Fluid-structure coupling with free surface

Summarized:

One has here the fluid coupling/structure if the fluid has a free surface. Surface elements free were established in *Code_Aster* to compute: the modes of ballotement of a fluid coupled to an elastic structure for a three-dimensional problem.

Contents

1 Notations

P	:	steady pressure in the fluid
p	:	fluctuating pressure in the fluid
$\mathbf{X_f}$	:	displacements in the fluid
$\mathbf{X_s}$	:	the field of displacements in structure
g	:	gravity
φ	:	potential of displacements of the fluid
ρ_f, ρ_s	:	density of the fluid, structure
$\mathbf{T}$	:	the tensor of the stresses in the fluid
$\boldsymbol{\sigma}$	:	the tensor of the stresses in structure
$\boldsymbol{\varepsilon}$	:	the tensor of the strains in structure
$\mathbf{C}$	:	the elasticity tensor
c	:	the speed of sound in the fluid
H	:	the height fluid (or average height)
$\mathbf{n}$	:	the norm external of the fluid.

2 Introduction

In order to studied the structure behavior filled with fluid, one can be resulted in taking into account the phenomena of shaking i.e. to add to the fluid system coupled - structure, the effect of gravity on the level of the free surface of the fluid. The structures concerned are, for example, the tanks of nuclear power plants of the fast reactor core, the swimming pools of fuel storage [bib4].

One thus supplemented the developments already carried out in fluid-structure coupling [bib3] by the introduction of new surface elements which take into account, in their formulation, the effect of gravity.

3 Theoretical formulation of the problem

the problem of interaction heavy structure-fluid amounts solving three problems simultaneously:

- the structure is subjected to a field of pressure P imposed by the fluid on the wall Σ ;
- the fluid is subjected to a field of displacement $\mathbf{X_s}$ imposed by structure on Σ ;
- gravity acts on free surface where $p=\rho g z$.

It will be considered initially that the fluid is nonheavy before introducing gravity in the paragraph [§3.2].

3.1 Recalls on fluid-structure coupling

In order to giving an account of the interaction fluid-structure well, we will analyze the equations separately governing the behavior of the fluid and those which govern that of structure, without considering in this chapter the boundary conditions relating to free surface.

3.1.1 Description of the fluid

One considers that the studied system is subjected to small disturbances around its state of equilibrium where the fluid and the structure are at rest: thus, $P=p_0+p$ and $\mathbf{X_s}=\mathbf{x_s}(x_0=0)$. What makes it possible to write [bib2]:

$$\rho=-\rho_f \operatorname{div}(\mathbf{x_f}) \text{ from where } p=-\rho_f c^2 \operatorname{div}(\mathbf{x_f}) .$$

With:

- p fluctuating pressure of the fluid,
- ρ the disturbance of density of the fluid, ρ_f density of the fluid at rest,
- $\mathbf{x_f}(\mathbf{r},t)$ the field of displacement of a fluid particle.

The fluid is:

- perfect (i.e. nonviscous)
- barotropic:

$$p=\rho c^2 \quad \text{éq 3.1.1-1}$$

- and irrotational: there exists a potential of displacements φ , such as $p=\rho_f \frac{\partial^2 \varphi}{\partial t^2}$

the behavior of the volume of eulerian fluid is thus described by the following equations:

- constitutive law:

$$\mathbf{T}_{ij}=-p\,\boldsymbol{\delta}_{ij}$$

- conservation equation of the linear momentum in the fluid in the absence of source:

$$\mathrm{div}(\overline{\overline{\mathbf{T}}})=\rho_f \frac{\partial^2 \mathbf{x_f}}{\partial t^2} \quad \text{éq 3.1.1-2}$$

- conservation equation of the mass:

$$\frac{\partial \rho}{\partial t}+\rho_f \,\mathrm{div}\left(\frac{\partial \mathbf{x_s}}{\partial t}\right)=0 \quad \text{éq 3.1.1-3}$$

