Constitutive law élasto (visco) plastic in large deformations with metallurgical transformations

Abstract

This document presents a model of thermo-élasto behavior (visco) plastic to isotropic hardening with effects of the metallurgical transformations writes in large deformations. This model can be used for three-dimensional, axisymmetric modelizations and in plane strains.

One presents the writing of this model and his digital processing.

To understand this document, it is practically essential to read the two notes [R5.03.21] and [R4.04.02] devoted to the written models of behavior, respectively, in large deformations without metallurgical effects and small strains with metallurgical effects.

Contents

1 Introduction

This document presents a thermo-élasto constitutive law (visco) plastic to isotropic hardening in large deformations which takes into account the effects of the metallurgical transformations. This model can be used for three-dimensional, axisymmetric problems and in plane strains.

This model represents a "assembly" of two models established in *Code_Aster*, namely a thermoelastoplastic model with isotropic hardening written in large deformations (factor key word `DEFORMATION : "SIMO_MIEHE"`, cf [R5.03.21]) and a model small strains thermo - élasto- (visco) plastic with effects of the metallurgical transformations (factor key word `"META_P_ ** _**"` or `"META_V_ ** _**"` of `COMP_INCR` of operator `STAT_NON_LINE`). The first model of large deformations was thus wide to take account of the consequences of the metallurgical transformations on the mechanics.

To understand this document, it is practically essential to read reference documents [R5.03.21] and [R4.04.02] which concerns, respectively, the model large deformations without metallurgical effects and the model small strains with metallurgical effects. Nevertheless, to facilitate the reading of this note, we make some recalls on these two models.

To justify the extension of the model written in large deformations to the model large deformations with metallurgical effects, we take again some theoretical aspects extracted from [bib1] related to the writing the model large deformations.

One presents then the behavior models of the complete model, his numerical integration and the forms of the tangent matrix (options `FULL_MECA` and `RIGI_MECA_TANG`).

2 Notations

One will note by:

$\mathbf{Id}$	stamp identity
$\mathrm{tr}\,A$	traces tensor $\mathbf{A}$
A^T	transposed of the determining $\mathbf{A}$
$\det A$	tensor of $\mathbf{A}$
$\langle X \rangle$	positive part of X
$\tilde{A}$	deviatoric part of the tensor $\mathbf{A}$ defined by $\tilde{A}=A-(\frac{1}{3}\mathrm{tr}\,A)\mathbf{Id}$
$:$	doubly contracted product: $A:B=\sum_{i,j} A_{ij}B_{ij}=\mathrm{tr}(\mathbf{AB^T})$
$\ddot{A}$	tensor product: $(A\ddot{A}B)_{ijkl}=A_{ij}B_{kl}$
A_{eq}	equivalent value of von Mises defined by $A_{eq}=\sqrt{\frac{3}{2}\tilde{A}:\tilde{A}}$
$\tilde{N}_X A$	gradient: $\tilde{N}_X A=\frac{\partial A}{\partial X}$
$\mathrm{div}_x A$	divergence: $(\mathrm{div}_x A)_i=\sum_j \frac{\partial A_{ij}}{\partial x_j}$
λ , μ	coefficients of Lamé: $\lambda=\frac{E\nu}{(1+\nu)(1-2\nu)}$, $m=\frac{E}{2(1+\nu)}$
E	Young modulus
ν	Poisson's ratio
K	modulates stiffness with compression: $3\mathrm{K}=3\lambda+2\mathrm{m}=\frac{E}{(1-2\nu)}$
T	temperature
T_{ref}	reference temperature
Z_g	proportion of austenite
Z_i	proportion of the four phases α : ferrite, pearlite, bainite and martensite

In addition, in the frame of a discretization in time, all the quantities evaluated at previous time are subscripted by $^-$, the quantities evaluated at time $t+\Delta t$ are not subscripted and the increments are indicated par. Δ One has as follows:

$$\Delta Q=Q-Q^-$$

3 Recalls of the metallurgical model and the Model model

3.1 large deformations with metallurgical transformations

We present only here the consequences of the metallurgical transformations on the structural mechanics behavior.

The determination of the mechanical evolution associated with a process bringing into play metallurgical transformations requires a thermo-metallurgical computation as a preliminary. This thermo-metallurgical computation is decoupled and allows the determination of the thermal evolutions then metallurgical. For the metallurgical models of behavior of steels, one will be able to consult the note [R4.04.01].

For the study of the metallurgical transformations of steel, there exist five metallurgical phases: ferrite, pearlite, the bainite, martensite (phases α) and austenite (phase γ).

The effects of the metallurgical transformations (at the solid state) are of four types:

- the mechanical characteristics of the material which undergoes the transformations are modified. More precisely, the elastic characteristics (modulus of YOUNG E and Poisson's ratio ν) are not very affected whereas the plastic characteristics, such as the elastic limit, are it strongly,
- the expansion or the voluminal contraction which accompanies the metallurgical transformations translates by a strain (spherical) of "transformation" which is superimposed on the purely thermal strain of origin. In general, one gathers this effect with that due to the modification of the thermal coefficient of thermal expansion α ,
- a transformation being held under stresses can give rise to an unrecoverable deformation and this, even for levels of stresses much lower than the elastic limit of the material. One calls "plasticity of transformation" this phenomenon. The total deflection ε is written then:

$$\varepsilon=\varepsilon^{e}+\varepsilon^{th}+\varepsilon^{p}+\varepsilon^{pt}$$

  where ε^{e} ε^{th} , ε^{p} and ε^{pt} are, respectively the elastic strain, thermal, plastic and of plasticity of transformation,
- one can have at the time of the metallurgical transformation a phenomenon of restoration of hardening. The hardening of the mother phase is not completely transmitted to the phases lately created. Those can then be born with a virgin state of hardening or inherit only one part, even totality, hardening of the mother phase. The cumulated plastic strain p is not then any more characteristic of the state of hardening and it is necessary to define other variables of hardening for each phase, noted r_k which take account of the restoration. The laws of evolution of these hardenings differ from the usual models so as to allow a "return towards zero" total, or partial, of these parameters during the transformations.

One will be able to find in the document [R4.04.02] the forms of the various behavior models.

