sqrt{\det G}\left(1 + x_3\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\right) \end{aligned}$$

where $R_{1,}R_2$ are the principal radii of curvature of ω at the point x_γ considered.

Note:

It is known indeed that the trace (tr) *of a tensor is an invariant (by basic change). One has the habit however to write the quantities in orthonormal physical base i.e. As follows: .*
$G_{\alpha\beta}^{phy}(x_\gamma, x_3)=\delta_{\alpha\beta}-2x_3 C_{\alpha\beta}^{phy}(x_\gamma)$

And if the base is principal of curvature: $C_{\alpha\beta}^{phy}=\frac{-1}{R_\beta}\delta_{\alpha\beta}$ *(without summation).*

One will note: $\frac{1}{R_1}+\frac{1}{R_2}=H_1$ and $\frac{1}{R_1}-\frac{1}{R_2}=H_2$.

One limited oneself here under the first order in x_3 $\mathrm{tr}\, C$; as it subsequently will be done. In practice, indeed, the thinness of the shell allows such a simplification. There will be advantage also to place oneself in a principal reference of curvature, orthonormal. The tensor g is then the identity, C is diagonal. It is what one will do henceforth.

2.1.2 Equation of heat

the equations of three-dimensional thermal conduction are written (for a rigid driver):

$$-\mathrm{div}(\mathbf{K}\,\overrightarrow{\mathrm{grad}}\,T)+\rho C_t T=r \quad \text{éq 2.1.2-1}$$

where $\mathbf{K}$ the tensor of conductivity indicates, ρC heat capacity and the r possible sources.

There is advantage to write the statement of the differential operator according to the metric one G generated by mean surface ω . One will indeed consider transverse isotropic tensors $\mathbf{K}$ of conductivity according to these axes of coordinates (multi-layer cf materials).

$$\mathbf{K}_j^i=\begin{pmatrix} k_\beta^\alpha & 0 \\ 0 & K \end{pmatrix}, k_\beta^\alpha \text{ et } K \text{ pouvant varier avec } x_{1,} x_{2,} x_3$$

The statement of the operator:

$$-\mathrm{div}(\mathbf{K}\,\overrightarrow{\mathrm{grad}}\,T)=-(\det G)^{-1/2}.\partial_i\left(\det G^{1/2}\mathbf{K}_j^i G^{ij}\partial_j T\right)$$

is written then with the first order in $x_3/\mathrm{tr}\, c$, for an orthotropic conductivity according to the principal directions of curvature:

$$-(1-x_3 H_1).\left[\partial_1\left[(1-x_3 H_2)k_{11}\partial_1 T\right]+\partial_2\left[(1+x_3 H_2)k_{22}\partial_2 T\right]+H_1 K\partial_3 T\right]-\partial_3\left[K\partial_3 T\right] \quad \text{éq 2.1.2-2}$$

If the curvatures are constant, this becomes:

$$-(1-2x_3/R_1).(\partial_1(k_{11}\partial_1 T))-(1-2x_3/R_2).(\partial_2(k_{22}\partial_2 T))-H_1(1-x_3 H_1)K\partial_3 T-\partial_3(K\partial_3 T)$$

The effect of the curvature is thus in the same way standard than a modified distribution of conductivity in the thickness.

2.1.3 Thermal for a thin structure

the equations of the steady thermal on the shell can be written in the form of a problem of minimization.

It is supposed in particular that the boundary conditions on the ends $\partial\omega\times I$ of the shell Ω are in the same way standard on all the thickness I. One partitionne $\partial\omega\times I$ in:

$\partial\omega_T\times I$ (zone with imposed temperature),

and $\partial\omega_\varphi\times I$ (zone on condition that exchange or imposed flux).

$$\begin{cases} \text{Trouver le champ de température T :} \\ T=\underset{\theta\in V}{\text{Arg Min}}\, J(\theta), \text{avec } J(\theta)=\frac{1}{2}A(\theta,\theta)-F(\theta), \text{avec :} \\ A(T,\theta)=\int_\Omega \mathbf{K}.\Delta T.\Delta\theta \, d\Omega+\int_{\omega^+\cup\omega^-}\lambda T.\theta \, d\omega^\pm+\int_{\partial\omega_\varphi\times I}\lambda T.\theta \, dS \\ F(\theta)=\int_{\omega^+\cup\omega^-}\varphi.\theta \, d\omega^\pm+\int_{\partial\omega_\varphi\times I}\varphi.\theta \, dS \end{cases} \quad \textbf{éq 2.1.3-1}$$

One notes:

- $V=\left\{\theta\in H^1(\omega\times I), \theta=0 \text{ sur } \partial\omega_T\times I\right\}$.
- The boundary conditions on $\omega^+\cup\omega^-\cup(\partial\omega_\varphi\times I)$ are of the type exchanges or imposed flux φ :

$\vec{\boldsymbol{\Phi}}\vec{n}=\lambda T-\varphi=-(\mathbf{K}\Delta T).\vec{n}$ λ being a coefficient of heat exchange.

The term of conductivity in $A(T,\theta)$ is written:

$$\int_\Omega \mathbf{K}\Delta T.\Delta\theta \, d\Omega$$
$$=\int_\omega\int_I\left[k_{\alpha\beta}\left(1-x_3\left(\frac{1}{R_\alpha}+\frac{1}{R_\beta}\right)\right)\partial_\alpha T\,\partial_\beta\theta+K\,\partial_3 T\,\partial_3\theta\right](1+x_3 H_1)\,dx_1\,dx_2\,dx_3$$

this, in an orthonormal principal reference of curvature of ω ($k_{\alpha\beta}$ and K are then the physical components of the tensor of conduction $\mathbf{K}$).

The terms of exchange on surfaces ω^+ and ω^- are:

$$\int_{\omega^+\cup\omega^-}\varphi.\theta \, d\omega^\pm=\int_\omega \lambda^\pm T^\pm\theta^\pm.(1\pm h.H_1)\,dx_1\,dx_2$$

The object of a model of thermal shell is thus to bring back from three to two variables of space the dependence of the field of temperature T in the statement of the differential operator corresponding to [eq 2.1.2-2] or [eq 2.1.3-1], realising the choice and the justification of suitable assumptions.

The model proposed in [§2.3] rests on the results of the asymptotic development of the equations of the thermal presented in [§2.2] hereafter.

2.2 Recall of the results resulting from the asymptotic development

2.2.1 The model limits obtained

One summarizes here the principal results got in [bib1] by an asymptotic technique of development. The case of a plate is considered: $\omega \times I_\varepsilon$, of thickness $2\varepsilon h$. The temperature is fixed at 0 on edge $\partial\omega \times I_\varepsilon$, and of flux φ^+, φ^- on the sides ω^+ and ω^-.

One seeks to study the dependence of the solution T^ε of the thermal problem [éq 2.1.3-1] screw - with - screw of the thickness of the plate $2\varepsilon h$. One uses for that a technique of change of open which brings back the problem to a fixed field $\omega \times I$, with $i =]-h, +h[$. The parameter ε appears then explicitly in the equations of the transported problem (P^ε), of $\omega \times I_\varepsilon$ with $\omega \times I$.

On $\omega \times I_\varepsilon$, the initial problem is written in variational form:

$$\begin{cases} \text{Trouver}: T^\varepsilon \in V = \{\theta \in H^1(\omega \times I_\varepsilon), \theta = 0 \text{ sur } \partial\omega \times I_\varepsilon\} \\ \text{tel que}: \\ \int_{\omega \times I_\varepsilon} (k_{\alpha\,\text{bêta}} . T_{,\alpha} . \theta_{,\beta} + k_{33} T_{,3} . \theta_{,3}) = \int_\omega (\varphi^+ \theta^+ + \varphi - \theta^-) \qquad \forall \theta \quad \mathbf{V}^\varepsilon \end{cases}$$ **éq the 2.2.1-1**

results of the asymptotic development [feeding-bottle 1] consist of the following properties checked by $T(\varepsilon)$, the solution of the transported problem (P^ε), posed on $\omega \times I$:

$$\begin{cases} (i) & \frac{1}{\varepsilon} T(\varepsilon) \text{ tend vers } T_1(x_\alpha; x_3) = T_1(x_\alpha) \text{ dans } H^1(\omega \times I) \\ & T_1(x_\alpha) \text{ apparaît comme une température moyenne sur l'épaisseur } I \text{, au point } x_\alpha \\ (ii) & \varepsilon T_{,3}(\varepsilon) \text{, qui est la dérivée de } \varepsilon T \text{ selon la variable d'épaisseur } x_3 \in I \text{, tend vers la dérivée selon } x_3 \\ & \text{du champ } \rho(x_\alpha; x_3) \text{ dans } L^2(\omega) \times H^1_m(I) \text{, où } H^1_m(I) \text{ désigne l'espace des fonctions de} \\ & H^1(I) \text{ à moyenne nulle.} \end{cases}$$

éq 2.2.1-2

In conclusion, the solution T^ε of the initial problem on $\omega \times I_\varepsilon$ can be represented by the first two terms of its development:

$$T^\varepsilon(x_\alpha, x_3^\varepsilon) = \frac{1}{\varepsilon} T_1(x_\alpha) + \varepsilon \rho(x_\alpha, x_3 = x_3^\varepsilon / \varepsilon) + ...$$ **éq 2.2.1-3**

However the gradient of T^ε is not represented by the gradient of the representation of T^ε. This situation is generic problems of singular disturbances encountered in the study of the thin structures (plates, beams…):

$$(\nabla T^\varepsilon)^\varepsilon (x_\alpha, x_3^\varepsilon) = \frac{1}{\varepsilon} T_1(x_\alpha)_{,\beta} \vec{e}_\beta + \rho(x_\alpha, x_3 = x_3^\varepsilon / \varepsilon)_{,3} \vec{e}_3$$ **éq 2.2.1-4**

the field of the “gradient of T^ε ” is thus not a field of gradient!

The fields T_1 and ρ are calculated on mean surface ω. If conductivity is homogeneous in the thickness, it a:

$$\begin{cases} T_1 \in H_0^1(\omega), \text{solution de :} \\ \int_\omega h\, k_{\alpha\beta} T_{1,\alpha} . \theta_{,\beta} = \int_\omega \frac{1}{2}(\varphi^+ + \varphi^-)\theta, \qquad \forall \theta \in H_0^1(\omega) \end{cases}$$ **éq 2.2.1-5**

$$\rho(x_\alpha, x_3) = \frac{\varphi^+(x_\alpha) + \varphi^-(x_\alpha)}{4K(x_\alpha)} . \left(\frac{x_3^2}{h} - \frac{h}{3} \right) + \frac{\varphi^+(x_\alpha) - \varphi^-(x_\alpha)}{2K(x_\alpha)} . x_3$$ **éq 2.2.1-6**

One is noted that T_1 is the solution of a problem posed on ω, whereas ρ is obtained explicitly according to imposed flux. These two equations constitute the model "limits" obtained by the asymptotic development.

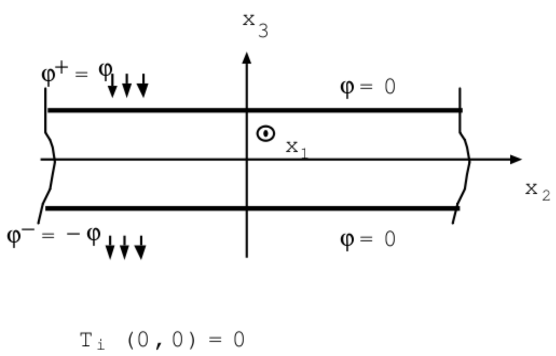
Note:

- *In a language more coloured and vaguer, the preceding results are interpreted by saying that for a thin plate, the average temperature is governed by received average flux and conduction in the plan of the plate. The distribution in the thickness is not function, in a given point, that flux imposed in this point on the sides higher and lower, it are not affected by the presence of the close points.*
- *The distribution of temperature in the thickness is "parabolic" according to the representation [éq 2.2.1-6].*

2.2.2 An application

One can illustrate the results of the asymptotic development for a simple example, which shows also the limitations of the model obtained by means of the representation of the temperature [éq 2.2.1-3] using the fields T_1 and ρ, [éq 2.2.1-5] and [éq 2.2.1-6].

One considers an infinite plate subjected on his half $x_2 < 0$ to a balanced constant flux couple $(\varphi^+ = \varphi, \varphi^- = -\varphi)$, and isolated on other half $x_2 > 0$.



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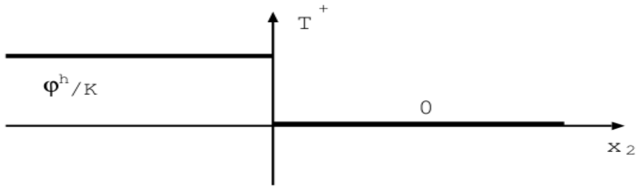
problem [eq 2.2.1-5] of determination of the average temperature T_1 is a differential equation here in x_2: $\frac{\partial^2 x_2}{\partial x_2^2} T_1(x_2) = 0$ since the average flux $j^+ + j^-$ is null. The solution is then $T_1 = 0$ everywhere.

