Vibro-acoustic elements

Abstract :

The elements described here make it possible to carry out computations of the frequencies and eigen modes of a structure coupled to a fluid. They also allow the acoustic computation of response.

After the formulation of the problem of fluid-structure coupling, this document describes the approach followed to implement in *Code_Aster* the new finite elements.

Contents

1 Introduction

the vibratory behavior of a structure is often modified if this one is in the presence of a fluid: it is what is called the vibro-acoustic coupling. One distinguishes the cases from coupling in two categories: either the fluid is infinite (it is the case of immersed structures), or the fluid is contained in a limited medium (it is the case of the tanks more or less filled with fluid).

The finite elements described here make it possible to solve the problems of coupling with a fluid of finished size.

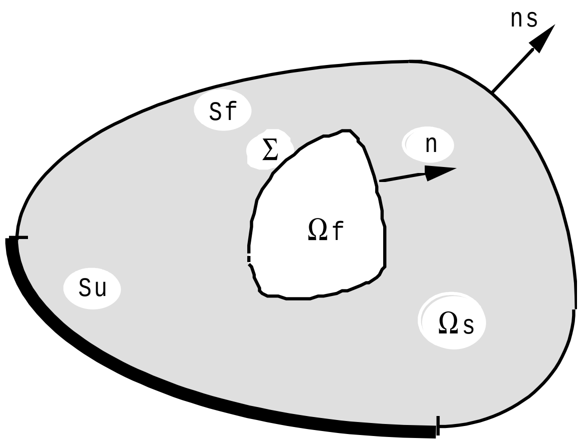
General notations:

P :	instantaneous stagnation pressure in a point of the fluid
p_0 :	pressure at rest
p :	acoustic pressure
ρ_t :	instantaneous total density in a point of the fluid
ρ_0 :	density of the fluid at rest
ρ :	acoustic density
ρ_S :	density of structure
$\mathbf{u}_f$:	acoustic displacement
$\mathbf{u}$:	displacement of structure
Φ :	gradient of acoustic displacements
$\omega \quad f$:	pulsation, frequency
c :	speed of sound in the fluid
$\lambda \quad k$:	wave length, wave number
σ :	tensor of the stresses of structure
ε :	tensor of the structural deformations
C :	elasticity tensor of structure
T :	tensor of the stresses of the fluid.

2 The vibro-acoustic coupling

2.1 Presentation

Is an elastic structure defined in a field Ω_s which vibrates in the presence of a true fluid, nonheavy, compressible, in isentropic evolution defined in a field Ω_f. One indicates by $\Sigma = \Sigma_f \cap \Sigma_S$, their common surface.
One notes n, the norm external with the fluid field Ω_f.



A a given time, the state of the fluid is defined by its field of pressure P and that of structure by its field of displacement $\mathbf{U}$.
It is considered that the coupled system is subjected to small disturbances around its state of equilibrium where the fluid and the structure are at rest.
As follows:
$P = p_0 + p$
$\mathbf{U} = \mathbf{u} (\mathbf{u}_0 = \mathbf{0})$

The problem of interaction fluid-structure then consists in solving two problems simultaneously:

- one in structure subjected, on Σ, to a field of pressure p imposed by the fluid
- the other in the fluid subjected to a field of displacement $\mathbf{u}$ of the wall Σ.

2.2 Formulation of the vibro-acoustic problem

2.2.1 Description of the structure

Assumption:

The structure is homogeneous and obeys the models of linear elasticity.

Taking into account this assumption, one can write the various following equations controlling the state of the structure [bib2].

2.2.1.1 Conservation equation of the linear momentum

the conservation equation of the linear momentum is written, in the absence of volume forces other than the inertia forces:

$$\sigma_{ij,j} - \rho_s \frac{d^2 u_i}{dt^2} = 0 \quad \text{[éq 2.2.1.1 - 1]}$$

where : ρ_s is the density of structure,
$\mathbf{u}$ is displacement,
σ_{ij} is the tensor of the stresses.

2.2.1.2 Relation of compatibility

$$\varepsilon_{kl} = \frac{1}{2}(u_{k,l} + u_{l,k}) \quad \text{[éq 2.2.1.2 - 1]}$$

where ε_{kl} is the tensor of the strains.