3.2 Model writes in large deformations

3.2.1 general Presentation

This model is a thermoelastoplastic eulerian constitutive law written in large deformations which was proposed by Simo and Miehe ([bib2]) which tends under the assumption of the small strains towards the model with isotropic hardening and criterion of von Mises describes in [R5.03.02]. It makes it possible to treat not only the large deformations, but also, in an exact way, large rotations.

The essential characteristics of this model are the following ones:

- just like in small strains, one supposes the existence of a slackened configuration, i.e. locally free of stress, which makes it possible to break up the total deflection into a thermo-elastic part and a plastic part,
- the decomposition of this thermo-elastic strain into cubes parts and plastic is not additive any more as in small strains (or for the model large deformations written in strain rate with for example a derivative of Jaumann) but multiplicative,
- as in small strains, the stresses depend only on the thermo-elastic strains,
- to write the constitutive law, one uses the tensor of the stresses of Kirchhoff τ which is connected to the tensor of Cauchy σ by the relation $J\boldsymbol{\sigma}=\boldsymbol{\tau}$ where J the variation represents of volume between the configurations initial and current,
- plastic strains are done with constant volume. The variation of volume is then only due to the thermoelastic strains,
- this model during leads its numerical integration to a model incrémentalement objective what makes it possible to obtain the exact solution in the presence of large rotations.

3.2.2 Kinematics

We make here some basic recalls of mechanics in large deformations and on the model of behavior.

Let us consider a solid subjected to large deformations. That is to say Ω_0 the field occupied by solid before strain and $\Omega(t)$ the field occupied at time t by deformed solid. In the initial configuration Ω_0, the position of any particle of solid is indicated by $\mathbf{X}$ (Lagrangian description). After strain, the position at the time t of the particle which occupied the position $\mathbf{X}$ before strain is given by the variable $\mathbf{x}$ (eulerian description).

The total motion of solid is defined, with $\mathbf{u}$ displacement, by:

$$\mathbf{x}=\hat{\mathbf{x}}(\mathbf{X},t)=\mathbf{X}+\mathbf{u}$$

To define the change of metric in the vicinity of a point, the tensor gradient of the transformation is introduced $\mathbf{F}$:

$$\mathbf{F}=\frac{\partial\hat{\mathbf{x}}}{\partial\mathbf{X}}=\mathbf{Id}+\nabla_X\mathbf{u}$$

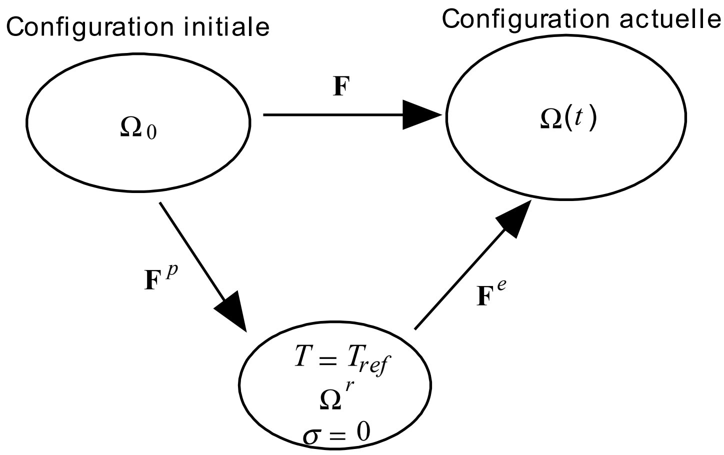
The transformations of the volume element and the density are worth:

$$d\,\Omega = J d\,\Omega_o \text{ with } J = \det \mathbf{F} = \frac{\rho_o}{\rho}$$

where ρ_o and ρ are respectively the density in the configurations initial and current.

To write large deformations now the model, one supposes the existence of a slackened configuration Ω^r , i.e. locally free of stress, which makes it possible then to break up the total deflection into cubes parts thermo-elastic and plastic, this decomposition being multiplicative.

One will note by $\mathbf{F}$ the tensor gradient which makes pass from the initial configuration Ω_0 to the present configuration $\Omega(t)$, by $\mathbf{F^p}$ the tensor gradient which makes pass from the configuration Ω_0 to the slackened configuration Ω^r , and $\mathbf{F^e}$ of the configuration Ω^r with $\Omega(t)$. The index p refers to the plastic part, the index e with the thermo-elastic part.



Appear 3.2.2-a: Decomposition of the tensor gradient $\mathbf{F}$ in an elastic and $\mathbf{F^e}$ plastic part $\mathbf{F^p}$

By composition of motions, one obtains following multiplicative decomposition:

$$\mathbf{F} = \mathbf{F^e}\,\mathbf{F^p}$$

The thermo-elastic strains are measured in the present configuration with the eulerian tensor of left Cauchy-Green $\mathbf{b^e}$ and the plastic strains in the initial configuration by the tensor $\mathbf{G^p}$ (Lagrangian description). These two tensors are defined by:

$$\mathbf{b^e} = \mathbf{F^e}\,\mathbf{F}^{eT} \text{ , } \mathbf{G^p} = (\mathbf{F}^{pT}\,\mathbf{F^p})^{-1} \text{ from where } \mathbf{b^e} = \mathbf{F}\mathbf{G}^p\,\mathbf{F^T}$$

The model presented is written of such way to distinguish the isochoric terms from the terms of change from volume. One introduces for that the two following tensors:

$$\bar{\mathbf{F}} = J^{-1/3}\,\mathbf{F} \text{ and } \bar{\mathbf{b}}^e = J^{-2/3}\,\mathbf{b^e} \text{ with } J = \det \mathbf{F}$$

By definition, one a: $\det \bar{\mathbf{F}}=1$ and $\det \bar{\mathbf{b}}^e=1$.

In this model, plastic strains are done with constant volume so that:

$$J^p=\det \mathbf{F}^\mathbf{p}=1 \text{ from where } J=J^e=\det \mathbf{F}^\mathbf{e}$$

One will find in the reference document ([R5.03.21]) the forms of the behavior models.

4 Extension of the model large deformations

the purpose of this paragraph is to justify the extension of the model written in large deformations to take account of the metallurgical transformations. In particular, to take account of the plasticity of transformation, we cannot add as in small strains an additional term with strain related to plasticity with transformation. In fact, on the kinematical aspect decomposition, the taking into account of the plasticity of transformation does not change anything. There is always decomposition $\mathbf{F}=\mathbf{F}^\mathbf{e}\mathbf{F}^\mathbf{p}$ where $\mathbf{F}^\mathbf{p}$ all information contains on the “unelastic” strain (thus including that related to the plasticity of transformation). It is only on the level behavior that is done, in particular, the processing of the plasticity of transformation.