The field $\rho(x_{2,} x_3)$ is calculated easily by [éq 2.2.1-6]: `

$$\rho(x_{2,} x_3)=\frac{\varphi}{K}.x_3 \quad \text{for } x_2<0 ,$$

$$\rho(x_{2,} x_3)=0 \quad \text{for } x_2>0 .$$

The discontinuity of the boundary condition of NEUMANN on $\omega^{\pm}$ thus refers directly on the field of temperature: opposite the higher temperature T is the following one:



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This discontinuity appears of more independent of the thickness h in this limiting model, once the flux φ brought standardized par. h

This limitation of the limiting model obtained by asymptotic development is inherent in the purely local determination of the parabolic complementary term $\rho(x_{\alpha}, x_3)$. Induced discontinuities will be awkward for the applications, in particular in thermomechanics.

One is thus brought differently the model to formulate thermal of shell, while keeping the results of this asymptotic development.

2.3 Formulation of the steady model of thermal of shell

One saw on the results of the asymptotic study of the three-dimensional equations on solid $\Omega=\omega\times I$, that the model limits obtained comprised an average temperature solution of a problem of the 2nd order posed on ω , and that the additional parabolic term was given only locally (point by point on ω). This thus had the disadvantage of providing discontinuous solutions when the thermal “loadings” are it.

One thus presents in this paragraph a representation of the temperature, always parabolic in the thickness, but avoiding the preceding pitfall. One describes the equations obtained, and their properties.

2.3.1 Equations of the model

Following the results of the asymptotic development, one **chooses** the representation in the following thickness on $\Omega=\omega\times I$:

$$T(x_\alpha,x_3)=T_1(x_\alpha)+T_2(x_\alpha).w_2(x_3)+T_3(x_\alpha).w_3(x_3)$$ **éq 2.3.1-1**

with $(w_1=1,w_{2,}w_3)$ a given base of the polynomials of degree 2.

One thus replaces the determination of the field T with three variables of space by that of three scalar fields T_1 T_2 , T_3 with 2 surface variables on ω . This decomposition [éq 2.3.1-1] is practical to show its restrain with asymptotic the model. But one will use another representation for numerical the model: to see [§ 2.3.5].

One will inject this representation of the temperature $T(x_\alpha,x_3)$ directly in the thermal problem [éq 2.1.3-1] posed on $\Omega=\omega\times I$.

From the definition of space $\mathbf{V}$ in [éq 2.1.3 - 1], one adopts for the fields T_i :

$$\mathbf{W}=\left\{\mathbf{V}=(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2,\boldsymbol{\theta}_3)\in H^1(\omega)^3,\boldsymbol{\theta}_i=0 \text{ sur } \partial\omega_T\right\}$$

By posing $\boldsymbol{T}=(T_{1,}T_{2,}T_3)$ the formulation of the thermal problem on Ω becomes:

$$\begin{cases}\text{Trouver } \mathbf{T}\in W\\ \boldsymbol{T}=\underset{V\in\mathbf{W}}{\text{Argmin}}\, J(\boldsymbol{\theta}), \text{avec } J(\boldsymbol{\theta})=\frac{1}{2}A(\theta,\theta)-F(\theta)\text{et}\\ A(\boldsymbol{T},\boldsymbol{\theta})=\int_\omega\left({}^t\nabla\boldsymbol{T}.\mathbf{A}.\nabla\boldsymbol{\theta}+{}^t\mathbf{T}.\boldsymbol{B}.\boldsymbol{\theta}\right)d\omega\\ F(\boldsymbol{\theta})=\int_\omega {}^t\mathbf{C}.\boldsymbol{\theta}\,d\,\omega+\int_{\partial\omega_\varphi}{}^t\boldsymbol{D}.\theta\,d\,s\end{cases}$$ **éq 2.3.1-2**

Indeed, from [éq 2.3.1-1] one deduces the statements:

$$\begin{cases} (\nabla T)^{\alpha}=\nabla \mathbf{T}_{\alpha}.\begin{pmatrix}1\\ w_2\\ w_3\end{pmatrix} \\ \partial_3 T=T.\begin{pmatrix}0\\ w'_2\\ w'_3\end{pmatrix}\end{cases}$$

The tensor $\boldsymbol{A}$ of order 4 corresponds to surface average conductivities:

$$\boldsymbol{A}_{\alpha\beta i j}(x_{\gamma})=\int_I k_{\alpha\beta} w_i . w_j .\left(1-x_3.\left(\frac{1}{R_{\alpha}}+\frac{1}{R_{\beta}}\right)\right)(1+x_3 H_1)\,d x_3 \qquad \textbf{éq 2.3.1-3}$$

(by means of metric sight in [éq 2.1.1-1]).

The dependence of $\boldsymbol{A}$ following (x_{γ}) comes from that of $k_{\alpha\beta}$ and that of the average curvature H_1 of surface ω .

The tensor $\boldsymbol{B}$ of order 2 described transverse conduction as well as the exchanges on the sides ω^+ and ω^- :

$$\begin{aligned}\boldsymbol{B}_{i j}(x_{\gamma})=\int_I K. w'_i . w'_j (1+x_3 H_1)\,d x_3+\lambda^+ w_i(h). w_j(h)(1+h H_1)\\ +\lambda^- w_i(-h). w_j(-h)(1-h H_1)\end{aligned} \qquad \textbf{éq 2.3.1-4}$$

With regard to the second member F , the vector $\boldsymbol{C}$ is:

$$\boldsymbol{C}(x_{\gamma})=\varphi_+\begin{pmatrix}1\\ w_2(h)\\ w_3(h)\end{pmatrix}(1+h H_1)+\varphi_-\begin{pmatrix}1\\ w_2(-h)\\ w_3(-h)\end{pmatrix}(1-h H_1) \qquad \textbf{éq 2.3.1-5}$$

(One supposes the absence of heat sources in the thickness to simplify.)

Finally:

$$\boldsymbol{D}(x_{\gamma})=\int_I \varphi\begin{pmatrix}1\\ w_2(x_3)\\ w_3(x_3)\end{pmatrix}.(1+x_3 H_1)\,d x_3,\ \text{pour}\ \ x_{\gamma}\in\partial\omega_{\varphi} \qquad \textbf{éq 2.3.1-6}$$

With the examination of the formulation [éq 2.3.1-2] obtained for the thermal of shell, one notes that the differential operator remains of order 2 , contrary with the mechanics where this one passes to 4 . In thermal the curvature of mean surface intervenes only in one modification of metric, and not directly in the operators, as a heterogeneity of conductivities in the thickness would do it.

2.3.2 Case of a homogeneous plate

If a plate is considered, or if one neglects the variation of metric in the thickness of the shell ($1 \gg h\,H_1$) and by supposing the homogeneous material in the thickness to simplify, one can propose the choice of a base $(1, w_2, w_3)$ of the polynomials of degree 2 (polynomials of Legendre), so that the tensors of conduction $\boldsymbol{A}$ and $\boldsymbol{B}$ are diagonalisent on the indices i, j (in U_j, V_j):

$$w_2(x_3)=x_3/h;\quad w_3(x_3)=\frac{3}{2}\left(\frac{x_3^2}{h^2}-\frac{1}{3}\right) \qquad \textbf{éq 2.3.2-1}$$

is: $w_i(h)=1\ \forall\, i; w_2(-h)=-1=-w_3(-h)$

$$\int_I w_2=0=\int_I w_3=\int_I w_2 . w_3=\int_I w_2^{'} . w_3^{'}$$

$$\text{et}:\int_I w_2^2=\frac{2h}{3};\int_I w_3^2=\frac{2h}{5};\int_I w_2^{'2}=\frac{2}{h};\int_I w_2^{'2}=\frac{6}{h}$$

Thus T_1 will be the average temperature, T_2 will be associated with the gradient in the thickness.

One finds then:

$$\boldsymbol{A}_{\alpha\beta}^{11}=2\,kh\,\delta_{\alpha\beta};\boldsymbol{A}_{\alpha\beta}^{22}=\frac{2}{3}kh\,\delta_{\alpha\beta};\boldsymbol{A}_{\alpha\beta}^{33}=\frac{2}{5}kh\,\delta_{\alpha\beta};\boldsymbol{A}_{\alpha\beta}^{ij}=0\ \text{ si }\ i\neq j$$

Moreover: $$\boldsymbol{B}=\frac{2K}{h}\begin{pmatrix}0&0&0\\0&1&0\\0&0&3\end{pmatrix}+(\lambda^{+}+\lambda^{-})\begin{pmatrix}1&0&1\\0&1&0\\1&0&1\end{pmatrix}+(\lambda^{+}-\lambda^{-})\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}$$

$$\boldsymbol{C}=(\varphi^{+}+\varphi^{-})\begin{pmatrix}1\\0\\1\end{pmatrix}+(\varphi^{+}-\varphi^{-})\begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$\boldsymbol{D}=\left.\begin{pmatrix}\int_I \varphi\\ \int_I \varphi . x_3/h\\ \int_I \varphi\frac{3}{2}(x_3^2/h^2-1/3)\end{pmatrix}\right|_{\text{sur}\,\partial\omega_\varphi}$$

By writing the variational formulation of the problem [éq 2.3.1-2]:

$$\begin{cases}\text{Trouver } U=\begin{pmatrix}T_1\\T_2\\T_3\end{pmatrix}\in H_1(\omega)^3 \text{ tel que } \forall\, V\in H_1(\omega)^3\\ \int_\omega \left({}^t\nabla U.\boldsymbol{A}.\nabla V+{}^tU\boldsymbol{B}.V\right)dx_1\,dx_2+\int_\omega {}^t\boldsymbol{C}\,dx_1 dx_2+\int_{\partial\omega_\varphi}{}^t\boldsymbol{D}.V\,ds\end{cases}$$

one establishes the local equations to solve in ω :

$$\begin{cases} -2\,kh\,\Delta T_1 + (\lambda^+ + \lambda^-)(T_1+T_3) + (\lambda^+ - \lambda^-)(T_2) = \varphi^+ + \varphi^- \\ \frac{-2}{3}\,kh\,\Delta T_2 + 2\frac{K}{h}T_2 + (\lambda^+ + \lambda^-)T_2 + (\lambda^+ - \lambda^-)(T_1+T_3) = \varphi^+ - \varphi^- \\ \frac{-2}{5}\,kh\,\Delta T_3 + \frac{6K}{h}T_3 + (\lambda^+ + \lambda^-)(T_1+T_3) + (\lambda^+ - \lambda^-)T_2 = \varphi^+ + \varphi^- \end{cases}$$ **éq 2.3.2-2**

with the following boundary conditions:

$$T_1, T_2, T_3 \text{ given on } \partial\omega_T$$

$$\begin{cases} T_{1,v} = \frac{1}{4k^2h^2}\int_I \varphi \\ T_{2,v} = \frac{9}{4k^2h^2}\int_I \varphi x_3/h \qquad \text{sur } \partial\omega_\varphi \\ T_{3,v} = \frac{25}{4k^2h^2}\int_I \varphi . \frac{3}{2}(x_3^2/h^2 - 1/3) \end{cases}$$

the equations [éq 2.3.2-2] are thus valid for the thin plates and the shells which one neglects the terms of curvature in the metric one $(1 \gg hH_1)$, and for a homogeneous material in the thickness.

The general solutions $[T_i]$ of [éq 2.3.2-2] comprise the exponential ones of type $e^{-|x|/l^a}$ with damping lengths l^a depending on the values on $\frac{k}{K}$ and $\frac{\lambda^\pm h}{k}$. For example, in the absence of conditions of the type exchanges on the walls $\omega+ \cup \omega-$ $(\lambda^\pm = 0)$, one obtains for the fields T_2 respective T_3 damping lengths and the:

$$l_2^a = h\sqrt{\frac{k}{3K}} \qquad l_3^a = h\sqrt{\frac{k}{15K}}$$

It in practice frequently arrives which to neglect the terms of curvature ($hH_1 \ll 1$) in the operator deteriorates only little the solution; on the other hand, one often may find it beneficial to keep the complete statement in the second member. Indeed this makes it possible to calculate the true quantity of heat brought by flux applied to the sides $\omega^\pm$ (cf example in [§3.1]). In this case, it is necessary to take $\boldsymbol{C}$ in [éq 2.3.1-2] and [éq 2.3.2-2].

$$\boldsymbol{C} = (\varphi^+ + \varphi^-)\begin{pmatrix} 1 \\ hH_1 \\ 1 \end{pmatrix} + (\varphi^+ - \varphi^-)\begin{pmatrix} hH_1 \\ 1 \\ hH_1 \end{pmatrix}$$ **éq 2.3.2-3**

2.3.3 Restrain with the model asymptotic

One can check easily that the model proposed here has well as a limit when the thickness εh tends towards 0 the results of the asymptotic development presented in [éq 2.2.1-5] and [éq 2.2.1-6].

Indeed the thickness h intervenes here explicitly in the coefficients of the differential operator in the local equations [éq 2.3.2-2], which are solved on mean surface ω .

