Pre and Postprocessing for the thin shells out of “composite” materials

Abstract:

One extends the results of the theory of the shell elements exposed in documentation [R3.07.03] to the case of the multi-layer orthotropic materials. Documentation suggested gathers the thermal aspects and thermo-élasto-mechanics. The use of these materials is theoretically valid only in the case of a geometrical symmetry compared to the average average of the plate. It is thus necessary that membrane-flexure coupling is null.

Contents

1 Introduction

The modelization of the thermomechanical behavior by a theory of shells of structures made up of stratified composites presents compared to the isotropic homogeneous case a certain number of characteristics:

- the coefficients intervening in the behavior models linear connecting the mechanical magnitudes and thermals defined on the mean surface of the shell must be calculated from the spatial distribution in the thickness of the various materials,
- the materials constitutive of the shell are in general orthotropic:
  - it is necessary to define, in each point of the mean surface of the shell, a material direction fixing the reference in which the behavior models are described,
  - the form of the anisotropy produced on the total behavior of the shell can be unspecified,
- finally couplings between quantities characterizing of the phenomena symmetric and skew-symmetric compared to the mean surface can appear (coupling bending - membrane, coupling temperature average MOYENNE-gradient in the thickness). Into thermo_mecanic the results presented are however theoretically valid only when membrane-flexure coupling is null,
- the analysis of the fracture or of the damage of these structures requires to return on a level of description finer than that provided by the models of shells: the criteria are formulated, layer by layer in the thickness, according to the “three-dimensional” stresses.

The preprocessing makes it possible to the user “to build” the quantities intervening in the theories of shells from a simple spatial description of the distribution of the various materials (position, thickness, directional sense).

Postprocessing intervenes once the structural analysis completed for providing, layer by layer, an evaluating of some rupture criteria or damage.

The bias here is to specify pre and postprocessings so that they are independent, in the frame of the models of shell selected, of the type of element chosen by the user to calculate the structural analysis. Indeed, the numerical difficulties of the computation of the shells and the representation of their geometry results in proposing according to the situations, several element types finished shell or of plate.

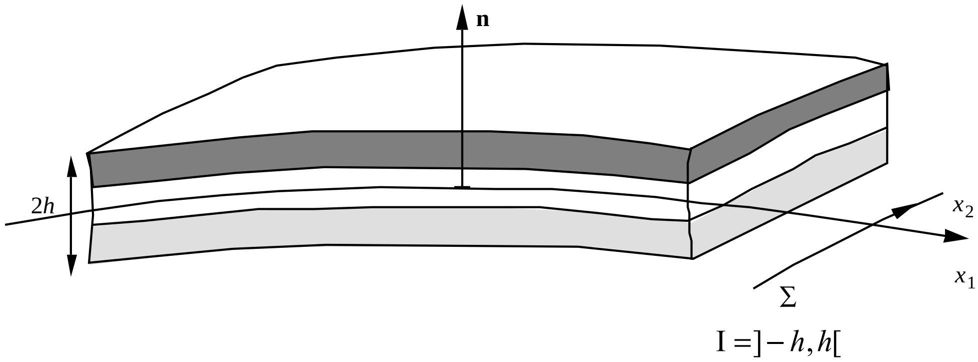
The note is divided into three parts. The first briefly points out the assumptions of the theory of shell used for thermomechanical computations and the statements of the coefficients homogenized to introduce. The second specifies the choices retained for the description of the directional sense of the materials compared to the elements like some notations. The last part details the application of these choices to the case of the shells made up of homogeneous layers.

To allow the use of the computation options available in *the Code_Aster*, it is thus necessary to define commands the pre one and postprocessing for stratified composites compatible with the existing commands.

2 Homogenized characteristics of a thin shell in thermoelasticity and thermal

2.1 Notations - Assumptions

the shell is made up various layers of orthotropic materials laid out parallel to mean surface Σ (cf [Figure 2.1-a]).



Appear 2.1-a

By noting (x_γ) the coordinates (x_1, x_2) on Σ and x_3 the normal coordinate on the surface $x_3 \in]-h, h[$. one can define the various characteristics of the materials intervening in Thermal and Thermo - elasticity. One will suppose moreover than one **of the axes of orthotropy coincides with the norm** $\boldsymbol{n}$ **at the point** (x_γ) **with the shell** Σ .

- Conductivity: $k_{\alpha\beta}(x_\gamma, x_3), k_{33}(x_\gamma, x_3)$
- Voluminal heat: $\rho c(x_\gamma, x_3)$
- Coefficients of thermal expansion: $d_{\alpha\beta}(x_\gamma, x_3)$
- Elastic stiffness (plane stress): $\Lambda_{\alpha\beta\lambda\mu}(x_\gamma, x_3)$
- Shear stiffness: $\Lambda_{\alpha 3\lambda 3}(x_\gamma, x_3)$
- Density: $\rho(x_\gamma, x_3)$

The Greek indices traverse {1, 2}. The system (x_γ) necessarily does not correspond to the axes of orthotropy of the materials in the tangent plane.

2.2 Thermal

One is placed in the frame of the model of thermal shell describes in [R3.11.01] and [bib1].

A field of temperature "shell" is represented by the three fields $\left(T^m, T^s, T^i\right)$ defined on Σ in the following way in the thickness

$$T\left(x_\gamma, x_3\right)=\sum_{j=1}^{3} T^j\left(x_\gamma\right) P_j\left(x_3\right)=T^m\left(x_\gamma\right) P_1\left(x_3\right)+T^s\left(x_\gamma\right) P_2\left(x_3\right)+T^i\left(x_\gamma\right) P_3\left(x_3\right) \qquad \textbf{éq 2.2-1}$$

where are P_j to them the polynomials of LAGRANGE $I=]-h,h[$:

$$P_1\left(x_3\right)=1-\left(x_3/h\right)^2$$

$$P_2\left(x_3\right)=\frac{x_3}{2\mathrm{h}}\left(1+x_3/h\right)$$

$$P_3\left(x_3\right)=-\frac{x_3}{2\mathrm{h}}\left(1-x_3/h\right)$$

the interpretation of the fields T^j is then the following one:

$T^m\left(x_\gamma\right)=T\left(x_\gamma,0\right)$
(temperature on the mean surface of the shell),

$T^s\left(x_\gamma\right)=T\left(x_\gamma,+h\right)$
(temperature on the upper surface of the shell),

$T^i\left(x_\gamma\right)=T\left(x_\gamma,-h\right)$
(temperature on the lower surface of the shell).