2.2.1.3 Constitutive law in isotropic linear elasticity

$$\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl} \quad \text{[éq 2.2.1.3 - 1]}$$

with the elasticity moduli C_{ijkl} checking the identities: $C_{ijkl} = C_{klij} = C_{jikl} = C_{ijlk}$
C being the elasticity tensor.

2.2.2 Description of the fluid

Assumption: the fluid obeys the models of the linear acoustics.

Taking into account this assumption, the equations controlling the state of the fluid are:

2.2.2.1 Conservation equation of the linear momentum

the conservation equation of the linear momentum is written, in the absence of sources:

$$T_{ij,j} - \rho_0 \frac{d^2 u_{fi}}{dt^2} = 0 \quad \text{[éq 2.2.2.1 - 1]}$$

where: T_{kl} is the tensor of the stresses in the fluid,
ρ_0 is the density of the fluid in a natural state,
x is the field of displacement of a fluid particle.

2.2.2.2 Conservation equation of the mass

To the first order and in the absence of acoustic sources, it is expressed by the relation:

$$\frac{\partial \rho}{\partial t} + \rho_0 \operatorname{div}\left(\frac{\partial \mathbf{u}_f}{\partial t}\right) = 0 \quad \text{[éq 2.2.2.2 - 1]}$$

2.2.2.3 Constitutive law

$$T_{ij}=-p\,\delta_{ij}$$ [éq 2.2.2.3 - 1]

the fluid is supposed in evolution barotropic (the pressure p is, for the fluid given, a known function of the only density) $p=\rho c_0^2$;

where c_0 is the speed of sound in the fluid at rest.

2.2.2.4 Equation of propagation of the waves or equation of Helmholtz

One deduces it by combination from the conservation equations from the mass [éq 2.2.2.2 - 1] and from the linear momentum [éq 2.2.2.1 - 1] written in harmonic mode, with the pulsation ω :

$$\Delta p+k^2 p=0$$ [éq 2.2.2.4 - 1]

where $k=\omega/c$ is the wave number.

2.2.3 Description of the interaction fluid-structure

To the interface fluid-structure (Σ), the fluid being nonviscous, it does not adhere to the wall. One thus writes:

- the continuity of the normal stresses:

$$\sigma_{ij}\cdot n_i=T_{ij}\cdot n_i=-p\,\delta_{ij}\cdot n_i$$ [éq 2.2.3-1]

- the continuity normal velocities:

$$\frac{du_i}{dt}\cdot n_i=\frac{dx_i}{dt}\cdot n_i$$ [éq 2.2.3-2]

2.2.4 Formulation of the problem coupled

Ultimately, the formulation of the problem of vibro-acoustic in terms of displacements for structure and pressure in the fluid led to the equations of the harmonic problem (P):

$$C_{ijkl}\cdot u_{k,lj}+\omega^2\rho_S u_i=0 \text{ in } \Omega_S$$ [éq 2.2.4-1]

$$\Delta p+k^2 p=0 \quad \text{in } \Omega_f$$

$$C_{ijkl}\cdot u_{k,l}\cdot n_i=-p\,\delta_{ij}\cdot n_i \quad \text{on } \Sigma$$

$$u_i\cdot n_i=\frac{1}{\rho_0\omega^2}\frac{\partial p}{\partial n} \quad \text{on } \Sigma$$

2.3 variational Equations associated with the problem

One solves the problem coupled by means of the finite element method from the weak formulation of the problem.

2.3.1 Variational equations associated with structure

Is δu , kinematically admissible in Ω_S , the equation [éq 2.2.4-1] can be written in the integral form:

$$\int_{\Omega_S} [C_{ijkl} \cdot u_{k,li} \delta u_i + \omega^2 \rho_S u_i \delta u_i] dV = 0$$

After integration by part, one obtains the weak formulation:

$$\int_{\Omega_S} [C_{ijkl} \cdot u_{k,l} \delta u_{i,j} - \omega^2 \rho_S u_i \delta u_i] dV - \int_{\Sigma} C_{ijkl} \delta u_i \delta u_{k,l} \cdot n_i^S dS = 0$$