In the case without heat exchange ($\lambda^{+}=\lambda^{-}=0$) considered in the asymptotic study, these equations [éq 2.3.2-2] have the form:

$$\begin{cases} -2k\,\Delta T_1=\dfrac{1}{\varepsilon h}(\varphi^{+}+\varphi^{-}) \\ \dfrac{-2}{3}k\,h^2\varepsilon^2\Delta T_2=\varepsilon h(\varphi^{+}-\varphi^{-}) \\ \dfrac{-2}{5}k\,h^2\varepsilon^2\Delta T_3=\varepsilon h(\varphi^{+}+\varphi^{-}) \end{cases}$$

After a formal asymptotic development of the solution (T_i) according to the thickness ε in these equations, one checks well that:

- εT_1 is solution of the problem [éq 2.2.1-5] giving the principal term of the asymptotic development (cf [§2.2]).
- $\dfrac{1}{\varepsilon}T_2$ and $\dfrac{1}{\varepsilon}T_3$ are:

$$\frac{1}{\varepsilon}T_2=h\frac{(\varphi^{+}-\varphi^{-})}{2K}\ ;\ \frac{1}{\varepsilon}T_3=h\frac{(\varphi^{+}+\varphi^{-})}{6K}$$

what corresponds well to the definition [éq 2.2.1-6] of the complementary field ρ .

The model [éq 2.3.1-1] to three scalar fields $T_{1,} T_{2,} T_3$, parabolic in the thickness, appears optimal to some extent the model with respect to the asymptotic behavior of the equations of the steady thermal in thin structures. The following diagram indicates the overlap of the various possible models, with their behavior when the thickness tends towards zero (deflections $\rightarrow$):

Model at 2 fields (refines)	Models asymptotic limit	Models at 3 fields (parabolic)	Model richer
$\hat{T}_1(x_\alpha)$ → +	$\frac{1}{\varepsilon}\tilde{T}_1(x_\alpha)$ +	← $T_1(x_\alpha)$ +	$T_1(x_\alpha)$ +
$\hat{T}_2(x_\alpha)\frac{x_3}{h}$	$\varepsilon\,\rho\left(x_\alpha,\frac{x_3^\varepsilon}{\varepsilon}\right)$ +	$\begin{cases} T_2(x_\alpha)w_2(x_\alpha) \\ + \\ T_3(x_\alpha)w_3(x_\alpha) \end{cases}$	$\begin{cases} T_2(x_\alpha)w_2(x_\alpha) \\ + \\ T_3(x_\alpha)w_3(x_\alpha) \end{cases}$ +
	$\varepsilon^2 \ldots + \ldots$		$T_i(x_\alpha)w_i(x_\alpha)$ $+\ldots$

One saw the interest of the additional term ρ to describe the evolutions of temperature in the thickness x_3 (whereas $\tilde{T}_1(x_\alpha)$ is constant on the thickness).

However the result preceding one proves that the term $\hat{T}_2$ of the model at 2 fields does not converge towards ρ : one needs at least a representation for 3 fields for that. However, knowing that the mechanical models of shells consider thermal strains closely connected in the thickness, one could have accepted sufficient a thermal model 2 fields. One will see in [§3.3] an example illustrating (for a thickness given) the effect of the parabolic term T_3 on the average temperature T_1 between the various models.

Other authors propose richer models of thermal (cf for example [bib9], [bib10], [bib2], probably interesting for thick shells, but whose terms higher than order 2 become useless for thin structures.

Indeed, as shows it the preceding diagram, the terms of a higher nature only come to correct (when $\varepsilon \neq 0$) statements of which the principal parts are given by T_1 on the one hand, T_2 and T_3 on the other hand. Qualitatively, they thus do not bring anything (contrary to T_2 and T_3), quantitatively their contribution quickly becomes negligible in general in front of the principal parts.

2.3.4 Generalization with the problems of thermal evolution

The model of thermal in the shells presented previously was justified starting from the results of the asymptotic development of the three-dimensional equations of the steady thermal. One does not have however of results on the problem evolution, except the convergence of the term in average temperature $<T>$ (cf [bib3]) (see also remark passed Ci - after in [éq 2.3.4-5]).

One can however give some indications on the resolution of the problem of evolution, in particular in the frame of a modal approach (contrary to a direct integration in time).

The three-dimensional equations are:

$$\begin{aligned}&-\boldsymbol{K}\,\Delta \mathrm{T}+\rho\,\mathrm{C}\,\partial_{\mathrm{i}}\mathrm{T}=\mathrm{r}\ \text{ sur }\ \Omega\\ &\text{avec}:\\ &\mathrm{T}=\mathrm{T}_{\mathrm{d}}\,\text{sur}\,\partial\Omega_{\mathrm{T}}\ \ ,\ \ -\mathrm{k}\,\partial_{\mathrm{n}}\mathrm{T}=\varphi\,\text{sur}\,\partial\,\Omega_{\varphi}\\ &\mathrm{T}(\mathrm{x},\mathrm{t}=0)=\mathrm{T}^{0}(\mathrm{x})\,\text{sur}\,\Omega\end{aligned} \qquad \textbf{éq 2.3.4-1}$$

One notes: $(\mu_q,\hat{\mathrm{T}}_{\mathrm{q}})$ eigenvalues and the eigenvectors of the following problem:

$$\boldsymbol{K}\,\Delta\hat{\mathrm{T}}+\mu\,\rho\,\mathrm{C}\hat{\mathrm{T}}=0\ \text{ sur }\ \Omega\ ;\ \hat{\mathrm{T}}=0\,\text{sur}\,\partial\,\Omega_{\mathrm{T}}\ ,\ \partial_{\mathrm{n}}\hat{\mathrm{T}}=0\,\text{sur}\,\partial\,\Omega_{\varphi} \qquad \textbf{éq 2.3.4-2}$$

the solution (three-dimensional) of [eq 2.3.4 - 1] is then given by:

$$\mathrm{T}(\mathrm{x},\mathrm{t})=\sum_{\mathrm{q}=1}^{\infty}\left[\left(\int_{\Omega}\mathrm{T}^{0}.\hat{\mathrm{T}}_{\mathrm{q}}\right)\mathrm{e}^{-\mu_{\mathrm{q}}\mathrm{t}}+\int_{0}^{\mathrm{t}}\left(\int_{\Omega}\mathrm{r}(\mathrm{s}).\hat{\mathrm{T}}_{\mathrm{q}}+\int_{\partial\,\Omega_{\varphi}}\varphi(\mathrm{s}).\hat{\mathrm{T}}_{\mathrm{q}}\right)\mathrm{e}^{-\mu_{\mathrm{q}}(\mathrm{t}-\mathrm{s})\mathrm{ds}}\right].\hat{\mathrm{T}}_{\mathrm{q}}(\mathrm{x}) \qquad \textbf{éq 2.3.4-3}$$

Them μ_{q}, opposite of relaxation time, is characteristic of the spatial modes of the problem [éq 2.3.4-2]. To solve the equations [éq 2.3.4-1] on a thin shell, one can adopt like in hover the representation [éq 2.3.1-1] for the field of temperature in the shell:

$$\mathrm{T}(\mathrm{x}_{\alpha};\mathrm{x}_{3,}\mathrm{t})=\sum_{\mathrm{i}=1}^{3}\mathrm{T}_{\mathrm{i}}(\mathrm{x}_{\alpha}).\mathrm{f}_{\mathrm{i}}(\mathrm{t}).\mathrm{w}^{\mathrm{i}}(\mathrm{x}_{3})$$

One then obtains the problem of eigen modes, posed on mean surface ω, in variational form:

$$\begin{cases}\text{Trouver }\left([\mu_{\mathrm{q}}],[\hat{\mathrm{T}}]_{\mathrm{q}}\right)\in\boldsymbol{R}_{+}^{3}\text{ tels que, }\forall\,\boldsymbol{\theta}\in\mathrm{H}_{1}(\omega)^{3}:\\ \int_{\omega}{}^{\mathrm{t}}\nabla[\hat{\mathrm{T}}]_{\mathrm{q}}.\mathbf{A}.\nabla\boldsymbol{\theta}+{}^{\mathrm{t}}[\hat{\mathrm{T}}]_{\mathrm{q}}.\left(\mathbf{B}-2\mathrm{h}\rho\,\mathrm{C}.\begin{bmatrix}\mu_{\mathrm{q}}^{1}&0&0\\0&\mu_{\mathrm{q}}^{2}&0\\0&0&\mu_{\mathrm{q}}^{3}\end{bmatrix}.\boldsymbol{\theta}\,\mathrm{d}\omega=0\right)\end{cases} \qquad \textbf{éq 2.3.4-4}$$

Note: the operator ${}^{t}\nabla(.).\boldsymbol{A}.\nabla(.)+(.).\boldsymbol{B}(.)$ is quite elliptic; one recalls that $\boldsymbol{B}$ described transverse conduction (coefficient k) as well as the exchanges on the two walls of the shell, whereas $\boldsymbol{A}$ corresponds to surface conduction (coefficient K). It was supposed here that ρc was homogeneous in the thickness.

For example, if one neglects the effect of curvature in the thickness, in the absence of condition of exchange on the walls ω^+, ω^- , and with a homogeneous material, one obtains the following partial derivative equations, to solve on ω (cf [éq 2.3.2-2]):

$$\begin{cases} \Delta \hat{T}_1 + \mu_1 \dfrac{\rho C}{k} \hat{T}_1 = 0 \\ \Delta \hat{T}_2 + 3 \dfrac{\rho C}{k} \left(\dfrac{-K}{h^2 \rho C} + \mu_2 \right) \hat{T}_2 = 0 \\ \Delta \hat{T}_3 + 5 \dfrac{\rho C}{k} \left(\dfrac{-3K}{h^2 \rho C} + \mu_3 \right) \hat{T}_3 = 0 \qquad \text{avec}\, \mu_i > 0 \end{cases}$$

éq 2.3.4-5

One notes here that the thickness h does not affect the modes of average temperature T_1 . On the other hand, a relative increase in transverse conductivity K/k or a reduction in thickness H causes to decrease times characteristic for the modes of "temperatures" T_2 and T_3 .

The total solution according to this representation thus appears in the form:

$$\begin{aligned} T(x_\alpha; x_3, t) &= \sum_{q=1}^{\infty} \left[\int_\omega 2h\, T_1^0 . \hat{T}_{1q} e^{-\mu_q^2 t} + \int_0^t \left(\int_\omega (\varphi^+(s) + \varphi^-(s)) . \hat{T}_{1q} \right) e^{-\mu_q^1 (t-s)} ds \right] . \hat{T}_{1q}(x_\alpha) \\ &+ \sum_{q=1}^{\infty} \left[\int_\omega \frac{2h}{3} T_2^0 . \hat{T}_{2q} e^{-\mu_q^2 t} + \int_0^t \left(\int_\omega (\varphi^+(s) + \varphi^-(s)) . \hat{T}_{2q} \right) e^{-\mu_q^1 (t-s)} ds \right] . \hat{T}_{2q}(x_\alpha) \frac{x_3}{h} \\ &+ \sum_{q=1}^{\infty} \left[\int_\omega \frac{2h}{5} T_3^0 . \hat{T}_{3q} e^{-\mu_q^3 t} + \int_0^t \left(\int_\omega (\varphi^+(s) + \varphi^-(s)) . \hat{T}_{3q} \right) e^{-\mu_q^3 (t-s)} ds \right] . \hat{T}_{3q}(x_\alpha) \frac{3}{2} \left(\frac{x_3^2}{h^2} - \frac{1}{3} \right) \end{aligned}$$

éq 2.3.4-6

where an initial temperature was considered:

$$T^0(x_\alpha; x_3) = T_i^0(x_\alpha).\, w_i(x_3)$$

and where one supposed the absence of heat sources in the thickness.

By comparing the solution 3D [éq 2.3.4-3] and the model of shell [éq 2.3.4-6], one notes that in this last the transverse modes $\hat{T}_q$ according to x_3 are represented only by the functions $w_i(x_3)$ given; what amounts truncating the series $\hat{T}_q$. But another limitation appears in the product of convolution for relaxation times $\frac{1}{\mu_q}$ characteristic of the transverse modes in [éq 2.3.4-3] which disappear in the model [éq 2.3.4-6] beyond from a parabolic “mode”.

In a diffusion of purely transverse heat (described by $(\hat{T}_{i0}, \mu_{i0})$ in the model [éq 2.3.4-6]), the lowest eigenvalue being $K/h^2 \rho C$ one can hope for a correct solution with the model of shell if relaxation times t_c of the loadings applied are such as:

$$t_c > \frac{\rho C}{K} h^2 \qquad \textbf{éq 2.3.4-7}$$

This inequality can be used as practical limit of application of the model.

2.3.5 Equations of the model with usual variables

the choice of the variables T_1, T_2, T_3 of the representation [éq 2.3.1-1] corresponded to the development of the temperature according to the thickness.

For the applications, it is however more convenient to replace them by the variables: T^m, T^s, T^i :

- T^m indicate the temperature on the mean surface of the shell,
- T^s the temperature on “external” surface $(x_3 = +h)$,
- T_i the temperature on “interior” surface $(x_3 = -h)$.