By combining the conservation equations of the linear momentum [éq 3.1.1-2] and the mass [éq 3.1.1-3] written in harmonic mode with the pulsation w, one obtains, thanks to [éq 3.1.1-1], the equation of Helmholtz:

$$\Delta p+\frac{\omega^2}{c^2}p=0$$

3.1.2 Description of structure

One considers that the structure is elastic linear and that one remains in the field of the small disturbances. Taking into account these assumptions, one writes:

- the constitutive law in linear elasticity:

$$\boldsymbol{\sigma}_{ij}=\mathbf{C}_{ijkl}\boldsymbol{\varepsilon}_{kl}$$

- the conservation equation of the linear momentum in structure in the absence of volume forces other than the inertia forces:

$$\mathrm{div}(\overline{\overline{\boldsymbol{\sigma}}})=\rho_s \frac{\partial^2 \mathbf{x_s}}{\partial t^2}$$

- the compatibility equation on the strain tensor:

$$\boldsymbol{\varepsilon}_{kl}=\frac{1}{2}(\mathbf{u}_{\mathbf{k},\mathbf{l}}+\mathbf{u}_{\mathbf{l},\mathbf{k}})$$

3.1.3 Description of the interface fluid-structure

To the interface (Σ) between the fluid and structure, like the fluid is not viscous, there is continuity of the normal stresses and normal velocities to the wall, and nullity of the shear stress (absence of viscous friction). These boundary conditions are written:

$$\begin{cases} \boldsymbol{\sigma}_{ij}\,\mathbf{n_i}=\mathbf{T}_{ij}\,\mathbf{n_i}=-p\,\boldsymbol{\delta}_{ij}\,\mathbf{n_i} \\ \dfrac{\partial\,\mathbf{x_f}}{\partial t}.\mathbf{n}=\dfrac{\partial\,\mathbf{x_s}}{\partial t}.\mathbf{n} \end{cases}$$

3.1.3.1 Formulation of the Finally coupled

problem, the equation of the problem coupled fluid-structure is written, while taking p as variable describing the field of pressure in the fluid and x_s the field of displacements in structure:

$$\begin{cases} \mathbf{C}_{ijkl}\,\mathbf{x}_{\mathbf{s}_{k,lj}}+\omega^2\rho_s\,\mathbf{x}_{\mathbf{s}_i}=0 & \text{dans}\,V_s \\ \Delta p+\dfrac{\omega^2}{c^2}p=0 & \text{dans}\,V_f \\ \boldsymbol{\sigma}_{ij}\,\mathbf{n_i}=\mathbf{C}_{ijkl}\,\mathbf{x}_{\mathbf{s}_{k,l}}\,\mathbf{n_i}=-p\,\boldsymbol{\delta}_{ij}\,\mathbf{n_i} & \text{sur}\,\Sigma \\ \dfrac{\partial p}{\partial n}=\rho_f\,\omega^2\,\mathbf{x}_{\mathbf{f}_i}\,\mathbf{n_i} & \text{sur}\,\Sigma \end{cases}$$

The fields of displacements $\mathbf{x_s}$ for structure and pressure p for the fluid sought minimize the functional calculus:

$$\mathbf{L}(\mathbf{x_s},p,z)=\frac{1}{2}\int_{V_s}\left[\boldsymbol{\sigma}_{ij}(\mathbf{x_s})\boldsymbol{\varepsilon}_{ij}(\mathbf{x_s})-\rho_s\,\omega^2\mathbf{x_s^2}\right]-\int_{\Sigma}p\,\mathbf{x_s}\,\mathbf{n}\,d\,\Sigma+\frac{1}{2\rho_f}\int_{V_f}\left[\frac{1}{\omega^2}(\mathrm{grad}\,p)^2-\frac{p^2}{c^2}\right]dV$$

3.2 Action of gravity on free surface

3.2.1 Formulation of the problem

One points out the linearized dynamic equations here describing small motions of a true fluid. One chooses an eulerian description of the fluid:

$$\mathrm{grad}\,P=\rho_f\left(\mathbf{g}-\frac{\partial^2\mathbf{x_s}}{\partial t^2}\right)\ \text{in}\ V_f$$

With the equilibrium the fluid particle was in M_0 and thus: $\mathrm{grad}\,P_0=\rho_f\,\mathbf{g}$ in V_f .