Thanks to the representation [éq 2.2-1], one calculates the bilinear form K_S^T of $\left(T^m, T^s, T^i\right)\equiv T$ from the form of the problem 3D (the indices ij take the values m, s, i):

$K_\Sigma^T(T,\tau)=\int_\Sigma\left(A_{\alpha\beta}^{ji}.T_{,\alpha}^i.\tau_{,\beta}^j+B^{ji}.T^i.\tau^j\right)d\,\Sigma$ (summation on the repeated indices),

where τ is a virtual field of temperature and where

$$\begin{cases} A_{\alpha\beta}^{ij}=A_{\alpha\beta}^{ji}=A_{\beta\alpha}^{ij}, \qquad B^{ij}=B^{ji} \\ A_{\alpha\beta}^{ij}\left(x_\gamma\right)=\int_I k_{\alpha\beta}\left(x_\gamma, x_3\right) P_i\left(x_3\right) P_j\left(x_3\right) dx_3 \\ B^{ij}\left(x_\gamma\right)=\int_I k_{33}\left(x_\gamma, x_3\right)\dfrac{\partial P_i}{\partial x_3}\left(x_3\right)\dfrac{\partial P_j}{\partial x_3}\left(x_3\right) dx_3 \end{cases} \qquad \textbf{éq 2.2-2}$$

the bilinear form related to voluminal heat in the problem of evolution is written:

$$M(T,\tau)=\int_S C^{ij}.T^i.t^j$$
$$C^{ij}(x_g)=\int_I \rho c(x_g,x_3)P_i(x_3)P_i(x_3)dx_3 \quad \text{éq 2.2-3}$$

2.3 Thermomechanical

One is placed in the frame of the modelization of shell of LOVE-KIRCHHOFF (shell thin) or REISSNER-MINDLIN (thick shell). In both cases, the sections are supposed to remain plane.

The strains of the tangent plane with Σ are thus expressed, in the thickness, using the strain tensors $E_{\alpha\beta}(x_\gamma)$, of variation of curvature $K_{\alpha\beta}(x_\gamma)$ and distortion $\gamma_\alpha(x_\gamma)$ of surface [bib2]:

$$e_{\alpha\beta}(x_\gamma,x_3)=E_{\alpha\beta}(x_\gamma)+x_3K_{\alpha\beta}(x_\gamma) \qquad e_{\alpha 3}(x_\gamma,x_3)=\frac{\gamma_\alpha(x_\gamma)}{2} \quad \text{éq 2.3-1}$$

the material undergoing a local strain of thermal origin given by ($T^{réf}$ is the reference temperature):

$$e^{th}_{\alpha\beta}(x_\gamma,x_3)=\left(T(x_\gamma,x_3)-T^{réf}\right)d_{\alpha\beta}(x_\gamma,x_3)$$

The local stress field is given by the thermo-elastic model in plane stresses:

$$s_{\alpha\beta}=L_{apha\,\beta\lambda\mu}\left(e_{\lambda\mu}-e^{th}_{\lambda\mu}\right)$$

maybe with the model preceding for T :

$$s_{\alpha\beta}(x_\gamma,x_3)=L_{\alpha\beta\lambda\mu}(x_\gamma,x_3)\left[E_{\lambda\mu}(x_\gamma)+x_3K_{\lambda\mu}(x_\gamma)-e^{th}_{\lambda\mu}(x_\gamma,x_3)\right]$$
$$\text{avec}\, e^{th}_{\lambda\mu}(x_\gamma,x_3)=\left(\sum_{j=1}^{3}T^j(x_\gamma).P_j(x_3)-T^{réf}\right)d_{\lambda\mu}(x_\gamma,x_3) \quad \text{éq the 2.3-2}$$

generalized forces (bending $M^{\alpha\beta}$ and membrane $N^{\alpha\beta}$) are related to σ by:

$$\begin{cases} M^{\alpha\beta}(x_\gamma)= & \int_I s^{\alpha\beta}(x_\gamma,x_3)x_3\,dx_3, \\ N^{\alpha\beta}(x_\gamma)= & \int_I s^{\alpha\beta}(x_\gamma,x_3)\,dx_3, \end{cases} \quad \text{éq 2.3-3}$$

so that the constitutive law of the shell is written with point: x_γ

$$\begin{cases} M^{\alpha\beta}= & P^{\alpha\beta\lambda\mu}K_{lambda\,\mu}+Q^{\alpha\beta\lambda\mu}E_{\lambda\mu}+M^{\alpha\beta}_{th} \\ N^{\alpha\beta}= & R^{\alpha\beta\lambda\mu}E_{\lambda\mu}+Q^{\alpha\beta\lambda\mu}K_{\lambda\mu}+N^{\alpha\beta}_{th} \end{cases} \quad \text{éq 2.3-4}$$

where

$$\begin{cases} P^{\alpha\beta\lambda\mu} = & +\int_I L^{\alpha\beta\lambda\mu}(x_3)\, x_3^2\, dx_3 \\ Q^{\alpha\beta\lambda\mu} = & +\int_I L^{\alpha\beta\lambda\mu}(x_3)\, x_3\, dx_3 \\ R^{\alpha\beta\lambda\mu} = & +\int_I L^{\alpha\beta\lambda\mu}(x_3)\, dx_3 \\ N_{th}^{\alpha\beta} = & -\int_I L^{\alpha\beta\lambda\mu} e_{\lambda\mu}^{th}\, dx_3 \\ M_{th}^{\alpha\beta} = & -\int_I L^{\alpha\beta\lambda\mu} e_{\lambda\mu}^{th}\, x_3\, dx_3 \end{cases} \quad \text{éq 2.3-5}$$

When the temperature is calculated by the model Thermal one can directly express the "Thermal" forces according to the three "components" $\left(T^m, T^s, T^i\right)$:

$$\begin{cases} M_{th}^{\alpha\beta} = -\left[\int_I L_{\alpha\beta\lambda\mu} d_{\lambda\mu}(x_3) P_j(x_3)\, x_3\, dx_3\right]\left(T^j - T^{réf}\right) = DM_j^{\alpha\beta}\left(T^j - T^{réf}\right) \\ N_{th}^{\alpha\beta} = -\left[\int_I L_{\alpha\beta\lambda\mu} d_{\lambda\mu}(x_3) P_j(x_3)\, dx_3\right]\left(T^j - T^{réf}\right) = DN_j^{\alpha\beta}\left(T^j - T^{réf}\right) \end{cases} \quad \text{éq the 2.3-6}$$

quantities DN and DM depend only on the materials constitutive of the shell and their distribution.