The representation in the thickness uses the polynomials of LAGRANGE then: P_1, P_2, P_3 :

$$T(x_\alpha; x_3) = T^m(x_\alpha).\, P_1(x_3) + T^s(x_\alpha).\, P_2(x_3) + T^i(x_\alpha).\, P_3(x_3)$$

with:

$$P_1(x_3) = 1 - (x_3/h)^2$$
$$P_2(x_3) = \frac{x_3}{2h}(1 + x_3/h)$$
$$P_3(x_3) = \frac{x_3}{2h}(1 - x_3/h)$$

The formulation of the thermal problem on Ω is similar to [éq 2.3.1-2], but where one considers:

$$\boldsymbol{T}(T^m, T^s, T^i)$$

$$(\nabla T)^{\alpha} = \nabla \boldsymbol{T}_{\alpha} . \begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix}; \quad {}_3T = \boldsymbol{T} . \begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix}$$

The tensor $\boldsymbol{A}$ of order 4 is written then:

$$\boldsymbol{A}_{\alpha\beta i j}(x_{\gamma}) = \int_I k_{\alpha\beta} P_i P_j (1 + x_3 . H_1) \left(1 - x_3 \left(\frac{1}{R_{\alpha}} + \frac{1}{R_{\beta}} \right) \right) d x_3$$

The tensor $\boldsymbol{B}$ of order 2 is:

$$\boldsymbol{B}_{i j}(x_{\gamma}) = \int_I K . P'_i . P'_j (1 + x_3 H_1) d x_3 + \lambda^+ P_i(h) . P_j(h)(1 + h H_1) + \lambda^- P_i(-h) . P_j(-h)(1 - h H_1)$$

For the second member, $\boldsymbol{C}$ becomes:

$$\boldsymbol{C}(x_{\gamma}) = \varphi_+ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} (1 + h H_1) + \varphi_- \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} (1 - h H_1) \quad \textbf{éq 2.3.1-5}$$

And $\boldsymbol{D}$:

$$\boldsymbol{D}(x_{\gamma}) = \int_I \varphi \begin{pmatrix} P_1(x_3) 1 \\ P_2(x_3) \\ P_3(x_3) \end{pmatrix} . (1 + x_3 H_1) d x_3, \text{ pour } x_{\gamma} \in \partial \omega_{\varphi}$$

2.3.5.1 Cases of a homogeneous plate

the various integrals on $I=]-h;h[$ necessary to the computation of $\boldsymbol{A}$ and $\boldsymbol{B}$ are gathered Ci - afterwards:

$$\int_I (P_1)^2 dx_3 = \frac{16h}{15} \quad ; \quad \int_I (P_2)^2 dx_3 = \int_I (P_3)^2 dx_3 = \frac{4h}{15}$$
$$\int_I (P_1 . P_2) dx_3 = \int_I (P_1 . P_3) dx_3 = \frac{2h}{15} \quad ; \quad \int_I (P_2 . P_3) dx_3 = \frac{-h}{15}$$
$$\int_I (P_1^{'})^2 dx_3 = \frac{8}{3h} \quad ; \quad \int_I (P_2^{'})^2 dx_3 = \int_I (P_3^{'})^2 dx_3 = \frac{7}{6h}$$
$$\int_I (P_1^{'} . P_2^{'}) dx_3 = \int_I (P_1^{'} . P_3^{'}) dx_3 = \frac{-4}{3h} \quad ; \quad \int_I (P_2^{'} . P_3^{'}) dx_3 = \frac{1}{6h}$$

One finds then (by neglecting the correction of curvature):

$$\boldsymbol{A}_{\alpha\beta}^{11} = \frac{16hk}{15} \delta_{\alpha\beta} \quad ; \quad \boldsymbol{A}_{\alpha\beta}^{22} = \boldsymbol{A}_{\alpha\beta}^{33} = \frac{4hk}{15} \delta_{\alpha\beta}$$
$$\mathbf{A}_{\alpha\beta}^{12} = \mathbf{A}_{\alpha\beta}^{21} = \mathbf{A}_{\alpha\beta}^{13} = \mathbf{A}_{\alpha\beta}^{31} = \frac{2hk}{15} \delta_{\alpha\beta} \quad ; \quad \mathbf{A}_{\alpha\beta}^{23} = \mathbf{A}_{\alpha\beta}^{32} = \frac{-hk}{15} \delta_{\alpha\beta}$$

Then:

$$\boldsymbol{B} = \frac{K}{6h} \begin{pmatrix} 16 & -8 & -8 \\ -8 & 7 & 1 \\ -8 & 1 & 7 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda^+ & 0 \\ 0 & 0 & \lambda^- \end{pmatrix}$$

$$\boldsymbol{C} = \begin{pmatrix} 0 \\ \varphi^+ \\ \varphi^- \end{pmatrix}$$

$$\boldsymbol{D} = \begin{pmatrix} \int_I \varphi [1-(x_3/h)^2] . dx_3 \\ \int_I \varphi . \frac{x_3}{2h} . (1+x_3/h) dx_3 \\ \int_I \varphi . \left(\frac{-x_3}{2h}\right) (1-x_3/h) dx_3 \end{pmatrix} \text{sur } \omega_\varphi$$

2.3.5.2 Relation between the variables of the two representations

$$\begin{cases} T^m(x_\alpha)=T_1(x_\alpha)-\frac{1}{2}T_3(x_\alpha) \\ T^s(x_\alpha)=T_1(x_\alpha)-T_2(x_\alpha)+T_3(x_\alpha) \\ T^i(x_\alpha)=T_1(x_\alpha)+T_2(x_\alpha)+T_3(x_\alpha) \end{cases}$$

and:

$$\begin{cases} T_1(x_\alpha)=\frac{1}{6}\left[4T^m(x_\alpha)+T^s(x_\alpha)+T^i(x_\alpha)\right] \\ T_2(x_\alpha)=\frac{1}{2}\left[\qquad +T^s(x_\alpha)+T^i(x_\alpha)\right] \\ T_3(x_\alpha)=\frac{1}{3}\left[-2T^m(x_\alpha)+T^s(x_\alpha)+T^i(x_\alpha)\right] \end{cases}$$

2.3.6 Synthesis

the problem to be solved on the shell ω , of thickness $2h$ is written:

To find $\boldsymbol{T}=(T^m,T^s,T^i)\in\mathbf{W}\left\{\boldsymbol{\theta}=(\boldsymbol{\theta}^{\mathbf{m}},\boldsymbol{\theta}^{s},\boldsymbol{\theta}^{i})\in H^1(\omega)^3,\boldsymbol{\theta}^{\mathbf{m}}=\boldsymbol{\theta}^{s}=\boldsymbol{\theta}^{\mathbf{i}}=0\,\mathrm{sur}\,\partial\omega_T\right\}$
such as:
$\int_\omega ({}^t\nabla\boldsymbol{T}.\boldsymbol{A}+\nabla\boldsymbol{\theta}+{}^t\boldsymbol{T}.\boldsymbol{B}.\boldsymbol{\theta}).\,d\omega+\int_\omega {}^t\boldsymbol{C}.\boldsymbol{\theta}\,d\omega+\int_{\partial\omega_\varphi} t\,\boldsymbol{D}.\boldsymbol{\theta}\,ds \quad , \forall\boldsymbol{\theta}\in\mathbf{W}$

with:

$$\begin{cases} \mathbf{A}_{\alpha\beta ij}(x_\gamma)=\int_{-h}^{h}\left[k_{\alpha\beta}P_i.P_j(1+x_3.H_1)\left(1-x_3\left(\frac{1}{R_\alpha}+\frac{1}{R_\beta}\right)\right)\right]dx_3 \\ \mathbf{B}_{ij}(x_\gamma)=\int_{-h}^{h}\left[K.P_i'.P_j'(1+x_3H_1)\right]dx_3+\lambda^{\pm}P_i(\pm h).P_j(\pm h)(1\pm h\,H_1) \\ \mathbf{C_i}(x_\gamma)=\varphi^{\pm}.P_i(\pm h)(1\pm hH_1)+\int_{-h}^{h} r.P_i(1+x_3H_1)dx_3 \\ \mathbf{D_i}(x_\gamma)=\int_{-h}^{h}\varphi.P_i(1+x_3H_1)dx_3\ , \quad \text{pour } x_\gamma\in\partial\omega_\varphi \end{cases}$$

and:

$$T(x_\gamma;x_3)=T^m(x_\gamma).P_1(x_3)+T^s(x_\gamma).P_2(x_3)+T^i(x_\gamma)+T^i(x_\gamma).P_3(x_3)$$

$P_i(x_3)$: three polynomials of LAGRANGE in the thickness $[-h,h]$:

$$P_1(x_3)=1-(x_3/h)^2\ ;\ P_2(x_3)=\frac{x_3}{2h}(1+x_3/h)\ ;\ P_1(x_3)=\frac{x_3}{2h}(1-x_3/h)$$

H_1:	average curvature: $H_1=\left(\frac{1}{R_1}+\frac{1}{R_2}\right)$;
(x_1, x_2) :	coordinate system orthonormal according to the principal curvatures of ω ; $d\omega=dx_1.dx_2$;
$k_{\alpha\beta}$:	surface components of the tensor $\boldsymbol{K}$ of conductibility;
K:	transverse component of the tensor $\boldsymbol{K}$ of conductibility;
$\lambda^{\pm}$:	coefficients of heat exchange on the sides ω^+ and ω^- ;
$\varphi^{\pm}$:	flux applied to the sides ω^+ and ω^- ;
r :	sources distributed in the thickness;
φ :	flux imposed on the end $\partial\omega_{\varphi}$ of the shell.

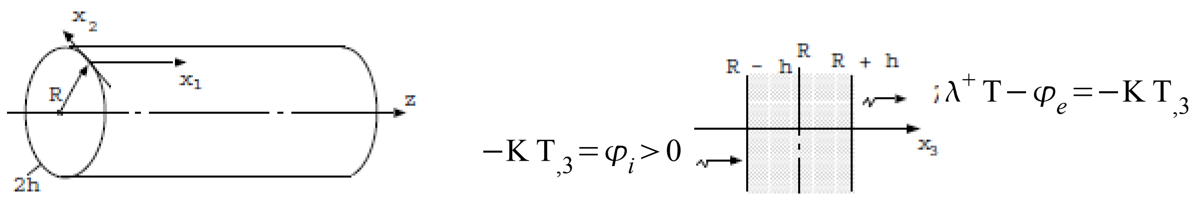
3 Validation of the model on some examples

One presents here applications on cylinders and plates. The first draft by way of a unidimensional case in the thickness and makes it possible to evaluate the effect of the terms of curvature, in particular in the second member of the equations. The others make it possible to judge the capacity of the model to treating the case of discontinuous thermal loadings, by reference to solutions 3D.

3.1 The infinite cylinder subjected to a uniform interior flux

One considers an infinite cylinder (radius R, thickness $2h$), subjected to a uniform flux inside: φ_i, and with a condition of exchange in external skin $\lambda^+(T-T_{ext})=\lambda^+ T-\varphi_{e^-}$.
One notes K the coefficient of transverse conductivity.
The analytical solution of this axisymmetric problem 1D is:

$$T(x_3)=\hat{T}_1+\hat{T}_0 \ln(1+x_3/R)$$



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with:

$$\hat{T}_0=-\frac{R\varphi_i}{K}\left(1-\frac{h}{R}\right)$$

$$\hat{T}_1=\frac{R\varphi_i}{K}\left(1-\frac{h}{R}\right)\ln\left(1+\frac{h}{R}\right)+\left(\varphi_i\frac{1-h/R}{1+h/R}+\varphi_e\right).\frac{1}{\lambda^+}$$

A development limited to the 2nd order in x_3/R is:

$$T(x_3)\approx\frac{\varphi^+ +\varphi^-}{\lambda^+}-\frac{2\varphi_i}{\lambda^+}.\frac{h}{R}\left[1-\frac{h}{R}-\frac{\lambda^+ R}{2k}\left(1-\frac{3h}{2R}\right)\right]-\varphi_i\frac{h}{K}\left(1-\frac{h}{R}\right)\left(\frac{x_3}{h}-\frac{x_3^2}{2h^2}.\frac{h}{R}\right)+...$$

Let us use now the model at 3 fields $\boldsymbol{T}=(T_1, T_2, T_3)$ defined in [§ 2.3.2]. Because of independence in x_1 and x_2 solution, one is reduced to the resolution of: $\boldsymbol{B}\,\boldsymbol{T}=\boldsymbol{C}$.

For the representation $T(x_3)=T_1+T_2\frac{x_3}{h}+\frac{3}{2}.T_3\left(\frac{x_3^2}{h^2}-\frac{1}{3}\right)$, one a:

$$\boldsymbol{B}=\frac{2K}{h}\begin{pmatrix}0 & 0 & 0\\ 0 & 1 & h/R\\ 0 & h/R & 3\end{pmatrix}+\lambda^+\left(1+\frac{h}{R}\right)\begin{pmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{pmatrix}$$

$$\boldsymbol{C}=\varphi_i\left(1-\frac{h}{R}\right)\begin{pmatrix}1\\ -1\\ 1\end{pmatrix}+\varphi_e\left(1+\frac{h}{R}\right)\begin{pmatrix}1\\ 1\\ 1\end{pmatrix}$$ for the second member.

If one neglects the intervention of the curvature in the metric one, one removes the terms in $\frac{h}{R}$ the preceding statements.

The solution is, if the curvature completely is neglected $(1\gg\frac{h}{R})$:

$$T(x_3)=\frac{\varphi_e+\varphi_i}{\lambda^+}+\frac{\varphi_i.h}{K}\left(1-\frac{x_3}{h}\right)$$ i.e. the solution of the plane “wall”.