One considers motions of low amplitude around the state of equilibrium (it is the assumption of the small disturbances): then $\mathbf{M}=\mathbf{M_0}+\mathbf{x_f}\left(\mathbf{M_0},\mathrm{t}\right)$

Are p the fluctuation in eulerian pressure and p_L the fluctuation in Lagrangian pressure, then:

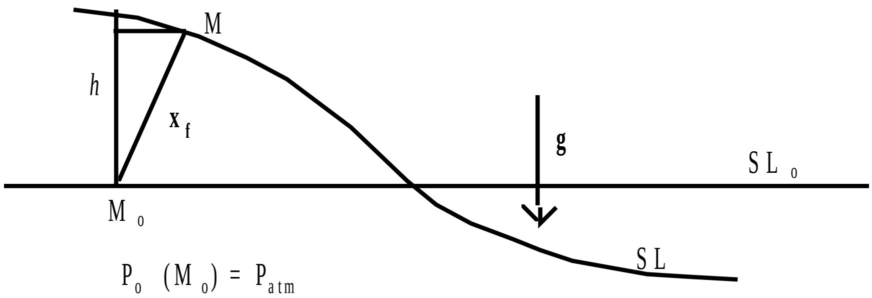
$$p(\mathbf{M},t)=P(\mathbf{M_0},t)-P_0(\mathbf{M_0})$$
$$p_L=P(\mathbf{M},t)-P_0(\mathbf{M_0})$$

Taking into account the assumption of small displacements:

$$p_L - p = \text{gradP}(\mathbf{M_o}, t)\mathbf{x_f}(\mathbf{M_o}, t) = -\rho_f g \mathbf{x_f}(\mathbf{M_o}, t)$$ éq 3.2.1-1

If one considers the case of a heavy fluid having a free surface in contact with a medium to constant pressure P_{atm}, one can write, by neglecting the effects of surface tension:

$P(\mathbf{M}, t) = P_{atm}$ on free surface SL i.e.: $p_L = 0$. Maybe, with [éq 3.2.1-1], $p = \rho_f g(\mathbf{x_f} . \mathbf{z})$
Taking into account the assumption of small motions, the instantaneous slope of the tangent plane is a first order infinitely small. $\mathbf{x_f} . \mathbf{z}$ thus merges with the second order near with vertical rise h.



Appear 3.2.1-a: approximation on free surface

Thus, if one adds with the boundary conditions the condition of gravity on free surface, that amounts considering in $z = H$ the linearized condition:

$$p = \rho_f g z$$ éq the 3.2.1-2

equations of the total problem become:

$$\begin{cases} \mathbf{C}_{ijkl} \mathbf{x}_{\mathbf{s}_{k,lj}} + \omega^2 \rho_s \mathbf{x}_{\mathbf{s}_i} = 0 & \text{dans } V_s \\ \Delta p + \dfrac{\omega^2}{c^2} p = 0 & \text{dans } V_f \\ \boldsymbol{\sigma}_{ij} \mathbf{n_i} = \mathbf{C}_{ijkl} \mathbf{x}_{\mathbf{s}_{k,l}} \mathbf{n_i} = -p \boldsymbol{\delta}_{ij} \mathbf{n_i} & \text{sur } \Sigma \\ \dfrac{\partial p}{\partial n} = \rho_f \omega^2 \mathbf{x}_{\mathbf{f}_i} \mathbf{n_i} & \text{sur } \Sigma \text{ et sur } SL \\ p = \rho_f g z & \text{sur } SL \end{cases}$$