Note:

When the provision of the materials is symmetric compared to Σ *, certain integrals, being sum of odd terms, are cancelled:* $Q^{\alpha\beta\lambda\mu}=0,\ DM_1^{\alpha\beta}=DM_3^{\alpha\beta}=0\ ;\ DN_2^{\alpha\beta}=0$.

The shears and shearing stresses transverse are obtained by writing of the local balance equations without volume force:

$$\sigma^{ij}_{,j}=0 \quad \text{where} \quad \{i, j\} \in \{1,2,3\}$$

what makes it possible to write:

$$\begin{aligned} & V^{\alpha}(x_\gamma) = M^{\alpha\beta}_{,\beta}(x_\gamma) \\ & \sigma^{\alpha 3}(x_\gamma, x_3) = -\int_{-h}^{x_3} \sigma^{\alpha\beta}_{,\beta}(x_\gamma, z)\, dz \end{aligned}$$

by means of the fact that $\sigma^{\alpha 3}(x_\gamma, +h) = \sigma^{\alpha 3}(x_\gamma, -h) = 0$.

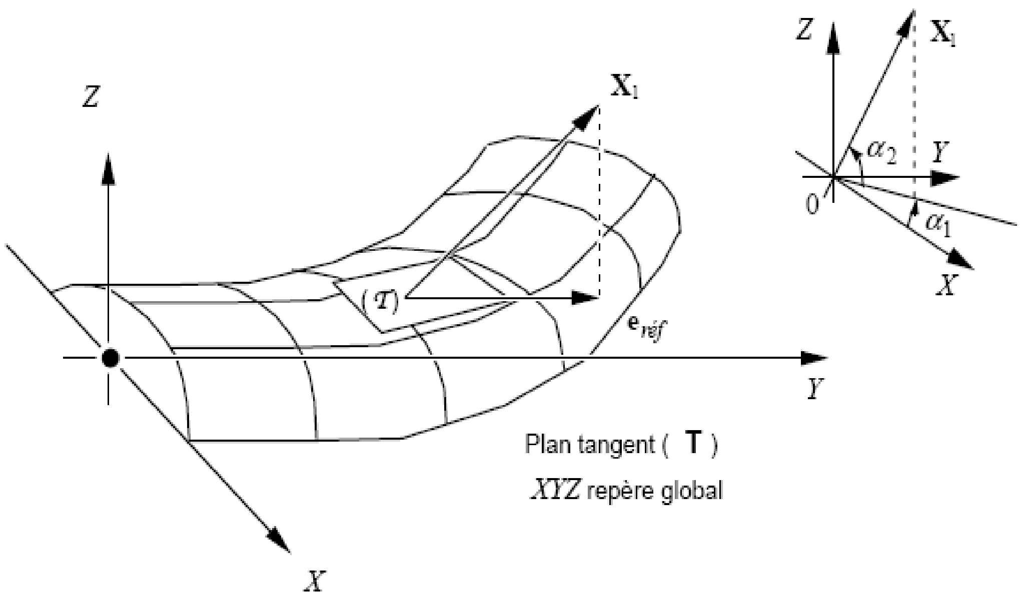
The role of the preprocessing is to calculate the various quantities A, B, C, P, Q, R, DM, DN , from the description of the material (number, directional sense and thickness of the various layers, local characteristics $\rho\mathrm{c}, \mathrm{k}, \rho, \mathrm{L}, \mathrm{d}$).

3 References in the tangent plane with the shell. Matric notation

3.1 References

One considers the total reference of structure (X, Y, Z) : to see figure [Figure 3.1-a]. In the case of the stratified composites the directional sense of full-course is defined compared to a direction of reference $\boldsymbol{e}_{réf}$ in the tangent plane (T).

This vector $\boldsymbol{e}_{réf}$ is determined by the projection of a vector $\boldsymbol{X}_1$, given by the user under key word `ANGL_REP` of `AFFE_CARA_ELEM [U4.24.01]`, on the tangent level (T) in an unspecified point of the shell.



Appear 3.1-the

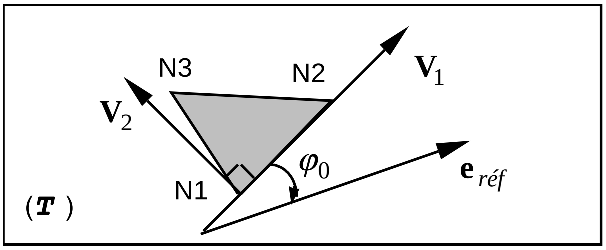
vector $\boldsymbol{X}_1$ is defined by the user by two directed angles:

α_1 : enter OX and $\boldsymbol{X}_{1\,\mathrm{proj}(X,\,Y)}$
α_2 : enter $\boldsymbol{X}_{1\,\mathrm{proj}(X,\,Y)}$ and X_1

α_1 : fact of passing from the direction $0\mathrm{X}$ to projection in the plane XOY of the vector $\boldsymbol{X}_1$.
α_2 : fact of passing from this projection to X_1 itself: to see figure [Figure 3.1-a].

Whenever in a given zone of the shell, (T) is orthogonal with $\boldsymbol{X}_1$, the user will have to define another vector (in practice for some meshes).

For a finite element of type facets planes, contained in the tangent plane (T), one defines the local orthonormal $\left(V_1, V_2\right)$ reference in the element using the classification of the tops. For example for the triangle:



Appear 3.1-b: Local coordinate system of the element $\left(V_1, V_2\right)$

the directed angle $j_0 = \left(V_1,\ e_{réf}\right)$ makes it possible to pass from the local coordinate system to the element to the reference of reference.

3.2 Matric notation

In thermal as into thermomechanical, the programming of the elements requires to express the operators of elasticity and conduction in the local coordinate system of the finite element $\left(V_1, V_2\right)$. One is used simplifying the representation of the tensorial quantities as follows.

3.2.1 Thermal

One represents the tensorial quantities in the reference $\left(V_1, V_2\right)$:

$$\left(A_{\alpha\beta}{}^{ij}\left(\left(i,\ j\right)\in m, s, i^2\left(\alpha\beta\right)\in 1{,}2^2\right)\right)$$

in a vectorial form with 6 vectors by taking account of symmetries [§2.2]:

$$A^{ij} = \begin{pmatrix} A_{11}^{ij} \\ A_{22}^{ij} \\ A_{12}^{ij} \end{pmatrix} = \int_{-h}^{h} P_i\left(x_3\right).P_j\left(x_3\right).\begin{pmatrix} k_{11} \\ k_{22} \\ k_{12} \end{pmatrix} dx_3$$

where $k = \begin{pmatrix} k_{11} \\ k_{22} \\ k_{12} \end{pmatrix}$ indicates the thermal vector conductivity built using the tensor $\begin{pmatrix} k_{\alpha\beta} & & 0 \\ & & 0 \\ 0 & 0 & k_{33} \end{pmatrix}$ (cf [§2.1]),

and of $P_i\left(x_3\right)$, the polynomials of LAGRANGE in the thickness. One makes in the same way for B^{ij}, C^{ij}.