Maybe, if one takes into account the boundary condition [eq 2.2.3 - 1]:

$$\int_{\Omega_S} [C_{ijkl} \cdot u_{k,l} \delta u_{i,j} - \omega^2 \rho_S u_i \delta u_i] dV - \int_{\Sigma} p \delta u_i \cdot n_i dS = 0 \quad \text{[éq 2.3.1-1]}$$

2.3.2 variational Equation associated with the equation of the fluid

Is δp , kinematically admissible in Ω_f. One writes in variational form the equation [éq 2.2.2.4 - 1]:

$$\int_{\Omega_f} [\Delta p \delta p + k^2 p \delta p] dV = 0$$

$$\Leftrightarrow \int_{\Omega_f} \operatorname{div}(\delta p \cdot \nabla p) dV - \int_{\Omega_f} \nabla p \cdot \nabla \delta p \, dV + k^2 \int_{\Omega_f} p \delta p \, dV = 0$$

$$\Leftrightarrow \int_{\Sigma_f} \delta p \cdot \frac{dp}{\partial n} dS - \int_{\Omega_f} \nabla p \cdot \nabla \delta p \, dV + k^2 \int_{\Omega_f} p \delta p \, dV = 0$$

Maybe, if one takes into account the boundary condition [éq 2.2.3-1]

$$\frac{1}{\rho_0 \omega^2} \int_{\Omega_f} [\nabla p \cdot \nabla \delta p - k^2 p \cdot \delta p] dV - \int_{\Sigma} u_n \cdot \delta p \, dS = 0 \quad \text{[éq 2.3.2-1]}$$

2.4 Discretization by finite elements

the approximation by finite elements of the complete problem leads then to the system:

$$\begin{bmatrix} \mathbf{K}-\omega^2\mathbf{M} & -\mathbf{C} \\ -\mathbf{C}^T & \dfrac{\mathbf{H}}{\rho_0\omega^2}-\dfrac{\mathbf{Q}}{\rho_0 c^2} \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ p \end{bmatrix}=0$$

maybe

$$\begin{bmatrix} K & -C \\ 0 & H \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} - \omega^2 \begin{bmatrix} M & 0 \\ \rho_0 C^T & \dfrac{Q}{c^2} \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix}=0$$

where: $\mathbf{K}$ and $\mathbf{M}$ are the stiffness matrixes and of mass of structure,
$\mathbf{H}$ and $\mathbf{Q}$ are the fluid matrixes obtained respectively starting from the bilinear forms:

$$\int_{\Omega_f} \nabla p \cdot \nabla \delta p \, dV \quad \text{and} \quad \int_{\Omega_f} p \cdot \delta p \, dV$$

$\mathbf{C}$ is the matrix of coupling obtained from the bilinear form $\int_{\Sigma_f} p \cdot u_n dS$.

The choice of the formulation led to an asymmetric system matric what does not make it possible to use the classical algorithms of resolution.

2.5 Choice of an additional variable for the description of the fluid

2.5.1 Formulation of the new problem

to obtain a symmetric problem, one associates with the variable of pressure, an additional variable. This new variable is, either the potential of displacement of the fluid Φ such as $\mathbf{u}_f=\mathbf{grad}\,\Phi$ [bib1], [bib4], [bib5], or the variable π [bib3] such as $\pi=-\rho \cdot \Phi$. The variable π makes it possible to directly take into account the fluids with variable density. However, it does not represent anything physically. This is why, the potential of displacements is preferred to him.
One thus replaces the displacement $\mathbf{u}_f$ of the fluid by $\mathbf{grad}\,\Phi$ in the equations of the problem (P) [§ 2.2.4]. One thus obtains the new problem to be solved (P') :

$C_{ijkl} \cdot u_{k,lj}+\omega^2 \rho_S u_i=0$ in Ω_s	
$\rho_0 \omega^2 \Delta \Phi + k^2 p=0 \quad \Omega_f$	[éq 2.5.1-1]
$p=\rho_0 \omega^2 \Phi$ in Ω_f	[éq 2.5.1-2]
$C_{ijkl} \cdot u_{k,l} \cdot n_i=-\rho_0 \omega^2 \Phi \delta_{ij} \cdot n_j$ on Σ	[éq 2.5.1-3]
$u_i \cdot n_i=\dfrac{\partial \Phi}{\partial n}$ to Σ	[éq 2.5.1-4]