Initially, we point out some theoretical elements which make it possible to write the model without metallurgical effects then we show the modifications to be made to take account of the metallurgical effects and the plasticity of transformation in particular.

4.1 Thermodynamic aspect

the writing of the constitutive law large deformations is from the thermodynamic frame with local variables. The thermodynamic formalism rests on two assumptions. First is that the free energy depends only on the elastic strain $\mathbf{b}^\mathbf{e}$ and the local variables related to the hardening of the material (here cumulated plastic strain associated with the isotropic variable of hardening R). This allows, thanks to the inequality of Clausius-Duhem, to obtain the state models. The second assumption is the principle of maximum dissipation, which corresponds to the data of a potential of dissipation, which then makes it possible to determine the laws of evolution of the local variables.

The free energy is given by:

$$\psi=\psi(\mathbf{b}^\mathbf{e}, p)=\psi^e(\mathbf{b}^\mathbf{e})+\psi^p(p)$$

One obtains by the first assumption, the state models, that is to say:

$$\boldsymbol{\tau}=2\rho_0\frac{\partial\psi^e}{\partial\mathbf{b}^\mathbf{e}}\mathbf{b}^\mathbf{e} \text{ and } R=\rho_0\frac{\partial\psi^p}{\partial p}$$

It remains for dissipation:

$$\boldsymbol{\tau}:(-\frac{1}{2}\mathbf{F}\dot{\mathbf{G}}^p\mathbf{F}^\mathbf{T}\mathbf{b}^{\mathbf{e}-\mathbf{1}})-R\dot{p}\geq 0$$

With the help of the introduction of a function threshold such as $f(\boldsymbol{\tau}, R) \leq 0$, the principle of maximum dissipation (or an equivalent way the data of a pseudopotential of dissipation [bib3]) makes it possible to deduce some, by the property of normality, the laws of evolution, that is to say:

$$-\frac{1}{2}\mathbf{F}\dot{\mathbf{G}}^{p}\mathbf{F}^{\mathbf{T}}\mathbf{b}^{\mathbf{e-1}}=\dot{\lambda}\frac{\partial f}{\partial \boldsymbol{\tau}} \quad \text{and} \quad \dot{p}=-\dot{\lambda}\frac{\partial f}{\partial R}$$

It is here about a model of associated plasticity.

4.2 Extension

For the restoration of hardening, it does not have there particular difficulties dependant on the large deformations. It is enough that the free energy depends, either to the cumulated plastic strain, but to the local variables of hardening r_k associated with the variables with hardenings $Z_k.R_k$ with each metallurgical phase.

To keep maintaining account of the strains due to the plasticity of transformation, one proposes to add an additional term in the model with flow of the plastic strain $\mathbf{G}^{\mathbf{p}}$ which derives from a potential of dissipation Ω .

One obtains thus for the state models:

$$\boldsymbol{\tau}=2\rho_0\frac{\partial\psi^e}{\partial\mathbf{b}^{\mathbf{e}}}\mathbf{b}^{\mathbf{e}} \quad \text{and} \quad Z_k.R_k=\rho_0\frac{\partial\psi^p}{\partial r_k}$$

for the laws of evolution:

$$-\frac{1}{2}\mathbf{F}\dot{\mathbf{G}}^{p}\mathbf{F}^{\mathbf{T}}\mathbf{b}^{\mathbf{e-1}}=\dot{\lambda}\frac{\partial f}{\partial\boldsymbol{\tau}}+\underbrace{\frac{\partial\Omega^{pt}}{\partial\boldsymbol{\tau}}}_{\text{plasticité de transformation}}$$

$$\dot{r}_k=-\dot{\lambda}\frac{\partial f}{\partial(Z_k.R_k)}-\underbrace{\frac{\partial\Omega^{r}}{\partial(Z_k.R_k)}}_{\substack{\text{restauration d'écrouissage}\\ \text{métallurgique et visqueux}}}$$

$$\Omega=\Omega^{pt}(t)+\Omega^{r}$$

One chooses the potentials Ω^{pt} and Ω^{r} , respectively related on the plasticity of transformation and the restoration of hardening, of such way to find, under the assumption of the small strains, the same laws of evolution that those of the model with metallurgical effects writes in small strains.

4.3 Behavior models

a linear isotropic hardening in the case of is placed.

The partition of the strains implies:

$$\bar{\mathbf{b}}^{e}=\bar{\mathbf{F}}\mathbf{G}^{\mathbf{p}}\bar{\mathbf{F}}^{T} \quad \text{with} \quad \bar{\mathbf{F}}=J^{-1/3}\mathbf{F}\ , \quad J=\det\mathbf{F} \quad \text{and the} \quad \bar{\mathbf{b}}^{e}=J^{-2/3}\mathbf{b}^{\mathbf{e}}$$

behavior models are given by:

- Relation thermo-elastic stress-strain:

$$\tilde{\boldsymbol{\tau}}=\mu\,\bar{\tilde{\mathbf{b}}}^e$$

$$\text{tr}\,\tau=\frac{3\text{K}}{2}(J^2-1)-\frac{9\text{K}}{2}\varepsilon^{th}(J+\frac{1}{J})$$

$$\epsilon^{th}=Z_\gamma\left[\alpha_\gamma(T-T_{ref})-(1-Z_\gamma^r)\Delta\varepsilon_{f\gamma}^{T\,ref}\right]+(\sum_{i=1}^{4}Z_i)\left[\alpha_f(T-T_{ref})+Z_\gamma^r\Delta\varepsilon_{f\gamma}^{Tref}\right]$$

where: Z_γ^r characterize the metallurgical phase of reference

$Z_\gamma^r=1$ when the phase of reference is the austenitic phase,

$Z_\gamma^r=0$ when the phase of reference is the ferritic phase.

$\Delta\varepsilon_{f\gamma}^{T_{ref}}=\varepsilon_f^{th}(T_{ref})-\varepsilon_\gamma^{th}(T_{ref})$ translated the difference in compactness between the phases ferritic and austenitic with the reference temperature T_{ref},

α_f is the coefficient of thermal expansion of the four ferritic phases and α_γ that of the austenitic phase.