While placing oneself in the reference of the element $(\boldsymbol{V}_1, \boldsymbol{V}_2)$, one uses the transition matrix $\boldsymbol{P}_k^{(m)}$ of the tensor of conductivity $k=\begin{pmatrix} k_{11} \\ k_{22} \\ k_{12} \end{pmatrix}$ of $(\boldsymbol{V}_1, \boldsymbol{V}_2)$ worms the reference associated with $\boldsymbol{e}_{réf}$ [bib3]:

$$\boldsymbol{P}_k^{(m)}=\begin{bmatrix} C^2 & S^2 & 2CS \\ S^2 & C^2 & -2CS \\ -CS & CS & C^2-S^2 \end{bmatrix} \quad \text{where} \quad \begin{matrix} C=\cos(\varphi_0) \\ S=\sin(\varphi_0) \end{matrix}$$

It results from it that the transition matrix $(\boldsymbol{P}_{\boldsymbol{k}}^{\boldsymbol{m}})^{-1}$ of the tensor of conductivity of the reference associated with $\boldsymbol{e}_{réf}$ worms $(\boldsymbol{V}_1, V_2)$ is given by:

$$\boldsymbol{P}_{k^{(m)^{-1}}}=\begin{bmatrix} C^2 & S^2 & -2CS \\ S^2 & C^2 & 2CS \\ CS & -CS & C^2-S^2 \end{bmatrix} \quad \text{where} \quad \begin{matrix} C=\cos(\varphi_0) \\ S=\sin(\varphi_0) \end{matrix}$$

3.2.2 Thermomechanical

One also represents in a vectorial form in the reference $(\boldsymbol{V}_1, \boldsymbol{V}_2)$:

- on the one hand, normal stresses σ_{11}, σ_{22}, shears σ_{12} in the plane and transverse shears σ_{13} and σ_{23} :

$$\sigma=\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix} \quad , \quad \tau=\begin{pmatrix} \sigma_{13} \\ \sigma_{23} \end{pmatrix}$$

- in addition, corresponding strains:

$$\varepsilon=\begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{pmatrix} \quad , \quad \frac{1}{2}\gamma=\begin{pmatrix} \varepsilon_{13} \\ \varepsilon_{23} \end{pmatrix} \qquad \gamma_{12}=2\varepsilon_{12}$$

who break up with the generalized strains of membrane $\boldsymbol{E}$ and bending $\boldsymbol{K}$:

$$\begin{gathered} \varepsilon(x_3)=\varepsilon(u)(x_3)-\varepsilon^{th}(x_3) \\ \text{avec}\, \varepsilon(u)(x_3)=\boldsymbol{E}+x_3\boldsymbol{K} \\ \varepsilon^{th}(x_3)=\boldsymbol{d}(x_3)\left(T_{(x_3)}-T^{réf}\right) \end{gathered}$$

for an Y-coordinate $x_3 \in]-h, h[$, and:

$$\boldsymbol{E}=\begin{pmatrix} E_{11} \\ E_{22} \\ E_{11} \end{pmatrix} \quad , \quad \boldsymbol{K}=\begin{pmatrix} K_{11} \\ K_{22} \\ K_{11} \end{pmatrix} \quad , \quad \boldsymbol{d}=\begin{pmatrix} d_{11} \\ d_{22} \\ d_{11} \end{pmatrix}$$

where $\boldsymbol{d}$ is the vector associated with the coefficients of thermal expansion thermal.
The vector forced σ is obtained using the stiffness matrix (3 X 3):
$\boldsymbol{\sigma}=\boldsymbol{R}.\left(\varepsilon(u)-\varepsilon^{th}\right)$ with $\boldsymbol{R}$, opposite of the matrix of flexibility (see in [§4.3]).
While placing oneself in the reference of the element $\left(\boldsymbol{V}_1, \boldsymbol{V}_2\right)$, one uses the transition matrix $\boldsymbol{P}^{(m)}$ of the strain tensor $\varepsilon=\begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{pmatrix}$ of $\left(\boldsymbol{V}_1, \boldsymbol{V}_2\right)$ worms the reference associated with $\boldsymbol{e}_{réf}$ [bib3]:

$$\boldsymbol{P}^{(m)}=\begin{bmatrix} C^2 & S^2 & CS \\ S^2 & C^2 & -CS \\ -2CS & 2CS & C^2-S^2 \end{bmatrix} \qquad \text{where} \quad \begin{matrix} C=\cos(\varphi_0) \\ S=\sin(\varphi_0) \end{matrix}$$

While placing oneself in the reference of the element $\left(\boldsymbol{V}_1, \boldsymbol{V}_2\right)$, one uses the transition matrix $\boldsymbol{P}_2^{(\boldsymbol{m})}$ of the strain tensor $\frac{1}{2}\gamma=\begin{pmatrix} \varepsilon_{13} \\ \varepsilon_{23} \end{pmatrix}$ of $\left(\boldsymbol{V}_1, \boldsymbol{V}_2\right)$ worms the reference associated with $\boldsymbol{e}_{réf}$:

$$P_2^{(m)}=\begin{bmatrix} C & S \\ -S & C \end{bmatrix} \qquad \text{where} \quad \begin{matrix} C=\cos(\varphi_0) \\ S=\sin(\varphi_0) \end{matrix}$$

In the same way, while being placed in the reference of the element $\left(\boldsymbol{V}_1, \boldsymbol{V}_2\right)$, the transition matrix $\boldsymbol{P}_s^{(m)}$ of the stress tensor $\sigma=\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix}$ of $\left(\boldsymbol{V}_1, \boldsymbol{V}_2\right)$ worms the reference associated with $\boldsymbol{e}_{réf}$ is worth:

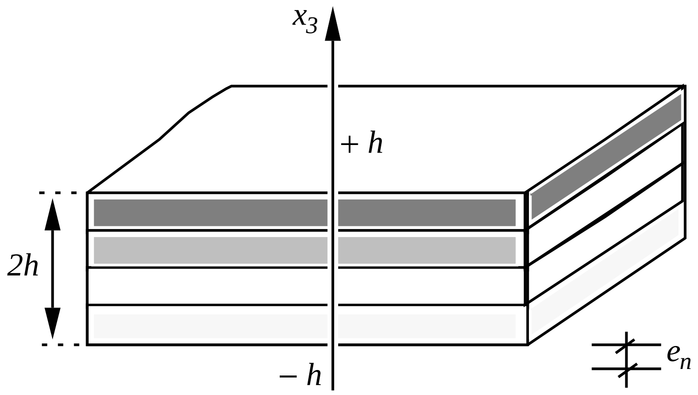
$$\boldsymbol{P}_s^{(m)}=\begin{bmatrix} C^2 & S^2 & 2CS \\ S^2 & C^2 & -2CS \\ -CS & CS & C^2-S^2 \end{bmatrix} \qquad \text{where} \quad \begin{matrix} C=\cos(\varphi_0) \\ S=\sin(\varphi_0) \end{matrix}$$

It results from it that the form of the transition matrix of the reference associated with $\boldsymbol{e}_{réf}$ worms the reference with the element $\left(\boldsymbol{V}_1, \boldsymbol{V}_2\right)$ for the stresses above is such as: $\boldsymbol{P}_{\sigma}^{(m)^{-1}}=\boldsymbol{P}_{\sigma}^{(m)}(-\varphi_0)={}^t\boldsymbol{P}^{(m)}$. This property will be particularly useful in the continuation of the talk.

4 Shells made up of homogeneous layers

4.1 Description of the layers

One considers the shell made up of a stacking of N_{couch} layers (parallel with the tangent plane) in the thickness $]-h,h[$ made up each one of one of M_{mater} the orthotropic homogeneous materials (stratified shell [Figure 4.1-a]).



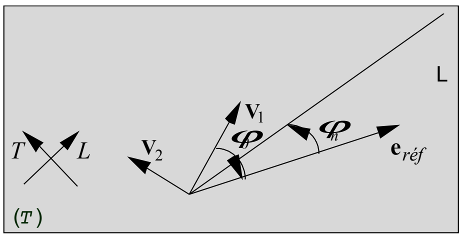
Appear 4.1-a

a layer N is defined by:

- its thickness e_n with the Y-coordinates of the interfaces lower and higher: $x_3^{n-1}=-h+\sum_{j=1}^{n-1} e_j ; x_3^n=x_3^{n-1}+e_n ;$
- the constitutive material m , and its physical characteristics,
- the angle φ_n of the first direction of orthotropy (noted L) in the tangent plane ($\boldsymbol{T}$) compared to the direction of reference $\boldsymbol{e}_{réf}$ (see figure [Figure 4.1-b]).

Note:

In the case of a layer made up of fibers in a resin matrix, the first direction of orthotropy corresponds to the direction of fibers.



Appear 4.1-b: On Thermal orthotropic

4.2 layer

the statement of the vectors $\boldsymbol{A}^{ij}\left((i,j)\in\{m,s,i\}^2, i\le j\right)$ defined in [§3.2.1] is obtained starting from conductivities $\boldsymbol{k}_m$ of the material m constituting the layers n.

In the cases of orthotropy (L, T) of the material m, the coefficients of conductivity are:

$$\boldsymbol{k}_{(L,T)}=\begin{pmatrix}k_L\\k_T\\0\end{pmatrix}$$

In the case of a transverse isotropic material the coefficient $\boldsymbol{k}_{33}$ is equal to $\boldsymbol{k}_T$.

To have the statement of $\boldsymbol{A}^{ij}$ in the reference of the element $(\boldsymbol{V}_1, \boldsymbol{V}_2)$ one must apply following rotation, from orthotropic reference towards the reference of the element, as clarified with [§3]:

$$\boldsymbol{k}^{(m)}=\begin{pmatrix}k_{11}\\k_{22}\\k_{12}\end{pmatrix}=\begin{pmatrix}C^2 & S^2\\S^2 & C^2\\CS & -CS\end{pmatrix}\begin{pmatrix}k_L\\k_T\end{pmatrix}_{(L,T)} \quad \text{with} \quad \begin{aligned}C&=\cos(\varphi_i+\varphi_0)\\S&=\sin(\varphi_i+\varphi_0)\end{aligned}$$

the vectors $\boldsymbol{A}^{ij}$ can then express itself by integration in the thickness of the contributions of layer:

$$\boldsymbol{A}^{ij}=\sum_{n=1}^{N_{couch}}\int_{x_3^{n-1}}^{x_3^n} P_i(x_3)\cdot P_j(x_3)\cdot \boldsymbol{k}_{(m)}\cdot dx_3 \qquad \textbf{éq the 4.2-1}$$

terms $\boldsymbol{B}^{ij}\left((i,j)\in\{2,3\}^2, i\le j\right)$ are:

$$\boldsymbol{B}^{ij}=\sum_{n=1}^{N_{couch}}\int_{x_3^{n-1}}^{x_3^n}\frac{\partial P_i(x_3)}{\partial x_3}\cdot\frac{\partial P_j(x_3)}{\partial x_3}\cdot k_{33(m)}\cdot dx_3$$

In the same way for $\boldsymbol{C}^{ij}$:

$$\boldsymbol{C}^{ij}=\sum_{n=1}^{N_{couch}}\int_{x_3^{n-1}}^{x_3^n} P_i(x_3)\cdot P_j(x_3)\cdot \rho C_{(m)}\cdot dx_3$$

4.3 Thermomechanical

4.3.1 Behavior model

In the case of the laminated shells, one shows that the relation between the strains $\boldsymbol{\varepsilon}$ and the stress $\boldsymbol{\sigma}$ in the "" layer n depends on the constants of the orthotropic material "" m :

That is to say:

$$\begin{cases} E_{\mathrm{LL}}^{(m)}, E_{TT}^{(m)}, \nu_{\mathrm{LT}}^{(m)}, G_{\mathrm{LT}}^{(m)}, G_{LZ}^{(m)}, G_{TZ}^{(m)} & \text{elastic coefficients} \\ d_{\mathrm{LL}}^{(m)}, d_{TT}^{(m)} & \text{coefficients of thermal expansion} \end{cases}$$