2.5.2 variational Formulation associated with the problem (P')

One applies to the equation [éq 2.5.1-1] the formula of GREEN:

$$\int_{\Omega_f} [\rho_0 \omega^2 \Delta \Phi + k^2 p] \Psi \, dV = 0 \quad \forall \Psi \, c.a.$$

$$\Leftrightarrow \int_{\Omega_f} [k^2 p \Psi - \rho_0 \omega^2 \, grad \Psi \, grad \Phi] dV + \int_{\Sigma_f} \Psi \rho_0 \omega^2 \frac{\partial \Phi}{\partial n} dS = 0 \quad \forall \Psi \, c.a.$$

Is, if one takes into account the boundary condition [éq 2.5.1-3]

$$\Leftrightarrow \int_{\Omega_f} \frac{p \Psi}{\rho_0 c^2} dV - \int_{\Omega_f} grad \Phi \, grad \Psi \, dV + \int_{\Sigma} \Psi u_n dS = 0 \quad \forall \Psi \, c.a.$$ [éq 2.5.2 - 1]

Moreover, one writes in weak form the equation [éq 2.5.1 - 2] for $\forall q \, c.a.$,

$$\int_{\Omega_f} (p - \rho_0 \omega^2 \Phi) q \, dV = 0 \Leftrightarrow \forall q \, c.a. \int_{\Omega_f} \frac{pq}{\rho_0 c^2} dV - \omega^2 \int_{\Omega_f} \frac{\Phi q}{c^2} dV = 0$$ [éq 2.5.2 – 2]

By adding the equations [éq 2.5.2 - 1] and [éq 2.5.2 - 2], one obtains the variational equation associated with the fluid:

$$\int_{\Omega_f} \frac{pq}{\rho_0 c^2} dV - \rho_0 \omega^2 \left[\int_{\Omega_f} \frac{\Phi q + p \Psi}{\rho_0 c^2} dV - \int_{\Omega_f} \frac{grad \Phi \, grad \Psi}{\rho_0 c^2} dV + \int \Psi u_n dS \right] = 0$$

$$\forall (q, \Psi) \, c.a.$$ [éq 2.5.2 - 3]

2.6 Discretization by finite elements

While proceeding with the same approach as that used in [§ 2.3], one is led to the following matric system:

$$\begin{bmatrix} \mathbf{K} & 0 & 0 \\ 0 & \frac{\mathbf{M}_f}{\rho_0 c^2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ p \\ \Phi \end{bmatrix} - \omega^2 \begin{bmatrix} \mathbf{M} & 0 & \rho_0 \mathbf{M}_\Sigma \\ 0 & 0 & \frac{\mathbf{M}_{fl}}{c^2} \\ \rho_0 \mathbf{M}_\Sigma^T & \frac{\mathbf{M}_{fl}^T}{c^2} & \rho_0 \mathbf{H} \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ p \\ \Phi \end{bmatrix} = 0$$

$\mathbf{K}$ and $\mathbf{M}$ being respectively stiffness matrixes and of mass of structure,
$\mathbf{M}_\Sigma$ being the matrix of fluid-structure coupling obtained from the bilinear form:

$$\int_\Sigma \Phi u \, dS$$

and

$\mathbf{M}_f$ $\mathbf{M}_{fl}$, $\mathbf{H}$ being fluid matrixes, respectively obtained starting from the bilinear forms:

$$\int_{\Omega_f} p^2 dV, \ \int_{\Omega_f} p \Phi \, dV \text{ and } \int_{\Omega_f} (grad \Phi)^2 dV$$

2.7 Computations of acoustic response

2.7.1 Velocity imposed on the fluid

On part Σ_v of the fluid border Σ_f, one can impose a limiting condition standard normal velocity v_0. The term of edge of the fluid is written then:

$$-\rho_0\omega^2\int_{\Sigma_f}\Psi\cdot u_n\,ds=-\rho_0\omega^2\int_{\Sigma_f-\Sigma_z}\Psi\cdot u_n\,ds+i\,\omega\,\rho_0\int_{\Sigma_v}\Psi\,v_0\,ds$$

2.7.2 Impedance imposed on a wall of the fluid

On part Σ_z of the fluid border Σ_f, one can impose a limiting condition of standard impedance Z :

$$p=Z\,v_n$$

where v_n is the outgoing normal velocity of the fluid.