If one takes account of the curvature in the second member like in the terms of exchange λ^+ (true surfaces of application of flux):

$$T(x_3)=\frac{\varphi_i+\varphi_e}{\lambda^+}-\frac{2\varphi_i}{\lambda^+}.\frac{h}{R}\left[1-\frac{h}{R}-\frac{\lambda^+R}{2k}\left(1-\frac{h}{R}\right)\right]-\varphi_i\frac{h}{K}\left(1-\frac{h}{R}\right).\frac{x_3}{h}+0$$

One finds the analytical solution developed with the 1st order in x_3/R. The taking into account of the curvature in the terms of conductivity in $\boldsymbol{B}$ would intervene on the level of the terms in $(x_3/R)^2$.

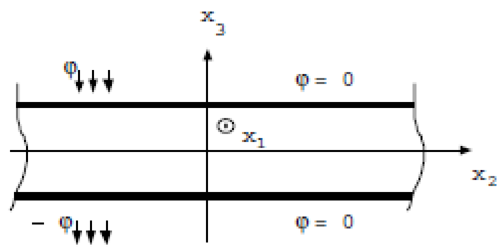
3.2 The infinite plate under a skew-symmetric flux couple

Let us take again the case of the infinite plate subjected on its half $x_2<0$ to a couple of constant flux $(\varphi^+=\varphi, \varphi^-=-\varphi)$ balanced, and adiabat on other half $x_2>0$.

The antisymetry of the loading imposes that: $T(x_1, x_2, 0)=0$. One can also show that T is linear in the thickness in $x_2=-\infty, 0, +\infty$.

The equations [éq 2.3.2-2] are reduced here to:

$$-2kh\begin{pmatrix}1 & 0 & 0\\0 & 1/3 & 0\\0 & 0 & 1/5\end{pmatrix}\begin{pmatrix}T_1\\T_2\\T_3\end{pmatrix}+\frac{2K}{h}\begin{pmatrix}0 & 0 & 0\\0 & 1 & 0\\0 & 0 & 3\end{pmatrix}\begin{pmatrix}T_1\\T_2\\T_3\end{pmatrix}=\varphi\begin{pmatrix}0\\2\\0\end{pmatrix}$$



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derivatives $T_{i,2}$ cancelling itself ad infinitum, T_1 and T_3 are identically null everywhere. It remains to determine T_2 (depending only on x_2) such as:

$-T_2''+\frac{3K}{kh^2}.T_2=\frac{3\varphi}{kh}$ whose solution is form:

$$\begin{cases}T_2(x_2)=a\,e^{\sqrt{3K/k}.x_2/h}+\frac{\varphi h}{K} & \text{si } x_2<0 ,\\ T_2(x_2)=b\,e^{-\sqrt{3K/k}.x_2/h} & \text{si } x_2>0 ,\end{cases}$$

The continuity of T_2 and T_2' 0 gives some:

$$\begin{cases}T_2(x_2)=\frac{\varphi h}{K}\left(2-e^{\sqrt{3K/k}.x_2/h}\right) & \text{si } x_2\le 0\\ T_2(x_2)=\frac{\varphi h}{K}\,e^{-\sqrt{3K/k}.x_2/h} & \text{si } x_2\ge 0\end{cases}$$

After change for the variables T^m, T^s, T^i , one finds:

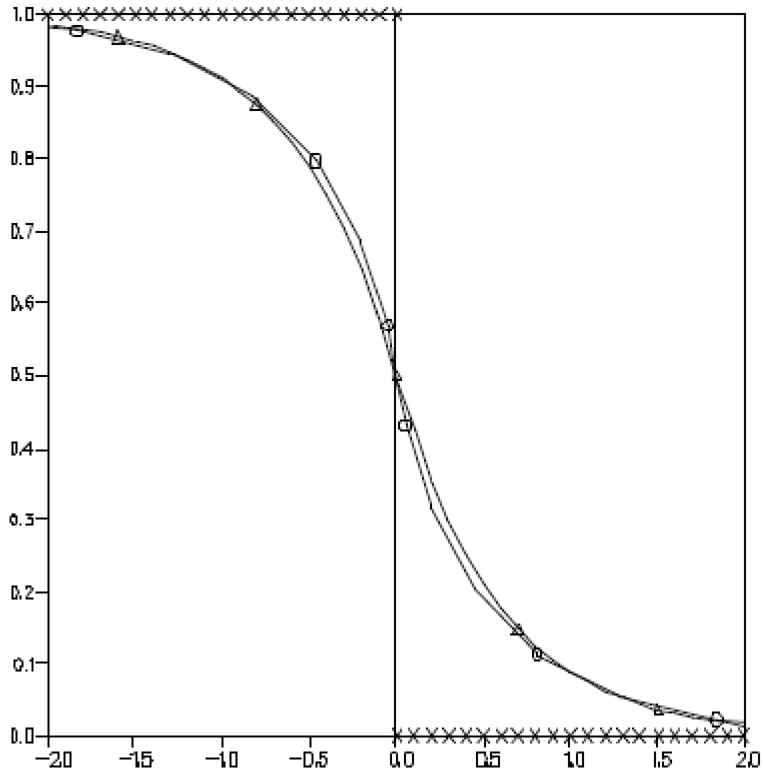
$$T^m(x_2)=0\ ;\ T^s(x_2)=T_2(x_2)\ ;\ T^i(x_2)=-T_2(x_2)$$

The temperature of the plate, calculated in the frame of this model is thus linear in the thickness and is expressed with $T^m(x_2)$ or $T^s(x_2)$ and $T^i(x_2)$ by:

$$T(x_{1,}x_{2,}x_3)=T_2(x_2).\frac{x_3}{h}=T^s(x_2).\frac{x_3}{2h}(1+x_3/h)-T^i(x_2).\frac{x_3}{2h}(1-x_3/h)$$

[Figure 3.2-a] allows to compare the temperatures in higher skin $(x_3=+h)$ of the plate under a standardized external flux ($\varphi=K/h$ with $k=K=1,h=1$), obtained by a numerical computation 3D (*Code Aster*), the model shell, and the model limits asymptotic (with the discontinuity observed in [§2.2]).

One notes the good capacity of the model to describe the boundary layer appearing in the vicinity of an external flux discontinuity.

Homogeneous plate H = 1; K = K = 1 Flux unit Opposed on X < 0 Flux null on X > 0 Caption: 0 = Δ models 3Δ = models Shell X = models Asymptot.	

Appear 3.2-b: Temperature compared in higher skin of the plate subjected to skew-symmetric flux.

3.3 The infinite plate under a symmetric flux couple

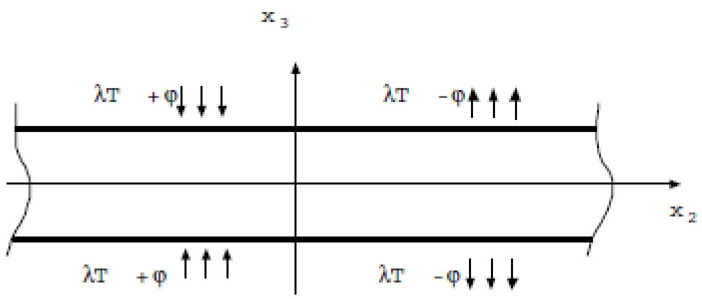
In the preceding example, the antisymetry of the loading involved the nullity of the even terms in $x_3(T_1=T_3=0)$. One treats now another case of loading, symmetric, (compared to $x_3=0$) allowing to judge the effect of the term T_3, in particular on T_1, which requires to take boundary conditions of the type exchanges, for dédiagonaliser $\boldsymbol{B}$ [§ 2.3.2].

In $x_3=+h$, one has like condition:

$$\begin{aligned} -KT &= -\lambda T+\varphi \text{ si } x_2<0 \\ &= -\lambda T-\varphi \text{ si } x_2>0 \end{aligned}$$

In $x_3=-h$, one a:

$$\begin{aligned} -KT &= -\lambda T+\varphi \text{ si } x_2<0 \\ &= -\lambda T-\varphi \text{ si } x_2>0 \end{aligned}$$



Appears: 3.2-the

conditions of symmetry and antisymetry force the solution to check:

- $-T(x_{1,}-x_{2,}x_3)=T(x_{1,}x_{2,}x_3)=T(x_{1,}x_{2,}-x_3)$

from where: $T(x_{1,}0,x_3)=0$, and $\partial_3 T(x_{1,}x_{2,}0)=0$.

The equations (18) are written in our case (T_i depends only on x_2):

$$\begin{cases} -kh.T_1''+\lambda T_1+\lambda T_3 & =\phi \quad \text{pour } x_2<0 \text{ ou} -\phi \text{ pour } x_2>0 \\ -kh/3.T_2''+\left(\dfrac{K}{h}+\lambda\right)T_2 & =0 \\ -kh/5.T_3''+\lambda T_1+\left(\dfrac{3K}{h}+\lambda\right)T_3 & =\phi \quad \text{pour } x_2<0 \text{ ou} -\phi \text{ pour } x_2>0 \end{cases}$$

T_2 is thus identically null (what is coherent with the conditions of symmetry).

The solutions T_1 and T_3 are given by (cf [Year 1]):

$$\begin{cases} T_1(x_1,x_2)=-\frac{\varphi}{\lambda}\left(1-\frac{\lambda/kh-s_2^2}{s_1^2-s_2^2}.e^{-s_1|x_2|}+\frac{\lambda/kh-s_1^2}{s_1^2-s_2^2}.e^{-s_2|x_2|}\right)\mathrm{sgn}(x_2) \\ T_3(x_1,x_2)=-\frac{\varphi}{\lambda}.\frac{kh}{\lambda}\frac{(\lambda/kh-s_2^2).(\lambda/kh-s_1^2)}{s_1^2-s_2^2}.(e^{-s_1|x_2|}-e^{-s_2|x_2|})\mathrm{sgn}(x_2) \end{cases}$$

s_1 and s_2 being positive roots of the characteristic polynomial.

After change for the variables T^m, T^s, T^i, one finds:

$$\begin{cases} T^m(x_1,x_2)=T_1(x_1,x_2) \\ T^s(x_1,x_2)=T_1(x_1,x_2)+T_3(x_1,x_2) \\ T^i(x_1,x_2)=T^s(x_1,x_2) \end{cases}$$

If one adopts to solve the thermal problem a model with 2 fields $(\tilde{T}_1,\tilde{T}_2)$, with a representation closely connected in the thickness, one obtains like solution:

$$\begin{cases} \tilde{T}_1(x_1,x_2)= & \varphi/\lambda\left(1-e^{\sqrt{\lambda/kh}.x_2}\right) \quad \text{si } (x_2<0) \\ & -\varphi/\lambda\left(1-e^{\sqrt{\lambda/kh}.x_2}\right) \quad \text{si } (x_2>0) \\ \tilde{T}_2(x_1,x_2)= & 0 \end{cases}$$

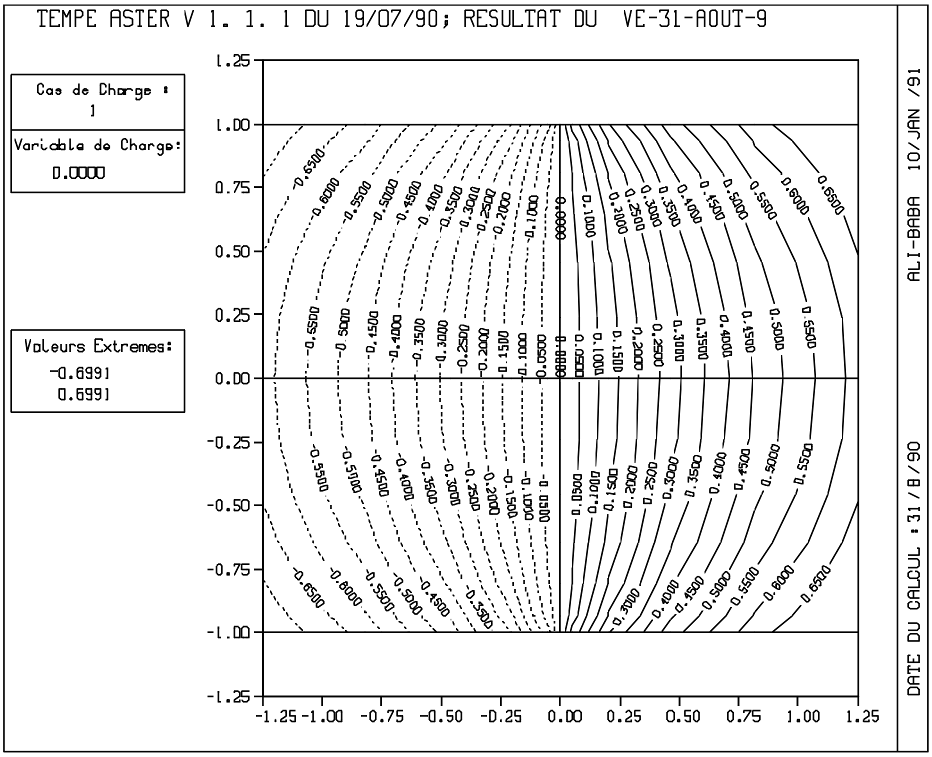
In such a model the temperature appears constant in the thickness. The model limits asymptotic produced the same solution.
One 3D compares the numerical solution and that of a model with 2 fields $(\tilde{T}_1)$. This last comparison makes it possible to judge effect of the parabolic term on the distribution of the average temperature. Indeed it is the latter which, in mechanical theory of the shells, generates a membrane strain.