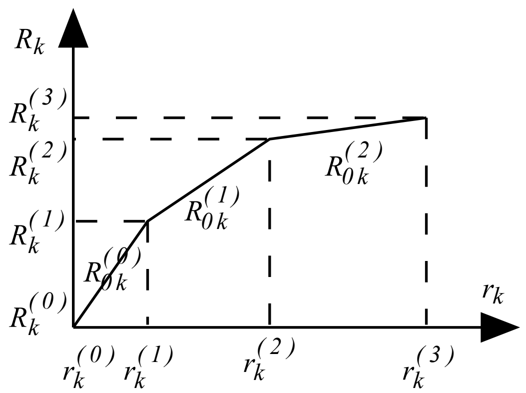
- Threshold of plasticity:

$$f=\tau_{eq}-R-\sigma_y$$

R is the variable of hardening of the multiphase material, which is written:

$$R=(1-\bar{f}(Z))R_\gamma+\frac{\bar{f}(Z)}{Z}\sum_{i=1}^{4}Z_i.R_i\ ,\ Z=\sum_{i=1}^{4}Z_i$$

where R_k is the variable of hardening of the phase k which can be linear or not linear compared to r_k and $\bar{f}(Z)$ a function depending on Z such as $\bar{f}(Z)\in[0,1]$.
In the linear case, one has $R_k=R_{0\text{k}}r_k$ where $R_{0\text{k}}$ is the hardening slope phase k .
In the nonlinear case, one writes: $R_k=R_k^{(i)}+R_{0\text{k}}^{(i)}(r-r_k^{(i)})$ where the meanings of $R_k^{(i)}$, $R_{0\text{k}}^{(i)}$ and $r_k^{(i)}$ are represented on the figure below.



Nonlinear curve of hardening

the elastic limit σ_y is worth:

If $Z \neq 0 \quad \sigma_y = (1-\bar{f}(Z))\sigma_{y\gamma} + \bar{f}(Z)\sigma_{y\alpha}$, $\sigma_{y\alpha} = \frac{\sum_{i=1}^{4} Z_i s_{y\alpha i}}{Z}$

If $Z=0$, $\sigma_y = \sigma_{y\gamma}$

where $\sigma_{y\alpha i}$ are the four elastic limits of the ferritic phases, $\sigma_{y\gamma}$ that of the austenitic phase.

- Laws of evolution:

$$\bar{\mathbf{F}}\dot{\mathbf{G}}^p\bar{\mathbf{F}}^T = -\dot{p}\frac{3}{\tau_{eq}}\tilde{\boldsymbol{\tau}}\bar{\mathbf{b}}^e - 3\tilde{\boldsymbol{\tau}}\bar{\mathbf{b}}^e \sum_{i=1}^{4} K_i F_i^{'}(1-Z_\gamma)\langle \dot{Z}_i \rangle$$

$$\dot{r}_\gamma = \dot{p} + \frac{\sum_{i=1}^{4} \langle -\dot{Z}_i \rangle (\theta_{i\gamma} r_i - r_\gamma)}{Z_\gamma} \underbrace{-(Cr_{moy})^m}_{\text{uniquement en viscosité}} \qquad \text{si} \quad Z_\gamma > 0$$

$$\dot{r}_i = \dot{p} + \frac{\langle \dot{Z}_i \rangle (\theta_{\gamma i} r_\gamma - r_i)}{Z_i} \underbrace{-(Cr_{moy})^m}_{\text{uniquement en viscosité}} \qquad \text{si} \quad Z_i > 0$$

$$r_{\text{moy}} = \sum_{k=1}^{5} Z_k r_k \quad C = \sum_{k=1}^{5} Z_k C_k \ , \ m = \sum_{k=1}^{5} Z_k m_k$$

where K_i $F_i^{'}$, C_i and m_i are data of the material associated with the phase i , $\theta_{\gamma i}$ the coefficient of restoration of hardening at the time of the transformation γ in i ($\theta_{\gamma i} \in [0,1]$) and $\theta_{i\gamma}$ the coefficient of restoration of hardening at the time of the transformation i in γ ($\theta_{i\gamma} \in [0,1]$).

All the material characteristics are indicated in operator `DEFI_MATERIAU` ([U4.43.01]) under different the key word factors `ELAS_META (_F0)` and `META_ **`.

For a model of plasticity, the plastic multiplier is obtained by writing the condition of coherence $\dot{f}=0$ and one a:

$$\dot{p} \geq 0, f \leq 0 \text{ et } \dot{p} f = 0$$

In the viscous case, $\dot{p}$ is written:

$$\dot{p} = \left(\frac{\langle f \rangle}{h} \right)^n$$

or in an equivalent way:

$$\langle f \rangle = (1-\bar{f}(Z))\eta_\gamma \dot{p}^{1/n_\gamma} + \frac{\bar{f}}{Z}\sum_{i=1}^{4} Z_i \eta_i \dot{p}^{1/n_i}$$

where n_i and η_i are the viscosity coefficients of the material associated with the phase i which possibly depend on the temperature.

The computation of $\bar{\mathbf{F}}\dot{\mathbf{G}}^p\bar{\mathbf{F}}^T$ gives:

$$\bar{\mathbf{F}}\dot{\mathbf{G}}^p\bar{\mathbf{F}}^T=-3(A\tau_{eq}+\dot{p})(\frac{1}{3}\operatorname{tr}\bar{\mathbf{b}}^e\frac{\tilde{\boldsymbol{\tau}}}{\tau_{eq}}+\frac{\tau_{eq}}{\mu}\frac{\tilde{\boldsymbol{\tau}}\tilde{\boldsymbol{\tau}}}{t_{eq}^2})$$

where one posed $A=\sum_{i=1}^{4}K_i F_i^{'}\langle\dot{Z}_i\rangle$.