By deferring this condition in the equation representing the conservation of the linear momentum [éq 2.2.2.1 - 1] and by taking account of the constitutive law of the fluid [éq 2.2.2.3 - 1], one a:

$$grad\;p+\frac{\rho_0}{z}\dot{p}=0 \qquad \text{[éq 2.7.2 - 1]}$$

to preserve the symmetry of the system, one expresses the equation [eq 2.7.2 - 1] according to the potential of displacement of the fluid Φ, one a:

$$grad\,\ddot{\Phi}+\frac{\rho_0}{z}\frac{\partial^3\phi}{\partial t^3}=0$$

Is, in harmonic:

The term of edge of the fluid is written then:

$$\rho_0\omega^2\int_{\Sigma_f}\Psi\cdot\frac{\partial\Psi}{\partial n}ds=\rho_0\omega^2\int_{\Sigma_f-\Sigma_z}\Psi\cdot\frac{\partial\Psi}{\partial n}ds+i\,\omega^3\int_{\Sigma_v}\frac{\rho_0^2}{z}\Phi\,\Psi\,ds$$

Ultimately, to impose an impedance on a wall of the fluid amounts introducing into the system a term of damping.

2.7.3 Discretization by finite elements

If one imposes limiting conditions of standard imposed velocity or impedance of wall imposed on the fluid, one is led to solve the following matric system:

$$\begin{bmatrix} \mathbf{K} & 0 & 0 \\ 0 & \frac{\mathbf{M}_f}{\rho_0 c^2} & 0 \\ 0 & 0 & 0 \end{bmatrix} - \omega^2 \begin{bmatrix} \mathbf{M} & 0 & \rho_0 \mathbf{M}_\Sigma \\ 0 & 0 & \frac{\mathbf{M}_{fl}}{c^2} \\ \rho_0 \mathbf{M}_\Sigma^T & \frac{\mathbf{M}_{fl}^T}{c^2} & \rho_0 \mathbf{H} \end{bmatrix} + i\omega^3 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{\rho_0^2}{Z}\mathbf{Q} \end{bmatrix} = \begin{bmatrix} 0 \\ i\omega \\ \mathbf{V} \end{bmatrix}$$

$\mathbf{Q}$ being the matrix obtained from the bilinear form: $\int_{\Sigma_z} \Phi^2 \, dS$ and $\mathbf{V}$, the vector obtained from $\int_{\Sigma_V} \rho_0 v_0 \Phi \, dS$.

3 Integration in *Aster*

the elements described previously belong, for the fluid part, with modelization "3D_FLUIDE" of the MECHANICAL phenomenon and, for the interface fluid-structure, with modelization "FLUI_STRU" of the same phenomenon.

They lead to voluminal or surface elements, for the fluid part, in pressure-potential of displacement and with surface elements for the interface fluid-structure in potential of displacement of fluid-displacement of structure.

4 Bibliography

1) T. BALANNEC, S. COURTIER-ARNOUX, E. LUZZATO, P. THOMAS: Confrontation of numerical tools of department AMV on a fluid test of coupling - structure. Internal report E.D.F. D.E.R. HP-61/91.012. P. GERMAIN

2) : Introduction to the mechanics of the continuums. Masson. R.J. GIBERT

3) : Vibrations of structures. Interactions with the fluids. Random sources of excitation. Collection of the Management of the Studies and Search for E.D.F. n°69. R. OHAYON, R.

4) VALID: True symmetric formulations of free vibrations of fluid - structure interaction. Applications and extensions. International conference one numerical method for coupled systems. P. THOMAS: Formulation

5) of fluid-structure coupling for the modal analysis in SIVA. Application to the three-dimensional elements. Internal report E.D.F. D.E.R. HP-35.82/259. Description of the versions

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