In the axes of orthotropy (L, T) of the material m, the matrix of flexibility $\boldsymbol{S}$ is expressed by:

$$\boldsymbol{S}_{(m)}\big|_{(L,T)} = \begin{bmatrix} \dfrac{1}{E_{\mathrm{LL}}} & -\dfrac{\nu_{LT}}{E_{TT}} & 0 \\ -\dfrac{\nu_{TL}}{E_{TT}} & \dfrac{1}{E_{TT}} & 0 \\ 0 & 0 & \dfrac{1}{G_{\mathrm{LT}}} \end{bmatrix}_{(m)}$$

with

$$\frac{\nu_{LT}}{E_{LL}} = \frac{\nu_{TL}}{E_{TT}}$$

the stiffness $\boldsymbol{\Lambda}_{(m)} = \boldsymbol{S}_{(m)}^{-1}$ being:

$$\boldsymbol{\Lambda}_{(m)}\big|_{(L,T)} = \begin{bmatrix} \dfrac{E_{\mathrm{LL}}}{1-\nu_{TL}\cdot\nu_{\mathrm{LT}}} & \dfrac{\nu_{TL}\cdot E_{\mathrm{LL}}}{1-\nu_{TL}\cdot\nu_{\mathrm{LT}}} & 0 \\ \dfrac{\nu_{LT}\cdot E_{TT}}{1-\nu_{TL}\cdot\nu_{\mathrm{LT}}} & \dfrac{E_{TT}}{1-\nu_{TL}\cdot\nu_{\mathrm{LT}}} & 0 \\ 0 & 0 & G_{\mathrm{LT}} \end{bmatrix}_{(m)}$$

The stiffness in transverse shears is expressed for its part in the following way:

$$\boldsymbol{\Lambda}_{\tau(m)}\big|_{(L,T)} = \begin{bmatrix} G_{LZ} & 0 \\ 0 & G_{TZ} \end{bmatrix}_{(m)}$$

While placing oneself in the reference of the element $(\boldsymbol{V}_1, \boldsymbol{V}_2)$, one uses the transition matrix $\boldsymbol{P}^{(m)}$ of the strain tensor defined in [§3] of $(\boldsymbol{V}_1, \boldsymbol{V}_2)$ worms orthotropic reference:

$$\boldsymbol{P}^{(m)}=\begin{bmatrix} C^2 & S^2 & 2CS \\ S^2 & C^2 & -2CS \\ -CS & CS & C^2-S^2 \end{bmatrix} \quad \text{where} \quad \begin{matrix} C=\cos(\varphi_i+\varphi_0) \\ S=\sin(\varphi_i+\varphi_0) \end{matrix}$$

In the same way the vector thermal expansion is expressed in the reference $(\boldsymbol{V}_1, \boldsymbol{V}_2)$:

$$\boldsymbol{d}^{(m)}=\begin{pmatrix} d_{11} \\ d_{22} \\ d_{12} \end{pmatrix}=\boldsymbol{P}^{m^{-1}}\begin{pmatrix} d_{LL} \\ d_{TT} \\ 0 \end{pmatrix}_{(L,T)}=\begin{pmatrix} C^2 & S^2 \\ S^2 & C^2 \\ 2CS & -2CS \end{pmatrix}\begin{pmatrix} d_{LL} \\ d_{TT} \end{pmatrix}_{(L,T)}$$

One thus has in the layer n (material: m), in x_3 :

$$s_{(n)}=\boldsymbol{P}_s^{(m)^{-1}}.\boldsymbol{\Lambda}|_{(L,T)}.\boldsymbol{P}^{(m)}.(e(u)-e^{th})={}^T\boldsymbol{P}^{(m)}.\boldsymbol{\Lambda}|_{(L,T)}.\boldsymbol{P}^{(m)}.(e(u)-e^{th})=\boldsymbol{\Lambda}_{(\boldsymbol{m})}(e(u)-e^{th})$$

with:

$$\varepsilon(u)=\begin{pmatrix} E_{11} \\ E_{22} \\ E_{12} \end{pmatrix}+x_3\begin{pmatrix} K_{11} \\ K_{22} \\ K_{12} \end{pmatrix} \text{et } \varepsilon^{th}=\begin{pmatrix} d_{11} \\ d_{22} \\ d_{12} \end{pmatrix}.(T(x_3)-T^{réf})$$

Note:

In the code, one chose to carry out the transition of orthotropic reference with the reference of the element in two stages. A first stage relates to the transition of orthotropic reference with the reference defined by `ANGL_REP` *. The data of* `DEFI_MATERIAU` *are thus transformed during this first transition. One treats then the equivalent material as one would do it with classical shell elements.*
The processing of thermal thermal expansion is made in the form of a contribution to the second member of the matric equation solve resulting from the principle of virtual work. This contribution is written: $\sigma^{th(n)}=-{}^T\boldsymbol{P}^{(m)}.\boldsymbol{\Lambda}|_{(L,T)}.\begin{Bmatrix} d_{LL}\Delta T \\ d_{TT}\Delta T \\ 0 \end{Bmatrix}$.

4.3.2 Transverse shears

the stiffness in transverse shears of each layer are written in the reference $(\boldsymbol{V}_1, \boldsymbol{V}_2)$ in the same way that thermal expansion:

$$\boldsymbol{\Lambda}_{t(m)}|_{(V_1,V_2)}={}^t\boldsymbol{P}_2^{(m)}.\boldsymbol{\Lambda}_{t(m)}.\boldsymbol{P}_2^{(m)}$$

with $\boldsymbol{P}_2^{(m)}=\begin{bmatrix} C & S \\ -S & C \end{bmatrix}$ vectorial transition matrix of $(\boldsymbol{V}_1, \boldsymbol{V}_2)$ worms orthotropic reference.

The stiffness in transverse shears total of the shell $\left[R_c\right]$ calculated so as to be equal to that given by the model of three-dimensional elasticity [bib2], the matrix $\left[\boldsymbol{R}_c\right]$ is defined so that the surface density of energy of transverse shears U_2 obtained for a three-dimensional distribution of the stresses σ_{13} and σ_{23} is identical to that associated with the model of plate of noted REISSNER-MINDLIN U_2 .

$$U_1=\frac{1}{2}\int_{-h}^{h}\langle\tau\rangle\left[\boldsymbol{\Lambda}_{\tau(m)}\right]^{-1}\{\tau\}\,d_3 \qquad \langle\tau\rangle=\langle\sigma_{13}\sigma_{23}\rangle$$

$$U_2=\frac{1}{2}\boldsymbol{V}\left[\boldsymbol{R}_c\right]^{-1}\boldsymbol{V}=\frac{1}{2}\left(\int_{-h}^{h}\{\tau\}\,d_3\right)\quad\left[\boldsymbol{H}_c\right]^{-1}\left(\int_{-h}^{h}\{\tau\}\,d_3\right)$$

with the balance equations: $\begin{cases}\sigma_{13}=-\int_{-h}^{x_3}\left(\sigma_{11,1}+\sigma_{12,2}\right)d_3\\ \sigma_{23}=-\int_{-h}^{x_3}\left(\sigma_{12,1}+s_{22,2}\right)d_3\end{cases}$

and the conditions: $0=\sigma_{13}=\sigma_{23}$ for $x_3=\pm h$.