Since $\|\tilde{\boldsymbol{\tau}}/\tau_{eq}\|\leq 1$ and $\|\tilde{\boldsymbol{\tau}}\tilde{\boldsymbol{\tau}}/\tau_{eq}^2\|\leq 1$, the second term of the statement above can be neglected (in front of 1) for metallic materials insofar as:

$$\frac{\tau_{eq}}{\mu}=\frac{R+\sigma_y}{\mu}\approx 10^{-3}\ll 1\leq\operatorname{tr}\bar{\boldsymbol{b}}^e$$

$\operatorname{tr}\bar{\mathbf{b}}^e\geq 1$ because the tensor $\bar{\mathbf{b}}^e$ is symmetric, definite positive and $\det\bar{\mathbf{b}}^e=1$.

It is this simplification of the law of evolution of $\mathbf{G^p}$ which makes it possible to integrate the constitutive law easily i.e. to bring back it to the solution of a nonlinear scalar equation. One will thus take thereafter:

$$\bar{\mathbf{F}}\dot{\mathbf{G}}^p\bar{\mathbf{F}}^T\approx-\left(\dot{p}+\tau_{eq}A\right)\frac{\operatorname{tr}\bar{\boldsymbol{b}}^e}{\tau_{eq}}\tilde{\tau}$$ éq the 4.3-1

4.4 various relations

In operator `STAT_NON_LINE`, one by means of reaches these various models the key word following factors:

```
| COMP_INCR: (
                 RELATION:          /"META_P_IL"
                                    /"META_P_INL"
                                    /"META_P_IL_PT"
                                    /"META_P_INL_PT"
                                    /"META_P_IL_RE"
                                    /"META_P_INL_RE"
                                    /"META_P_IL_PT_RE"
                                    /"META_P_INL_PT_RE"
                                    /"META_V_IL"
                                    /"META_V_INL"
                                    /"META_V_IL_PT"
                                    /"META_V_INL_PT"
                                    /"META_V_IL_RE"
                                    /"META_V_INL_RE"
                                    /"META_V_IL_PT_RE"
                                    /"META_V_INL_PT_RE"

                 DEFORMATION: /"SIMO_MIEHE"

               )
```

We point out only here the meaning of the letters for behaviors `META` :

- `P_IL` : plasticity with linear isotropic hardening,
- `P_INL` : plasticity with nonlinear isotropic hardening,
- `V_IL` : viscoplasticity with linear isotropic hardening,
- `V_INL` : viscoplasticity with nonlinear isotropic hardening,
- `Pt` : plasticity of transformation,
- `RE` : restoration of metallurgical hardening of origin.

Example: "META_V_INL_RE" = elastoviscoplastic model with nonlinear isotropic hardening with restoration of hardening but without taking into account of the plasticity of transformation

the various characteristics of the material are given in operator DEFI_MATERIAU. One returns the reader to the note [R5.04.02] for the meaning of the key word factors of this operator.

Caution:

If isotropic hardening is linear, one informs under key word META_ECR0_LINE of DEFI_MATERIAU, the hardening modulus i.e. the slope in the stress-strain plane.
On the other hand, if isotropic hardening is nonlinear, one gives directly under key word META_TRACTION of DEFI_MATERIAU, curved isotropic hardening R *(* $R=\tau-\sigma_y$ *) according to the cumulated plastic strain* p *(* $p=\varepsilon-\frac{\tau}{E}$ *).*

Note:

The user must make sure well that "experimental" curve of tension used to deduce some the hardening slope is well given in the plane forced rational $\sigma=F/S$ *- logarithmic strain* $\ln(1+\Delta l/l_0)$ where l_0 *is the initial length of the useful part of the test-tube,* Δl *the variation length after strain,* F *the applied force and* S *current surface. It will be noticed that* $\sigma=F/S=\frac{F}{S_0}\frac{l}{l_0}\frac{1}{J}$ *from where* $\tau=J\sigma=\frac{F}{S_0}\frac{l}{l_0}$ *. In general, it is well the quantity* $\frac{F}{S_0}\frac{l}{l_0}$ *which is measured by the experimenters and this gives the stress of Kirchhoff directly used in the model of Simo and Miehe.*

4.5 Stresses and local variables

the stresses of output are the stresses of Cauchy σ, therefore measured on the present configuration.
For all relations META_ **, the local variables produced in *Code_Aster* are:

- V1 : r_1 variable of hardening for ferrite,
- V2 : r_2 variable of hardening for the pearlite,
- V3 : r_3 variable of hardening for bainite,
- V4 : r_4 variable of hardening for martensite,
- V5 : r_5 variable of hardening for austenite,
- V6 : indicator of plasticity (0 if the last calculated increment is elastic; 1 if not),
- V7 : R the isotropic term of hardening of the function threshold,
- V8: the trace divided by three of the elastic strain tensor $\bar{b}^e$ is $\frac{1}{3}tr\,\bar{\boldsymbol{b}}^e$.

5 Numerical formulation

For the variational formulation, it acts of same as that given in the note [R5.03.21] and which refers to the constitutive law large deformations. We point out only that it is about an eulerian formulation, with reactualization of the geometry to each increment and each iteration, and that one takes account of the stiffness of behavior and the geometrical stiffness.
We present the numerical integration of the constitutive law now and give the form of the tangent matrix (options FULL_MECA and RIGI_MECA_TANG).

5.1 Integration of the various behavior models

In the case of an incremental behavior, factor key word COMP_INCR, knowing the tensor of the stresses $\boldsymbol{\sigma}^-$, the local variables r_k^-, the trace divided by three of the strain tensor elastics $\frac{1}{3} tr\ \bar{\boldsymbol{b}}^e$, displacements $\mathbf{u}^-$ and $\Delta \mathbf{u}$, the temperatures T^- and T, and the proportions of the various metallurgical phases Z_k ΔZ_k, one seeks to determine $(\sigma, r_k, \frac{1}{3} tr\ \bar{\boldsymbol{b}}^e)$.