To express the functional calculus, one uses the constitutive law on free surface. By considering an acceptable field of displacement dz one obtains [bib2]:

$$\int_{SL} \rho g z \delta z \, ds = \int_{SL} p \delta z \, ds$$

Maybe, finally, the functional calculus of the total system fluid structure subjected to gravity:

$$\mathrm{L}(\mathbf{x_s}, p, z) = \frac{1}{2} \int_{V_s} \left[\boldsymbol{\sigma}_{ij}(\mathbf{x_s}) \boldsymbol{\varepsilon}_{ij}(\mathbf{x_s}) - \rho_s \omega^2 \mathbf{x_s^2} \right] - \int_{\Sigma} p \mathbf{x_s} . \mathbf{n} \, d\Sigma + \frac{1}{2} \rho_f ¿ \int_{V_f} \left[\frac{1}{\omega^2} (\text{grad } p)^2 - \frac{p^2}{c^2} \right] dV + \frac{1}{2} \int_{SL} \rho g z^2 dS - \int_{SL} p z \, dS$$

This taking into account of gravity implies two additional terms in the functional calculus describing the fluid:

- a term of potential energy related to free surface: $\frac{1}{2}\int_{SL} \rho g z^2 ds$
- a term due to the work of the hydrodynamic pressure in the displacement of free surface: $\int_{SL} p z ds$

However it should be noted that it is not the single effect of gravity since in any point of the wall Σ is exerted a permanent pressure $-\rho g z$ (where z is the altitude of the point M considered: one supposes that $z=0$ on the level of free surface to the equilibrium). The point M is actuated by an infinitesimal X_s movement, the surface element $d\Sigma$ thus varies and the force due to the permanent pressure too. This force is responsible for an additional term of stiffness being added to the structure stiffness in the system. It could cause a buckling of structure by cancelling the structural stiffness. This effect is negligible on the vibratory characteristics ([bib2], [bib1]), one thus does not take it into account.

3.2.2 Discretization by finite elements

to obtain the discretized form of the functional calculus, one replaces each integral by a sum of integrals on each element i of the discretized system, then one uses an approximation by finite elements of the unknown functions of displacement and pressure on each element i [bib18].

The unknowns are $\mathbf{X_s}(u, v, w), p, z$, one has then by posing Ni the shape functions (or nodal interpolation functions on the element i):

$$\begin{cases} \boldsymbol{\sigma}=\mathbf{D}\,\boldsymbol{\varepsilon} \\ \boldsymbol{\varepsilon}=\mathbf{B}\,\mathbf{u} \end{cases} \quad \begin{cases} \mathbf{x_s}=\mathbf{N_i}\,\mathbf{u} \\ \mathbf{p}(x, y, z)=\mathbf{N_i}\,\mathbf{p} \\ \nabla\,\mathbf{p}=\overline{\mathbf{N_i}}\,\mathbf{p} \end{cases} \quad \begin{cases} \mathbf{x_s}.\mathbf{n}=\mathbf{N}_{\Sigma i}\,\mathbf{u} \\ \mathbf{z}=\mathbf{N}_{Si}\,\mathbf{z} \end{cases}$$

where δ, p, are the unknowns with the nodes structures and the fluid nodes, and the z unknown at free surface.

From where the discretized statement of the functional calculus associated with problem:

$$\mathrm{L}=\mathbf{u^t}(\mathbf{K}-\omega^2\mathbf{M})\mathbf{u}+\mathbf{p^t}(\frac{\mathbf{H}}{\rho_f\omega^2}-\frac{\mathbf{Q}}{\rho_f c^2})\mathbf{p}+\mathbf{z^t}\rho_f g\,\mathbf{K_z}\mathbf{z}-2\,\mathbf{p^t}\mathbf{M_z}\mathbf{z}-2\,\mathbf{p^t}\mathbf{C}\,\mathbf{u}$$

with

$\mathbf{K}=\sum_i \int_{Vi} \mathbf{N_i^t}\mathbf{B_i^t}\mathbf{D_i}\mathbf{B_i}\mathbf{N_i}\,dV_i$ matrix stiffness of the structure