The plane stresses $\sigma_{11},\sigma_{22},\sigma_{12}$ are expressed according to the forces resulting by making the assumption from pure bending and absence from coupling membrane/bending. It results from it that:

$$\sigma\left(x_3\right)=x_3.\boldsymbol{\Lambda}_{(m)}(x_3)\boldsymbol{P}^{-1}.\boldsymbol{M} \quad\text{and}\quad \boldsymbol{A}\left(x_3\right)=\boldsymbol{\Lambda}_{(m)}(x_3)\,\boldsymbol{P}^{-1}$$

where $\boldsymbol{P}$ is the stiffness matrix of bending of the group of multi-layer defined by [éq 2.3-5].
These computations, as well as the following are to be carried out in a single reference. One chooses in *the Code_Aster* the intrinsic reference with the element. It is thus necessary A to transform the matrix in this reference.

One has then: $\left\{\tau\left(x_3\right)\right\}=\boldsymbol{D}_1\left(x_3\right)\boldsymbol{V}+\boldsymbol{D}_2\left(x_3\right)\{\lambda\}$

with $\boldsymbol{V}=\langle M_{11,1}+M_{12,2};M_{12,1}+M_{22,2}\rangle$

$\langle\lambda\rangle=\langle M_{11,1}-M_{12,2};M_{12,1}-M_{22,2};M_{22,1};M_{11,2}\rangle$

and

$$\boldsymbol{D}_1=\int_{-x_3}^{h}-\frac{z}{2}\begin{bmatrix}A_{11}+A_{33} & A_{13}+A_{32}\\ A_{31}+A_{23} & A_{22}+A_{33}\end{bmatrix}dz$$

$$\boldsymbol{D}_2=\int_{-x_3}^{h}-\frac{z}{2}\begin{bmatrix}A_{11}-A_{33} & A_{13}-A_{32} & 2\mathrm{A}_{12} & 2\mathrm{A}_{31}\\ A_{31}-A_{23} & A_{33}-A_{22} & 2\mathrm{A}_{32} & 2\mathrm{A}_{21}\end{bmatrix}dz$$

U_1 is thus written: $U_1=\frac{1}{2}\langle\boldsymbol{V}|\lambda\rangle\begin{bmatrix}\boldsymbol{C}_{11} & \boldsymbol{C}_{12}\\ \boldsymbol{C}_{12^T} & \boldsymbol{C}_{22}\end{bmatrix}\begin{Bmatrix}\boldsymbol{V}\\ \hline \lambda\end{Bmatrix}$

with
$$\underset{2\times2}{\mathbf{C}_{11}}=\int_{-h}^{h}\mathbf{D}_{1^T}\boldsymbol{\Lambda}_{\tau(m)}^{-1}\mathbf{D}_1 d_3$$
$$\underset{2\times4}{\mathbf{C}_{12}}=\int_{-h}^{h}\mathbf{D}_{1^T}\boldsymbol{\Lambda}_{\tau(m)}^{-1}\mathbf{D}_2 d_3$$
$$\underset{4\times4}{\mathbf{C}_{22}}=\int_{-h}^{h}\mathbf{D}_{2^T}\boldsymbol{\Lambda}_{\tau(m)}^{-1}\mathbf{D}_2 d_3$$

from where

$$U_1=U_2\Leftrightarrow\langle V|\lambda\rangle\begin{bmatrix}\mathbf{C}_{11}-\mathbf{H}_c^{-1} & \mathbf{C}_{12}\\ \mathbf{C}_{12^T} & \mathbf{C}_{22}\end{bmatrix}\begin{Bmatrix}V\\ \lambda\end{Bmatrix}=0\quad\forall V,\{\lambda\}$$

one thus suggests the solution $\boldsymbol{H}_c=\boldsymbol{C}_{11^{-1}}$.

The coefficients of transverse correction of shears correspond to the ratio of the terms of $\boldsymbol{H}_c$ to the integral on the thickness of the laminate of the terms of $\boldsymbol{\Lambda}_{\tau(m)}$.

4.3.3 Generalized forces

the forces generalized definite with [§1.3] and put in a vectorial form are obtained by integration in the thickness of the shell by adding the contributions of the layers (of thickness $\boldsymbol{e}_n=x_3^n-x_3^{n-1}$):

$$\mathbf{M}=\begin{pmatrix}M_{11}\\ M_{22}\\ M_{12}\end{pmatrix}=\int_l\sigma\cdot x_3\cdot dx_3=\sum_{n=1}^{N_{couch}}\int_{x_3^{n-1}}^{x_3^n}\sigma_{(n)}\cdot x_3\cdot dx_3$$

If one expresses like previously (with m

$$\mathbf{N}=\begin{pmatrix}N_{11}\\ N_{22}\\ N_{12}\end{pmatrix}=\int_l\sigma\cdot dx_3=\sum_{n=1}^{N_{couch}}\int_{x_3^{n-1}}^{x_3^n}\sigma_{(n)}\cdot dx_3$$

material of the layer n):

$$\boldsymbol{\sigma}_{(n)}=\boldsymbol{\Lambda}_{(m)}\cdot\left(\mathbf{E}+x_3\cdot\mathbf{K}-\mathbf{d}_{(\mathbf{m})}\left(T(x_3)-T^{réf}\right)\right)$$

one can note the forces generalized in the form: (cf [§1.3])

$$\mathbf{M}-\mathbf{M}^{th}=\mathbf{P}\cdot\mathbf{K}+\mathbf{Q}\cdot\mathbf{E}$$
$$\mathbf{N}-\mathbf{N}^{th}=\mathbf{Q}\cdot\mathbf{K}+\mathbf{R}\cdot\mathbf{E}$$

with $\boldsymbol{P},\boldsymbol{Q},\boldsymbol{R}$ matrixes 3 X 3 being expressed by:

$$\mathbf{P}=\sum_{n=1}^{N_{couch}} \boldsymbol{\Lambda}_{(m)} \int_{x_3^{n-1}}^{x_3^n} x_3^2 \cdot dx_3 \quad =\sum_{n=1}^{N_{couch}} \boldsymbol{\Lambda}_{(m)} \cdot \frac{1}{3} \cdot \left(\left(x_3^n \right)^3 - \left(x_3^{n-1} \right)^3 \right)$$

$$\mathbf{Q}=\sum_{n=1}^{N_{couch}} \boldsymbol{\Lambda}_{(m)} \int_{x_3^{n-1}}^{x_3^n} x_3 \cdot dx_3 \quad =\sum_{n=1}^{N_{couch}} \boldsymbol{\Lambda}_{(m)} \cdot \frac{1}{2} \cdot \left(\left(x_3^n \right)^2 \cdot \left(x_3^{n-1} \right)^2 \right)$$

$$\mathbf{R}=\sum_{n=1}^{N_{couch}} \boldsymbol{\Lambda}_{(m)} \cdot \left(x_3^n - x_3^{n-1} \right) \quad =\sum_{n=1}^{N_{couch}} \boldsymbol{\Lambda}_{(m)} \cdot e_n$$

the shears $\boldsymbol{V}$ are obtained by derivative of the moment [§4.3.2].