These comparisons are made for unit values of $k, K, \lambda, h, \varphi$ in units If. One 3D visualizes the isovaleurs in the thickness on [Figure 3.3-a].

The average temperatures T_1 and $\tilde{T}_1$ are represented [Figure 3.3-b]. Finally it [Figure 3.3-c] the change of the temperature in higher skin () $x_3=+h$ of the plate shows, for the three solutions considered, as by that of the asymptotic limiting model; [Figure 3.3-d] the same comparison for the average average of the plate presents.

One notes on these results the good adequacy between the solution supplements 3D (points 0) and that obtained with the model from shells at 3 fields (points Δ), whereas the model at 2 fields (points $+$) appears insufficient.

These observations remain valid for other choices of $k, K, \lambda, h, \varphi$, since the problem is linear in φ , and that the variable of space x_2 appears normalisable by $\sqrt{\frac{\lambda}{kh}}$ in the equations.



Appear: 3.3-a: Isovaleurs of temperature by numerical computation 3D

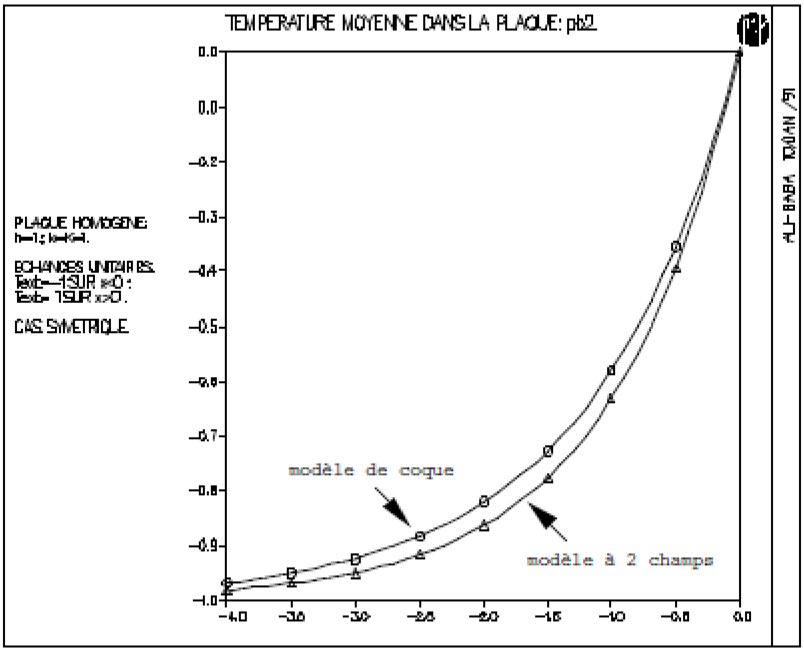
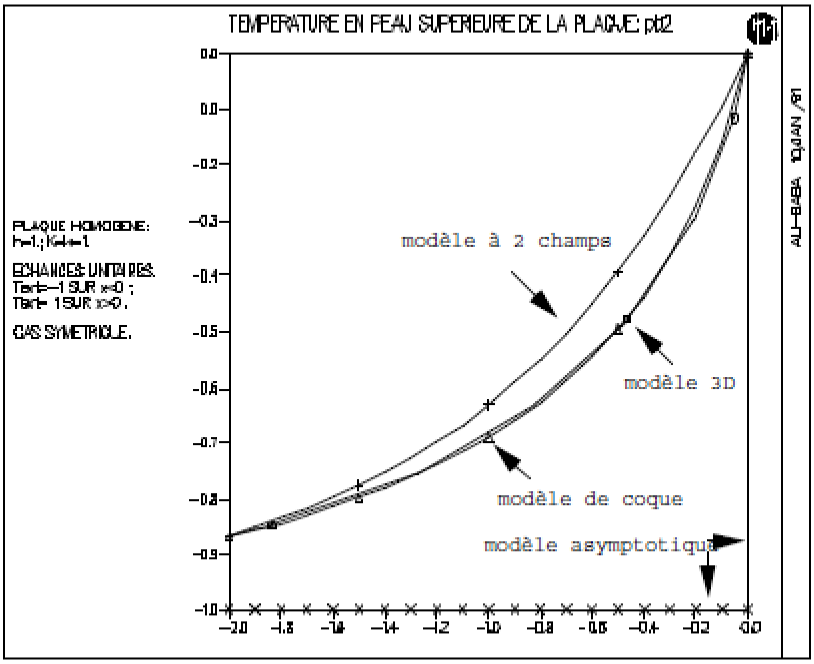
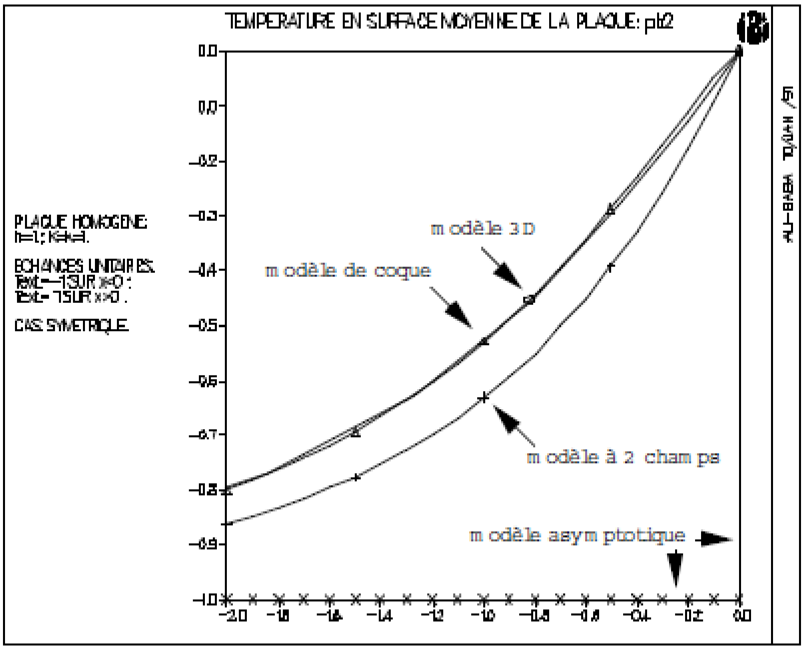


Figure: 3.3-b: Comparison of the average temperatures: effect of the parabolic term



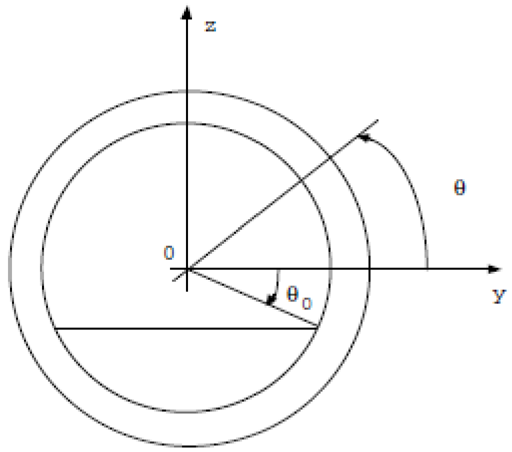
Appears: 3.3-c: Temperature compared in higher skin of the plate subjected to symmetric exchanges



Appears: 3.3-d: Temperature compared on the average average of the plate subjected to symmetric exchanges

3.4 the infinite cylinder subjected to a horizontal stratification

One is interested in this paragraph in a situation more industrial than the preceding cases. It is about a thermal problem of stratification in a horizontal pipe [bib12]. Under certain thermohydraulic conditions, the temperature of the fluid can vary very quickly with the dimension z (cf appears Ci - below). One can practically consider that there exist two zones with constant temperatures on both sides of a horizontal interface.



Appear: 3.4-a

- Characteristic geometrical:

$R=1,0\,m$ $h=0,075\,m$ $\theta_0=-30°$

- Physical characteristics:

Conductivity $k=17\,W/m/°C$

Exchanges: outside (air) $=\lambda^e$ $=12\,W/m^2/°C$
interior (hot water) $=\lambda^c$ $=1000\,W/m^2/°C$
interior (cool water) $=\lambda^h$ $=1000\,W/m^2/°C$

Temperatures: external: $25\,°C$
interior: cold $250\,°C$ heat $50\,°C$

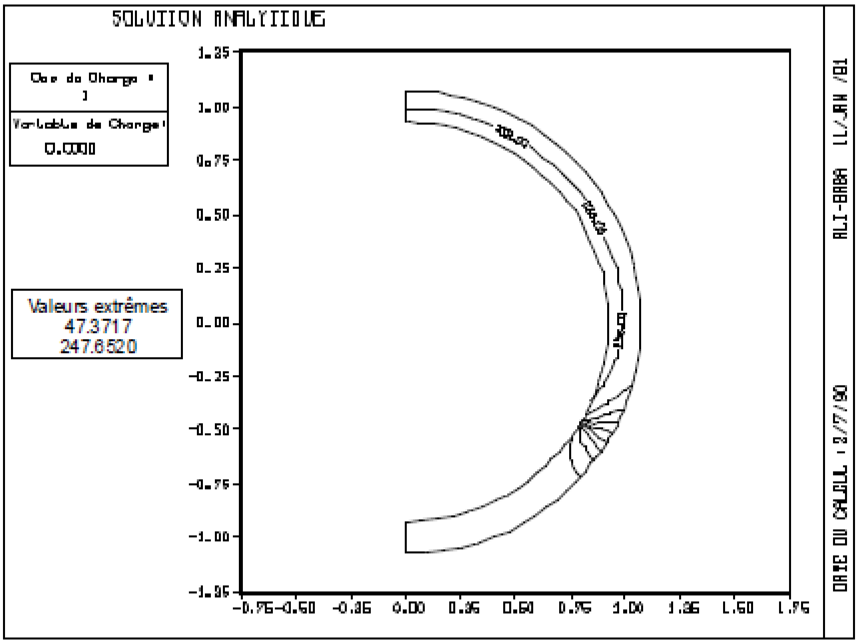
the determination of the temperature in the pipe is of two interests, first is to be able to lead to the distribution of stress in the vicinity of the stratification, second is to estimate the heat transfers between the “cold” water zone and the “hot” water zone via conduction in the tube.

The problem being independent of the variable x, it becomes unidimensional in the frame of the model of shell. To solve it, one first of all seeks the general solutions of the equation without second member on each segment $]-\frac{\pi}{2},\theta_0[$ and $]\theta_{0,}\frac{\pi}{2}[$:

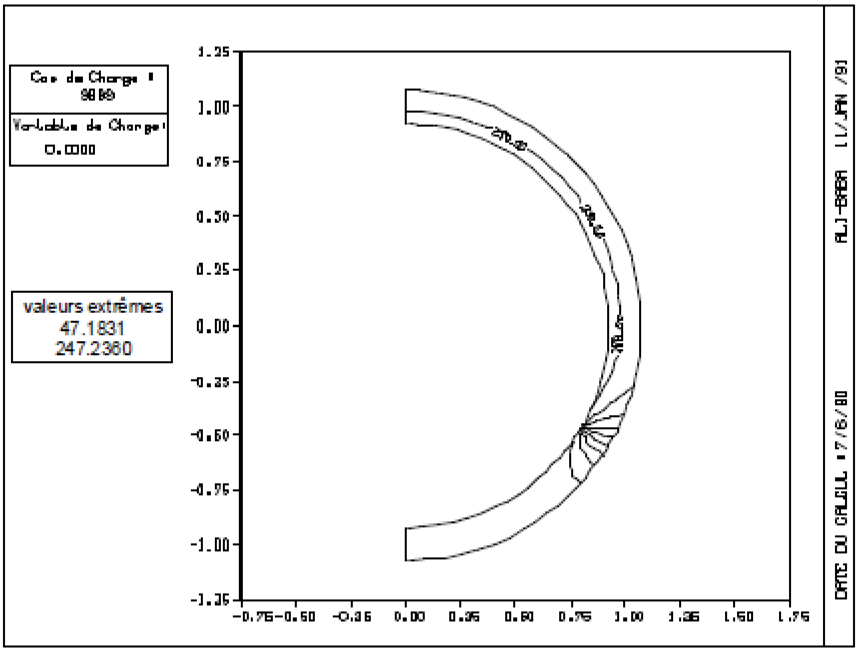
$$-\boldsymbol{A}\begin{bmatrix}\Delta T_1\\\Delta T_2\\\Delta T_3\end{bmatrix}+\boldsymbol{B}\begin{bmatrix}T_1\\T_2\\T_3\end{bmatrix}=0$$

Solving for that numerically a cubic equation (characteristic polynomial in s), one writes then the conditions of continuity of the fields T_i and their tangential derivatives with the interface, by expressing those by the combination of the general solutions and the particular solutions in each field. The linear system to solve (12×12) is brought back to a system of dimension 6×6 by considerations of symmetries, then solved numerically.

One has an semi-analytical solution thus [bib4] (numerical resolution of the linear system and cubic equation) although the situation is complex. The comparison with a computation 2D by finite elements is given on [Figure 3.4-b] and [Figure 3.4-c]: the difference between the two solutions is indistinguishable.