$\mathbf{M}=\sum_i \int_{Si} \mathbf{N_i^t}\rho_f\mathbf{N_i}\,dV_i$ mass matrix of the structure and

$$\mathbf{Q}=\sum_i \int_{Vi_{fl}} \mathbf{N_i^t}\mathbf{N_i}\,dV_i \qquad \mathbf{M_z}=\sum_i \int_{Si} \mathbf{N_{\Sigma i}^t}\mathbf{N_{\Sigma i}}\,dS_i$$

$$\mathbf{K_z}=\sum_i \int_{Si} \mathbf{N_{Si}^t}\mathbf{N_{Si}}\,dS_i \qquad \mathbf{M_z}=\sum_i \int_{Si} \mathbf{N_i^t}\mathbf{N_{Si}}\,dS_i$$

$$\mathbf{H}=\sum_i \int_{Vi_{fl}} \overline{\mathbf{N_i^t}}\,\overline{\mathbf{N_i}}\,dV_i$$

where c is the speed of sound in the fluid, ρ_f the density of the fluid and where $\mathbf{K_f}$ corresponds to the potential energy of the fluid, $\mathbf{K_z}$ with the potential energy of free surface, $\mathbf{H}$ the kinetic energy of the fluid, $\mathbf{M}$ the coupling fluid-solid and $\mathbf{M_z}$ the coupling $p-z$ on free surface.

The approximation by finite elements of the complete problem leads then to the following matric system:

$$\begin{bmatrix} \mathbf{K} & -\mathbf{C} & 0 \\ 0 & \mathbf{H} & 0 \\ 0 & -\mathbf{M_z} & \mathbf{K_z} \end{bmatrix} \begin{Bmatrix} \mathbf{u} \\ \mathbf{p} \\ \mathbf{z} \end{Bmatrix} - \omega^2 \begin{bmatrix} \mathbf{M} & 0 & 0 \\ \rho_f \mathbf{C} & \dfrac{\mathbf{Q}}{c^2} & \rho_f \mathbf{M_z} \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \mathbf{u} \\ \mathbf{p} \\ \mathbf{z} \end{Bmatrix} = 0$$

The first equation corresponds to the motion of structure subjected to the compressive forces, the second with that of the motion of the fluid coupled with structure and at free surface, the third is the equation of free surface.

However the written problem of the kind has asymmetric mass matrixes and stiffness what prevents the use of the classical algorithms of resolution of *Code_Aster*.

3.2.3 Introduction of an additional variable

to make the problem symmetric and to be able to use the classical methods of resolution, one introduces an additional variable: potential of displacements in the fluid j [bib2].

$$\mathbf{X_f}=\mathrm{grad}\,\varphi \text{ i.e formula } \rho\,\omega^2\varphi=p$$

unknown additional is related to the unknowns of the problem, which leads to a stiffness matrix singular. One

reformulates the problem coupled heavy structure-fluid: What

$$\begin{cases} \mathbf{C}_{ijkl}\,\mathbf{x}_{\mathbf{s}_{k,lj}}+\omega^2\rho_s\mathbf{x}_{\mathbf{s}_i}=0 & \text{dans } V_s \\ \Delta\varphi+\dfrac{\omega^2}{\rho_f c^2}\,p=0 & \text{dans } V_f \\ p=\rho_f\omega^2\varphi & \text{dans } V_f \\ \boldsymbol{\sigma}_{ij}\,\mathbf{n_i}=\mathbf{C}_{ijkl}\,\mathbf{x}_{\mathbf{s}_{k,l}}\mathbf{n_i}=-\rho_f\,\omega^2\varphi\,\boldsymbol{\delta}_{ij}\,\mathbf{n_i} & \text{sur } S \\ \dfrac{\partial\varphi}{\partial n}=\mathbf{x}_{\mathbf{f}_i}\mathbf{n_i} & \text{sur } S \\ p=\rho_f\,g\,z & \text{sur } SL \end{cases}$$