The generalized forces of thermal origin are calculated directly:

$$\boldsymbol{M}^{th}=\sum_{n=1}^{N_{couch}} \boldsymbol{\Lambda}_{(m)} . \int_{x_3^{n-1}}^{x_3^n} x_3 . \left(T(x_3)-T^{réf}\right) . \boldsymbol{d}_{(m)} . dx_3$$

$$\boldsymbol{N}^{th}=\sum_{n=1}^{N_{couch}} \boldsymbol{\Lambda}_{(m)} . \int_{x_3^{n-1}}^{x_3^n} \left(T(x_3)-T^{réf}\right) . \boldsymbol{d}_{(m)} . dx_3$$

4.3.4 Localization of the stresses (postprocessing)

Conversely, following a computation by finite element and of obtaining the strains $\boldsymbol{E}$ and variations of curvature $\boldsymbol{K}$, one can then calculate the stress field $\sigma_{(n)}(n=1, N_{couch})$ in each layer of the element.

It is necessary to calculate in each layer (n), the matrix $\boldsymbol{\Lambda}_{(m)}$ and the terms $\left(T(x_3)-T^{réf}\right) . \boldsymbol{d}_{(m)}$ (cf [§3.2]) ($m=mat_n$ represents the characteristics material of the layer n).

The stresses $\sigma_{\alpha\beta}$ with an Y-coordinate $x_3 \in]x_3^{n-1}, x_3^n[$ in the layer (n) are then:

$$\sigma_{(n)}(x_3)=\boldsymbol{\Lambda}_{(m)} . \left[\boldsymbol{E}+x_3 . \boldsymbol{K}-\boldsymbol{d}_{(m)}\left(T(x_3)-T_{réf}\right)\right]$$

and transverse shears:

$$\tau_{(n)}(x_3)=D_1(x_3) . \boldsymbol{V}+D_2(x_3) . \lambda \qquad \textbf{éq 4.3.4-1}$$

Note::

In the code postprocessings of the shell elements are generally defined in the reference associated with `ANGL_REP` . The stresses in the intrinsic reference of the element are thus brought back in the reference of the variety. One a:

$$\begin{pmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{pmatrix}_{eref}=\begin{pmatrix}C^2 & S^2 & +2CS\\ S^2 & C^2 & -2CS\\ -CS & +CS & C^2-S^2\end{pmatrix}\begin{pmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{pmatrix}_n$$

where $C=\cos(\varphi_0)$, $S=\sin(\varphi_0)$ (cf.[§4.1])

where φ_0 *is the angle enters* $\boldsymbol{V}_1$ *and* $\boldsymbol{e}_{réf}$

4.3.5 Computation of the rupture criteria in the layers (postprocessing)

the limiting values of breaking stresses depend on the material of the layer, the direction and the meaning of the request (for a group of elements corresponding to the same field material):

mat_n:

- X : limite en traction dans le sens L (1ère direction orthotropie : sens des fibres)
- X' : limite en compression dans le sens L (1ère direction orthotropie : sens des fibres)
- Y : limite en traction dans le sens T (2ème direction orthogonale à la 1ère)
- Y' : limite en compression dans le sens T (2ème direction orthogonale à la 1ère)
- S : limite en cisaillement dans le sens LT

It is necessary to calculate the stresses in the reference of the layer (defined by the axes of orthotropy) starting from the stresses in the reference of the element:

the angle enters $\mathbf{V}_1$ and $\boldsymbol{e}_{réf}$ is φ_0 , and that enters $\boldsymbol{e}_{réf}$ and orthotropic reference is φ_n :

$$\begin{pmatrix}\sigma_L\\ \sigma_T\\ \sigma_{¿}\end{pmatrix}_n=\begin{pmatrix}C^2 & S^2 & +2\,CS\\ S^2 & C^2 & -2\,CS\\ -CS & +CS & C^2-S^2\end{pmatrix}\begin{pmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{pmatrix}_n \quad \text{where} \quad \begin{array}{c}C=\cos\left(\varphi_0\right)\\ S=\sin\left(\varphi_0\right)\quad\left(\text{cf}.\left[\S 4.1\right]\right)\end{array}$$

maximum Criterion of stress:

The 5 following criteria are calculated by layer: ($n=1,\ N-couch$)

$$\frac{\sigma_{L_{(n)}}}{X_{(mat_n)}}\left(\text{si }\sigma_{L_{(n)}}>0\right)\qquad\frac{\sigma_{L_{(n)}}}{X'_{(mat_n)}}\left(\text{si}\,\sigma_{L_{(n)}}<0\right)$$

$$\frac{\sigma_{T_{(n)}}}{Y_{(mat_n)}}\left(\text{si}\,\sigma_{T_{(n)}}>0\right)\qquad\frac{\sigma_{T_{(n)}}}{Y'_{(mat_n)}}\left(\text{si}\,\sigma_{T_{(n)}}<0\right)$$

$$\frac{\left|\sigma_{LT_{(n)}}\right|}{S_{(mat_n)}}$$

Criterion of TSAI-HILL: this criterion is written in each layer in the following way:

$$C_{TH}=\frac{\sigma^2_{L(n)}}{X^2_{(mat_n)}}-\frac{\sigma_{L(n)}.\sigma_{T(n)}}{X^2_{(mat_n)}}+\frac{\sigma^2_{T(n)}}{Y^2_{(mat_n)}}+\frac{\sigma^2_{LT(n)}}{S^2_{(mat_n)}}$$

The material is broken when $C_{TH}\geq 1$.

The values X and Y are replaced by X' and Y' when the corresponding $\left(\sigma_{L(n)}\ ,\ \sigma_{T(n)}\right)$ stresses are negative.

5 Bibliography

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6 Description of the versions of the document

Version Aster	Author (S) Organization (S)	Description of modifications
6.3	P. MASSIN, F. NAGOT, F. VOLDOIRE EDF-R&D/AMA	initial Text

7.4	P. MASSIN, J.M.PROIX-R&D/AMA	