Displacements being known, the gradients of the transformation of Ω_0 with Ω^-, noted $\mathbf{F}^-$, and of Ω^- with $\Omega(t)$, noted $\Delta \mathbf{F}$, are known.
One will pose thereafter:

$$DA=\sum_{i=1}^{4} K_i F_i' \langle DZ_i \rangle \ , \quad \Delta G_\gamma = \frac{\sum_{i=1}^{4} \langle -\Delta Z_i \rangle (\theta_{i\gamma} r_i^- - r_\gamma^-)}{Z_\gamma} \quad \text{and}$$

$$\Delta G_i = \frac{\langle \Delta Z_i \rangle (\theta_{\gamma i} r_\gamma^- - r_i^-)}{Z_i} \quad (i=1,4)$$

the implicit **discretization** of the model gives:

$$\mathbf{F}=\Delta \mathbf{F}\ \mathbf{F}^-$$
$$J=\det \mathbf{F}$$
$$\bar{\mathbf{F}}=J^{-1/3}\mathbf{F}$$
$$\bar{\mathbf{b}}^e=J^{-2/3}\mathbf{b}^{\mathbf{e}}$$
$$J\boldsymbol{\sigma}=\boldsymbol{\tau}$$
$$\tilde{\boldsymbol{\tau}}=\mu \tilde{\bar{\mathbf{b}}}^e$$
$$\mathrm{tr}\,\boldsymbol{\tau}=\frac{3\mathrm{K}}{2}(J^2-1)-\frac{9\mathrm{K}}{2}\varepsilon^{th}(J+\frac{1}{J})$$
$$\varepsilon^{th}=Z_\gamma\left[\alpha_\gamma(T-T_{ref})-(1-Z_\gamma^r)\Delta\varepsilon_{f\gamma}^{T\,ref}\right]+(\sum_{i=1}^{4} Z_i)\left[\alpha_f(T-T_{ref})+Z_\gamma^r\Delta\varepsilon_{f\gamma}^{T\,ref}\right]$$
$$f=\tau_{eq}-(1-\bar{f})R_\gamma-\frac{\bar{f}}{Z}\sum_{i=1}^{4} Z_i R_i-\sigma_y$$
$$\bar{\mathbf{b}}^e=\bar{\mathbf{F}}\mathbf{G}^{\mathbf{p}}\bar{\mathbf{F}}^T=\bar{\mathbf{F}}\mathbf{G}^{\mathbf{p}-}\bar{\mathbf{F}}^T-\frac{\Delta p\,\mathrm{tr}\,\bar{\mathbf{b}}^e}{\tau_{eq}}\tilde{\boldsymbol{\tau}}-\Delta A\,\mathrm{tr}\,\bar{\boldsymbol{b}}^e\tilde{\boldsymbol{\tau}}$$

So $Z_\gamma>0$ then $\Delta r_\gamma=\Delta p+\Delta G_\gamma-\underbrace{\Delta t(Cr_{moy}^-)^m}_{\text{uniquement en viscosité}}$, if not $r_\gamma^-=0$ and $\Delta r_\gamma=0$

If $Z_i>0$ $\Delta r_i=\Delta p+\Delta G_i-\underbrace{\Delta t(Cr_{moy}^-)^m}_{\text{uniquement en viscosité}}$, if not $r_i^-=0$ and $\Delta r_i=0$

In the resolution of this system, only the deviatoric stress $\tilde{\boldsymbol{\tau}}$ is unknown because the trace of $\boldsymbol{\tau}$ is function only of J (known).

One introduces $\boldsymbol{\tau}^{Tr}$, the tensor of Kirchhoff which results from an elastic prediction (Tr : trial, in English test):

$$\tilde{\boldsymbol{\tau}}^{Tr}=\mu\,\tilde{\bar{\mathbf{b}}}^{eTr}$$

where

$$\bar{\mathbf{b}}^{eTr}=\bar{\mathbf{F}}\,\mathbf{G}^{\mathbf{p}-}\,\bar{\mathbf{F}}^{T}=\Delta\,\bar{\mathbf{F}}\,\bar{\mathbf{b}}^{e-}\,\Delta\,\bar{\mathbf{F}}^{T}\ ,\ \Delta\,\bar{\mathbf{F}}=(\Delta\,J)^{-1/3}\,\Delta\,\mathbf{F}\ \text{ and }\ \Delta\,J=\det(\Delta\,\mathbf{F})$$

One obtains $\bar{\mathbf{b}}^{e-}$ starting from the stresses $\boldsymbol{\tau}^{-}$ by the thermo-elastic relation stress-strain and from the trace of the tensor of the elastic strain.

$$\bar{\mathbf{b}}^{e-}=\frac{\tilde{\boldsymbol{\tau}}^{-}}{\mu^{-}}+\frac{1}{3}\,tr\ \boldsymbol{b}^{\bar{e}-}$$

One obtains for the tensor of Kirchhoff $\tilde{\boldsymbol{\tau}}$:

$$\tilde{\boldsymbol{\tau}}=\mu\,\tilde{\bar{\mathbf{b}}}^{eTr}-\mu\,\Delta\,p\frac{\mathrm{tr}\ \mathbf{b}^{\bar{e}Tr}}{\tau_{eq}}\tilde{\boldsymbol{\tau}}-\mu\,\Delta\,\mathrm{tr}\ \boldsymbol{b}^{\bar{e}Tr}\,\tilde{\boldsymbol{\tau}}$$

If $f<0$, one has then $\Delta\,p=0$ and:

$$\tilde{\boldsymbol{\tau}}=\frac{\tilde{\boldsymbol{\tau}}^{Tr}}{1+\mu\,\Delta\,\mathrm{A}\ \mathrm{tr}\ \boldsymbol{b}^{\bar{e}Tr}}$$

if not one obtains:

$$\mathrm{tr}\ \bar{\boldsymbol{b}}^{e}=\mathrm{tr}\ \boldsymbol{b}^{\bar{e}Tr}$$

$$1+\mu\,\Delta\,p\frac{\mathrm{tr}\ \bar{\boldsymbol{b}}^{eTr}}{\tau_{eq}}$$

¿

$$\tilde{\boldsymbol{\tau}}\left[+\mu\,\Delta\,\mathrm{tr}\ \bar{\boldsymbol{b}}¿¿eTr\right]=\tilde{\tau}^{Tr}$$

By calculating the equivalent stress, one obtains the scalar equation into $\Delta\,p$ following:

$$\tau_{eq}+\mu\,\Delta\,p\,\mathrm{tr}\ \boldsymbol{b}^{\bar{e}Tr}+\mu\,\Delta\,A\,\tau_{eq}\,\mathrm{tr}\ \boldsymbol{b}^{\bar{e}Tr}=\tau_{eq}^{Tr}$$

Statement of τ_{eq} :