Appear 3.4-b: Stratified pipework: analytical solution by the model of thermal shell



Appears 3.4-c: Stratified pipework: solution finite elements thermal 2D *Aster*

4 Remarks on the numerical discretization

In this paragraph one is limited to some observations as for the numerical resolution of the equations of the thermal model of shell: first of all on the use of a method of finite elements and then on numerical blocking appearing when the thickness $2\mathrm{h}$ is low. This last comes from the intervention from h to powers different in the coefficients from the equations.

4.1 Resolution by finite elements

The model of thermal of shell describes in [§ 2.3] shows the following characteristics:

- it leads to an operator of a nature 2 acting on the three scalar fields $\boldsymbol{T}=(\mathrm{T}^{\mathrm{m}},\mathrm{T}^{\mathrm{s}},\mathrm{T}^{\mathrm{i}})$;
- these three fields are defined on a surface ω field, plunged in $\mathbb{R}^3$;
- the curvature of surface ω intervenes, possibly, only in the statement of the coefficients $\boldsymbol{A},\boldsymbol{B},\boldsymbol{C},\boldsymbol{D}$.

In the general case of a shell of an unspecified form plunged in $\mathbb{R}^3$, one can discretize the geometry of his mean surface ω by a mesh in plane triangular elements (this method presents certainly the default not to be able to explicitly take into account the curvature of ω).

The thermal problem (see [§ 2.3.6]) being scalar, at 3 fields, of the second order, one proposes the usual finite elements: plane triangles $P1$ (with 3 nodes) or $P2$ (with 6 nodes).

Their formulation is the same one, that ω plane or is curved: one thus neglects the corrections of metric in the operators of stiffness $\boldsymbol{A}$ and $\boldsymbol{B}$, (one saw in the cases of validation that had little effect in practice). On the other hand the user, if he knows the statement of the curvature, will may find it beneficial to take of it account in the values of the coefficients $\lambda^{\pm}$ and flux $\varphi^{\pm}$, as in the statements [éq 2.3.1-4] and [éq 2.3.1-5].

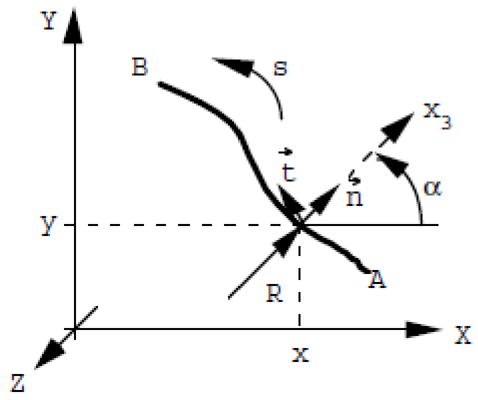
In the case of composites (just as if one wanted to take account of the curvature), one has to envisage a preprocessing providing the coefficients $\boldsymbol{A},\boldsymbol{B},\boldsymbol{C},\boldsymbol{D}$, as well as a post - processing allowing to reconstitute the temperature and the flux in any point of the thickness.

There exist situations where the problem does not depend any more but on one variable of space: it is the shells of axisymmetric revolution loading, or the "slices of shells", axis $\vec{e}_3$.

The geometry is then represented by a generator: (see [Figure 4.1-a]). The average curve is then:

- case revolution: $H_1=\left(\dfrac{1}{R}+\dfrac{\cos\alpha}{x}\right)$
- case "slices", or arc: $H_1=\dfrac{1}{R}$

where R the radius of curvature of meridian indicates line $A\,B$.



Appear 4.1-a.

For these types of problems, one proposes also a finite element $P2$ with 3 nodes, using the same formulation, where one neglects the correction of metric in the thickness, for the coefficients $\mathbf{A}$ and $\mathbf{B}$. One Gauss points uses a formula of squaring to 4.

This element is associated exactly with that proposed in mechanics for chained thermomechanical studies [R3.07.02].

4.2 Numerical blocking of a finite element of thermal shell

blocking is a phenomenon appearing in the numerical resolution by finite elements of certain problems such as that of the thin shells or the arcs when the element is curved (blocking of membrane), that of the shells or the beams with taking into account of the shears (blocking of shears), or that of plasticity (plastic blocking of incompressibility [bib7]). It was met initially in mechanics of the incompressible fluids and it is in this frame that its theoretical study began [bib6].

This phenomenon of blocking appears by a very great loss of accuracy and important oscillations on certain calculated quantities when a physical parameter of the model becomes “small”. The illustration of these nuisances is given in note HI-71/7131, (§4.2). The origin of these problems lies in the difference in order of magnitude which appears between certain components of the bilinear form of “stiffness” when the physical or geometrical parameter tends towards zero (thickness of the shell for the blocking of membrane, opposite of the tangent modulus of compressibility for plastic blocking for example). Here, it is the thickness of the shell which will play the part of small parameter.

Let us take again the equations of the steady thermal problem posed on a plate in variational form; let us note $2\varepsilon\mathrm{h}$ its thickness (ε reality without dimension):

$$\begin{cases} \text{Trouver } \mathbf{T}=(\mathrm{T}_1,\mathrm{T}_2,\mathrm{T}_3)\in\mathbf{W}\left(=\left[\mathrm{H}_0^1(\omega)\right]^3\right) \text{ tel que} \\ \mathrm{A}(\boldsymbol{T},\boldsymbol{\theta})+\frac{1}{\varepsilon}\mathrm{B}(\boldsymbol{T},\boldsymbol{\theta})=\mathrm{F}(\boldsymbol{\theta}) \qquad \forall\boldsymbol{\theta}=(\theta_1,\theta_2,\theta_3)\in\mathbf{W} \end{cases} \qquad \text{éq 4.2-1}$$

with:

$$\mathrm{A}(\boldsymbol{T},\boldsymbol{\theta})=2\mathrm{kh}\int_\omega[\nabla\boldsymbol{T}].\begin{bmatrix}1 & 0 & 0\\ 0 & \frac{1}{3} & 0\\ 0 & 0 & \frac{1}{5}\end{bmatrix}.[\nabla\boldsymbol{\theta}]\mathrm{d}\omega$$

. Indicating the usual scalar product,

$$B(\boldsymbol{T},\boldsymbol{\theta})=\frac{2K}{h}\int_{\omega}[\boldsymbol{T}].\begin{bmatrix}1&0&0\\0&1&0\\0&0&3\end{bmatrix}.[\boldsymbol{\theta}]d\omega$$

$$[\nabla\boldsymbol{\theta}]=\left(\vec{\nabla}\theta_1,\vec{\nabla}\theta_2,\vec{\nabla}\theta_3\right)\quad\text{(gradients surfaciques)}$$

an equivalent mixed formulation of this problem is obtained with the variables q_2 and q_3, heat flux in the thickness (cf [Year 2]): one notes $\theta=(L^2(\omega))^2$.

$$\begin{cases}\text{Trouver }(\boldsymbol{T},[p])\in\mathbf{W}\times Q\quad\text{tels que}\\ \begin{cases}A(\boldsymbol{T},\boldsymbol{\theta})-M(\boldsymbol{\theta},p)=F(\boldsymbol{\theta})&\forall\boldsymbol{\theta}\in\mathbf{W}\\-\varepsilon\bar{B}(p,q)-M(\boldsymbol{\theta},q)=0&\forall q\in Q\end{cases}\end{cases}\qquad\text{éq 4.2-2}$$

where:

$$[q]=(q_2,q_3)$$

$$M(\boldsymbol{\theta},q)=\int_{\omega}(\theta_2q_2+\theta_3q_3)d\omega$$

$$\bar{B}(p,q)=\frac{h}{v}2K\int_{\omega}[p]\begin{bmatrix}1&0\\0&\frac{1}{3}\end{bmatrix}[q]d\omega$$

On this formulation, numerical blocking appears clearly (at least formally). Indeed, the discretization of $\mathbf{W}\times Q$ being carried out, (it is noted $\mathbf{W_d}\times Q_d$), the problem tends formally when ε tends towards zero towards the following problem:

$$\begin{cases}A(\boldsymbol{T_d},\boldsymbol{\theta})-M(\boldsymbol{\theta},p_d)&=F(\boldsymbol{\theta})\forall\boldsymbol{\theta}\in\mathbf{W_d}\\M(\boldsymbol{T_d},q)&=0\quad\forall q\in Q_d\end{cases}$$

What amounts solving on the core of M :

$$A(\boldsymbol{T_d},\boldsymbol{\theta})=F(\boldsymbol{\theta})\forall\boldsymbol{\theta}\in\boldsymbol{W_d}\qquad\text{éq 4.2-3}$$

blocking appears when the core discretized of M is too small or reduced to zero: the resolution of [éq 4.2-3] is done on a very small space even reduced to zero. Even if the mesh is fine, the solution is then of very poor quality.

The core of M in $\boldsymbol{W_d}$ being by definition, space:

$$\text{Ker}\,M_d=\left\{\boldsymbol{\theta}\in\mathbf{W_d}\middle|M(\boldsymbol{\theta},q)=0\quad\forall q\in Q_d\right\}$$

It is seen that the choice of the discretization of Q is not innocent and strongly conditions the behavior of the solution when ε tends towards zero. There exists a condition of convergence relating to spaces $\boldsymbol{W_d}$ and Q_d which ensures the good numerical behavior of the solution with ε small, it is condition known as LBB discrete, version adapted to the discrete case of continuous condition LBB

(LADYJENSKAIA - BREZZI - BABUCHKA). We return to [feeding-bottle 7] for a case study (plasticity) and a bibliography on this subject.

Parallel to the theoretical studies quickly mentioned above, a practical remedy for blocking, appearing once the discretization into $\boldsymbol{W}_d$ selected if this choice were unhappy, consists under-integrating the term “blocking” in the construction of the stiffness, i.e. the term B here. Certain choices of under-integration, in the primal formulation [éq 4.2-1] are interpreted like choices of interpolation of $\boldsymbol{W}_d$ and $\mathrm{Q_d}$ in the mixed formulation and can thus, via the checking (sometimes hard) of discrete condition LBB, being justified on the theoretical level.

Let us consider indeed, a triangular finite element with 3 nodes and interpolation $P1$ to solve the problem [éq 4.2-2]. Let us choose then for discretization of Q a discontinuous $P0$ interpolation, i.e. a representation of $[q]$ constant per element.

The second equation of [éq 4.2-2] is then a local equation, i.e. to solve on each element separately since p is unspecified on each element E .

$$-\frac{\varepsilon \mathrm{h}}{2\mathrm{K}}\int_{\mathrm{E}}(\mathrm{p_2 q_2}+\frac{1}{3}\mathrm{p_3 q_3})-\int_{\mathrm{E}}\mathrm{T_2 q_2}+\mathrm{T_3 Q_3}=0 \quad \forall(\mathrm{q_{2,} q_3})$$

from where the immediate solution if $|\mathrm{E}|$ is the surface of E .

$$\begin{cases} \mathrm{p_2}=-\dfrac{2\mathrm{K}}{\varepsilon \mathrm{h}}\dfrac{1}{|\mathrm{E}|}\displaystyle\int_{\mathrm{E}}\mathrm{T_2} \\ \mathrm{p_3}=-\dfrac{6\mathrm{K}}{\varepsilon \mathrm{h}}\dfrac{1}{|\mathrm{E}|}\displaystyle\int_{\mathrm{E}}\mathrm{T_3} \end{cases}$$

By deferring these results in the form M , one has on the element E :

$$\mathrm{M}(\boldsymbol{\theta},\mathrm{p})=\frac{2\mathrm{K}}{\varepsilon \mathrm{h}}\left[\frac{1}{|\mathrm{E}|}\int_{\mathrm{E}}\mathrm{T_2}\int_{\mathrm{E}}\boldsymbol{\theta}_2+\frac{3}{|\mathrm{E}|}\int_{\mathrm{E}}\mathrm{T_3}\int_{\mathrm{E}}\boldsymbol{\theta}_3\right] \text{ for all } \boldsymbol{\theta}\in\boldsymbol{W}_d$$

having thus eliminated p , one is brought back to a primal formulation on T only:

$$\mathrm{M}(\boldsymbol{\theta},\mathrm{p})=\frac{2\mathrm{K}}{\varepsilon \mathrm{h}}\left[\frac{1}{|\mathrm{E}|}\int_{\mathrm{E}}\mathrm{T_2}\int_{\mathrm{E}}\boldsymbol{\theta_2}+\frac{3}{|\mathrm{E}|}\int_{\mathrm{E}}\mathrm{T_3}\int_{\mathrm{E}}\boldsymbol{\theta_3}\right] \text{ pour tout } \boldsymbol{\theta}\in\mathbf{W_d}$$

who corresponds very exactly to the formulation [éq 4.2-1] in which the elementary term:

$$\mathrm{B_E}(\boldsymbol{T},\boldsymbol{\theta})=\frac{2\mathrm{K}}{\varepsilon \mathrm{h}}\int_{\mathrm{E}}(\mathrm{T_2}\boldsymbol{\theta_2}+3\,\mathrm{T_3}\boldsymbol{\theta_3})$$

under-is integrated by a diagram into a Gauss point:

$$\int_{\mathrm{E}}\mathrm{f.g}=\frac{1}{|\mathrm{E}|}.\int_{\mathrm{E}}\mathrm{f}.\int_{\mathrm{E}}\mathrm{g}.$$

The examination of discrete condition LBB remains to be made for this discretization in order to conclude with its convergence (cf [bib7]).