leads to the functional calculus of the coupled system: Maybe

$$L(\mathbf{x_s}, p, j, z) = \frac{1}{2}\int_{V_s}\left[\boldsymbol{\sigma}_{ij}(\mathbf{x_s})\boldsymbol{\varepsilon}_{ij}(\mathbf{x_s}) - \rho_s\omega^2\mathbf{x_s^2}\right] + \frac{1}{2}\int_{V_f}\frac{p^2}{c^2}dV + \frac{1}{2}\int_{SL}\rho g z^2 ds$$
$$-\omega^2\left[\int_{\Sigma}\rho_f\varphi\,\mathbf{x_s}\mathbf{n}\,d\Sigma + \int_{SL}\rho_f\varphi z\,ds + \int_{V_f}\left[\frac{\rho_f}{2}(\text{grad}\,\varphi)^2 + \frac{p\varphi}{c^2}\right]dV\right]$$

while discretizing: What

$$\mathrm{L} = \boldsymbol{\delta}^{\mathbf{t}}(\mathbf{K} - \omega^2\mathbf{M})\boldsymbol{\delta} + \frac{1}{\rho_f c^2}\mathbf{p}^{\mathbf{t}}\mathbf{Q}\,\mathbf{p} + \rho_f g\,\mathbf{z}^{\mathbf{t}}\mathbf{K_z}\mathbf{z}$$
$$-2\omega^2\left[\frac{\rho_f}{2}\boldsymbol{\varphi}^{\mathbf{t}}\mathbf{H}\,\boldsymbol{\varphi} + \rho_f\boldsymbol{\varphi}^{\mathbf{t}}\mathbf{C}\,\boldsymbol{\delta} + \rho_f\boldsymbol{\varphi}^{\mathbf{t}}\mathbf{M_z}\mathbf{z} + \frac{1}{c^2}\mathbf{p}^{\mathbf{t}}\mathbf{Q}\boldsymbol{\varphi}\right]$$

is written, in matric form: Establishment

$$\begin{bmatrix}\mathbf{K} & 0 & 0 & 0\\ 0 & \frac{\mathbf{Q}}{\rho_f c^2} & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & \rho_f g\,\mathbf{K_z}\end{bmatrix}\begin{bmatrix}\boldsymbol{\delta}\\ \mathbf{p}\\ \boldsymbol{\varphi}\\ \mathbf{z}\end{bmatrix} - \omega^2\begin{bmatrix}\mathbf{M} & 0 & \rho_f\mathbf{C} & 0\\ 0 & 0 & \frac{\mathbf{Q}}{c^2} & 0\\ \rho_f\mathbf{C}^{\mathbf{t}} & \frac{\mathbf{Q}^{\mathbf{t}}}{c^2} & \rho_f\mathbf{H} & \rho_f\mathbf{M_z}\\ 0 & 0 & \rho_f\mathbf{M_z^t} & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{\delta}\\ \mathbf{p}\\ \boldsymbol{\varphi}\\ \mathbf{z}\end{bmatrix} = 0$$

4 in Code_Aster

the library of the finite elements of Code_Aster was enriched by five isoparametric surface elements having like degrees of freedom the deflection of free surface and the potential of displacements of the fluid at free surface. They are compatible with the elements 3D which deal with the problem of coupling fluid/structure [bib3] One

names: MEFP

_FACE3 and MEFP _FACE6 respectively triangles with 3 or 6 nodes, MEFP _FACE4, MEFP _FACE8 and MEFP _FACE9 respectively quadrangles with 4,8 or 9 nodes. These

elements belong to modelization 2D_FLUI_PESA of the MECHANICAL phenomenon . Bibliography

5 J.

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6 of the versions of the document Version

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