In plasticity : $\tau_{eq}=\sigma_y+R' Dp+D(r^- ;T,Z)$
with

$$R'=(1-\bar{f})R_{0\gamma}+\frac{\bar{f}}{Z}\sum_{i=1}^{4} Z_i R_{0i}$$

and $D(r^- ;\mathrm{T,Z})=[1-\bar{f}]R_\gamma(r_\gamma^-+\Delta G_\gamma)+\frac{\bar{f}}{Z}\sum_{i=1}^{4} Z_i R_i(r_i^-+\Delta G_i)$

In viscosity :

$$\tau_{eq}=\sigma_y+R' \Delta p+D(r^- ;T,Z)+(1-\bar{f}(Z))\eta_\gamma(\Delta p/\Delta t)^{1/n_\gamma}+\frac{\bar{f}}{Z}\sum_{i=1}^{4} Z_i \eta_i(\Delta p/\Delta t)^{1/n_i}$$

with

$$D(r^- ;\mathrm{T,Z})=[1-\bar{f}]R_\gamma(r_\gamma^-+DG_\gamma-\Delta t(\mathrm{Cr}_{\mathrm{moy}}^-)^m)+\frac{\bar{f}}{Z}\sum_{i=1}^{4} Z_i R_i(r_i^-+\Delta G_i-\Delta t(\mathrm{Cr}_{\mathrm{moy}}^-)^m)$$

Δp checks:

$$(1-\bar{f}(Z))\eta_\gamma(\Delta p/\Delta t)^{1/n_\gamma}+\frac{\bar{f}}{Z}\sum_{i=1}^{4} Z_i \eta_i(\Delta p/\Delta t)^{1/n_i}=\frac{\tau_{eq}^{Tr}-\mu\Delta p\ \mathrm{tr}\ \bar{\boldsymbol{b}}^{eTr}}{1+\mu\Delta A\,'\mathrm{tr}\ \bar{\boldsymbol{b}}^{eTr}}-D(r^- ;T,Z)-\sigma_y-R'\Delta p$$

The resolution is made in *Code_Aster* by a method of the secants with interval of search [bib4].

Note:

In the case of a nonlinear isotropic hardening, the slopes of hardening R_{0k} *and the hardenings* R_k *in the statements of* R' *and* $D(r^- ;T,Z)$ *correspond to the variables* r_k *taken at time* t *, i.e* $r_k=r_k^-+\Delta G_k+\Delta p-\Delta t(Cr_{moy}^-)^m$ *. However, as one does not know a priori the value of these variables* r_k *, one solves the equation and the* Δp *by taking some* R_{0k} *the slopes hardenings* R_k *for the quantities* $r_k^-+\Delta G_k-\Delta t(Cr_{moy}^-)^m$ *. Once solved the equation in* Δp *, one checks, for each phase, which one is well in the good interval during the computation of hardening and the slope. In the contrary case, for the phases concerned, one takes the following interval and one solves the equation again in* Dp *. One continues this process until finding the good interval for all the phases.*

One finds then for the deviator of the stresses:

$$\tilde{\boldsymbol{\tau}}=\frac{1}{1+\mu\,\Delta A\ \mathrm{tr}\ \bar{\mathbf{b}}^{eTr}}\left[1-\mu\frac{\Delta p\ \mathrm{tr}\ \bar{\mathbf{b}}^{eTr}}{\tau_{eq}^{Tr}}\right]\tilde{\boldsymbol{\tau}}^{Tr}$$

Once calculated the cumulated plastic strain, the tensor of the stresses and the tangent matrix, one carries out a correction on the trail of tensor of the elastic strain $\bar{\mathbf{b}}^e$ to take account of the plastic incompressibility, which is not preserved with the simplification made on the flow model [éq 4.3.1]. This correction is by means of carried out a relation between the invariants of $\bar{\mathbf{b}}^e$ and $\tilde{\bar{\mathbf{b}}}^e$ by exploiting the plastic condition of incompressibility $J^p=1$ (or in an equivalent way $\det \bar{\mathbf{b}}^e=1$). This relation is written:

$$x^3-\bar{J}_2^e x-(1-\bar{J}_3^e)=0$$

with $\bar{J}_2^e=\frac{1}{2}(\tilde{\bar{b}}^e)_{\text{eq}}^2=\frac{(\tau_{\text{eq}})^2}{2(\mu)^2}$, $\bar{J}_3^e=\det \tilde{\bar{\boldsymbol{b}}}^e=\det \frac{\tilde{\boldsymbol{\tau}}}{\mu}$ and $x=\frac{1}{3} tr\ \bar{\boldsymbol{b}}^e$

the solution of this cubic equation makes it possible to obtain $tr\ \bar{\boldsymbol{b}}^e$ and consequently the thermo-elastic strain $\bar{\mathbf{b}}^{e-}$ with time step according to. If this equation admits several solutions, one takes the solution nearest to the solution of time step preceding. It is besides why one stores in a local variable $\frac{1}{3} tr\ \bar{\boldsymbol{b}}^e$.

Note:

If the plasticity of transformation is not taken into account, the statements obtained are the same ones while taking $\Delta A=0$.
If it is the restoration of hardening which is neglected then one also has the same statements but by taking all θ the equal ones to 1.

5.2 Form of the tangent matrix

We give only here the forms of the tangent matrix (option `FULL_MECA` during iterations of Newton, option `RIGI_MECA_TANG` for the first iteration). For the assumptions concerning the metallurgical part, they are the same ones as those of the document [R4.04.02]. For the large deformations part, one will find in appendix of [bib1], the detail of the linearization of the constitutive law.