5 Conclusion

an asymptotic analysis of the equations of the thermal in a thin structure when the thickness tends towards zero leads to a limiting model characterized by an average temperature, solution of a problem in extreme cases, and a parabolic complementary term in the thickness, defined locally.

One deduced the formulation from it from a model at 3 scalar fields definite on the mean surface of the shell, giving a parabolic representation of the temperature in the thickness. The differential operator obtained is of order 2 ; the thickness of the shell appears in its coefficients.

This model seems “optimal” for thin structures:

- its limit when the thickness tends towards zero is identical to the asymptotic limiting model;
- possible additional terms would tend towards zero with the thickness.

In a standard version the curvature of the mean surface of the shell does not intervene directly. Test examples show a good adequacy of the temperature obtained with complete three-dimensional solutions.

This model thus appears completely entitled with:

- to be used in a finite elements formulation to compute: the temperature in a thin shell of an unspecified form; the solution obtained being able to be easily injected into a thermomechanical computation of the shell; one thus proposes surface and linear elements for the cases where a variable of space does not intervene;
- to be introduced directly (or by coupling) into a method of resolution of the equations governing the thermohydraulic state of a pipework for example, in order to take account of the thermal restitution of the wall on the fluid;
- to be used as model integrated in the resolution of problems of identification (inverse problems) from experimental measurements (for example for stratified conduits);
- to seek analytical solutions in cases with simple geometry.

The model described here can also be used in the problems of thermal evolution, provided that the thermal loadings do not vary too quickly.

Lastly, it remains to study the numerical methods to use to avoid the blocking which could appear in a computation by finite elements, when the thickness becomes low.

6 References

1) ANDRIEUX S., MARIGO. J.J. : Application of the asymptotic methods to the problem of thermal conduction in the thin plates. Note EDF-DER-MMN: HI-71/5963, 1987.

2) BLANCHARD J.L, CARNOY E.: Finite element of thermal shell for the thermomechanical analysis of thin shells. In "C.F.A.O and Structural analyzes". FRANKFURT

3) G.: Asymptotic transient thermoelastic behavior in thermo-mechanical couplings in solids. IUTAM. Bui and Guyen Eds. Elsevier 1987. FAZOUANNE

4) A.: Study of an opposite thermal problem. Ratio of internship scientific, ENPC. 1990. GREEN

5) A.E., NAGHDI TOKEN ENTRY: One thermal effects in the theory of shells. Proc. Roy. London plowshare. A365, A367. 1979. HUGHES

6) T.J.R., MALKUS D.S.: "Mixed Finite Element Methods Reduced and selective integration techniques. Meth computer. Appl. Mech. Eng. 15-1, pp. 63-81, 1978. MIALON P.

7) , THOMAS B.: Incompressibility in plasticity: under-integration and other digital techniques. Note EDF-DER-MMN: HI-72/6404, of January 19th, 1990. See also: Bull DER Series C, n°3, 1991. RUBIN M.B

8) . : Heat conduction in punts and shells with emphasis one has conical Shell, in Int. J. of Solids and Structures, vol. 22, N° 5, pp. 527-551. 1986. SURANA K.

9) , ABUSALEH G.: Curved Shell elements for heat conduction with P-approximation in the Shell thickness direction. Computers and Structures, vol. 34, N° 6,1990. SURANA K.

10) S., ORTH N.J. : Axisymmetric Shell elements for heat conduction with p - approximation in the thickness direction, Computers and Structures, vol. 33, N° 3, pp. 689-705. 1989. SURANA K.

11) , PHILIPS R.: Three dimensional curved Shell finite elements for heat conduction. Computers and Structures, vol. 25, N° 5. 1987. SABATON MR.

12) , BIMONT G., PURPLE P.L., VOLDOIRE F., MASSON J.C., GRATTIER J.: Stratifications in the pipework of the pressurized water reactors. Note EDF-DER: HP/109/88/01. February 1988. Description

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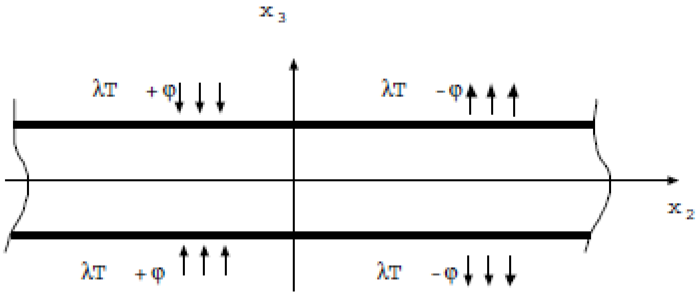
Annexe 1 under a symmetric flux couple In,

the boundary conditions $x_3=+h$ are: In, one

$$-K\partial_3 T \quad =\lambda T+\varphi \text{ si } x_2<0$$
$$=\lambda T-\varphi \text{ si } x_2>0$$

a: $x_3=-h$ the conditions

$$-K\partial_3 T \quad =-\lambda T+\varphi \text{ si } x_2<0$$
$$=-\lambda T-\varphi \text{ si } x_2>0$$



$T_i(0,0)=0$

of symmetry and antisymetry force the solution to check: and thus:

$$-T(x_1,-x_2,x_3)=T(x_1,x_2,x_3)=T(x_1,x_2,-x_3)$$

. $T(x_1,0,x_3)=0$ The equations $\partial_3 T(x_1,x_2,0)=0$

[éq 2.3.2-2] are written in our case: is thus

$$\begin{cases}-kh.T_1''+\lambda T_1+\lambda T_3 & =\phi \quad \text{pour } x_2<0 \text{ ou} -\phi \text{ pour } x_2>0\\ -kh/3.T_2''+\left(\dfrac{K}{h}+\lambda\right)T_2 & =0\\ -kh/5.T_3''+\lambda T_1+\left(\dfrac{3K}{h}+\lambda\right)T_3 & =\phi \quad \text{pour } x_2<0 \text{ ou} -\phi \text{ pour } x_2>0\end{cases}$$

T_2 identically null (what is coherent with the conditions of symmetry). The preceding system admits like particular solution: The characteristic polynomial

$$\begin{cases}T_1^p(x_1,x_2)=\dfrac{j}{\lambda} \quad \text{si} \quad x_2<0 \quad \text{et} \quad -\dfrac{j}{\lambda} \quad \text{si} \quad x_2>0\\ T_3^p(x_1,x_2)=0 \text{ sur } R^2\end{cases}$$

in S of the homogeneous system is: , of which

$$\frac{k^2h^2}{5}s^4-k\left(\frac{6h\lambda}{5}+3K\right)s^2+\frac{3\lambda K}{h}=0$$ the 4 roots are: s_i The solutions

$$s_i=\pm\frac{1}{h}\cdot\sqrt{\frac{3}{k}}\cdot\sqrt{\left(\lambda h+\frac{5}{2}K\right)\pm\sqrt{\lambda^2h^2+\frac{25K^2}{4}+\frac{10K\lambda h}{3}}}\ ,\quad s_1>s_2>0>s_3>s_4$$

and, finished $T_1(x_1,x_2)$ $T_3(x_1,x_2)$ in, are thus expressed $|x_2|=\infty$: The conditions

$$\begin{aligned}
T_1(x_1,x_2) &= \frac{\varphi}{\lambda}+\alpha\,e^{s_1x_2}+\beta\,e^{s_2x_2} && \text{pour}\,x_2>0\\
&= -\frac{\varphi}{\lambda}-\alpha\,e^{-s_1x_2}-\beta\,e^{-s_2x_2} && \text{pour}\,x_2>0\\
T_3(x_1,x_2) &= \gamma\,e^{s_1x_2}+\delta\,e^{s_2x_2} && \text{pour}\,x_2>0\\
&= -\gamma\,e^{-s_1x_2}-\delta\,e^{-s_2x_2} && \text{pour}\,x_2>0
\end{aligned}$$

of connection are naturally expressed $x_2=0$ by it by the conditions of antisymetry of, already T used above. The four constants are determined $\alpha,\beta,\gamma,\delta$ by: From where:

$$\begin{cases}
\left.\begin{array}{l}\alpha+\beta=-\varphi/\lambda\\ \gamma+\delta=0\end{array}\right\}\ \text{nullité de T en}\ x_2=0\\
\left.\begin{array}{l}\alpha(\lambda-kh\,s_1^2)+\gamma\,\lambda=0\\ \beta(\lambda-kh\,s_2^2)+\delta\,\lambda=0\end{array}\right\}\ \text{modes}\ T_1-T_3\ \text{associés à}\ s_1,s_2
\end{cases}$$

The solutions

$$\begin{cases}
\alpha=-\dfrac{\varphi}{\lambda}\cdot\dfrac{(\lambda/kh-s_2^2)}{s_1^2-s_2^2}\\
\beta=\dfrac{\varphi}{\lambda}\cdot\dfrac{(\lambda/kh-s_1^2)}{s_1^2-s_2^2}\\
\gamma=\dfrac{\varphi.kh}{\lambda^2}\cdot\dfrac{(\lambda/kh-s_2^2).(\lambda/kh-s_1^2)}{s_1^2-s_2^2}\\
\delta=\dfrac{\varphi.kh}{\lambda^2}\cdot\dfrac{(\lambda/kh-s_2^2).(\lambda/kh-s_1^2)}{s_1^2-s_2^2}
\end{cases}$$

and are thus written T_1 T_3 : Mixed

$$\begin{cases}
T_1(x_1,x_2)=-\dfrac{\varphi}{\lambda}\cdot\left(1-\dfrac{\lambda/kh-s_2^2}{s_1^2-s_2^2}e^{-s_1|x_2|}+\dfrac{\lambda/kh-s_1^2}{s_1^2-s_2^2}e^{-s_2|x_2|}\right)\text{sgn}(x_2)\\
T_3(x_1,x_2)=-\dfrac{\varphi.kh}{\lambda^2}\cdot\dfrac{(\lambda/kh-s_2^2).(\lambda/kh-s_1^2)}{s_1^2-s_2^2}.(e^{-s_1|x_2|}-e^{-s_2|x_2|})\text{sgn}(x_2)
\end{cases}$$

Annexe 2 formulation of the steady problem for the plate In the case

of the plate, the variational problem [éq 4.2-1] is equivalent to a problem of minimization (while revealing explicitly the thickness in) $\varepsilon\,h$ functional calculus B : To obtain

$$J(\theta)=\frac{1}{2}A(\theta,\theta)+\frac{1}{2\varepsilon}B(\theta,\theta)-F(\theta)$$

the mixed formulation, let us notice that: Proposal

: Demonstration

$$\frac{1}{2\varepsilon}B(\theta,\theta)=\sup_{(q_2,q_3)\in\left[L^2(\omega)\right]^2}\left[-\int_\omega (q_2\theta_2+q_3\theta_3)-\frac{\varepsilon}{4}\frac{h}{K}\int_\omega\left(q_1^2+\frac{1}{3}q_3^2\right)\right]$$

: Let us write

the condition of extremality of the functional calculus between hooks (its opposite is strictly convex, coercive and semi-continuous in a lower position) and note p the couple where the sup is reached: from where:

$$q_2\theta_2+q_3\theta_3+\frac{\varepsilon}{2}\frac{h}{K}\left(q_2p_2+\frac{1}{3}q_3p_3\right)=0 \qquad \forall q$$

The value $\begin{cases} p_2=-\dfrac{2K}{\varepsilon h}\theta_2 \\ p_3=-\dfrac{6K}{\varepsilon h}\theta_3 \end{cases}$

of the functional calculus in this point is thus: that is to say

$$\int_\omega +\frac{2K}{\varepsilon h}\theta_2^2+\frac{6K}{\varepsilon h}\theta_3^2-\frac{\varepsilon h}{4K}\left[\frac{4K^2}{\varepsilon^2h^2}\theta_2^2+\frac{36K^2}{3\varepsilon^2h^2}\theta_3^2\right]=\frac{1}{\varepsilon h}\left[\frac{2K}{h}\theta_2^2+\frac{3K}{h}\theta_3^2\right]$$

result announced indeed. There is thus

a formulation equivalent to the minimization of on: while J noting $\mathbf{W}$

$$\underset{\theta\in\mathbf{W}}{\mathrm{Min}}\ \underset{q\in Q}{\mathrm{Max}}\left[\frac{1}{2}A(\theta,\theta)-\frac{\varepsilon}{2}\bar{B}(q,q)-M(\theta,q)-F(\theta).\right]$$

: The condition

$$M(\theta,q)=\int_\omega q_2\theta_2+q_3\theta_3 \quad , \quad \bar{B}(q,q)=\int_\omega\frac{h}{2K}\left(p_2q_2+\frac{1}{3}p_3q_3\right)$$

of POINT-saddles of this Lagrangian led to the formulation [éq 4.2 - 2]:

$$\begin{cases} A(\mathbf{T},\theta)-M(\theta,p)=F(\theta) & \forall\theta\in\mathbf{W} \\ -\varepsilon\bar{B}(p,q)-M(\mathbf{T},q)=0 & \forall q\in Q \end{cases}$$