One poses:

$$J=\det\mathbf{F}\ ,\ J^{-}=\det\mathbf{F}^{-}\ \text{ and }\ \Delta J=\det\Delta\mathbf{F}$$

- For the option `FULL_MECA` , one a:

$$\begin{aligned}\bar{\mathbf{A}}=\frac{\partial\boldsymbol{\sigma}}{\partial\Delta\mathbf{F}}=&\frac{(\Delta J)^{-1/3}}{J}\mathbf{H}-\frac{1}{3J\Delta J}(\mathbf{H}\Delta\bar{\mathbf{F}})\otimes\mathbf{B}-\frac{J^{-}}{J^{2}}\boldsymbol{\tau}\otimes\mathbf{B}\\&+\frac{J^{-}}{J}\left[KJ-\frac{3}{2}K\varepsilon^{th}(1-J^{-2})\right]\mathbf{Id}\otimes\mathbf{B}\end{aligned}$$

where $\mathbf{B}$ is worth:

$$\begin{aligned}B_{11}&=\Delta F_{22}\Delta F_{33}-\Delta F_{23}\Delta F_{32}\\B_{22}&=\Delta F_{11}\Delta F_{33}-\Delta F_{13}\Delta F_{31}\\B_{33}&=\Delta F_{11}\Delta F_{22}-\Delta F_{12}\Delta F_{21}\\B_{12}&=\Delta F_{31}\Delta F_{23}-\Delta F_{33}\Delta F_{21}\\B_{21}&=\Delta F_{13}\Delta F_{32}-\Delta F_{33}\Delta F_{12}\\B_{13}&=\Delta F_{21}\Delta F_{32}-\Delta F_{22}\Delta F_{31}\\B_{31}&=\Delta F_{12}\Delta F_{23}-\Delta F_{22}\Delta F_{13}\\B_{23}&=\Delta F_{31}\Delta F_{12}-\Delta F_{11}\Delta F_{32}\\B_{32}&=\Delta F_{13}\Delta F_{21}-\Delta F_{11}\Delta F_{23}\end{aligned}$$

and where $\mathbf{H}$ and $\mathbf{H}\Delta\bar{\mathbf{F}}$ are given by:

In the elastic case ($f<0$):

$$H_{ijkl}=\frac{\mu}{(1+\mu\Delta A\,\mathrm{tr}\,\boldsymbol{b}^{\bar{e}Tr})}(\delta_{ik}\bar{b}^{e-}_{lp}\Delta\bar{F}_{jp}+\Delta\bar{F}_{ip}\bar{b}^{e-}_{pl}\delta_{jk}-\frac{2}{3}\delta_{ij}\Delta\bar{F}_{kp}\bar{b}^{e-}_{lp}-2\Delta A\tilde{\tau}_{ij}\Delta\bar{F}_{kp}\bar{b}^{e-}_{pl})$$

and

$$\mathbf{H}\Delta\bar{\mathbf{F}}=\frac{2\mu}{(1+\mu\Delta A\,\mathrm{tr}\,\boldsymbol{b}^{\bar{e}Tr})}(\tilde{\bar{\mathbf{b}}}^{eTr}-\Delta A\,\mathrm{tr}\,\boldsymbol{b}^{\bar{e}Tr}\tilde{\boldsymbol{\tau}})$$

if not in plastic or viscoplastic load, one a:

$$
\begin{aligned}
H_{ijkl}=&\frac{\mu}{a}(\delta_{ik}\bar{b}^{e-}_{lp}\Delta\bar{F}_{jp}+\Delta\bar{F}_{ip}\bar{b}^{e-}_{pl}\delta_{jk})\\
&-2\mu\left[\frac{\delta_{ij}}{3a}+\frac{\bar{R}'(\tau_{eq}\Delta A+\Delta p)\tilde{\tau}_{ij}}{\tau_{eq}(\bar{R}'+\mu\,\mathrm{tr}\,\bar{\boldsymbol{b}}^{eTr}(1+\bar{R}'\Delta A))}\right]\Delta\bar{F}_{kp}\bar{b}^{e-}_{lp}\\
&+\frac{3\mu^2\,\mathrm{tr}\;\bar{\boldsymbol{b}}^{eTr}(\bar{R}'\Delta p-\tau_{eq})}{a\tau^3_{eq}(\bar{R}'+\mu\,\mathrm{tr}\,\bar{\boldsymbol{b}}^{eTr}(1+\bar{R}'\Delta A))}\tilde{\tau}_{ij}\tilde{\tau}_{kq}\Delta\bar{F}_{qp}\bar{b}^{e-}_{lp}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbf{H}\Delta\bar{\mathbf{F}}=&\frac{2\mu}{a}\bar{\mathbf{b}}^{eTr}-2\mu\,\mathrm{tr}\;\bar{\mathbf{b}}^{eTr}\left[\frac{\mathbf{Id}}{3a}+\frac{\bar{R}'(\tau_{eq}\Delta A+\Delta p)\tilde{\boldsymbol{\tau}}}{\tau_{eq}(\bar{R}'+\mu\,\mathrm{tr}\;\bar{\mathbf{b}}^{eTr}(1+\bar{R}'\Delta A))}\right]\\
&+\frac{3\mu^2\,\mathrm{tr}\,\bar{\mathbf{b}}^{eTr}(\bar{R}'\Delta p-\tau_{eq})}{a\tau^3_{eq}(\bar{R}'+\mu\,\mathrm{tr}\,\bar{\boldsymbol{b}}^{eTr}(1+\bar{R}'\Delta A))}(\tilde{\boldsymbol{\tau}}:\bar{\mathbf{b}}^{eTr})\tilde{\boldsymbol{\tau}}
\end{aligned}
$$

with

$$
\bar{R}'=(1-\bar{f})R_{0\gamma}+\frac{\bar{f}}{Z}\sum_{i=1,4}Z_i R_{0i}+\underbrace{(1-\bar{f}(Z))\eta_\gamma(\Delta p/\Delta t)^{(1-n_\gamma)/n_\gamma}/n_\gamma\Delta t+\frac{\bar{f}}{Z}\sum_{i=1}^{4}Z_i\eta_i(\Delta p/\Delta t)^{(1-n_i)/n_i}/n_i\Delta t}_{\text{uniquement en viscosite}}
$$

$$
a=\frac{\tau^{Tr}_{eq}}{\tau_{eq}}
$$

- For the option `RIGI_MECA_TANG`

  for the model plastic: it is the same statements as those given for `FULL_MECA` but with $\Delta p=0$ and $\Delta A=0$, all the variables and coefficients of the material being taken at time t^- . In particular, one will have $\Delta\bar{\mathbf{F}}=\mathbf{Id}$.

  for the model viscous: one takes only the statements of `FULL_MECA` in the elastic case with $\Delta A=0$, all the variables being taken at time t^- .

6 Bibliography

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7 of the document Version Aster Author

(S) Organization	**(S) Description of the modifications**	**5 V.CANO, E.LORENTZ**
- EDF	- R&D/AMA initial Text 6.3 V	.CANO EDF-R&D/
AMA	Corrections	